Power Set:

Recall the recursion meta:

1. Solve explicitly for small instances
2. Assume your program works on all smaller inputs & use that to build a solution to the larger instance.

$$n! = n \cdot (n-1)!$$

$$\gcd(a,b) = \gcd(b, a\%b)$$

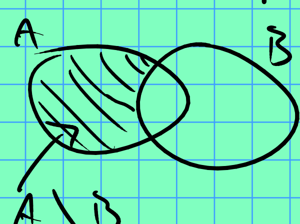
$$x^n = x(x^{n-1}) \quad \ldots$$

$$H \subseteq S \implies \mathcal{P}(H) \subseteq \mathcal{P}(S)$$

(Base case: $\{\} \longrightarrow \{\{\}\}$)

Now if $S \neq \{\}$, then $\exists\, x \in S$.

Now set $S_x = S \setminus \{x\}$ (Aside: $A \setminus B$ is the "difference" $= \{x \in A \mid x \notin B\}$)



Clearly $S_x \subseteq S$, so

$\mathcal{P}(S_x) \subseteq \mathcal{P}(S)$.

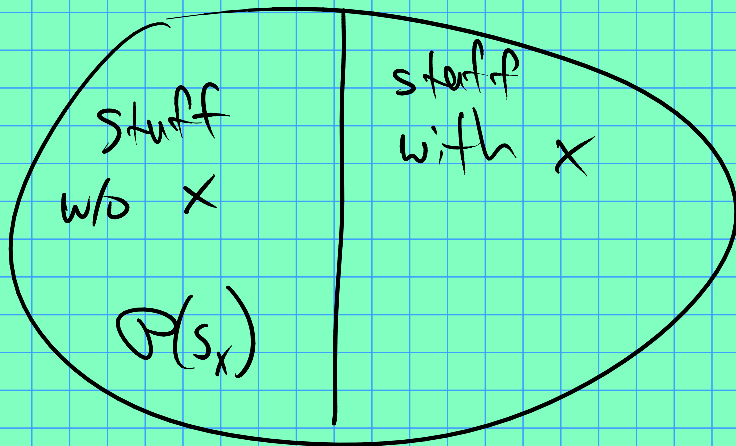
Question: what is $\mathcal{P}(S_x)$ lacking?

(what do we need to add to get $\mathcal{P}(S)$?)

There's an easy way to describe $\mathcal{P}(S) \setminus \mathcal{P}(S_x)$:

$\mathcal{P}(S) \setminus \mathcal{P}(S_x)$ = "all subsets with x as an element"!

$\mathcal{P}(S)$



$$\mathcal{P}(S) = \mathcal{P}(S_x) \cup \boxed{?}$$

"stuff with an x..."

Example: Say $S = \{0,1,2\}$, & $x = 2$, so $S' = \{0,1\}$.

$$\mathcal{P}(S_2) = \{\{\}, \{0\}, \{1\}, \{0,1\}\}$$

$$\mathcal{P}(S) = \{\{\}, \{0\}, \{1\}, \{0,1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

add x to everything in $\mathcal{P}(S_2)$

$\downarrow$ = "add an x"

A bit more formally:

Define $S^x = \{A \subseteq S \mid x \in A\}$

$$S^x = \{A \cup \{x\} \mid A \in \mathcal{P}(S_x)\} \quad (\ast)$$

And $\mathcal{P}(S) = \mathcal{P}(S_x) \,\dot\cup\, S^x$.

Trace on $S = \{0,1,2\}$:

$\{0,1,2\} = \{\underline{\{\}, \{0\}, \{1\}, \{0,1\}},\ \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$ $(x=2)$

$\downarrow \qquad \uparrow$

$\{0,1\} = \{\underline{\{\}, \{0\}}, \{1\}, \{0,1\}\}$ $(x=1)$

$\downarrow \qquad \uparrow$

$\{0\} = \{\underline{\{\}}, \{0\}\}$ $(x=0)$

$\downarrow \qquad \uparrow$

$\{\} = \{\{\}\}$ (base case)

Note: here's some maybe better notation:

$\mathcal{P}_x = \{A \subseteq S \mid x \notin A\}$.

$\mathcal{P}^x = \{A \subseteq S \mid x \in A\}$.

Then observe that

① $\mathcal{P}(S) = \mathcal{P}_x \,\dot\cup\, \mathcal{P}^x$ ($\dot\cup$ = <u>disjoint</u> union)

② $\mathcal{P}_x = \mathcal{P}(S \setminus \{x\})$

③ $\mathcal{P}^x = \{ A \cup \{x\} \mid A \in \mathcal{P}_x \}$

This gives us a way to describe $\mathcal{P}(S)$ in terms of the powerset of something <u>smaller</u> ($\mathcal{P}_x$).