Path Connected Inverse Limits

Joseph Camacho

Brigham Young University

February 26, 2021

Overview

I am looking at when inverse limits of normal covering spaces over the Hawaiian Earring are path connected. I have been researching for just over a month, so there's still much to explore.

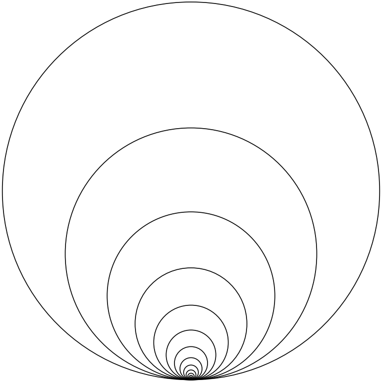
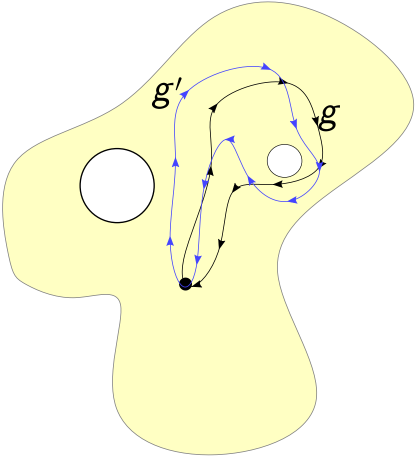


Figure: Hawaiian Earring

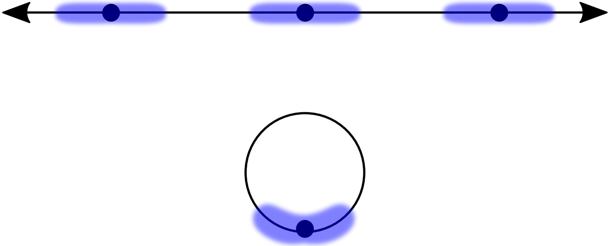
Fundamental Group

- Equivalence classes of loops
- Group operation is concatenation end to end
- Denoted $\pi_1(X, x_0)$ or $\pi_1(X)$.



Covering Space I

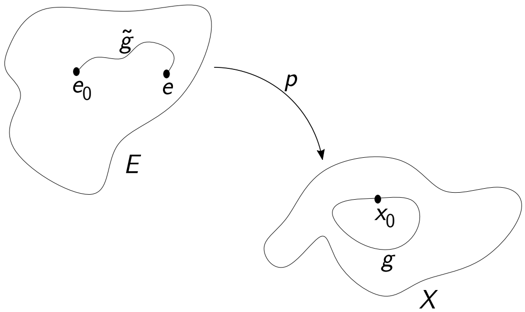
A cover $p : E \to X$ is a way of "unrolling" a space X.



Covering Space II

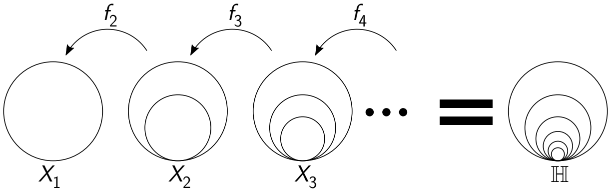
Properties:

- If X is path connected, so is E.
- Paths in X starting at x_0 have a unique lift starting at e_0.
- p_* is an inclusion from $\pi_1(E)$ to $\pi_1(X)$.



Inverse Limit I

- Way of successively approximating a space
- Example: Hawaiian Earring is the inverse limit of bouquets of circles:

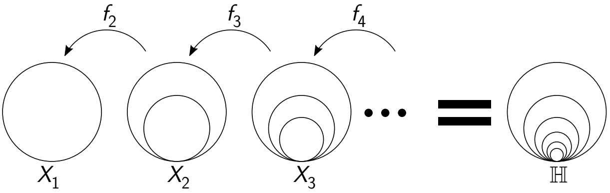


- When path connected, the inverse limit of covering spaces behaves much like a covering space.

Inverse Limit II

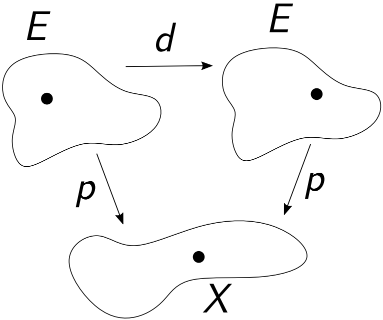
Inverse limits also exist for groups.

- Similar definition to inverse limit for topological spaces
- $E = \varprojlim X_n$ does **not** imply that $\pi_1(E) \cong \varprojlim \pi_1(X_n)$.
- Example: As above, let X_n be the bouquet of n circles. Then $\varprojlim X_n = \mathbb{H}$, but $\varprojlim \pi_1(X_n) \ncong \pi_1(\mathbb{H})$.



Deck Transformation I

- An automorphism on E such that $p \circ d = p$.
- Forms a group (through composition) $\mathrm{Aut}(E)$.

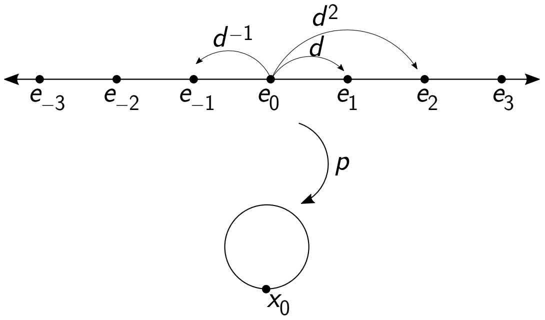


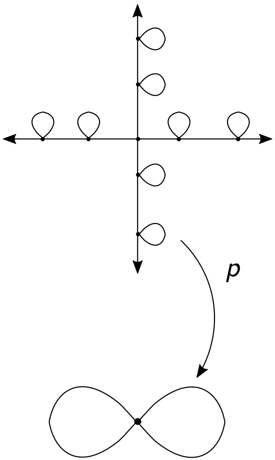
Deck Transformation II

A covering space is *normal* if for each point e in the fiber of x_0 there is a deck transformation that sends e_0 to e. Properties:

- All fiber points look alike.
- $\pi_1(E, e_0)$ is a normal subgroup of $\pi_1(X, x_0)$.
- $\mathrm{Aut}(E)$ is isomorphic to $\pi_1(X)/\pi_1(E)$.

Deck Transformation III





The Main Question

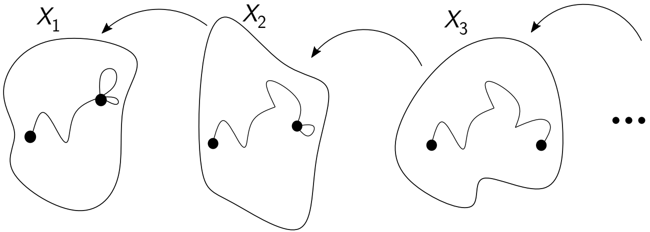
Let $X_0 = \mathbb{H}$ be the Hawaiian Earring. For $n \in \mathbb{Z}_+$, let $p_n : X_n \to X_{n-1}$ be a normal cover. Furthermore, assume that X_n is a covering space of $\mathbb{H}$. When is $\varprojlim X_n$ path connected?

Relation to Fundamental Groups

Lemma: Let $\phi : \pi_1(\mathbb{H}) \to \varprojlim \pi_1(X)/\pi_1(X_n)$ be defined by

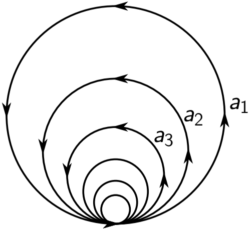
$$\phi(g) = (g\pi_1(X_1), g\pi_1(X_2), \dots).$$

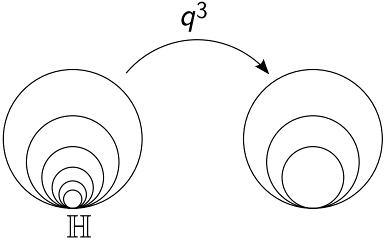
Then $\varprojlim X_n$ is path connected if and only if ϕ is surjective.



Example 1 (Free Groups) I

For $n = 1, 2, 3, \ldots$, let α_n be a loop counterclockwise around the nth circle of the Hawaiian Earring. Let $a_n = [\alpha_n]$.
Let q^n be the function that collapses all but the first n circles.





Example 1 (Free Groups) II

There exists a covering space X_n with fundamental group $\ker(q_*^n)$. It looks like the Cayley graph of the free group with n generators.

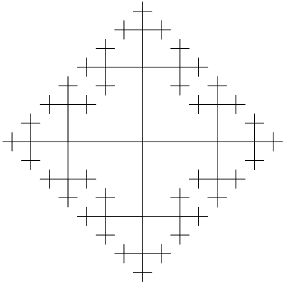


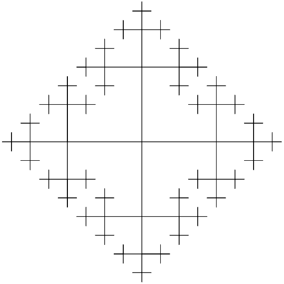
Figure: X_2

Example 1 (Free Groups) III

$\varprojlim X_n$ is **not** path connected. Why?
You can construct coherent sequences that would require going around the first loop infinitely many times. For instance, the element

$$(1\pi_1(X_1), [a_1, a_2]\pi_1(X_2), [a_1, a_2][a_1, a_3]\pi_1(X_3), \dots) \in \varprojlim \pi_1(\mathbb{H})/\pi_1(X_n)$$

is unattainable by ϕ.



Example 2 (Wireframe) I

Define a_n and q_n as in the previous example. Let

$$K_n = \langle\langle \ker(q_*), [a_i, a_j] : 1 \leq i < j \leq n \rangle\rangle.$$

There exists a covering space X_n of $\mathbb{H}$ with fundamental group K_n. The commutator $[a_i, a_j]$ has the effect of joining together the two ends of $a_i a_j$ and $a_j a_i$. The fundamental group looks like a wireframe:

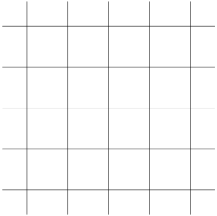


Figure: X_2

Example 2 (Wireframe) II

$\varprojlim X_n$ is path connected. Why?
Let

$$(w_1\pi_1(X_1), w_2\pi_1(X_2), \dots) \in \varprojlim \pi_1(\mathbb{H})/\pi_1(X_n).$$

Because all the a_i commute with each other modulo $\pi_1(X_n)$, we can rearrange the w_n into the form

$$\begin{aligned}
w_1\pi_1(X_1) &= a_1^{k_1}\pi_1(X_1),\\
w_2\pi_1(X_2) &= a_1^{k_1}a_2^{k_2}\pi_1(X_2),\\
w_3\pi_1(X_3) &= a_1^{k_1}a_2^{k_2}a_3^{k_3}\pi_1(X_3),\\
&\dots.
\end{aligned}$$

Let g be a path that in the interval $[2^{-n}, 2^{-n+1}]$ wraps around the ith loop k_i times. Under ϕ this path maps to the desired element.

Summary

These examples show that some sense of commutativity is required in order for the inverse limit to be path connected. My research aims to discover exactly when that is. Right now, I conjecture that $\varprojlim X_n$ is path connected if and only if

$$[a_i, a_j]\pi_1(X_n) \in \ker(q^i_*)\pi_1(X_n)$$

for all integers $i < j$ and n.