Ratios and Rates

Contents

§1 Introduction

Ratios help us establish relationships between two quantities. We will use this to solve many problems on the AMC 8 and AMC 10. We will also cover rates which help us understand how fast something is moving and help us relate distance, speed and time. We will explore inverse rates and look at classic work problems.

§2 Ratios

Ratios help us compare two quantities. For example if we are told 10 pencils are 15 cents we can establish a relation that 10 pencils are worth 15 cents. So we can say that the ratio between number of pencils to cents is 10 to 15. This is called a direct proportion. Ratios can be written as a fraction or can be written with a colon. With our example we can write

$$\frac{\text{number of pencils}}{\text{cents}} = \frac{10}{15}$$

or

$$\text{number of pencils} : \text{cents} = 10 : 15.$$

We will use these interchangeable throughout the chapter. Here is a simple example that will introduce the concepts of ratios.

§2.1 Direct Rates

These following examples are called direct proportions. That means the relation between quantities whose ratio is constant. We will start by looking at some examples.

> **Example 2.1** — The ratio of men to women at a international conference is $1 : 2$ respectively. If there are a total of 99 people at the conference, how many are men and how many are women?

Solution. For every 1 man there are 2 women. So 1 out of 3 people are men. Hence there are $99 \cdot \frac{1}{3} = \boxed{33}$ men. Then there are $99 - 33 = \boxed{66}$ women. □

> **Example 2.2** — (Source: Mathcounts)
> If the ratio of $a : b$ is $2 : 5$, what is the ratio of $b : 10a$?

Solution. We can write $\frac{a}{b} = \frac{2}{5}$ and we want to find $\frac{b}{10a}$. Taking the reciprocal of the first fraction we know that $\frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{5}{2}$. Hence

$$\frac{b}{10a} = \frac{1}{10} \cdot \frac{5}{2} = \boxed{\frac{1}{4}}.$$

□

The last two examples were direct and easy. Now we will draw upon what we learned last chapter and use that to solve this problem.

Example 2.3 — (Source: AoPS)
The ratio of two numbers is $3:7$. When 5 is subtracted from each of those numbers, the new ratio is $1:3$. What is the smaller of those two numbers before 5 is subtracted?

Solution. Let the two numbers be x, y. So we know that

$$\frac{x}{y} = \frac{3}{7}.$$

We also know that

$$\frac{x-5}{y-5} = \frac{1}{3}.$$

From the first equality we get $7x = 3y$. From the second equality we get $3x - 15 = y - 5$. Using what we learned in the last chapter we can solve this system of equations and we see that $x = 15$ and $y = 35$. Hence the answer is $\boxed{15}$. □

Sidenote 2.4

When there are a lot of words the best way to approach is just to convert the words to equations.

Practice 2.5 — (Source: Mathcounts)
Jim's stride measures $2\frac{1}{2}$ feet, and Jeffrey's stride measures $2\frac{2}{3}$ feet. There are 5280 feet in a mile. If Jim and Jeffrey each walk one mile, what is the ratio of the number of strides Jim takes to the number of strides Jeffrey takes?

Practice 2.6 — (Source: AMC 10)
The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of w to y?

§2.2 Inverse Proportions

We have an inverse proportion when the two quantities of interest have a constant product. For example a rectangle with area 100, has $lw = 100$. When we increase the factor of l by k, we decrease the factor of w by k for the product to remain constant.

Example 2.7 — If a and b are inversely proportional and $a = 12$ when $b = 6$, what is a when $b = 2$?

Solution. By definition we know that if a and b are inversely proportional we have $ab = c$ for some constant c. When $a = 12$ and $b = 6$ we know that our constant $c = 12 \cdot 6 = 72$. Hence we have $ab = 72$. So when $b = 2$ we have $2a = 72$ or $\boxed{a = 36}$. □

Example 2.8 — (Source: AoPS)
John believes that the amount of sleep he gets the night before a test and his score on that test are inversely related. On his first exam, he got eight hours of sleep and scored 70 on the exam. To the nearest tenth, how many hours does John believe he must sleep the night before his second exam so that the average of his two exam

scores is an 80?

Solution. To achieve a score of 80, John must score 90 on his next text. By definition of inverse proportion, we have that $70 \cdot 8 = 90 \cdot s$ where s is the number of hours he sleeps. Therefore we have

$$s = \frac{70 \cdot 8}{90} = \boxed{6.2}.$$

□

Practice 2.9 — (Source: AoPS)
The variables a and b are inversely proportional. When the sum of a and b is 24, their difference is 6. What is b when a equals 5?

Practice 2.10 — (Source: Mathcounts)
Each term of a sequence, after the first term, is inversely proportional to the term preceding it, and the constant of proportionality stays the same. If the first term is 2 and the second term is 5, what is the 12th term?

§2.3 Work Problems

I think the best way to solve these types of problems is to draw a table and write the information.

Example 2.11 — (Source: AoPS)
If 5 hens can lay 24 eggs in 5 days, how many days are needed for 8 hens to lay 20 eggs?

Solution. Consider the following table.

Hens	Eggs	Day
5	24	5
1	$24/5$	5
1	$24/25$	1
8	$(24 \cdot 8)/25$	1
8	1	$25/192$
8	20	$(25 \cdot 20)/192$
8	20	$\boxed{125/48}$

□

The table is a powerful way to show what is going on. Ask yourself the question "If 5 hens can lay 24 eggs in 5 days, how many can 1 hen lay in 5 day?" Questions like this will help you fill the table and arrive at your answer. Try to build a table and answer the following practice problem.

Practice 2.12 — (Source: MAO)
The wages of 3 men for 4 weeks is 108 dollars. At the same rate of pay, how many weeks will 5 men work for 135 dollars?

§3 Percents

There isn't much content in this section. We will just go over the definition and do some examples. There is will a few pratice problems for you to practice.

Definition 3.1
A percent means "per hundred". For example $x\%$ means $x/100$.

Example 3.2 — Answer the following questions. Write 3/5 as a percent.

Write $\frac{25}{4}\%$ as a decimal and as a fraction.

Solution. It should be easy to see that $\frac{3}{5} = \frac{60}{100}$ so that means $3/5 = \boxed{60\%}$.

We note that 25/4% as a fraction and decimal is

$$\frac{25/4}{100} = \boxed{\frac{1}{16}} = \boxed{0.0625}.$$

□

This is a hard AIME problem. We will use concepts convered in this section and the last section to solve it.

Example 3.3 — (Source: AIME)
Jar A contains four liters of a solution that is 45% acid. Jar B contains five liters of a solution that is 48% acid. Jar C contains one liter of a solution that is $k\%$ acid. From jar C, $\frac{m}{n}$ liters of the solution is added to jar A, and the remainder of the solution in jar C is added to jar B. At the end both jar A and jar B contain solutions that are 50% acid. Given that m and n are relatively prime positive integers, find $k + m + n$.

Solution. (Credits: ihatemath123)
That is a lot of words. The best way to approach these types of problems is just to write the equations.

Jar A contains $\frac{11}{5}$ liters of water, and $\frac{9}{5}$ liters of acid; jar B contains $\frac{13}{5}$ liters of water and $\frac{12}{5}$ liters of acid.

The gap between the amount of water and acid in the first jar, $\frac{2}{5}$, is double that of the gap in the second jar, $\frac{1}{5}$. Therefore, we must add twice as much of jar C into the jar A over jar B. So, we must add $\frac{2}{3}$ of jar C into jar A, so $m = 2, n = 3$.

Since jar C contains 1 liter of solution, we are adding $\frac{2}{3}$ of a liter of solution to jar A. In order to close the gap between water and acid, there must be $\frac{2}{5}$ more liters of acid than liters of water in these $\frac{2}{3}$ liters of solution. So, in the $\frac{2}{3}$ liters of solution, there are $\frac{2}{15}$ liters of water, and $\frac{8}{15}$ liters of acid. So, 80

Therefore, our answer is $80 + 2 + 3 = \boxed{85}$. □

Practice 3.4 — (Source: MAO)
A chemist has 80 ml of a solution containing 20% acid. How many ml must be removed and replaced by pure acid in order to obtain a 40% solution?

Practice 3.5 — (Source: Mathcounts)
A test has two parts. The first part is worth 60% and the second part is worth 40%. If a student gets 95% of part one correct, what exact percent correct must the student achieve on part two to average 90% for the whole test?

Practice 3.6 — (Source: AHSME)
Ms. A owns a house worth 10000 dollars. She sells it to Mr. B at 10% profit. Mr. B sells the house back to Ms. A at a 10% loss. How much money does Ms. A make?

§4 Rates

Rate measures the speed of something whether it be the speed of a car, or how fast one does a job. In any rate problem pay close attention to the units because they are usually the biggest pitfalls in these type of problems.

§4.1 Units of Measurement

This is a useful table of conversion factors that we should keep in mind while tackling rate problems.

Time

1 Day	24 Hours
1 Hour	60 Minutes
1 Minute	60 Seconds
1 Second	1000 Miliseconds

Distance (US)

1 Mile	5280 Feet or 1760 Yards
1 Yard	3 Feet
1 Foot	12 Inches

Distance (Metric)

1 Kilometer	1000 Meters
1 Meter	100 Centimeters
1 Centimeter	10 Millimeters

Distance (US to Metric) (these are approximations)

1 Mile	1.6 Kilometers
40 Inches	1 Meter
1 Inch	2.54 Centimeters

Sidenote 4.1
You do not need to memorize all these conversion factors. These are just the common ones that are good to know. By using them again and again you will soon find them

engrained in your head.

§4.2 The Ruling Formula

In all distance = rate x time problems, there is one ruling equation

Definition 4.2

Distance = Rate × Time

Most rate problems will use this equation in one form or another. Be careful to make sure the units are correct. For example if the problem states the rate is in meters per second and the time is given in hours, be sure to convert the rate or the time to one common unit.

Example 4.3 — (Source: Classic)

(a) Find the distance traveled by a car traveling at 45 miles per hour for 3 hours

(b) Find the time it took for a person to bike 20 miles at 4 miles per hour

(c) Find the speed a car needs to travel to cover 2550 miles in 50 hours.

Solution. (a) Substituting the speed and the time, we find that the distance is

$$45 \cdot 3 = \boxed{135}.$$

(b) Note that the distance is 20 miles and the speed is 4 miles per hour. This means that the time needed is $\frac{20}{4} = \boxed{5}$.

(c) Note that the distance traveled is 2550 miles and the speed is 50 miles per hour. Thus, the speed is

$$\frac{2550}{50} = \boxed{51}.$$

□

Sidenote 4.4

Note that when using the distance, speed, and time formula, all three have to be in the same unit of measurement. To make this easier to understand, we can write miles per hour as $\frac{\text{miles}}{\text{hour}}$, and hours as hour. Therefore, a car traveling at 30 miles per hour for 3 hours would cover a distance of

$$\left(30 \cdot \frac{\text{mile}}{\text{hour}}\right) \cdot (3 \cdot (\text{hour})) = \boxed{90 \text{ miles}}.$$

On the other hand, a car traveling at 30 miles per hour for 20 minutes would cover a distance of

$$\left(30 \cdot \frac{\text{miles}}{\text{hour}}\right) \cdot (20 \cdot (\text{minutes})) = \left(30 \cdot \frac{\text{miles}}{\text{hour}}\right) \cdot \left(20 \cdot (\text{minutes}) \cdot \frac{\text{hour}}{60 \text{ minutes}}\right) =$$

$$\left(30 \cdot \frac{\text{miles}}{\text{hour}}\right) \cdot \left(\frac{\text{hour}}{3}\right) = \boxed{10 \text{ miles}}.$$

To emphasize this further we will look at another example. This is commonly called Dimensional Analysis or DA.

Example 4.5 — For the following questions, round to the nearest integer. You may refer to the table provided above.

(a) Find the distance traveled **in miles** of a person traveling 10000 feet per minute for 3 hours.

(b) Find the speed in **yards per milliseconds** if a spaceship traveled 1 billion miles in 13 days.

(c) Find the time in **days** it takes for a big to travel 3.5 million kilometers if it travels 19 centimeters per millisecond.

Solution. (a) The answer is

$$10000 \cdot \frac{\text{feet}}{\text{minute}} \cdot 3 \cdot (\text{hour}) = 30000 \cdot \frac{\frac{\text{miles}}{5280}}{\frac{\text{hour}}{60}} \cdot (\text{hour}) = \frac{30000 \cdot 60}{5280} \cdot \frac{\text{miles}}{\text{hour}} \cdot (\text{hour}) \approx \boxed{341 \text{ miles}}.$$

(b) The answer is

$$\frac{1000000000 \cdot (\text{miles})}{13 \cdot (\text{days})} = \frac{1000000000 \cdot 1760 \cdot (\text{yards})}{13 \cdot 24 \cdot 60 \cdot 60 \cdot 1000 \cdot (\text{milliseconds})} \approx \boxed{1567 \ \frac{\text{yards}}{\text{milliseconds}}}$$

(c) The answer is

$$\frac{3500000 \cdot (\text{kilometers})}{19 \cdot \frac{\text{centimeters}}{\text{milliseconds}}} = \frac{3500000 \cdot 1000 \cdot 100 \cdot (\text{centimeters})}{19 \cdot \frac{\text{centimeters}}{\frac{\text{day}}{1000 \cdot 60 \cdot 60 \cdot 24}}} \approx \boxed{213 \text{ days}}$$

□

This is usually a mind numbing exercise but do make sure that you know how to do these types of problems as they appear is physics, chemistry and other sciences. For formality there is a practice problem.

Practice 4.6 — (Source: Mathcounts)
The mural of your school mascot is 4 feet by 12 feet and is to be completely framed using a single row of square tiles each 2 inches on an edge. If the tiles are \$0.10 each, find the cost, in dollars, of the tiles needed to frame the mural.

§4.3 More Rate Problems

There are a few more things that we need to be aware of before we can dive into solving problems. The first is average speed.

Definition 4.7

(Average Speed)

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}.$$

Note that average speed IS NOT the average of the total speeds but rather the total distance travelled divided by the total time.

Example 4.8 — Adam travels 40 miles to his mom's house in 1 hour. On the return trip he takes 90 minutes due to a traffic jam. What was his average speed during the trip in mph?

Solution. Clearly the total distance is 40 miles on the trip to his mom's house and 40 miles back. So Total Distance = 40 + 40 = 80 miles. Since the problem is asking for the average speed in mph, we know that the time must be in hours. Since 90 minutes is 1.5 hours, we know that the Total Time = 1 + 1.5 = 2.5 hours. Hence

$$\text{Average Speed} = \frac{80 \text{ miles}}{2.5 \text{ hours}} = \boxed{32 \text{ mph}}.$$

□

That was an easy example. Here are two harder examples, in which we can formulate the information in a more creative way.

Example 4.9 (Source: AHSME) — A car travels 120 miles from A to B at 30 miles per hour but returns the same distance at 40 miles per hour. Find the average speed of the round trip.

Solution. Consider the following table:

	Distance	Speed	Time
Trip To	120	30	**4**
Return Trip	120	40	**3**
Round Trip	240	$\boxed{240/7}$	7

□

Example 4.10 (Source: AHSME) — An automobile went up a hill at a speed of 10 miles an hour and down the same distance at a speed of 20 miles an hour. Find the average speed of the round trip

Solution. Consider the following table:

	Distance	Speed	Time
Trip To	d	10	$d/10$
Return Trip	d	20	$d/20$
Round Trip	2d	$\boxed{40/3}$	$3d/20$

□

We have yet to solve a rate problem with algebra. Here is a particularly nice one.

Example 4.11 — (Source: Mathcounts)
Agent 020 chases a fugitive who is initially 600 meters ahead of her. The fugitive runs at a constant speed of 4 m/s. Agent 020 also runs at a constant speed, and takes 5 minutes to catch the fugitive. What is Agent 020's speed, in meters per second?

Solution. Suppose that Agent 020's speed is s. Now, Agent 020 catches up to the fugitive at a speed of $s-4$. Now, in order to travel 600 meters in 5 minutes, you would have to travel at a speed of 2 meters per second. Thus, $s-4=2 \implies s=\boxed{6}$. □

We will look at one more example, an infamous example due to its placement.

Example 4.12 — Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

Solution. Suppose that the boat travelled t meters in the time that Emily travelled 210 steps. Then, it would have travelled $\frac{1}{5}t$ meters in the time Emily travelled 42 steps. The two scenarios give us the equations

$$\ell - \frac{1}{5}t = 42$$
$$t + \ell = 210$$

which solves to get $6\ell = 420$, implying $\ell = \boxed{70}$.

Credits to HamstPan7 □

Since rates are a core concept in AMC 8 and AMC 10 here are more practice problems for you to work on.

Practice 4.13 — (Source: AMC 8)
Qiang drives 15 miles at an average speed of 30 miles per hour. How many additional miles will he have to drive at 55 miles per hour to average 50 miles per hour for the entire trip?

Practice 4.14 — (Source: AMC 8)
Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?

Practice 4.15 — (Source: AMC 10)
Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all,

it took her 44 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia walk?

Practice 4.16 — (Source: AMC 10)
Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

Practice 4.17 — (Source: AMC 12)
Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

Practice 4.18 — (Source: AMC 10)
Mr. Earl E. Bird gets up every day at 8:00 AM to go to work. If he drives at an average speed of 40 miles per hour, he will be late by 3 minutes. If he drives at an average speed of 60 miles per hour, he will be early by 3 minutes. How many miles per hour does Mr. Bird need to drive to get to work exactly on time?

Practice 4.19 — (Source: AMC 12)
A power boat and a raft both left dock A on a river and headed downstream. The raft drifted at the speed of the river current. The power boat maintained a constant speed with respect to the river. The power boat reached dock B downriver, then immediately turned and traveled back upriver. It eventually met the raft on the river 9 hours after leaving dock A. How many hours did it take the power boat to go from A to B?

Practice 4.20 — (Source: AIME)
Points A, B, and C lie in that order along a straight path where the distance from A to C is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at A and running toward C, Paul starting at B and running toward C, and Eve starting at C and running toward A. When Paul meets Eve, he turns around and runs toward A. Paul and Ina both arrive at B at the same time. Find the number of meters from A to B.

Practice 4.21 — (Source: AIME)
The AIME Triathlon consists of a half-mile swim, a 30-mile bicycle ride, and an eight-mile run. Tom swims, bicycles, and runs at constant rates. He runs fives times as fast as he swims, and he bicycles twice as fast as he runs. Tom completes the AIME Triathlon in four and a quarter hours. How many minutes does he spend

bicycling?

§4.4 Clock Problems

Clock problems also appear sometimes in the AMC's. There isn't much of an introduction that needs to be given. We will just dive right into some example problems. We will start with a simple example.

Example 4.22 — Find the smaller angle formed by the hour and minute hand at 12 : 36.

Solution. At 12 : 36, the hour hand as an angle of $\frac{36}{60} \cdot \frac{1}{12} \cdot 360° = 18°$. The minute hand has an angle $\frac{36}{60} \cdot 360 = 216°$. Thus, the angle formed by the hour and minute hand is $216° - 18° = 198°$. Since this angle is greater than 180°, we must subtract it from 360° to find the answer to be $\boxed{162°}$. □

Example 4.23 — Find all integral times such that the smaller angle formed by the hour and minute hand is 72°.

Solution. Suppose the time is $a : b$, such that $a \leq 12$. At this time, the hour hand is at an angle of

$$\frac{a}{12} \cdot 360 + \frac{b}{60} \cdot \frac{1}{12} \cdot 360 = \left(30a + \frac{b}{2}\right)^{\circ}.$$

The angle formed by the minute hand is $\frac{b}{60} \cdot 360 = (6b)°$. Thus, we have to have

$$\left|30a + \frac{b}{2} - 6b\right| = \left|30a - \frac{11b}{2}\right| = 72 textor 288.$$

We will split this into 4 cases now.

Case 1: $30a - \frac{11b}{2} = 72$

Note that $0 \leq a \leq 12$. Taking $\pmod{11}$, we find that $a = 9$. Thus, $b = 36 \implies \boxed{9 : 36}$.

Case 2: $30a - \frac{11b}{2} = -72$

Note that $0 \leq a \leq 12$. Taking $\pmod{11}$, we find that $a = 2$. Thus, $b = 24 \implies \boxed{2 : 24}$.

Case 3: $30a - \frac{11b}{2} = -288$

Note that $0 \leq a \leq 12$. Taking $\pmod{11}$, we find that $a = 8$. Thus, $b = 96$, and thus this case is extraneous.

Case 3: $30a - \frac{11b}{2} = 288$

Note that $0 \leq a \leq 12$. Taking $\pmod{11}$, we find that $a = 3$. Thus, $b = 36$, and thus this case is extraneous.

We conclude that the only two integral times that satisfy the conditions of the problem is $\boxed{9 : 36 \text{ and } 2 : 24}$. □

There is a formula that helps with solving problems like the above but you will have to derive it to use it. This will be left as your practice problem.

Practice 4.24 — Prove that the angle between the hour hand and the minute hand is

$$|30h - 5.5m|$$

where m is the minutes and h is the hours.

§5 Homework

In this chapter we learned what ratios, percents and rates are. In each section we explored multiple examples and left multiple practice problems for the reader to solve. For that reason there is no other problems for the homework section.