Heapsort

web.cs.wpi.edu/~cs2223/d13/Lectures/Week2/Heapsort.ppt

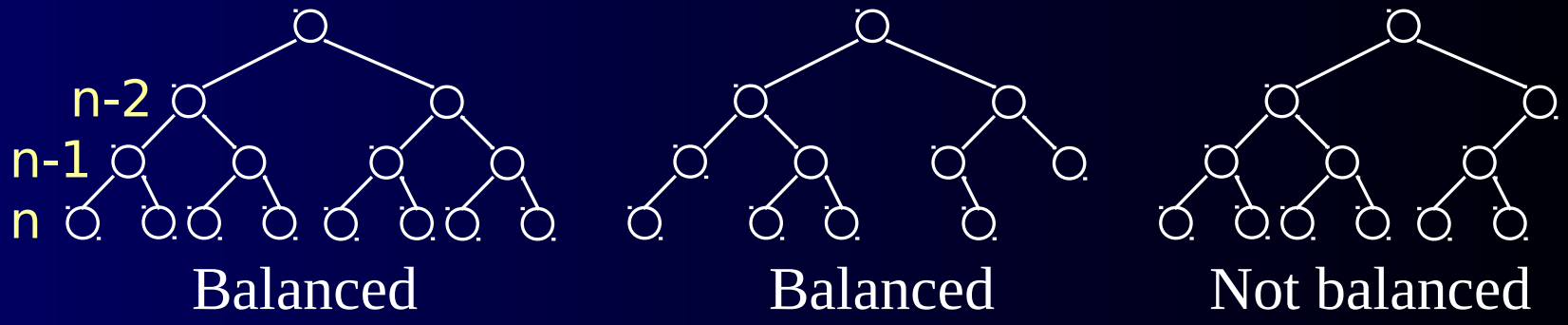
What is a “heap”?

- Definitions of heap:

  A balanced, left-justified binary tree in which no node has a value greater than the value in its parent

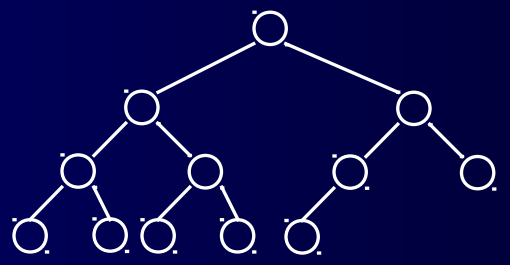
Balanced binary trees

- Recall:
  - The depth of a node is its distance from the root
  - The depth of a tree is the depth of the deepest node
- A binary tree of depth n is balanced if all the nodes at depths 0 through n-2 have two children

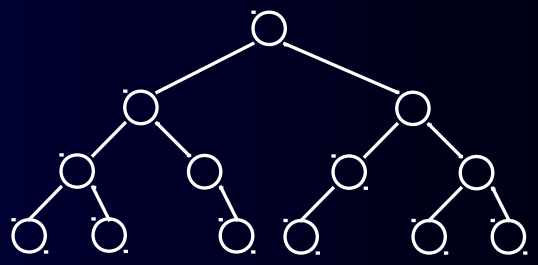


Left-justified binary trees

- A balanced binary tree is left-justified if:
  - all the leaves are at the same depth, or
  - all the leaves at depth `n+1` are to the left of all the nodes at depth `n`



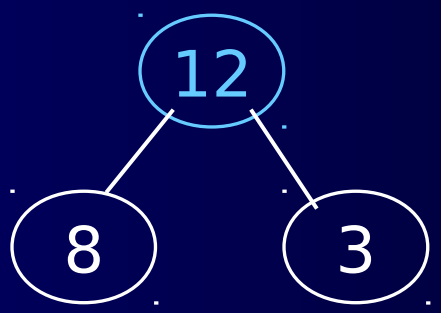
Left-justified



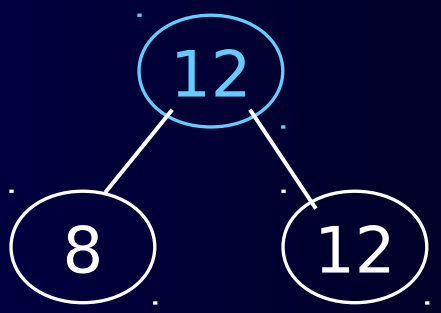
Not left-justified

The heap property

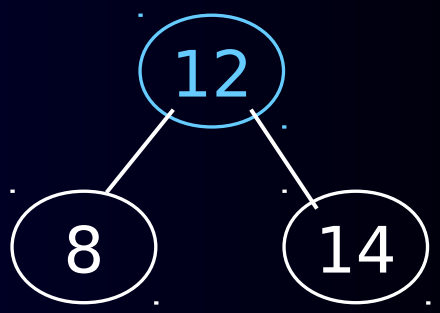
- A node has the heap property if the value in the node is as large as or larger than the values in its children



Blue node has heap property



Blue node has heap property



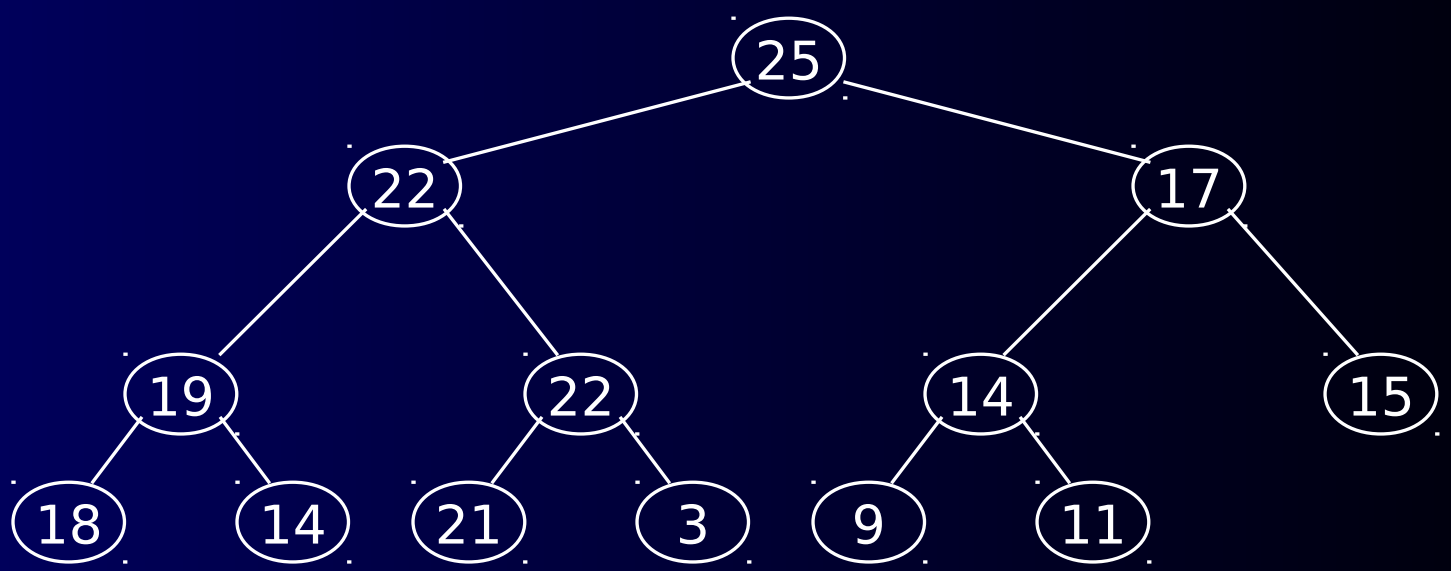
Blue node does not have heap property

All leaf nodes automatically have the heap property

A binary tree is a heap if *all* nodes in it have the heap property

A sample heap

- Here's a sample binary tree after it has been heapified

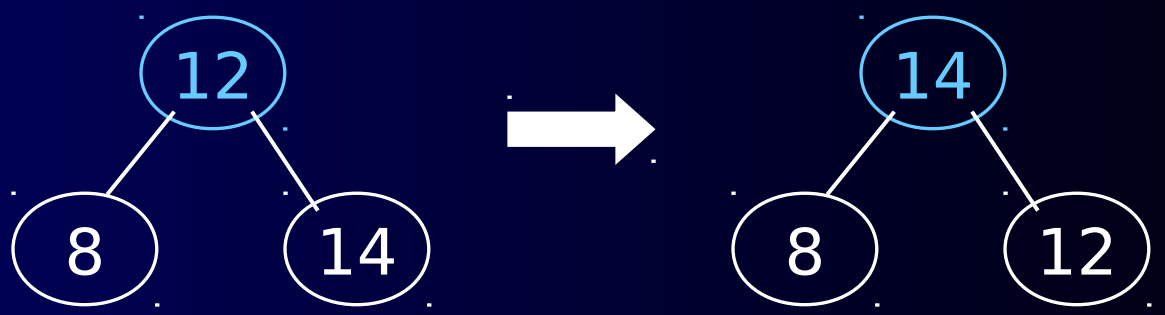


Notice that heapified does *not* mean sorted

Heapifying does *not* change the shape of the binary tree; this binary tree is balanced and left-justified because it started out that way

siftUp

- Given a node that does not have the heap property, you can give it the heap property by exchanging its value with the value of the larger child



Blue node does not have heap property

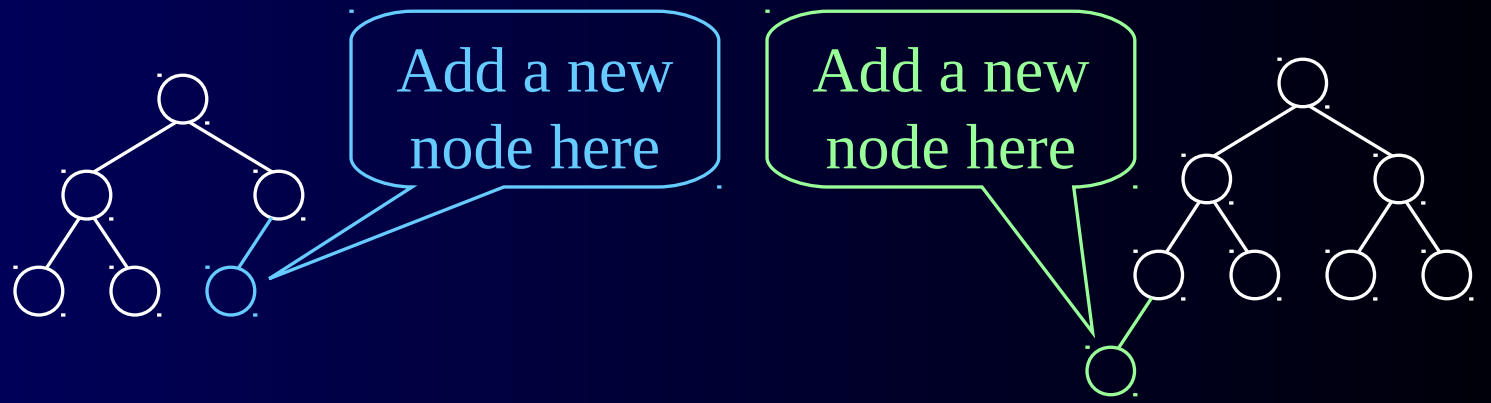
Blue node has heap property

This is sometimes called sifting up

Notice that the child may have *lost* the heap property

Insertion

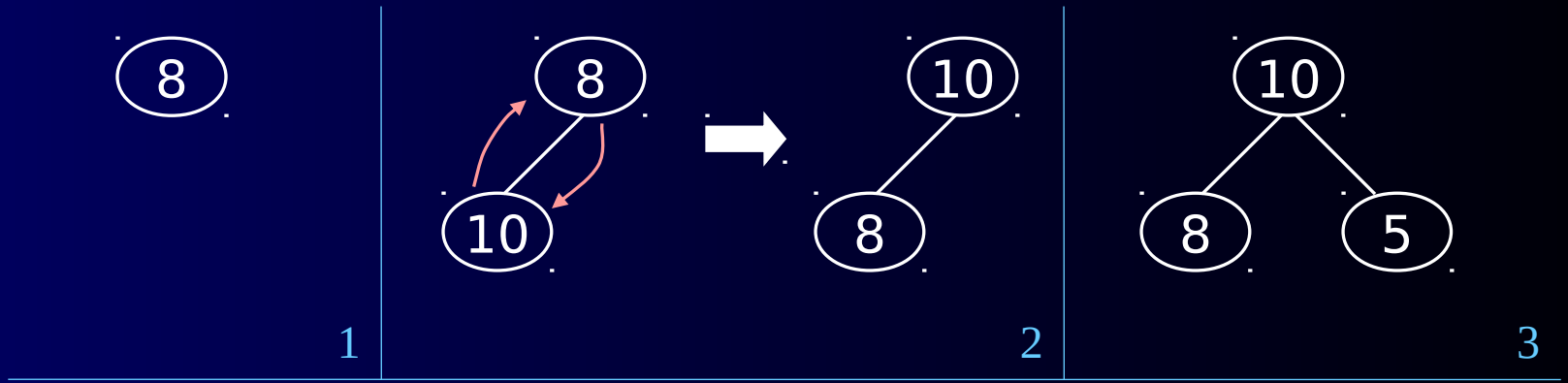
- A tree consisting of a single node is automatically a heap
  - Add the node just to the right of the rightmost node in the deepest level
  - If the deepest level is full, start a new level
- Examples:

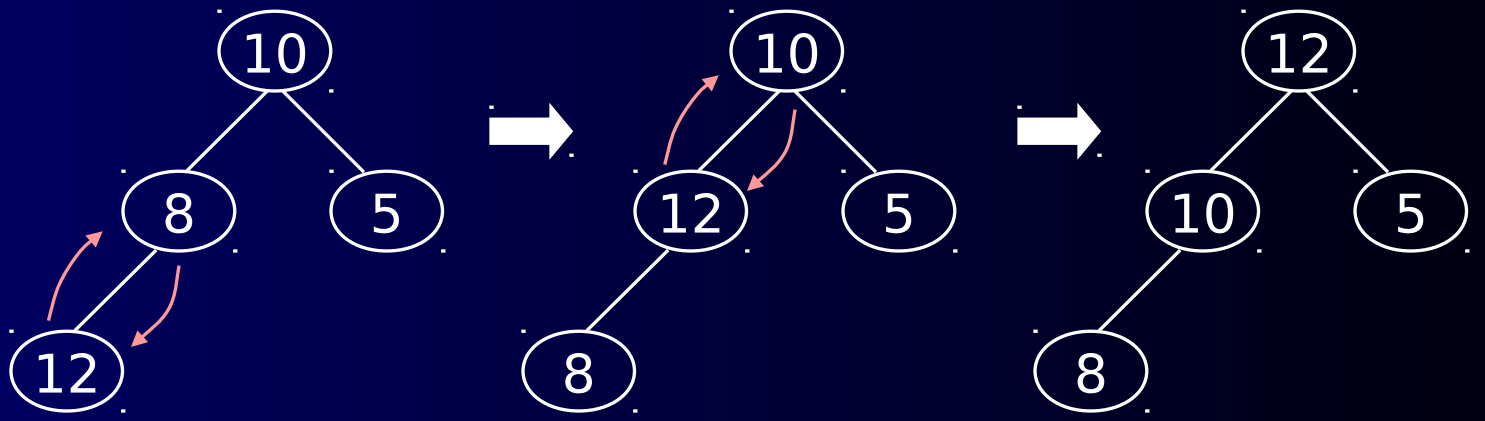


Insertion ..

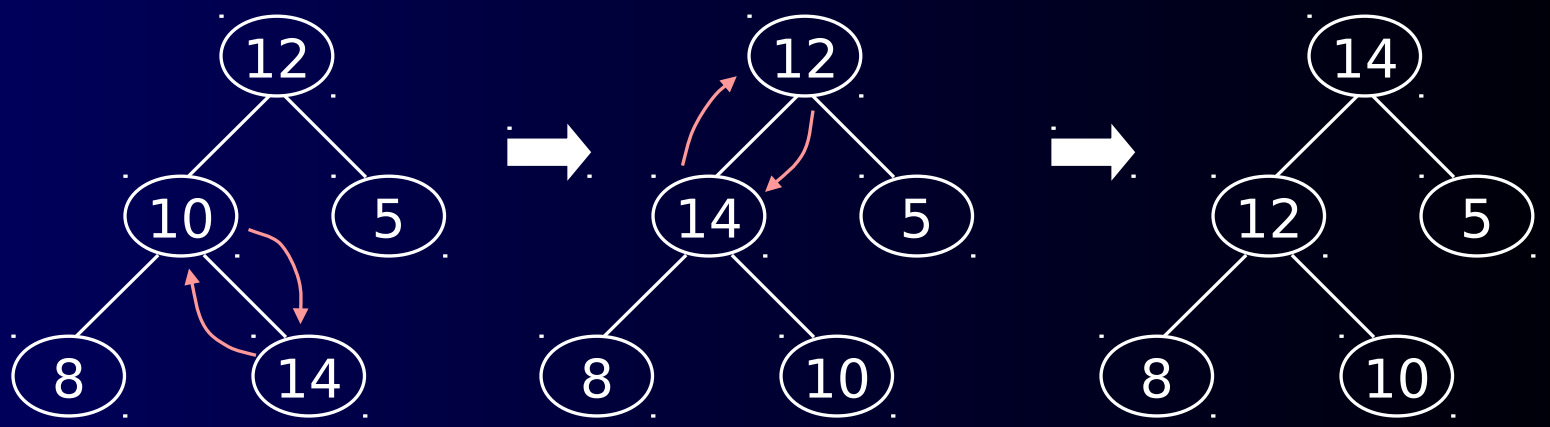
- Each time we add a node, we may destroy the heap property of its parent node
- To fix this, we sift up
- But each time we sift up, the value of the topmost node in the sift may increase, and this may destroy the heap property of *its* parent node
- We repeat the sifting up process, moving up in the tree, until either
  - We reach nodes whose values don't need to be swapped (because the parent is *still* larger than both children), or
  - We reach the root

Insertion ..





Insertion ..



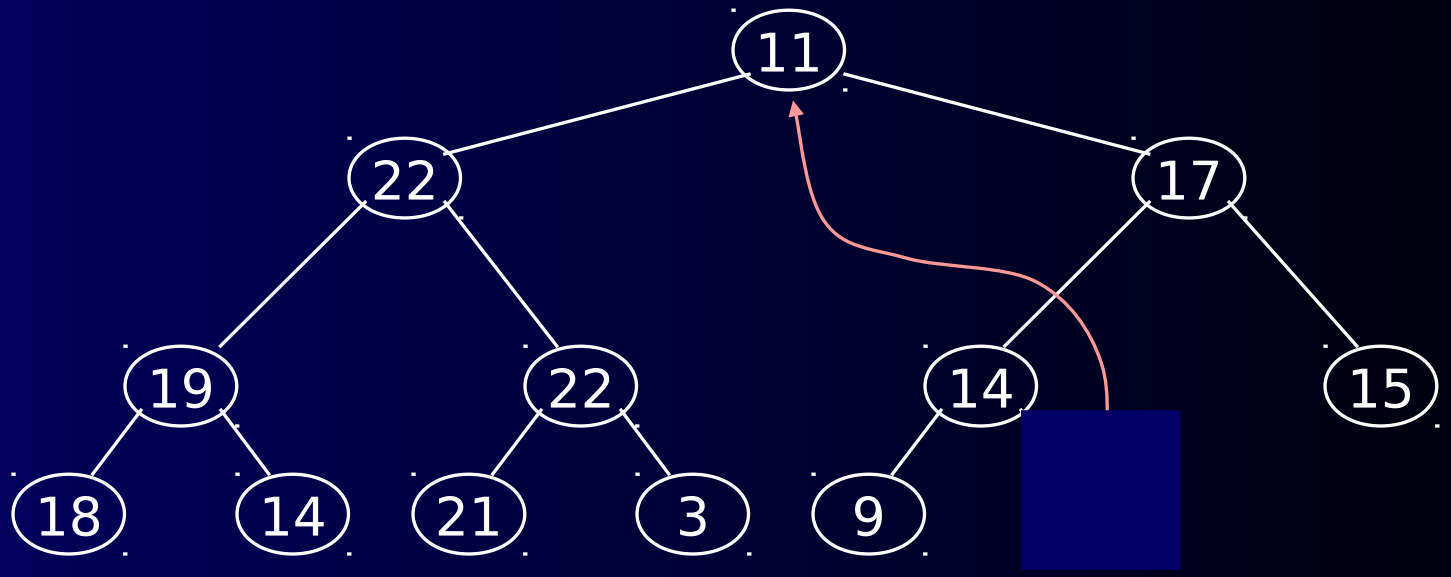
- The node containing 8 is not affected because its parent gets larger, not smaller

The node containing 5 is not affected because its parent gets larger, not smaller

The node containing 8 is still not affected because, although its parent got smaller, its parent is still greater than it was originally

Reheap on deletion

- Notice that the largest number is now in the root
- Suppose we *discard* the root:

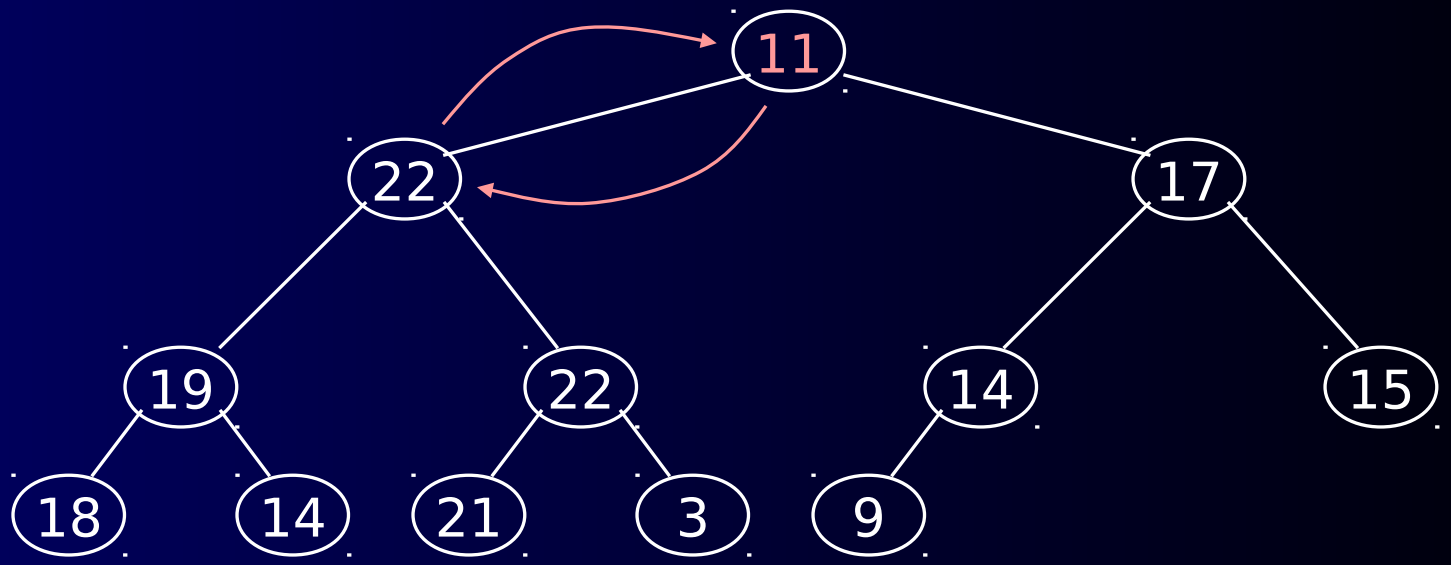


How can we fix the binary tree so it is once again *balanced and left-justified?*

Solution: remove the rightmost leaf at the deepest level and use it for the new root

The reHeap method

- Our tree is balanced and left-justified, but no longer a heap
- However, *only the root* lacks the heap property

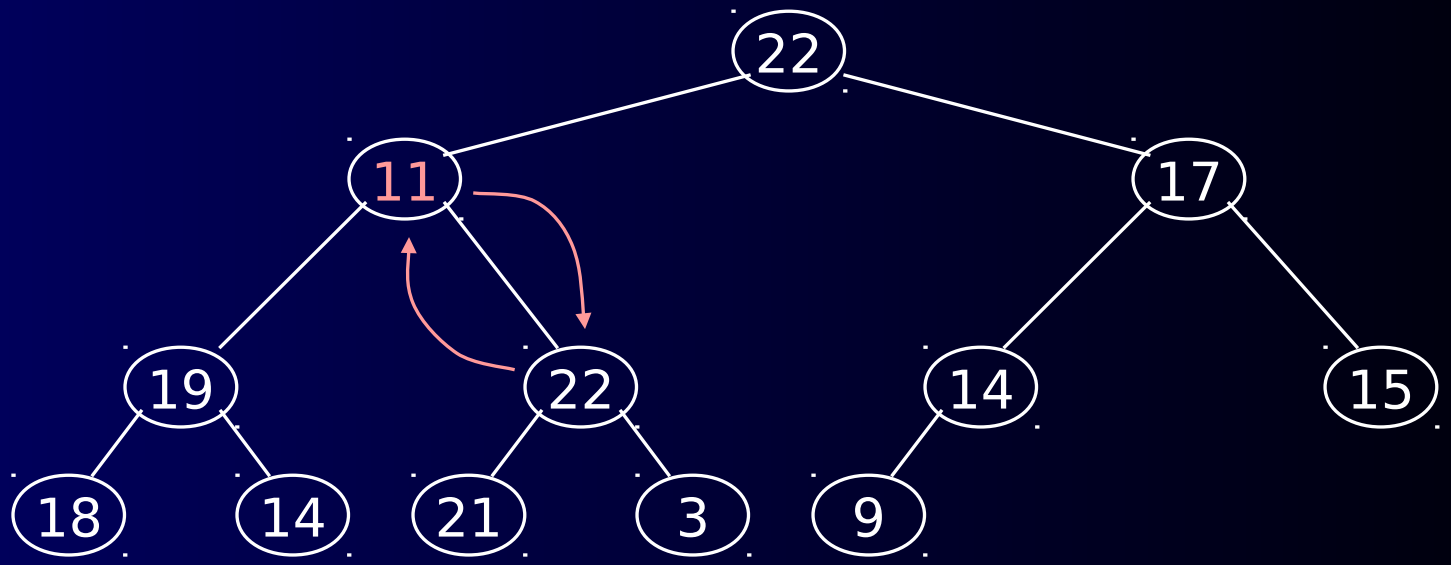


We can siftUp() the root

After doing this, one and only one of its children may have lost the heap property

The reHeap method ..

- Now the left child of the root (still the number 11) lacks the heap property

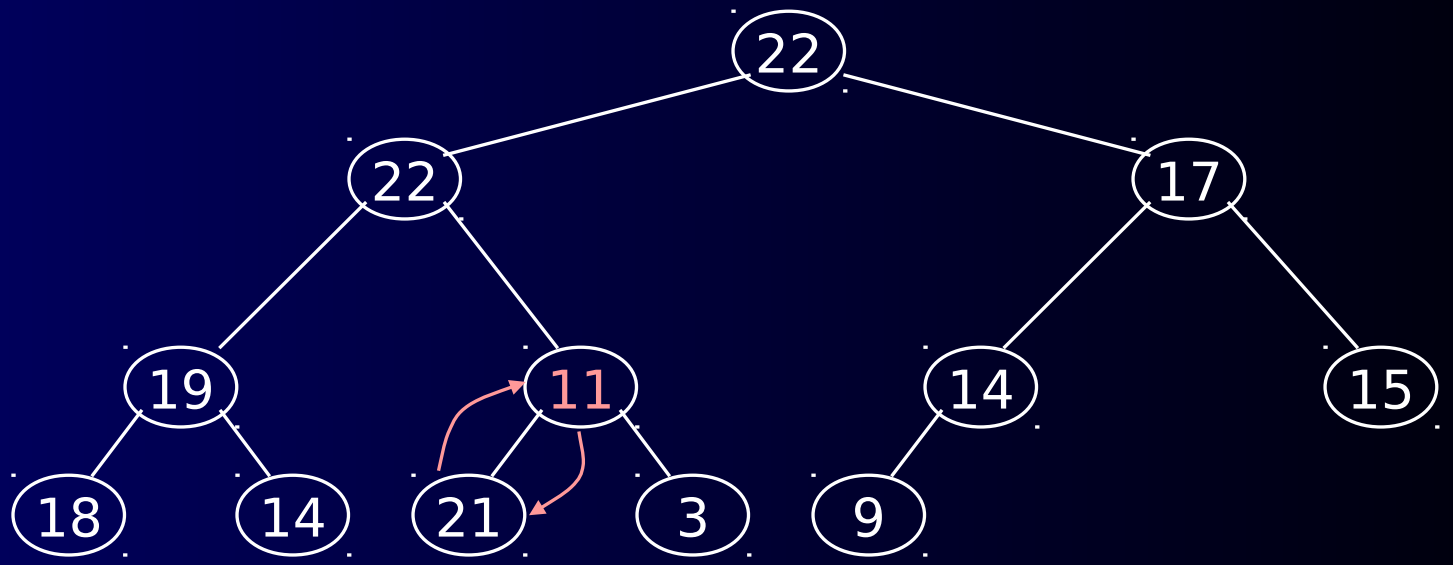


We can siftUp() this node

After doing this, one and only one of its children may have lost the heap property

The reHeap method ..

- Now the right child of the left child of the root (still the number 11) lacks the heap property:

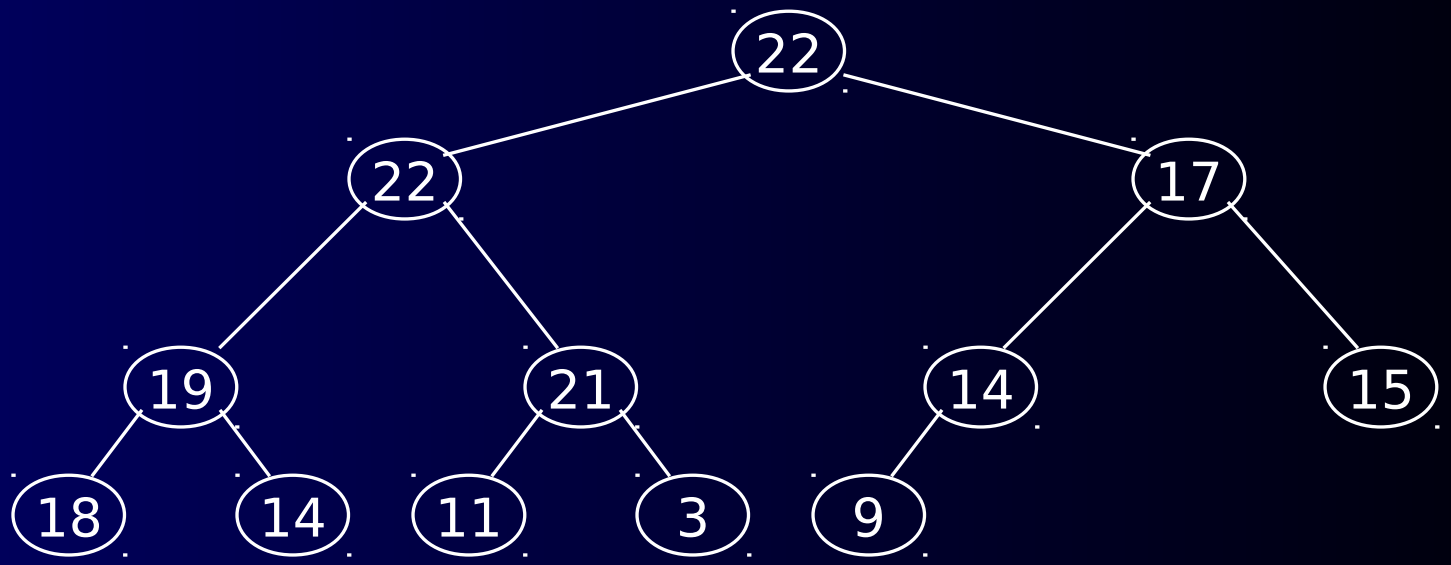


We can siftUp() this node

After doing this, one and only one of its children may have lost the heap property —but it doesn't, because it's a leaf

The reHeap method ..

- Our tree is once again a heap, because every node in it has the heap property



Once again, the largest (or *a* largest) value is in the root

We can repeat this process until the tree becomes empty

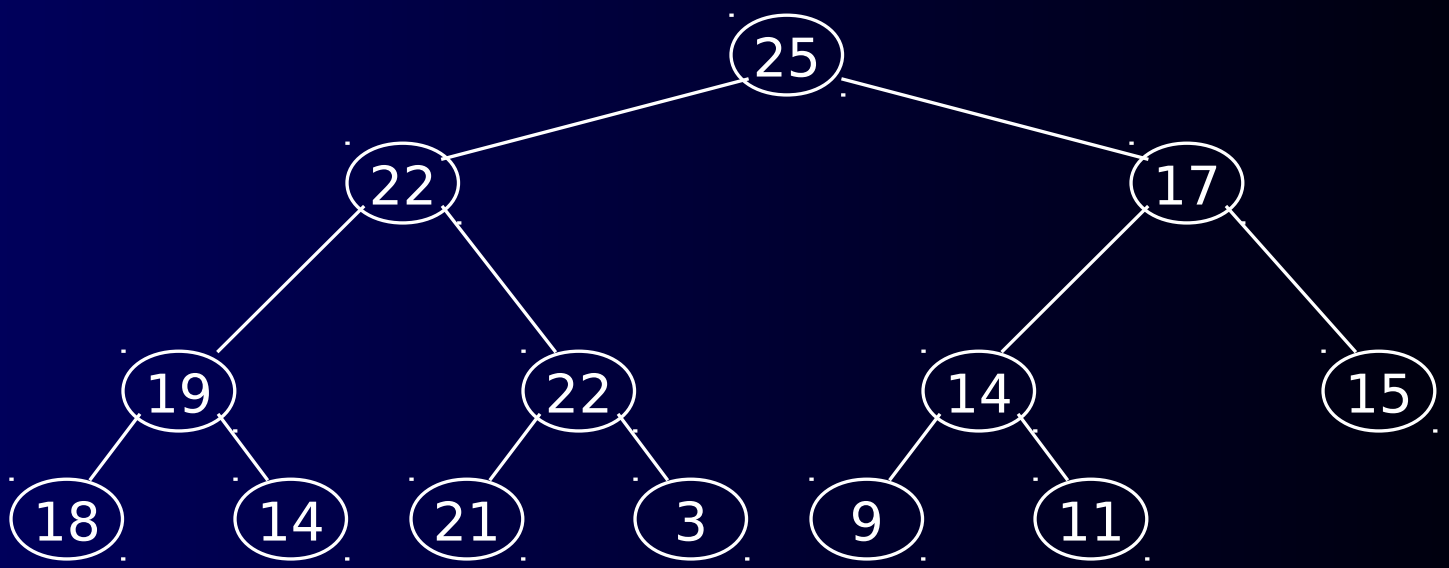
This produces a sequence of values in order largest to smallest

Sorting

- What do heaps have to do with sorting an array?
- Here's the neat part:
  - Because the binary tree is *balanced* and *left justified*, it can be represented as an array
  - All our operations on binary trees can be represented as operations on *arrays*
  - To sort:

```
heapify the array;
while the array isn't empty {
    remove and replace the root;
    reheap the new root node;
}
```

Mapping into an array

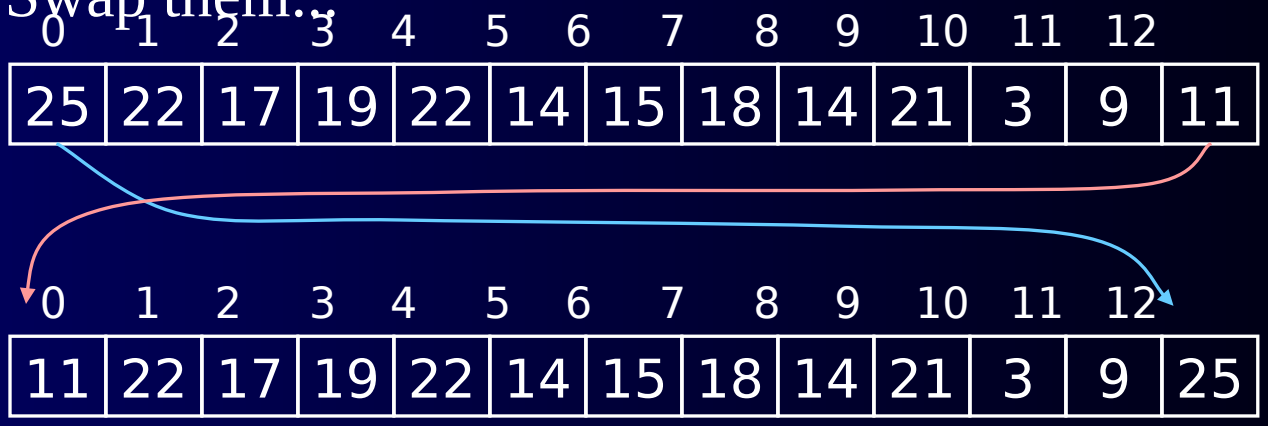


0	1	2	3	4	5	6	7	8	9	10	11	12
25	22	17	19	22	14	15	18	14	21	3	9	11

- Notice:
  - The left child of index i is at index 2*i+1
  - The right child of index i is at index 2*i+2
  - Example: the children of node 3 (19) are 7 (18) and 8 (14)

Removing and replacing the root

- The “root” is the first element in the array
- The “rightmost node at the deepest level” is the last element
- Swap them...



...And pretend that the last element in the array no longer exists—that is, the “last index” is `11` (9)

Reheap and repeat

- Reheap the root node (index 0, containing 11)...

...And again, remove and replace the root node

Remember, though, that the “last” array index is changed

Repeat until the last becomes first, and the array is sorted!

Naïve Build Heap

- Here's how the algorithm starts:

  heapify the array;

- Heapifying the array: we add each of n nodes
  - Each node has to be sifted up, possibly as far as the root
    - Since the binary tree is perfectly balanced, sifting up a single node takes O(log n) time
  - Since we do this n times, heapifying takes n*O(log n) time, that is, O(n log n) time

Build Heap O(n)

Build Max Heap can be done in O(n)

Note: Leaves of a heap lie in index $\lfloor n/2 \rfloor + 1$ to n.
If root has index 1 then nodes i has its parent at index $\lfloor i/2 \rfloor$
Thus the last element has its parent at $\lfloor n/2 \rfloor$
Hence node $\lfloor n/2 \rfloor + 1$ till n are leaves with no children.

```
BuildMaxHeap
{
    for i= |_ n/2 _| to 1
            MaxHeapify( A,i)
}
```

Linear BuildMaxHeap

$T(n)=cn/4+2cn/8+3cn/16+.....+ \log n\ c\ .1$
$=cn/2.\ (1/2 + 2/4 + 3/8 + 4/16 + \ldots)$

$1/2 + 2/4 + 3/8 + 4/16 + \ldots$
$= 1/2 + 1/4 + 1/8 + 1/16 + \ldots (= 1)$
$+ 1/4 + 1/8 + 1/16 + \ldots (= 1/2)$
$+ 1/8 + 1/16 + \ldots (= 1/4)$
$+ 1/16 + \ldots (= 1/8) ....$