Feynman Technique For Integrals

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§1 Introduction

Solving integrals by normal tricks such as Integration by Parts, Trig Substitutions, U-substitution only works for a certain category of integrals. There are other integrals that just can't be solved using these techniques. We will understand how to tackle problems of these sort of problems using the Feynman Technique. The Feynman Technique is also known as Leibniz rule's or Integration under the sign.

Instead of explaining the technique, we will dive into an example which will outline the approach to use Feynman Technique and then we can solidify our definition.

We will use some partial derivatives so if you aren't familiar with partial derivatives, please do get a quick understanding of it so you can understand this paper completely.

§2 An Integral Example

We will highlight this technique using an famous example that appears over and over again in complex analysis.

Example 2.1

Evaluate

$$\int_0^\infty \frac{\sin(x)}{x}\,dx.$$

Solution. Trying to solve this with usual techniques such as Integration by Parts, Trig and U substitution, Partial Fractions yields to no avail. So we will try Feynman's Technique for this problem.

We will try to make the denominator cancel out because the integral is very hard to work with. To do this we can try to introduce a parameter p. Let

$$f(p) = \int_0^\infty \frac{\sin(x)}{x}e^{-px}\,dx.$$

Note that our integral actually converges. Basically we just want to find what $f(0)$ is since this makes $e^0 = 1$. We will take the partial derivative with respect to p on both sides

$$f'(p) = \int_0^\infty \frac{\partial}{\partial p} \frac{\sin(x)}{x} e^{-px}\, dx$$
$$= -\int_0^\infty \sin(x) e^{-px}\, dx.$$

We can use IBP twice now or we can notice that $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$ where $i = \sqrt{-1}$. We will stick to using IBP twice, but the interested reader should try out the sine substitution method. Using IBP twice we see that

$$f'(p) = -\frac{1}{1+p^2}.$$

The computation is left as an exercise to the reader.

Taking the integral again, we see that $f(p) = \int -\frac{1}{1+p^2}$. Note that this is a standard integral for $\arctan(p)$. Thus we see that $f(p) = -\arctan(p) + C$ where C is a constant. Since $f(p) \to 0$ as $p \to \infty$, and

$$\lim_{p\to\infty} \arctan(p) = \frac{\pi}{2}.$$

Note that

$$\lim_{p\to\infty} C - \arctan(p) = 0$$

which means $C = \frac{\pi}{2}$. Thus $f(p) = \frac{\pi}{2} - \arctan(p)$. Thus we see that $f(0) = \frac{\pi}{2}$.

Therefore

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \boxed{\frac{\pi}{2}}.$$

□

§3 The Gaussian Integral

The Gaussian Integral is also know as the Euler–Poisson integral is known for its difficulty. The applications of the Gaussian integral is seen is many places. It is used to compute the normalized constant of the normal model, it is closely related to the error function, and it frequently appears in quantum mechanics.

Example 3.1

Evaluate

$$\int_{-\infty}^\infty e^{-x^2}\, dx$$

The usual solution is to use polar coordinates but we will use Feynman's Technique for this problem. The interested reader is encouraged to explore more using the polar coordinate technique.

Solution. Let

$$\int_{-\infty}^\infty e^{-x^2}\, dx = 2G.$$

We will take of the advantage of the symmetry in the problem: we can just evaluate

$$2\int_0^\infty e^{-x^2}\,dx.$$

This following statement is the hardest part of the problem:

$$f(t)=\int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$

This is one of those things in math that you just have to know by seeing different types of problems. Differentiating both sides with respect to t, we get

$$f'(t)=\int_0^\infty \frac{\partial}{\partial t}\frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx=\int_0^\infty \frac{-2te^{-t^2(1+x^2)}(1+x^2)}{1+x^2}\,dx=-2te^{-t^2}\int_0^\infty e^{-t^2x^2}\,dx.$$

Now we can exploit a u-substitution. Let $u = tx$ which implies that $\frac{1}{t}\,du = dx$. Substituting this into the integral, we get

$$f'(t)=-2e^{-t^2}\int_0^\infty e^{-u^2}\,du=-2e^{-t^2}G.$$

Now we can integrate on both sides and doing so we get

$$\int_0^\infty f'(t)=-2G\int_0^\infty e^{-t^2}\,dt\Rightarrow \lim_{b\to\infty} f(b)-f(0)=-2G^2$$

the latter coming from the substitution of the Gaussian Integral. Note that

$$f(0)=\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx=\lim_{b\to\infty}\arctan(x)|_0^b=\frac{\pi}{2}.$$

Now we also see that

$$\lim_{b\to\infty} f(b)=\lim_{b\to\infty}\int_0^b \frac{e^{-b^2(1+x^2)}}{1+x^2}=0.$$

Thus we know that

$$-2G^2=0-\frac{\pi}{2}\Rightarrow G=\frac{\sqrt{\pi}}{2}.$$

Thus

$$2G=\boxed{\sqrt{\pi}}.$$

□

Remark 3.2. There is also a arbitrary Gaussian integral which states

$$\int_{-\infty}^\infty e^{-a(x+b)^2}\,dx=\sqrt{\frac{\pi}{a}}.$$

§4 Concrete Definition

Now that we have looked and solved some problems using the Feynman Method, it is time to bring in the full concrete definition. You may skip over this part if you get the big picture highlighted in the two example problems as this is just for completeness sake.

Definition 4.1. If $f(x,t)$ is a continuous and differentiable function at all points from $x \in [a,b]$ where a,b are the limits of the integration then,

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial}{\partial t} f(x,t)\,dx.$$

This is the basic definition of the Feynman's Technique for Integrals. Once we have established this, it is up to the solver to make the correct substitution and this usually comes with practice.

§5 Practice Problems

The following two problems are for practice so the reader can actually get dirty and understand it better. The answers have been provided under each problem so the reader can verify if they are correct.

Example 5.1

Compute

$$\int_0^1 (x\ln(x))^{50}\,dx.$$

Answer: $\frac{50!}{51^{51}}$.

Example 5.2

Compute

$$\int_0^1 \frac{x^7-1}{\ln(x)}\,dx.$$

Answer: $\ln(8)$.

§6 Conclusion

As the two examples highlighted, the Feynman Technique for Integrals is quite powerful to help you solve problems that can't be solved with the usual techniques such as Integration by Parts, trig substitution, etc.

We usually integrate under the sign, when powers of e and $\ln(x)$ are present or when we have made no progress with the usual technique . With practice, Feynman's Technique will become easier and more natural.

If you have any questions, concerns or corrections please reach out to me by email at sambhu.ganesan150@gmail.com.

§7 References

1. https://web.williams.edu/Mathematics/lg5/Feynman.pdf.

2. https://medium.com/@rthvik.07/solving-the-gaussian-integral-using-the-feynman-integration-method-215cf3cd6236

3. https://brilliant.org/wiki/differentiate-through-the-integral/