

# Introduction to time series analysis

## Objectives for this chapter

1. Discuss some basic motivations for the topic of time series analysis.
2. Introduce some fundamental concepts for time series analysis: stationarity, autocorrelation, autoregressive models, moving average models, autoregressive-moving average (ARMA) models, state-space models. These will be covered in more detail later.

# 1.1 Overview

- Time series data are, simply, data collected at many different times.
- This is a common type of data! Observations at similar time points are often more similar than more distant observations.
- This immediately forces us to think beyond the independent, identically distributed assumptions fundamental to much basic statistical theory and practice.
- Time series dependence is an introduction to more complicated dependence structures: space, space/time, networks (social/economic/communication), ...

# 1.2 Looking for trends and relationships in dependent data

The first half of this course focuses on:

1. Quantifying dependence in time series data.
2. Finding statistical arguments for the presence or absence of associations that are valid in situations with dependence.

Example questions: Does Michigan show evidence for global warming? Does Michigan follow global trends, or is there evidence for regional variation? What is a good prediction interval for weather in the next year or two?

# 1.3 Modeling and statistical inference for dynamic systems

The second half of this course focuses on:

1. Building models for dynamic systems, which may or may not be linear and Gaussian.
2. Using time series data to carry out statistical inference on these models.

Example questions: Can we develop a better model for understanding variability of financial markets (known in finance as volatility)? How do we assess our model and decide whether it is indeed an improvement?

## 1.4 A simple example: Winter in Michigan

The previous winter was mild by Michigan standards. What should we expect this year? Is there a noticeable trend? Let's look at some data.

I downloaded from [www.usclimatedata.com](http://www.usclimatedata.com) and put in [ann\\_arbor\\_weather.csv](#).

- You can get this file from the course website ([ionides.github.io/531w20](https://ionides.github.io/531w20)).
- Better, you can set up a local git repository that will give you an up-to-date copy of all the data, notes, code, homeworks and solutions for this course. See Homework 0.

```
y <- read.table(file="ann_arbor_weather.csv", header=1)
```

# 1.5 Rmarkdown

Here, we use Rmarkdown to combine source code with text. This gives a nice way to generate statistical analysis that is

1. Reproducible,
2. Easily modified or extended.

These two properties are useful for developing your own statistical research projects. Also, they are useful for teaching and learning statistical methodology, since they make it easy for you to replicate and adapt analysis presented in class.

## 1.5.1 Question: How many of you already know Rmarkdown?

20% no  
70% yes

/ 5% not Latex.

To get a first look at our dataset, `str` summarizes its structure:

```
str(y)
```

```
## 'data.frame': 120 obs. of 12 variables:
## $ Year : int 1900 1901 1902 1903 1904 1905 1906 1907 1908 1909 ...
## $ Low : num -7 -7 -4 -7 -11 -3 11 -8 -8 -1 ...
## $ High : num 50 48 41 50 38 47 62 61 42 61 ...
## $ Hi_min : num 36 37 27 36 31 32 53 38 32 50 ...
## $ Lo_max : num 12 20 11 12 6 14 20 11 15 13 ...
## $ Avg_min : num 18 17 15 15.1 8.2 10.9 25.8 17.2 17.6 20 ...
## $ Avg_max : num 34.7 31.8 30.4 29.6 22.9 25.9 38.8 31.8 28.9 34.7 ...
## $ Mean : num 26.3 24.4 22.7 22.4 15.3 18.4 32.3 24.5 23.2 27.4 ...
## $ Precip : num 1.06 1.45 0.6 1.27 2.51 1.64 1.91 4.68 1.06 2.5 ...
## $ Snow : num 4 10.1 6 7.3 11 7.9 3.6 16.1 4.3 8.7 ...
## $ Hi_Pricip: num 0.28 0.4 0.25 0.4 0.67 0.84 0.43 1.27 0.63 1.27 ...
## $ Hi_Snow : num 1.1 3.2 2.5 3.2 2.1 2.5 2 5 1.3 7 ...
```

We focus on `Low`, which is the lowest temperature, in Fahrenheit, for January.

- Climate change: have polar vortex events recently brought unusually cold weather?
- Agriculture: can I grow ginseng in Ann Arbor?
- Energy: assess the cost-effectiveness of installing extra home insulation.

- As statisticians, we want an uncertainty estimate. We want to know how reliable our estimate is, since it is based on only a limited amount of data.
- The data are  $y_1, \dots, y_N$ , which we also write as  $y_{1:N}$ .
- Basic estimates of the mean and standard deviation are

$$\hat{\mu}_1 = \frac{1}{N} \sum_{n=1}^N y_n, \quad \hat{\sigma}_1 = \sqrt{\frac{1}{N-1} \sum_{n=1}^N (y_n - \hat{\mu}_1)^2}.$$

- This suggests an approximate confidence interval for  $\mu$  of  $\hat{\mu}_1 \pm 1.96 \hat{\sigma}_1 / \sqrt{N}$ .



- 1955 has missing data, coded as NA, requiring a minor modification. So, we compute  $\hat{\mu}_1$  and  $SE_1 = \hat{\sigma}_1 / \sqrt{N}$  as

```
mu1 <- mean(y$Low, na.rm=TRUE)
se1 <- sd(y$Low, na.rm=TRUE)/sqrt(sum(!is.na(y$Low)))
cat("mu1 =", mu1, ", se1 =", se1, "\n")
```

```
## mu1 = -2.932773 , se1 = 0.6813215
```

**1.8.1 Question:** What are the assumptions behind the resulting confidence interval,  $-2.93 \pm 1.34$ . *95% C.I.*

*normal distribution for observations.  $\rightarrow$  iid: independent, identically distributed common mean & variance. a central limit theorem helps.*

**1.8.2 Question:** When, in practice, is it reasonable to present this confidence interval? Is it reasonable here?

*enough data for a CLT? Probably yes.  
dependence structure? rising trend?*

**1.8.3 Question:** How would you proceed?

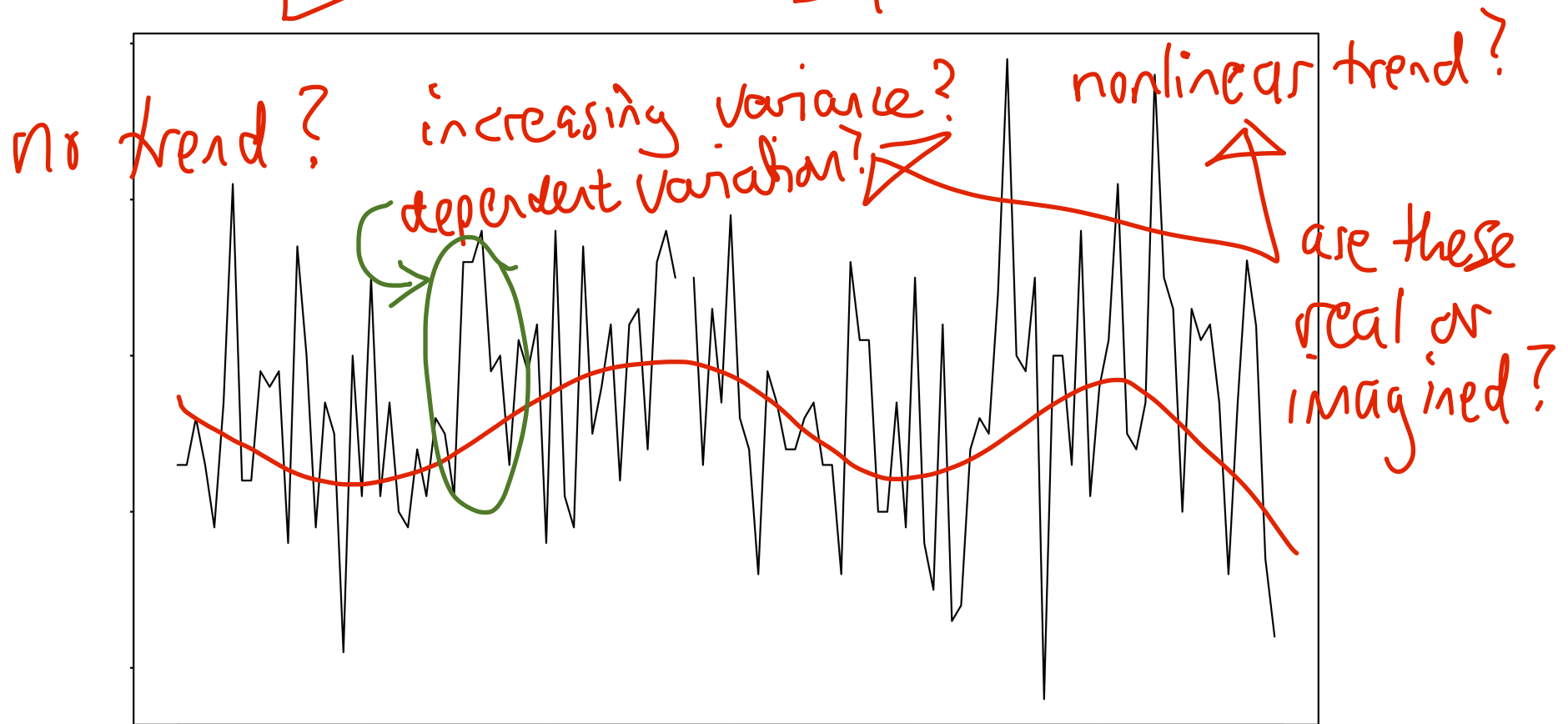
*First, look at the data.*

# 1.9 Some data analysis

- The first rule of data analysis is to plot the data in as many ways as you can think of! For time series, we usually start with a time plot

```
plot(Low~Year, data=y, ty="l")
```

usually, we add a line connecting points.



## 1.10 ARMA models

Another basic thing to do is to fit an **autoregressive-moving average** (ARMA) model. We'll look at ARMA models in much more detail later in the course. Let's fit an ARMA model given by

$$Y_n = \mu + \alpha(Y_{n-1} - \mu) + \epsilon_n + \beta\epsilon_{n-1}.$$

This has a one-lag autoregressive term,  $\alpha(Y_{n-1} - \mu)$ , and a one-lag moving average term,  $\beta\epsilon_{n-1}$ . It is therefore called an ARMA(1,1) model. These lags give the model some time dependence.

- If  $\alpha = \beta = 0$ , we get back to the basic independent model,  $Y_n = \mu + \epsilon_n$ .
- If  $\alpha = 0$  we have a moving average model with one lag, MA(1).
- If  $\beta = 0$ , we have an autoregressive model with one lag, AR(1).

We model  $\epsilon_1, \dots, \epsilon_N$  to be an independent, identically distributed sequence. To be concrete, let's specify a model where they are normally distributed with mean zero and variance  $\sigma^2$ .

## 1.11 A note on notation:

- In this course, capital Roman letters, e.g.,  $X$ ,  $Y$ ,  $Z$ , denote random variables. We may also use  $\epsilon$ ,  $\eta$ ,  $\xi$ ,  $\zeta$  for random noise processes. Thus, these symbols are used to build models.
- We use lower case Roman letters ( $x$ ,  $y$ ,  $z$ , . . .) to denote numbers. These are not random variables. Often,  $y$  will denote a data point.
- “We must be careful not to confuse data with the abstractions we use to analyze them.” (William James, 1842-1910).
- Other Greek letters will usually be parameters, i.e., real numbers that form part of the model.

# 1.12 Maximum likelihood

We can readily fit the ARMA(1,1) model by maximum likelihood,

```
armall <- arima(y$Low, order=c(1,0,1))
```

We can see a summary of the fitted model, where  $\alpha$  is called ar1,  $\beta$  is called ma1, and  $\mu$  is called intercept.

```
armall
```

```
##
## Call:
## arima(x = y$Low, order = c(1, 0, 1))
##
## Coefficients:
##          ar1          ma1  intercept
##         0.7852    -0.7414    -2.9680
## s.e.    0.3184     0.3429     0.8115
##
## sigma^2 estimated as 54.52:  log likelihood = -406.77,  aic = 821.55
```

We will write the ARMA(1,1) estimate of  $\mu$  as  $\hat{\mu}_2$ , and its standard error as  $SE_2$ .

# 1.13 Investigating R objects

Some poking around is required to extract the quantities of primary interest from the fitted ARMA model in R.

```
names(arma11)
```

```
## [1] "coef"      "sigma2"    "var.coef"  "mask"      "loglik"
## [6] "aic"       "arma"      "residuals" "call"      "series"
## [11] "code"      "n.cond"    "nobs"      "model"
```

```
mu2 <- arma11$coef["intercept"]
se2 <- sqrt(arma11$var.coef["intercept","intercept"])
cat("mu2 =", mu2, ", se2 =", se2, "\n")
```

```
## mu2 = -2.96805 , se2 = 0.8115071
```

## 1.14 Comparing the iid estimate with the ARMA estimate

We see that the two estimates,  $\hat{\mu}_1 = -2.93$  and  $\hat{\mu}_2 = -2.97$ , are close.

- However,  $SE_1 = 0.68$  is an underestimate of error, compared to the better estimate  $SE_2 = 0.81$ .
- The naive standard error needs to be inflated by  $100(SE_2/SE_1 - 1) = 19.1$  percent.

Exactly how the ARMA(1,1) model is fitted and the standard errors computed will be covered later.

**1.14.1 Question: How do we know if the ARMA analysis is more trustworthy?**

AIC, F test (likelihood ratio test).  
MSE? Out-of-the data. Residual analysis.

## 1.15 Model diagnostic analysis

- We should do **diagnostic analysis**. The first thing to do is to look at the residuals. For an ARMA model, the residual  $r_n$  at time  $t_n$  is defined to be the difference between the data,  $y_n$ , and a one-step ahead prediction of  $y_n$  based on  $y_{1:n-1}$ , written as  $y_n^P$ . From the ARMA(1,1) definition,

$$Y_n = \mu + \alpha(Y_{n-1} - \mu) + \epsilon_n + \beta\epsilon_{n-1},$$

a basic one-step-ahead predicted value corresponding to parameter estimates  $\hat{\mu}$  and  $\hat{\alpha}$  could be

$$y_n^P = \hat{\mu} + \hat{\alpha}(y_{n-1} - \hat{\mu}).$$

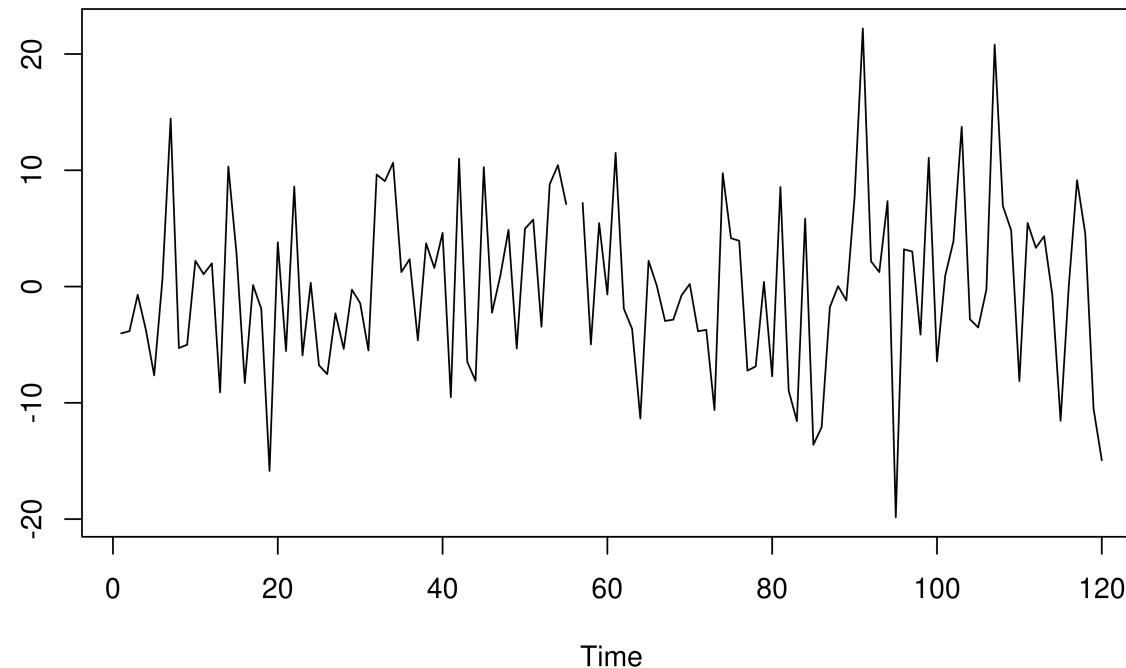
A so-called residual time series,  $\{r_n\}$ , is then given by

$$r_n = y_n - y_n^P.$$

In fact, R does something slightly more sophisticated.



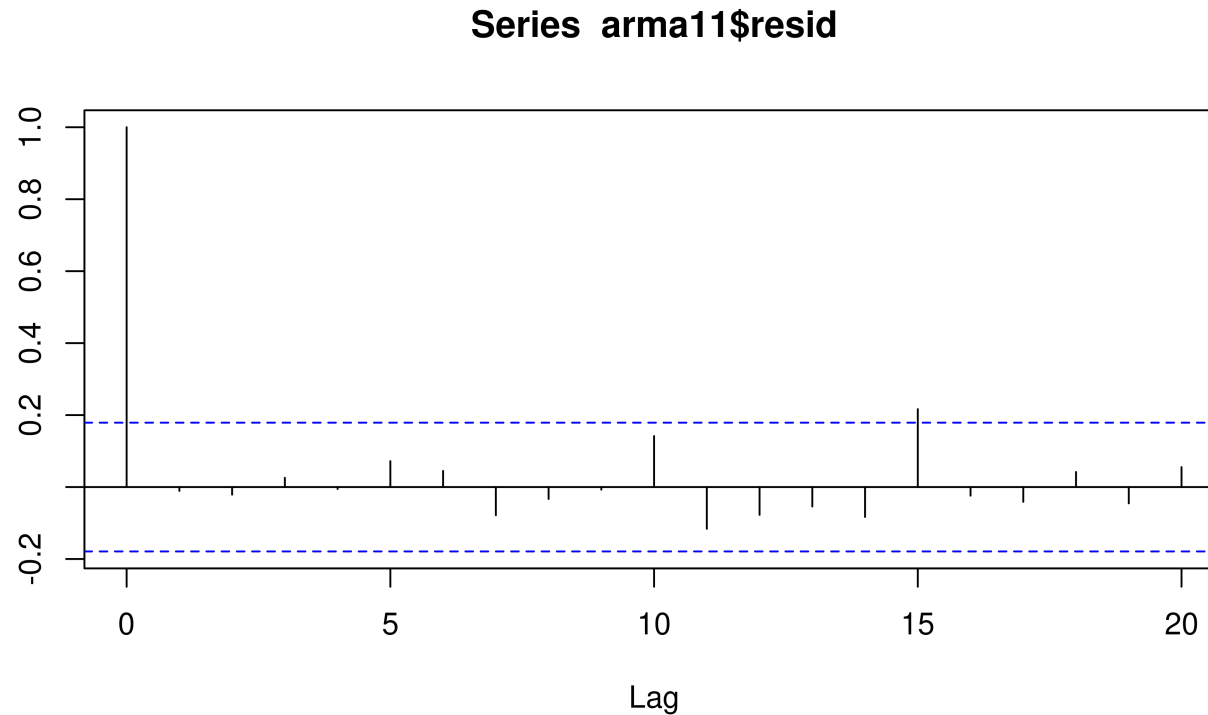
```
plot(armall$resid)
```



We see some slow variation in the residuals, over a decadal time scale. However, the residuals  $r_{1:N}$  are close to uncorrelated. We see this by plotting their pairwise sample correlations at a range of lags. This is the **sample autocorrelation function**, or **sample ACF**, written for each lag  $h$  as

$$\hat{\rho}_h = \frac{\frac{1}{N} \sum_{n=1}^{N-h} r_n r_{n+h}}{\frac{1}{N} \sum_{n=1}^N r_n^2}.$$

```
acf(arma11$resid, na.action=na.pass)
```



This shows no substantial autocorrelation. An ARMA model may not be a good way to describe the slow variation present in the residuals of the ARMA(1,1) model.

This is a toy example. However, inadequate models giving poor statistical uncertainty estimates is a real concern when working with time series data.

# 1.18 Quantifying uncertainty for scientific reproducibility

Usually, omitted dependency in the model will give overconfident (too small) standard errors.

- This leads to scientific reproducibility problems, where chance variation is too often assigned statistical significance.
- It can also lead to improper pricing of risk in financial markets, a factor in the US financial crisis of 2007-2008.

# 1.19 Models dynamic systems: State-space models

Scientists and engineers often have equations in mind to describe a system they're interested in. Often, we have a model for how the state of a stochastic dynamic system evolves through time, and another model for how imperfect measurements on this system gives rise to a time series of observations.

This is called a **state-space model**. The **state** models the quantities that we think determine how the system changes with time. However, these idealized state variables are not usually directly and perfectly measurable.

Statistical analysis of time series data on a system should be able to

1. Help scientists choose between rival hypotheses.
2. Estimate unknown parameters in the model equations.

We will look at examples from a wide range of scientific applications. The dynamic model may be linear or nonlinear, Gaussian or non-Gaussian.

## 1.20 A finance example: fitting a model for volatility of a stock market index

- Let  $\{y_n, n = 1, \dots, N\}$  be the daily returns on a stock market index, such as the S&P 500.
- Since the stock market is notoriously unpredictable, it is often unproductive to predict the mean of the returns and instead there is much emphasis on predicting the variability of the returns, known as the **volatility**.
- Volatility is critical to assessing the risk of financial investments.

Financial mathematicians have postulated the following model. We do not need to understand it in detail right now. The point is that investigators find it useful to develop models for how dynamic systems progress through time, and this gives rise to the time series analysis goals of estimating unknown parameters and assessing how successfully the fitted model describes the data.

$$Y_n = \exp\left\{\frac{H_n}{2}\right\}\epsilon_n, \quad G_n = G_{n-1} + \nu_n,$$

$$H_n = \mu_h(1 - \phi) + \phi H_{n-1}$$

$$+ Y_{n-1}\sigma_\eta\sqrt{1 - \phi^2} \tanh(G_{n-1} + \nu_n) \exp\left\{\frac{-H_{n-1}}{2}\right\} + \omega_n.$$

- $\{\epsilon_n\}$  is iid  $N(0, 1)$ ,  $\{\nu_n\}$  is iid  $N(0, \sigma_\nu^2)$ ,  $\{\omega_n\}$  is iid  $N(0, \sigma_\omega^2)$ .
- $H_n$  is unobserved volatility at time  $t_n$ . We only observe the return,  $Y_n$ .
- $H_n$  has auto-regressive behavior and dependence on  $Y_{n-1}$  and a slowly varying process  $G_n$ .

## 1.22 Questions to be addressed later in the course

This is an example of a **mechanistic model**, where scientific or engineering considerations lead to a model of interest. Now there is data and a model of interest, it is time to recruit a statistician!

1. How can we get good estimates of the parameters,  $\mu_h$ ,  $\phi$ ,  $\sigma_\nu$ ,  $\sigma_\omega$ , together with their uncertainties?
2. Does this model fit better than alternative models? So far as it does, what have we learned?
3. Does the data analysis suggest new models, or the collection of new data?

Likelihood-based inference for this partially observed stochastic dynamic system is possible, and enables addressing these questions (Bretó 2014). Carrying out such an analysis is facilitated by modern algorithms (Ionides et al. 2015). The R package system and Rmarkdown make state-of-the-art statistical analysis reproducible and extendable by Masters level statisticians.

# 1.23 Acknowledgments

- Previous versions of this material have been presented in 2016 and 2018.
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# References

Bretó, C. 2014. On idiosyncratic stochasticity of financial leverage effects. *Statistics & Probability Letters* 91:20–26.

Ionides, E. L., D. Nguyen, Y. Atchadé, S. Stoev, and A. A. King. 2015. Inference for dynamic and latent variable models via iterated, perturbed Bayes maps. *Proceedings of the National Academy of Sciences of the U.S.A.* 112:719–724.