Implementation of Sublinear Clustering Algorithm

James Camacho and Joseph Camacho

December 14, 2022

Introduction

A (+/-)-graph contains plus-edges and minus-edges, corresponding to positive or negative correlation between two vertices. For example, in a graph of a thesaurus there would be plus-edges between synonyms and minus-edges between antonyms. The clustering problem involves assigning labels to each vertex, trying to assign more highly correlated vertices with the same label. More precisely, a cost function is introduced, with one unit of cost for every pair of correlated vertices assigned different labels and anticorrelated vertices with the same label. In very large, very dense graphs, simply reading every edge can take a very long time, $O(n^2)$ in the number of vertices. We cannot do better than an $\Omega(n)$ algorithm, as every vertex must be assigned a label. This clustering problem is NP, but polynomial approximation schemes are known to exist. In this paper we implement one such approximation scheme by Assadi and Wang that runs in $\tilde{O}(n)$ time. The running time is $O(n \log^2 n\varepsilon^{-2})$ for some input ε, and it yields an $O(\varepsilon^{-2})$-factor approximation (in terms of cost).

Unfortunately, their theoretical bounds only hold when $\varepsilon < \frac{1}{360}$, which means the algorithm could be off by a factor of 10^5, not a very practical result. We explore empirically what happens for larger ε, and find more efficient and accurate clustering for $\varepsilon > \frac{1}{4}$.

Overview

Intuitively, minus-edges are not as relevant as plus-edges, because they only increase the cost if a cluster has formed, but that will only occur when there are already many plus-edges. The algorithm then only considers the plus-edges. From now on, assume all edges and vertices are from the plus-subgraph. The main steps for the algorithm are as follows:

1. Sample a constant number of edges from each vertex.
2. Choose a random (sublinear) number of vertices and query every edge of these vertices.
3. Filter these vertices using the sampled edges from step one to find the densest ones.
4. Create candidates for the clusters based on the dense vertices and their neighbors.
5. Find approximate clusters from the candidates.

If the constants are chosen right, there should be a high probability that the candidate clusters in step four form what Assadi and Wang call a *laminar collection*, a collection of sets for which any two sets are either disjoint or one is a superset of the other. Then, step five consists of assigning each vertex to the largest candidate it's contained in. If it isn't contained in any candidate clusters, it is assumed to be pretty isolated and assigned to its own cluster.

We can define a cost for the clustering—each time two vertices are put in different clusters but have a plus-edge between them, the cost is increased by one. Similarly if two vertices share a minus-edge but get put in the same cluster, the cost increases by one. It turns out the minus-edges are not too relevant, as two vertices with a minus-edge should only be clustered together if they share a lot of neighbors in the plus-subgraph, in which case the cost would increase whether or not they get clustered together. So, by only using the plus-subgraph, it is possible to determine good clusters with an $O(1)$ approximation to the optimum.

The accuracy of the algorithm depends on the details. How many edges do you sample from each vertex? What is the probability you sample all the edges for a particular vertex? What bounds do you use to filter dense vertices?

The paper suggests sampling $t = O(\varepsilon^{-2} \log n)$ vertices for some small constant ε. The particular ε is never explicitly chosen in their paper, but certain theorems of theirs are proved in the specific case that $\varepsilon < 1/360$.

In step two, a vertex v is chosen with probability $O(\log n / \deg(v))$. Note that this rewards sparser vertices—in a complete graph with n vertices you would only expect $O(\log n)$ vertices, and thus $O(n \log n)$ edges to be sampled, while a tree would have nearly every vertex chosen. Overall, steps one and two only require us to sample $\tilde{O}(n\varepsilon^{-2})$ edges.

The filtering in step three is a little more complicated. First, all vertices that have many neighbors that are denser than them should be filtered out. This is according to the formula

$$\varepsilon \deg(v) < |\{u \text{ neighbors } v : \deg(u) > (1+\varepsilon)\deg(v)\}|.$$

Then, sparse vertices are filtered out. Call a neighbor u of v *low* if

$$\deg(u) < (1+7\varepsilon)\deg(v).$$

A low neighbor u is *isolated* if

$$|\text{sampled neighbors}(u) \cap \text{low}(v)| < (1-4\varepsilon)t,$$

i.e. very few of its neighbors are also low neighbors of v. Note that when $\varepsilon > 0.25$, there are no isolated neighbors. When $\varepsilon < 0.25$, we call a vertex *sparse* if at least $2\varepsilon \deg(v)$ of its low neighbors are isolated. Any not-light and not-sparse vertex should be pretty dense.

From here, we use some more magic numbers to build candidate clusters. For each dense vertex v, we build a candidate set, including a low neighbor u if

$$1 + 22\varepsilon > \frac{\deg(u)}{\deg(v)} > \frac{(1-67\varepsilon)t}{|\text{sampled neighbors}(u) \cap \text{low}(v)|}.$$

Recall that there are t sampled neighbors of u, so the right inequality means that most neighbors of u are also low neighbors of v. In most of our analysis we have $\varepsilon > 1/67$, so this inequality doesn't do much, but it will serve to increase accuracy as ε gets very small.

Assadi and Wang claim that, with high probability, these candidate clusters will form a *laminar set family*, a group of sets where any two elements are either disjoint, or one is a subset of the other. If that holds, then we can take the largest disjoint subsets as the clusters. Any vertices not contained in a subset are assigned as their own cluster.

Our Python implementation can be found on GitHub. To measure operations, we create a counter. Every time a function is called, we add the number of operations that call takes within a constant factor. For example, a set 'x AND y' operation is known to take $O(\min(x, y))$ time, so we would add $\min(x, y)$ to the counter. To measure error ratios, we find an upper bound for the cost when generating the graph and divide the algorithm's output with this upper bound. This allows larger graphs to be comparable to smaller ones. In the case of the thesaurus, where we do not generate the graph, we simply give the cost. Finally, to determine how far off from laminar the candidate clusters are, we add the sizes of the symmetric differences of every pair of vertices ('len(x XOR y)'), and normalize by dividing by total size and number of vertices.

In this paper we explore more how the choice of ε and constant multiplier in $t = O(\varepsilon^{-2} \log n)$ affect the operation count and accuracy of the algorithm. We answer questions like, exactly how close are the candidate clusters to a laminar collection? How many intersections can we expect? Do we *really* need $\varepsilon < 1/360$ in practice, or would larger values like $\varepsilon = 1/64$, $\varepsilon = 1/4$ or $\varepsilon \approx 1$ fare equally well? How do sparser or denser datasets change these results?

Choice of ε

Assadi and Wang's main result is that their algorithm results in a constant-approximation of the optimal clustering in sublinear time (in the number of edges). However, the quality of this constant depends very much on their choice of ε, which as we showed above is around 10^5 when $\varepsilon = 1/360$, the theoretical maximum that guarantees the algorithm works. However, the number of edges we randomly sample per vertex is also proportional to ε^{-2}, so until we have more than $\sim 10^5$ vertices and $\sim 10^{10}$ edges, the algorithm runs as slow as reading every single edge, and still isn't guaranteed to do better than a 10^5 factor approximation. This is terrible!

Luckily, in practice ε can be raised to much larger values while maintaining sublinear time in edges, vastly improving the approximation of the clustering. In order to test this, we use two datasets. The first is a graph of English words with plus-edges between synonyms and minus-edges between antonyms. The second is a random graph with ten thousand vertices, one million edges, and ten clusters; edges within a cluster are assigned a plus while edges between clusters are labelled with a minus, however we randomly flip between 10% and 45% of these correlations.

For both datasets running time improved as ε increased, as fewer edges needed to be considered. However, the thesaurus dataset differed from the random dataset with the errors. As ε transitioned from less than 0.25 to more than 0.25 we saw a pronounced increase in the number of errors on the real-world data, while a marked improvement in accurracy for the random dataset. Note that determining the optimum for the thesaurus dataset is NP-hard, so we instead give the error count.

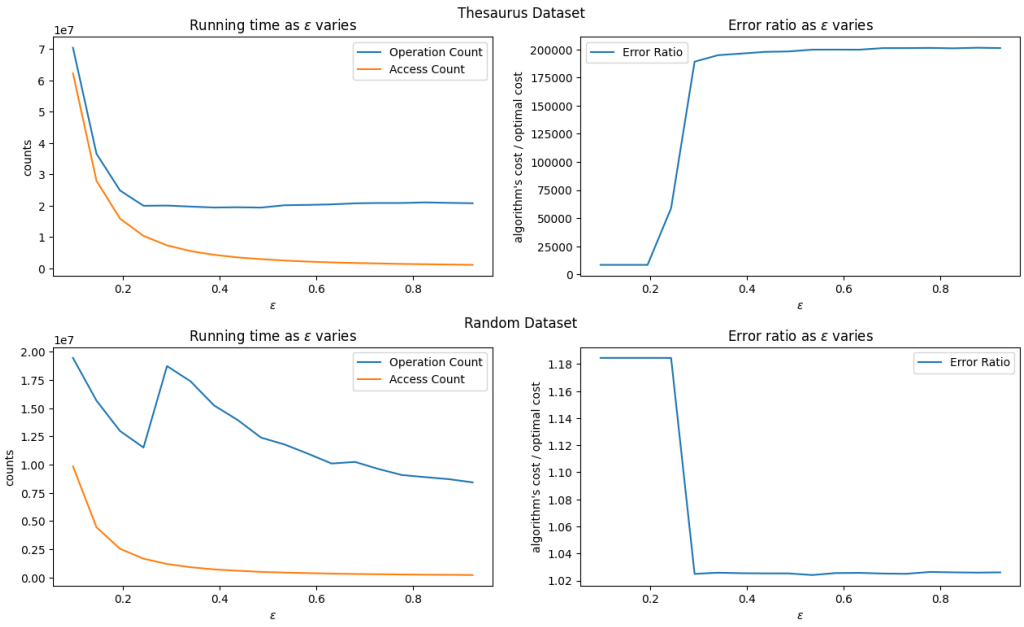


Figure 1: When run on the graph built from a thesaurus (top), the error dramatically increases as ε transitions from less than 0.25 to more than 0.25. The exact opposite trend is observed on a dataset built from random clusters (bottom).

This difference likely arises as there are only ~8,000 antonyms compared to ~200,000 synonyms in the thesaurus. Putting all words into one cluster is guaranteed to have very few errors, while separating into multiple clusters risks synonyms being in different clusters.

For the random dataset, we see this rapid improvement at $\varepsilon = 0.25$ even as we change the fraction of edge correlations that are flipped, how many clusters there are, or the size of the graph. For example, the below graphs show the algorithm run with $\varepsilon = 0.2$ and $\varepsilon = 0.3$, and a varying number of edges, 10% of which are

flipped. Observe that even the least competitive result when $\varepsilon = 0.3$ is still better than the most competitive result when $\varepsilon = 0.2$.

$\varepsilon = 0.2$ $\varepsilon = 0.3$

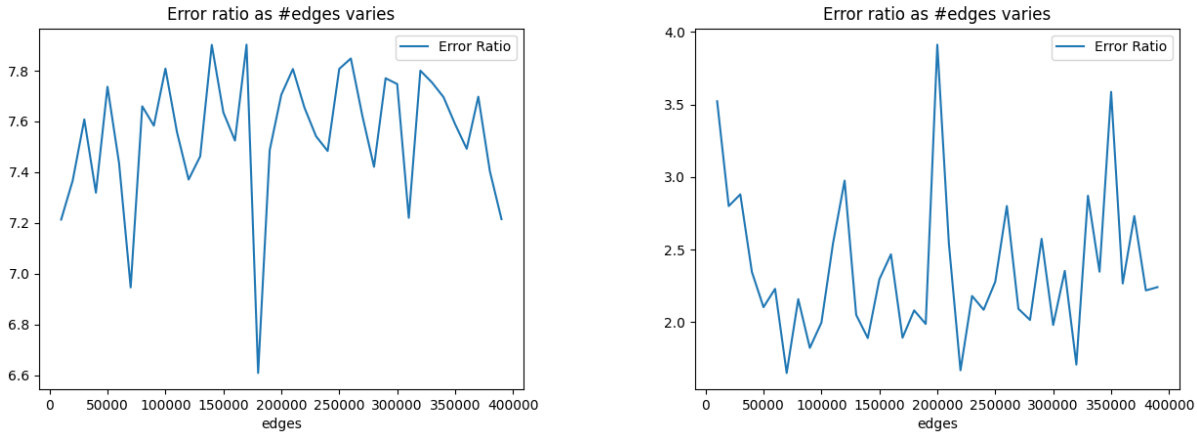


Figure 2: For both $\varepsilon = 0.2$ (left) and $\varepsilon = 0.3$ (right), the algorithm remains about the same competitive as the number of edges increases. However, larger ε does consistently yield a better competitive ratio.

Finally, we note that choosing an extremely small ε gives no improvement on both the randomly constructed graph and the graph based on the thesaurus. This could be because such effects are impossible to observe except in very large graphs, since the number of edges sampled per node is proportional to ε^2. Nevertheless, we think it's plausible that even in large graphs the optimal choice of ε is around 0.25. Whether ε should be above or below 0.25, though, appears to depend on the structure of the graph.

$\varepsilon = 0.2$ $\varepsilon = 0.3$

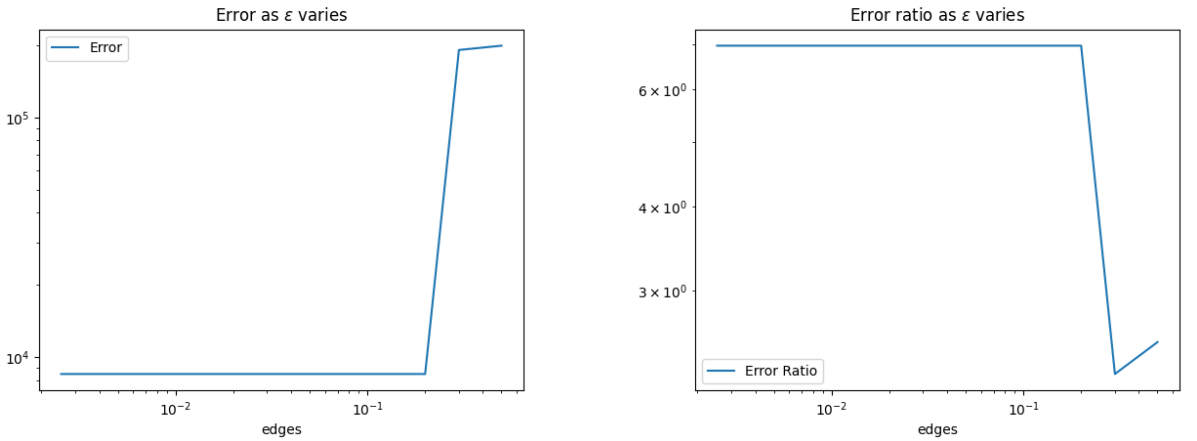


Figure 3: In both the thesaurus graph (left) and random graph (right), the error ratio fails to improve as ε is decreased, even when it passes criticaldatasets, the error ratio is significantly better for large ε For both $\varepsilon = 0.2$ (left) and $\varepsilon = 0.3$ (right), the algorithm remains about the same competitive as the number of edges increases. However, larger ε does consistently yield a better competitive ratio.

Word Vector Dataset

To get a denser model based on real-world data, we create a new dataset using word vectors. It is well known their dot products are a measure of correlation, with very positive dot products being highly correlated while negative dot products are anticorrelated, so we assign plus-edges to the top percentiles of these dot products, and minus-edges to the bottom. Changing this fraction allows for sparser or denser graphs.

For all levels of sparsity, we see an error trend similar to that of the random graphs. Both have optimal clustering when ε is slightly larger than 0.25.

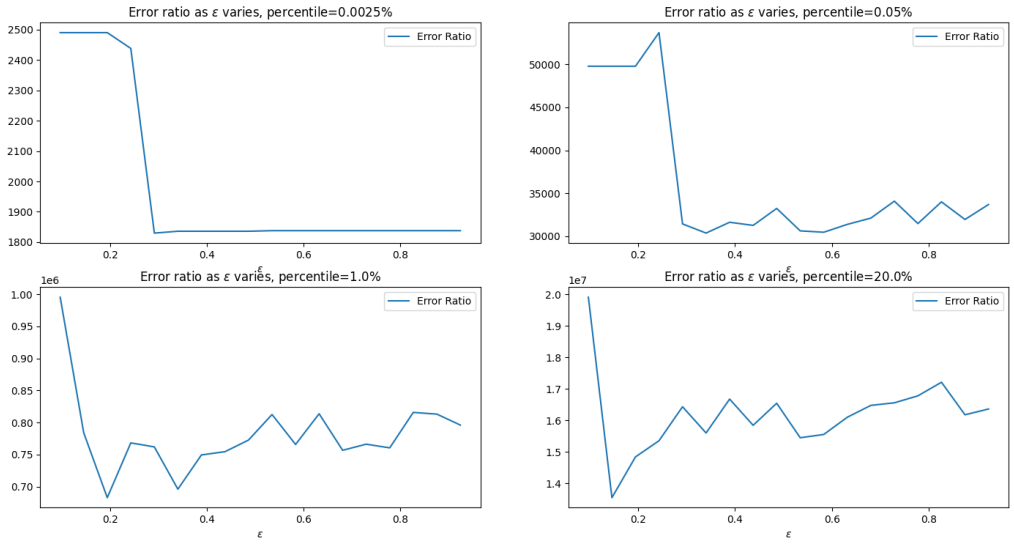


Figure 4: For a wide range of percentiles, the error trend is similar to that of the randomly constructed graphs.

Likewise, the running time improves as ε increases.

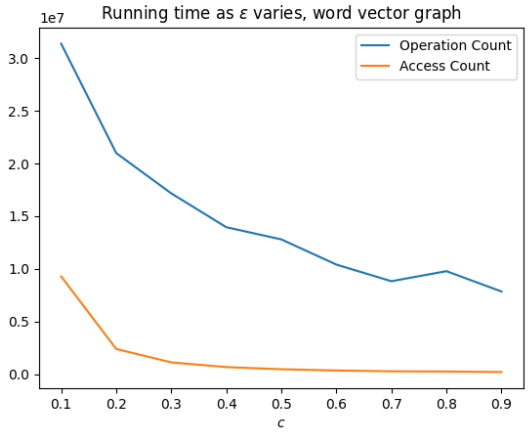


Figure 5: As ε increases, the running time decreases.

This suggests several things, most notably that filtering by isolated neighbors is actually detrimental to the performance of the algorithm! Although it's theoretically necessary to do this filtering to gurantee an $O(1)$ approximation (as the thesaurus dataset somewhat shows), in practice it's not only faster to not do so, but also more accurate.

In conclusion, dense graphs from real world data often look similar to the random dataset we built, and for these types of data, setting ε to be slightly more than 0.25 is a good choice.

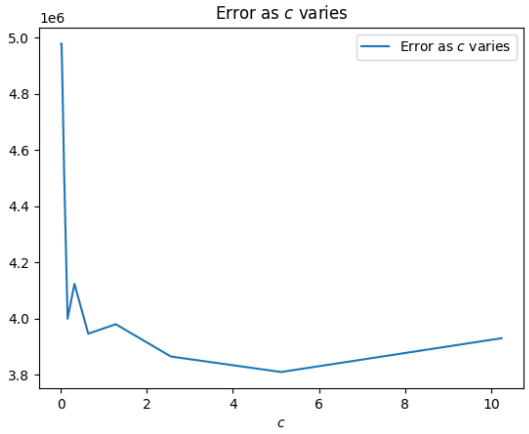
Choice of c

In their paper, Assadi and Wang state that "there is an absolute constant $c > 0$" for which their algorithm works, but they never specify exactly what c has to be. In fact, they only imply what restrictions c might have by constraints on other variables. These constraints imply that c to be on the order of at least ε^2. Since ε^2 is rather small, this means that c can be just about anything.

Nevertheless, setting $c = \varepsilon^2$ is hardly useful. While lowering c speeds up the algorithm by requiring fewer graph accesses, it also lowers its performance because the algorithm has less information to make use of. In this section, we explore the tradeoffs of what c should be, and demonstrate that $c \approx 2$ yields good results with competitive results.

In the previous section, we showed that a good choice of ε is not much more than 0.25. For this section, we set $\varepsilon = 0.3$. We find a steep decline in error until a steady increase after $c > 2$.

Word Vector



Random

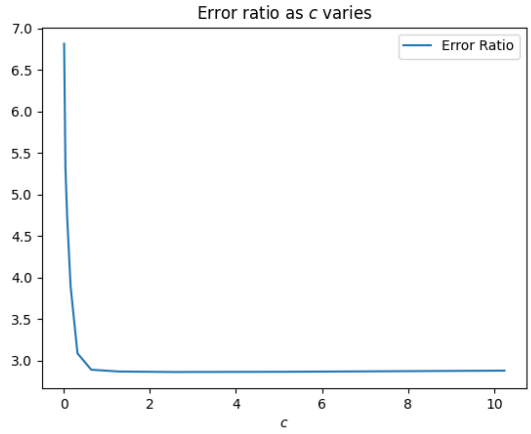


Figure 6: In both the word vector graph (left) and random graph (right), the error ratio has a sharp downturn followed by a very slow increase.

In both cases, the running time also scales linearly with c. A good trade off, then, between runtime and error ratio appears to be setting $c \approx 2$.

Sublinear Time

In their paper, Assadi and Wang mathematically proved that their algorithm runs in sublinear time in the number of edges for small enough ε. Specifically, they showed it ran in $O(|V| \log^2 |V|)$ time, where $|V|$ is the number of vertices of the graph. In this section, we explore whether this same relation holds for much larger ε—ε around 0.25—and a fixed $c = 2$. (Unfortunately, we don't have the compute available to empirically show their bound holds true for $\varepsilon < 1/360$.)

To do this, we observe the runtimes of the algorithm on two graphs with 10,000 vertices and a varying number

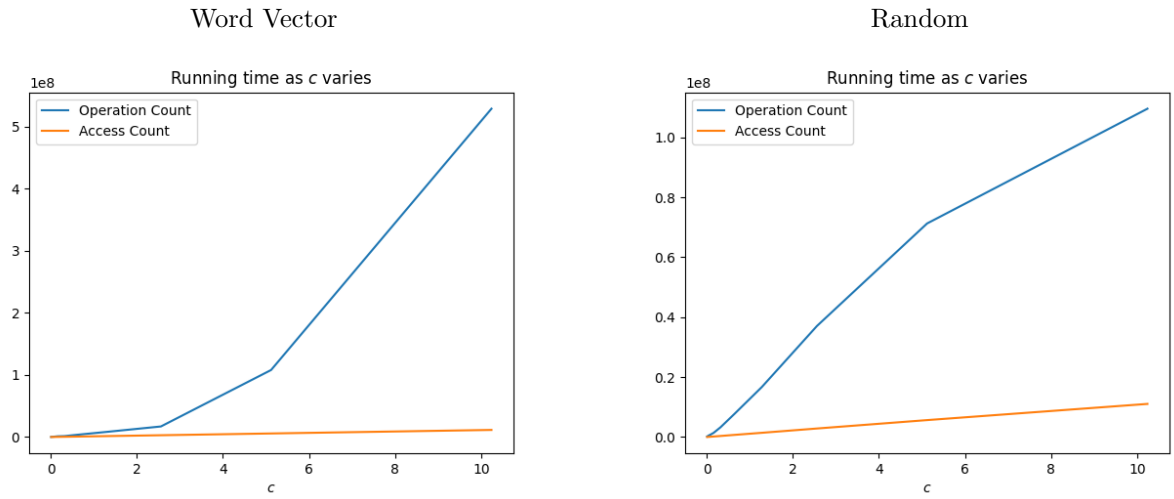


Figure 7: As c varies, the running time increases linearly.

of edges—one random graph and one graph made from word vectors. We see a what might be a concave trend, although it's difficult to tell whether it truly is sublinear at this scale.

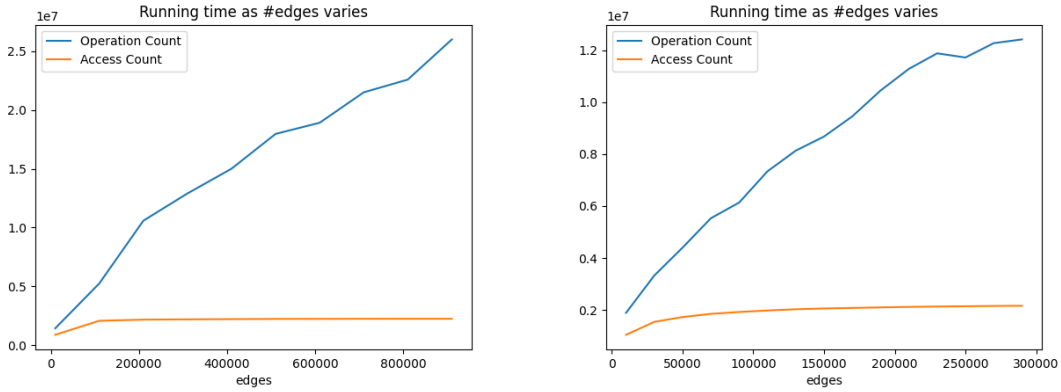


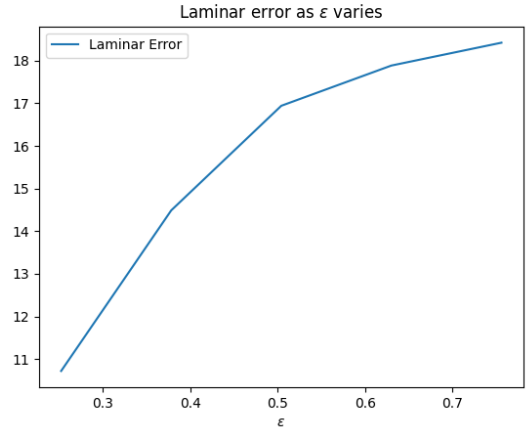
Figure 8: As the number of edges vary in the random graph (left) and word vector graph (right), the running time grows, possibly sub-linearly.

Laminar Candidate Clusters

Assadi and Wang make the claim that the candidate clusters have a high probability of forming a laminar set family. Every pair of candidates should either be disjoint, or one is a subset of the other. Suppose two candidates do not satisfy this laminar property. Their symmetric difference is all elements contained in exactly one subset but not the other, and so the sum of the symmetric differences is a measure of how far off from laminar the candidates are. Normalizing by number of vertices and number of candidate sets gives the graphs in figures 9 and 10.

For both graphs we see an approximately linear increase in laminar error as c increases, and sublinear increase for ε. From our earlier analysis, for graphs of this size we want $\varepsilon \approx 0.3$ and $c \approx 2$, which would give the average vertex at most ten potential clusters, around 1% of the total number of clusters, narrowing down potential clusters tremendously. In our implementation we assigned vertices to one of these clusters arbitrarily, but it would be interesting for future work to create a better algorithm for assigning clusters in the final step.

$c = 2$ is fixed, ε varies.



$\varepsilon = 0.3$ is fixed, c varies.

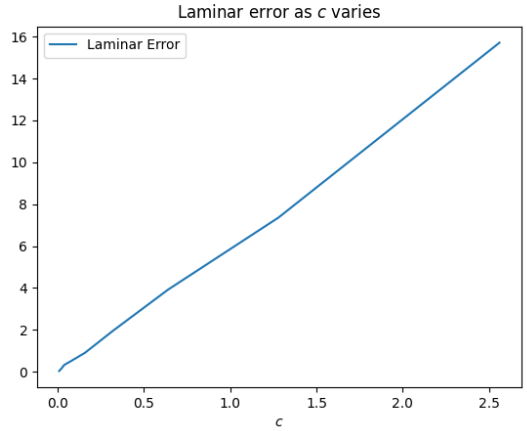
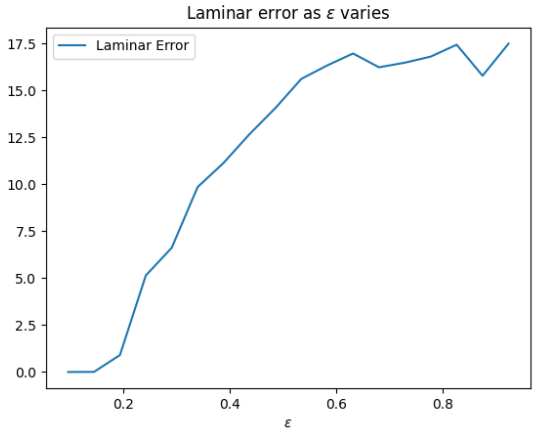


Figure 9: As we increase ε (left) or c (right), the laminar error increases for our random graph with 10^4 vertices, 10^6 edges, ten clusters, and 10% edge flip probability.

$c = 2$ is fixed, ε varies.



$\varepsilon = 0.3$ is fixed, c varies.

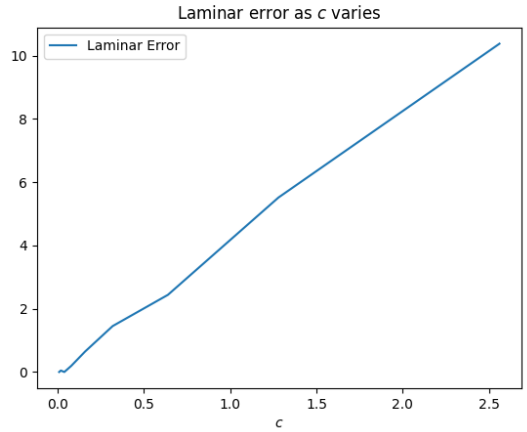


Figure 10: As we increase ε (left) or c (right), the laminar error increases for our word vector graph with 10^4 vertices and $5 \cdot 10^6$ edges.

Conclusion

In this paper, we implemented Sepehr Assadi and Chen Wang's algorithm for correlation clustering. Although they only proved theoretical results when $\varepsilon < 1/360$, we empirically found a dip in error just above $\varepsilon = 0.25$. This suggests the sparse vertex filtering from step three dramatically reduces the accuracy for a small speedup in running time. When $\varepsilon \approx 0.3$ we found an error ratio of approximately ten times the cost for our graph with 10 million edges, much better than the theoretical 10^5-factor for $\varepsilon = 1/360$. For this ε, we found that setting $c = 2$ gave a good tradeoff between accuracy and run time. In addition, each vertex had approximately ten different clusters it could belong to. These are incredibly good results, sorting vertices into the top 1% of all clusters. It would be interesting to see a future algorithm employ a recursive strategy here. Altogether, these results hint that there's a yet-to-be-discovered algorithm with near optimal cost and sublinear running time in the number of edges.