LINEAR EQUATIONS

Contents

§1 Introduction

Solving systems of equations is key to unlocking many questions on the AMC 8 and AMC 10. Today we will cover how to solve simple systems involving 2 variables and will eventually make our way to harder systems of equations with multiple variables. Then we will turn our attention to logic problem's and explore multiple knights and knaves problem's and others.

§2 How To Solve An Equation

Though you may already know this, it is best to do a quick review, thus this section will be short.

Example 2.1 — Solve for x:

$$6x + 7 = 91$$

Solution. The first step is to subtract 7 on both sides. Doing so, we get $6x + 7 - 7 = 91 - 7 \Rightarrow 6x = 84$. Now we will divide by 6 on both sides. Doing so we get $\frac{6x}{6} = \frac{84}{6} \Rightarrow x = \boxed{14}$. □

§3 Methods of Solving System Of Two Equations

There are many different ways to solve a system of two equation. Classically, there is elimination and substitution. But we'll also explore some more cooler and faster ways to solve. We'll present the definition of elimination first, followed by an example using that method.

§3.1 Classical Methods

§3.1.1 Elimination

Definition 3.1 (Elimination)

This method is the process of eliminating one of the variables in the system of linear equations using the addition or subtraction. Usually it involves multiplication or division of the coefficients of a variable.

This is a bit wordy but after an example it should be pretty easy to understand this method.

Example 3.2 — Find all ordered pairs of (x, y) that satisfy the equations:

$$7x + 9y = 63$$
$$x + 3y = 105$$

Solution. We will solve this question via the elimination method. We'll eliminate the y-variable. To avoid messy fraction, we multiply the second equation by 3. Doing so we get $3x + 9y = 315$. Now the system of equations reads

$$7x + 9y = 63$$
$$3x + 9y = 315$$

Subtracting the two equations we get $4x = 63 - 315 = -252$. Dividing by 4 on both sides, we get $x = -63$. Now we can quickly substitute $x = -63$ back into our second equation. Hence, we are left to solve a single variable equation, which we know how to do. Plugging back $x = -63$ into our second equation we get

$$-63 + 3y = 105$$

which means that $3y = 168$ or $y = 56$. Hence the answer is $\boxed{(x, y) = (-63, 56).}$ □

Sidenote 3.3

We can also solve this problem by eliminating the x-variable. Since it is extremely similar this is left as an exercise to the reader.

§3.1.2 Substitution

Definition 3.4 (Substitution)

In this method, the elimination of the variable can be performed by substituting the value of another variable in an equation.

Yet again, it should be much easier to understand once we see an example using the substitution method. For the sake of consistency, we will use the same example as the elimination method.

Example 3.5 — Find all ordered pairs of (x, y) that satisfy the equations:

$$\begin{aligned} 7x + 9y &= 63 \\ x + 3y &= 105 \end{aligned}$$

Solution. We will solve this question via the substitution method. Analyzing the second equation, we see that by subtracting $3y$ on both sides we get $x = 105 - 3y$. Now we will simply **substitute** $x = 105 - 3y$ into the first equation to solve for y. Doing that we get

$$7(105 - 3y) + 9y = 63.$$

Expanding and simplifying we get $735 - 12y = 63$ or $12y = 672$. Dividing by 12 we get $y = 56$. Now we can substitute $y = 56$, into the second equation. So we get

$$x = 105 - 3y = 105 - 3(56) = -63.$$

Thus our answer is $\boxed{(x, y) = (-63, 56)}$. □

Sidenote 3.6

We solved the problem in two different ways and got the same answer. This is usually a strong indication that our answer is correct!

We'll also present an example that was too good to miss.

Example 3.7 — Source: AMC 10
Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is T/N?
(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution. Let the children's age as of now be A, B, C. Let Tom's age as of now be T. From the first statement of the problem, we know that

$$T = A + B + C.$$

By the second statement, Tom's age N years ago was

$$T - N = 2(A - N + B - N + C - N).$$

So we see that $T - N = 2(T - 3N) \Rightarrow 5N = T$. Thus we see that $\frac{T}{N} = 5$ which means the answer is $\boxed{D}$. □

Practice 3.8 — Source: MathLeague.org
If $3a = b + 8$, find $12a - 4b + 9$

Practice 3.9 — Source: AoPS
Solve for x, and y.

$$x + y + 3xy = 5$$

$$2x + 2y + 4xy = 8.$$

Practice 3.10 — Source: Purple Comet
Andrea is three times as old as Jim was when Jim was twice as old as he was when the sum of their ages was 47. If Andrea is 29 years older than Jim, what is the sum of their ages now?

Sidenote 3.11

These types of problems are quite common on AMC's but they usually just require converting the words to equations.

§3.2 Other Methods

This method is a bit more complicated but learning it makes solving a system of two linear equations very easy.

Definition 3.12

Consider a system of equation:

$$ax + by = c$$
$$mx + ny = p.$$

Then the value of

$$x = \frac{nc - bp}{na - mb}.$$

We will first look into how this works with an example using numbers, and then we'll look into **why** it's true. For consistency's sake, we will again use the same example as before. For fun we will call this is the **cross method**.

Example 3.13 — Find all ordered pairs of (x, y) that satisfy the equations:

$$7x + 9y = 63$$
$$x + 3y = 105$$

Solution. We will solve this question via the cross method. Following the definition we see that

$$x = \frac{3 \cdot 63 - 9 \cdot 105}{3 \cdot 7 - 1 \cdot 9} = -\frac{756}{12} = -63.$$

Substituting this into the second equation, we see that $y = 56$. Thus our answer is $\boxed{(x, y) = (-63, 56)}$. Thankfully, we have gotten the same answer again! □

Now we will prove this surprising method. Recall that the definition is:

Definition 3.14

Consider a system of equation:

$$ax + by = c$$
$$mx + ny = p.$$

Then the value of

$$x = \frac{nc - bp}{na - mb}.$$

Proof. We will prove this using the substitution method. From the second equation we extract that $y = \frac{p-mx}{n}$. Substituting this into the first equation we get

$$\frac{anx}{n} + \frac{bp - bmx}{n} = c.$$

So this means that $anx + bp - bmx = nc$. Thus we see that $x = \frac{nc-bp}{na-mb}$ as desired. □

§4 Difference of Squares

The most important theorem any contestant should know is difference of squares. It pops up in so many places, sometimes being the main step or sometimes being the intermediate step. It is very important to know how to use difference of squares so we'll go over a couple of example problems highlighting the uses.

Definition 4.1 (Difference of Squares)
Let a, b be integers. Then we have

$$a^2 - b^2 = (a+b)(a-b).$$

Proof. This is a very simple proof. We can just expand and see that this is true. □

We will start with a very simple example that involves ideas from the last section.

Example 4.2 — Source: MathCounts
If $x + y = 10$ and $x^2 - y^2 = 40$, what is the value of $2x + 3y$?

Solution. Right off the bat, we see that $x^2 - y^2 = 40$ reminds us of difference of squares. Using difference of square we see that $(x+y)(x-y) = 40$. Substituting $x + y = 10$, we get $x - y = 4$. Now we can solve this system of equations in any method we like. We will just use elimination. Adding the two equations we get $2x = 14 \Rightarrow x = 7$. So we get $y = 3$. Thus

$$2x + 3y = 2(7) + 3(3) = 14 + 9 = \boxed{23}.$$

□

We'll also present a much harder difference of squares problem. Practicing these types of problems will really help you hone your skills in.

Example 4.3 — Source: AMC 10
Suppose that real number x satisfies

$$\sqrt{49 - x^2} - \sqrt{25 - x^2} = 3$$

What is the value of $\sqrt{49 - x^2} + \sqrt{25 - x^2}$?
(A) 8 **(B)** $\sqrt{33} + 8$ **(C)** 9 **(D)** $2\sqrt{10} + 4$ **(E)** 12

Solution. This problem is quite challenging for a beginner to difference of square because the key to the problem may seem to lack motivation.

Let $\sqrt{49 - x^2} + \sqrt{25 - x^2} = a$. The key step is to note that

$$(\sqrt{49 - x^2} + \sqrt{25 - x^2})(\sqrt{49 - x^2} - \sqrt{25 - x^2}) = 3a.$$

So by difference of squares, we see that $3a = (49 - x^2) - (25 - x^2) = 24$. Thus we see that $a = 8$. Thus the answer is $\boxed{A}$. □

This practice isn't very hard, as it is a direct application of difference of squares.

Practice 4.4 — Source: AMC 8
What is the largest power of 2 that is a divisor of $13^4 - 11^4$?
(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

Sidenote 4.5

Most of the beauty of mathematics lies on visual proofs. Watch this short 23 second video for a beautiful proof of difference of squares.

§5 Logic

Logic problems are often thought as not very important for contests, but there have been multiple instances where a logic problems come up every so often. While mathematical knowledge is important in its own aspect, begin able to think logically in seeming real-life problems is very important. After all, the purpose of doing math is to increase your logical reasoning and creativity.

§5.1 Knights and Knaves

Knights and knaves is a fancy way to saying truths and liars. The most common strategy is to start by assuming that a person is a knight or a knave and work towards that with to the answer. In the land of Knights and Knaves, knights always speak the truth while knaves always lie. Outsiders can both lie, and tell the truth.

We will start off with an easy example:

Example 5.1 — Source: Mathematical Circle Diaries
While visiting the Knights and Knaves Island, you meet two islanders, James and Peter. James tells you that at least one of the two is a knave. Are James and Peter knights or knaves.

Solution. We start of by assuming James is a knave. James tell you that at least one of them is a knave. Since we assumed that James is a knave, we know what James said is true. However, that can't be the case as James always lies, hence a contradiction. Thus, James has to be a knight leaving Peter to be a knave. □

Example 5.2 — Source: Mathematical Circle Diaries
Tao, a knight from the Island of Knights and Knaves (This island is known for its unusual animals), once stated: "All three-headed animals on our planet are very smart." Based on this information, which of the following are true statements, and which are not?

"All smart animals on the Island of Knights and Knaves have three heads."

2. "If an animal from the Island of Knights and Knaves is not smart, it definitely does not have three heads."

3. "If an animal is not three-headed, then it is not smart."

Solution. Note that since Tao is a knight, he always tells the truth. Now we will go statement by statement to see which of the statements are true.

1. Clearly this is $\boxed{\text{false}}$. Tao only stated three-headed animals are smart, not that all smart animals are three-headed. For all we know there ***could*** be a 1000-headed animal that is just as smart as a three-headed animal.
2. This is $\boxed{\text{true}}$. We know that all three-headed animals are smart. So a not smart animal can't possibly be three-headed.
3. This is $\boxed{\text{false}}$. This is the same as statement 1 but worded differently. For example, a four-headed animal ***can*** be smart.

□

The following problem has a new twist. Instead of being given the story, you make it! This may have different answers and it is to test your creativity!

Practice 5.3 — Source: Mathematical Circle Diaries
While visiting the Knights and Knaves island, I had a conversation with a local knight. I asked him the same question twice, and he gave me two different answers. What was my question?

§5.2 Magic Squares

Magic squares rely on a very simple rule:

- The sum of the row, column, and diagonal are the same.

But these can also get quite brain wrenching. It takes some experimentation and algebra to get the right answer.

Example 5.4 — Source: AMC 10
In the magic square shown, the sums of the numbers in each row, column, and diagonal are the same. Five of these numbers are represented by v, w, x, y, and z. Find $y + z$.

v	24	w
18	x	y
25	z	21

(A) 43 **(B)** 44 **(C)** 45 **(D)** 46 **(E)** 47

Solution. Since it is a magic square, we know that

$$25 + 18 + v = v + 24 + w.$$

Thus we know that $w = 19$. By examining the second column and the left-right diagonal, we see that

$$24 + x + z = 25 + x + 19$$

which means that $z = 20$. Thus we see that $z = 20$. Now by examining the third row and the third column we know

$$25 + 20 + 21 = 21 + y + 19$$

we see that $y = 26$. Thus

$$y + z = 26 + 20 = 46 = \boxed{D}.$$

□

We will end this section with a short practice problem.

Practice 5.5 — Source: AMC 10
The 25 integers from -10 to 14, inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?
(A) 2 **(B)** 5 **(C)** 10 **(D)** 25 **(E)** 50

§5.3 Logic Problems on AMC's

In this section we'll just go over a couple of examples of logic problem that have shown up on the AMC's. These are very similar to the knights and knaves problems but they often rely logical thinking and the process of elimination.

Example 5.6 — Source: AMC 10
In a magical swamp there are two species of talking amphibians: toads, whose statements are always true, and frogs, whose statements are always false. Four amphibians, Brian, Chris, LeRoy, and Mike live together in this swamp, and they make the following statements.
Brian: "Mike and I are different species."
Chris: "LeRoy is a frog."
LeRoy: "Chris is a frog."
Mike: "Of the four of us, at least two are toads."
How many of these amphibians are frogs?
(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution. We will start with Brian. Suppose Brian is a toad. So this means that Mike is a frog. Now suppose Brain is a frog. This means both Mike and Brian must be the same species implying that that Mike is a frog. Thus no matter what, Mike must be a frog. So since his statement is false, there must be at most 1 toad. We know that Brian can't be the toad, because then both Chris and LeRoy have contradicting statements. So we will assume that Chris is the toad. Then LeRoy has to be a frog and Brian is forced to be a frog. Thus the answer is 3 frogs or $\boxed{D}$. □

Sidenote 5.7

We didn't have to assume that Chris is the toad. We could have assumed that LeRoy is the toad and the answer would come out to be the same.

We'll go over one more example which is worded in a similar fashion. But both the examples presented bring in different ideas to the table and showcase two very power techniques

- Cases (from the first problem)
- Elimination (from this problem)

Example 5.8 — Source: AMC 10
Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that all of his happy snakes can add, none of his purple snakes can subtract, and all of his snakes that can't subtract also can't add. Which of these conclusions can be drawn about Tom's snakes?
(A) Purple snakes can add.
(B) Purple snakes are happy.
(C) Snakes that can add are purple.
(D) Happy snakes are not purple.
(E) Happy snakes can't subtract.

Solution. For this problem we'll just use process of elimination from the answer options. For the sake of completion we'll go over each option.

- Answer A is wrong because purple snakes can't add because the don't know how to subtract.
- Answer B is wrong because if purple snakes were happy then they would be able to add which contradicts A.
- Answer C is wrong because we already established that no purple snake can add.
- Answer E is wrong because purple snakes are not happy, which means happy snakes can subtract while purple snakes can't.

Thus we are left with D as the right answer. □

We'll end of this with a simple practice question.

Practice 5.9 — Source: AMC 8
Aaron, Darren, Karen, Maren, and Sharon rode on a small train that has five cars that seat one person each. Maren sat in the last car. Aaron sat directly behind Sharon. Darren sat in one of the cars in front of Aaron. At least one person sat between Karen and Darren. Who sat in the middle car?
(A) Aaron **(B)** Darren **(C)** Karen **(D)** Maren **(E)** Sharon

§6 Homework

Homework 6.1. (Source: AMC 8) (5)
Henry the donkey has a very long piece of pasta. He takes a number of bites of pasta, each time eating 3 inches of pasta from the middle of one piece. In the end, he has 10 pieces of pasta whose total length is 17 inches. How long, in inches, was the piece of pasta he started with?
(A) 34 **(B)** 38 **(C)** 41 **(D)** 44 **(E)** 47

Homework 6.2. (Source: AMC 10) (5) Assuming $a \neq 3$, $b \neq 4$, and $c \neq 5$, what is the value in simplest form of the following expression?

$$\frac{a-3}{5-c} \cdot \frac{b-4}{3-a} \cdot \frac{c-5}{4-b}$$

(A) -1 **(B)** 1 **(C)** $\frac{abc}{60}$ **(D)** $\frac{1}{abc} - \frac{1}{60}$ **(E)** $\frac{1}{60} - \frac{1}{abc}$

Homework 6.3. (Source: AMC 8) (9)
Before the district play, the Unicorns had won 45 of their basketball games. During district play, they won six more games and lost two, to finish the season having won half their games. How many games did the Unicorns play in all?
(A) 48 **(B)** 50 **(C)** 52 **(D)** 54 **(E)** 60

Homework 6.4. (Source: AMC 8) (10)
How many positive integers can fill the blank in the sentence below?
"One positive integer is ______ more than twice another, and the sum of the two numbers is 28."
(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Homework 6.5. (Source: AMC 10) (10)
Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away." It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?
(A) $(0, 4)$ **(B)** $(4, 5)$ **(C)** $(4, 6)$ **(D)** $(5, 6)$ **(E)** $(5, \infty)$

Homework 6.6. (Source: AoPS) (12)
If $725x + 727y = 1500$ and $729x + 731y = 1508$, what is the value of $x - y$?

Homework 6.7. (Source: Mathematical Circle Diaries) (15)
While visiting the Knights and Knaves Island, you come to a party. Every single person at this party tells you that there are some liars in the room. What is really happening? How many knights and knaves are there at this party? (Remember that there can also be tourists, such as you. Tourists sometimes lie and sometimes tell the truth.)

Homework 6.8. (Source: AMC 10) (15)
A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?
(A) 9 **(B)** 18 **(C)** 27 **(D)** 36 **(E)** 77

Homework 6.9. (Source: AMC 10) (17)
The knights in a certain kingdom come in two colors. $\frac{2}{7}$ of them are red, and the rest are blue. Furthermore, $\frac{1}{6}$ of the knights are magical, and the fraction of red knights who are magical is 2 times the fraction of blue knights who are magical. What fraction of red knights are magical?
(A) $\frac{2}{9}$ **(B)** $\frac{3}{13}$ **(C)** $\frac{7}{27}$ **(D)** $\frac{2}{7}$ **(E)** $\frac{1}{3}$

Homework 6.10. (Source: Brilliant) (20)
Simplify
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right).$$

Homework 6.11. (Source: AMC 10) (25)
Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?
(A) 10 **(B)** 13 **(C)** 15 **(D)** 17 **(E)** 20

Homework 6.12. (Source: Brilliant) (27)
Simplify
$$(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7})$$

Homework 6.13. (Source: AIME) (30)
The table below displays some of the results of last summer's Frostbite Falls Fishing Festival, showing how many contestants caught n fish for various values of n.

n	0	1	2	3	...	13	14	15
number of contestants who caught n fish	9	5	7	23	...	5	2	1

In the newspaper story covering the event, it was reported that
(a) the winner caught 15 fish;
(b) those who caught 3 or more fish averaged 6 fish each;
(c) those who caught 12 or fewer fish averaged 5 fish each.
What was the total number of fish caught during the festival?