Problem 1. (The SFGK zero-intelligence model and market impact)

In the Smith Farmer (SFGK) zero-intelligence model of the order book, market orders arrive at the rate μ, limit orders arrive at each price level at the rate α, and orders are canceled at the rate δ.

(a) Write down the dimensions of each of the parameters μ, α and δ.

(b) Recall the square-root formula for the market impact associated with the purchase of Q shares:

$$\frac{\Delta S}{S} \propto \sigma \sqrt{\frac{Q}{V}}$$

where S is the current stock price, σ is daily volatility, and V is average daily volume.

Using dimensional analysis, re-express this formula in terms of the SFGK parameters μ, α, and δ (and of course Q).

(c) Provide intuition for the sensitivities of your final formula to the parameters.

(d) Consider an order book with continuous order density $\rho(\Delta)$ where Δ is the distance (in dollars) to the best quote m. Write down an expression for the cumulative quantity $Q(\Delta)$ available in the order book up to size $m + \Delta$.

(e) Recall that *virtual market impact* is the price impact caused by a market order eating up existing limit orders in the order book.

Show that if the order density is continuous and linear in distance to the best quote, virtual market impact is square-root in the trade size.

Solution

(a) Dimensions are as follows:

Parameter	Dimensions
μ	$shares/T$
α	$shares/(ticks\, T)$
δ	$1/T$

(b) Write

$$\text{Impact} \propto A \sqrt{Q}.$$

Market impact has dimensions of ticks. $\sqrt{Q}$ has dimensions of $shares^{1/2}$. Thus A must have dimensions $ticks/\sqrt{shares}$. We then have

$$\frac{ticks}{\sqrt{shares}} = \frac{ticks\, T}{shares} \sqrt{\frac{shares}{T}} \frac{1}{\sqrt{T}}$$

so we must have

$$A \propto \frac{1}{\alpha} \sqrt{\mu\, \delta}$$

and so

$$\text{Impact} \propto \frac{1}{\alpha} \sqrt{\mu\, \delta} \sqrt{Q}.$$

(c) Impact decreases as the rate of arrival α of limit orders increases and increases as the rate of depletion of the order book via market orders and cancelations increases.

(d) The cumulative quantity $Q(\Delta)$ available in the order book up to size $m+\Delta$ is given by

$$Q(\Delta) = \int_0^{\Delta} \rho(x)\, dx.$$

(e) In the linear case, the order density $\rho(\delta) = \alpha\,\Delta$ for some $\alpha > 0$. Then

$$Q(\Delta) = \frac{1}{2}\alpha\,\Delta^2.$$

It follows that

$$\Delta = \sqrt{\frac{2\,Q(\Delta)}{\alpha}}$$

which is square-root in the quantity Q.

Problem 2. (Avellaneda and Stoikov (2008))
Consider a market maker $\mathcal{M}$ with CARA utility

$$U(W_T) = 1 - e^{-\alpha W_T}.$$

$\mathcal{M}$'s wealth W_t at time t consists of x dollars (in cash) and q shares of a stock whose current price is S_t. Suppose further that the stock price follows arithmetic Brownian motion: $dS_t = \sigma\, dZ_t$.

(a) Show that maximizing $U(W_T)$ is equivalent to maximizing

$$V(x,q,S) = \mathbb{E}\left[W_T\right] - \frac{\alpha}{2}\,\mathrm{Var}[W_T].$$

(b) Consider the "frozen inventory limit" where $\mathcal{M}$ holds inventory q until time T without trading.

Compute $\mathbb{E}\left[W_T\right]$ and $\mathrm{Var}[W_T]$ as of time $t < T$.

(c) With current inventory q shares, what are the market maker's indifference (or *reservation*) bid and ask prices, r_B and r_A respectively, for 1 share? What is the reservation bid-offer spread?

(d) Avellaneda and Stoikov go on to show that optimal bid and ask prices are given by

$$\begin{aligned} B &= r_B - \frac{1}{\alpha}\log\left(1+\alpha\,\frac{\lambda_B(B)}{\lambda_B{}'(B)}\right) \\ A &= r_A + \frac{1}{\alpha}\log\left(1-\alpha\,\frac{\lambda_A(A)}{\lambda_A{}'(A)}\right) \end{aligned}$$

where λ_B and λ_A are the arrival rates of market sell and market buy orders respectively.

Suppose now that

$$\lambda_B(B) = \frac{\eta}{(S-B)^\gamma}$$

for some $\gamma > 0$. For S sufficiently close to B show that in this case,

$$B \approx S - \frac{\alpha}{2 - 2/\gamma}\,(2\,q+1)\,\sigma^2\,(T-t).$$

Solution.

(a) $W = x + q\,S$ is normally distributed so

$$\mathbb{E}\left[U(W)\right] = \mathbb{E}\left[1 - e^{-\alpha W}\right] = 1 - \exp\left\{-\alpha\,\mathbb{E}\left[W_T\right] + \alpha^2/2\,\mathrm{Var}[W_T]\right\}.$$

Thus, maximizing utility is equivalent to maximizing

$$V(x,q,S) = \mathbb{E}\left[W\right] - \frac{\alpha}{2}\,\mathrm{Var}[W].$$

(b)

$$\mathbb{E}\left[W_T\right] = x + q\,S_t; \quad \mathrm{Var}[W_T] = q^2\,\sigma^2\,(T-t).$$

(c) For the dealer to be indifferent, we must have

$$V(x - r_B, q+1, S) = V(x,q,S)$$

so

$$x - r_B + (q+1)\,S - \frac{\alpha}{2}\,(q+1)^2\,\sigma^2\,(T-t) = x + q\,S - \frac{\alpha}{2}\,q^2\,\sigma^2\,(T-t)$$

from which it follows that

$$r_B = S - \frac{\alpha}{2}\,\left[(q+1)^2 - q^2\right]\,\sigma^2\,(T-t) = S - \frac{\alpha}{2}\,\left[2\,q+1\right]\,\sigma^2\,(T-t).$$

Similarly,

$$r_A = S + \frac{\alpha}{2}\,\left[(q-1)^2 - q^2\right]\,\sigma^2\,(T-t) = S - \frac{\alpha}{2}\,\left[-2\,q+1\right]\,\sigma^2\,(T-t).$$

(d) If

$$\lambda_B(B) = \frac{\eta}{(S-B)^\gamma}$$

then

$$\log\lambda_B(B) = \log\eta - \gamma\,\log(S-B)$$

and

$$\partial_B \log\lambda_B(B) = \frac{\lambda_B'(B)}{\lambda_B(B)} = \frac{\gamma}{S-B}.$$

...ituting back into the formula gives

$$\begin{aligned} B &= r_B - \frac{1}{\alpha}\log\left(1+\alpha\frac{S-B}{\gamma}\right) \\ &\approx r_B - \frac{S-B}{3} \\ &= S - \frac{\alpha}{2}\left[2q+1\right]\sigma^2(T-t) - \frac{1}{\gamma}(S-B). \end{aligned}$$

Rearranging gives

$$B \approx S - \frac{\alpha}{2-2/\gamma}\,(2q+1)\,\sigma^2\,(T-t).$$

Problem 3. (The Kyle (1985) model)

In the Kyle (1985) model, all traders know the distribution of $\tilde{S}$: $\tilde{S} \sim N(\mu, \sigma^2)$. The monopolistic informed trader $\mathcal{I}$ knows what the true value $\tilde{S}$ will be at the end of the trading period. Total demand y is the sum of the demand y_i of the informed trader and uninformed demand $y_u \sim N(0, \sigma_u^2)$ which is assumed independent of everything else.

(a) If $\mathcal{I}$ assumes that the market makers adopt a price function of the form $p = \mu + \lambda\, y$, find the informed trader's optimal (profit maximizing) demand y_i.

(b) The market makers $\mathcal{M}$ conjecture that $\mathcal{I}$'s demand function is of the form $y_i = a + b\,\tilde{S}$. Find a and b.

(c) Under perfect competition, the market makers set the pricing function as

$$p(y) = \mu + \lambda\, y = \mathbf{E}[\tilde{S}|y]$$

Deduce the formula for the Kyle lambda:

$$\lambda = \frac{\sigma}{2\,\sigma_u}.$$

(d) Why is σ_u a proxy for the level of noise trading?

(e) Some argue that insider trading is a very efficient way for private information to be impounded in prices. Why do regulators put so much effort into discouraging insider trading? Refer to the Kyle lambda in your argument.

Solution.

(a) $\mathcal{I}$ chooses his demand function y_i so as to maximize his expected profit:

$$\mathbf{E}\left[y_i\,(\tilde{S} - p(y))|\tilde{S}\right] = \mathbf{E}\left[y_i\,(\tilde{S} - \mu - \lambda(y_i + y_u))|\tilde{S}\right] = y_i\,(\tilde{S} - \mu - \lambda\, y_i)$$

Maximizing wrt y_i then gives $\mathcal{I}$'s optimal demand as

$$y_i = \frac{\tilde{S} - \mu}{2\,\lambda}.$$

(b) Then , if $y_i = a + b\,\tilde{S}$, we deduce that

$$a = -\frac{\mu}{2\,\lambda} \text{ and } b = \frac{1}{2\,\lambda}.$$

(c) Now regress $\tilde{S}$ against y (normal regression):

$$\tilde{S} = \mathbb{E}\left[\tilde{S}\right] + \theta\,(y - \mathbb{E}[y]) + \epsilon$$

to deduce that

$$\mathbb{E}[\tilde{S}|y] = \mu + \frac{\mathrm{Cov}[\tilde{S}, y]}{\mathrm{Var}(y)}\,(y - \mathbb{E}[y]) = \mu + \frac{b\,\sigma^2}{\sigma_u^2 + b^2\,\sigma^2}\,(y - a - b\,\mu)$$

Now $\forall y$,

$$\mu + \lambda\,y = \mu + \frac{b\,\sigma^2}{\sigma_u^2 + b^2\,\sigma^2}\,(y - a - b\,\mu)$$

so

$$\lambda = \frac{b\,\sigma^2}{\sigma_u^2 + b^2\,\sigma^2} = \frac{1}{2\,b}$$

or rearranging,

$$2\,b^2\,\sigma^2 = \sigma_u^2 + b^2\,\sigma^2$$

Solving for b gives

$$b = \frac{\sigma_u}{\sigma} \text{ and } \lambda = \frac{1}{2\,b} = \frac{\sigma}{2\,\sigma_u}.$$

(d) $y_u \sim N(0, \sigma_u^2)$ so

$$\mathbb{E}\left[|y_u|\right] = \sqrt{\frac{2}{\pi}}\,\sigma_u \propto \sigma_u.$$

Thus σ_u is a measure of the number of noise traders.

(e) Regulators try to minimize market impact (which is effectively the cost of trading) by promoting the participation σ_u of uninformed traders. In the extreme case, if insider trading is too prevalent, uninformed traders will cease to participate and $\lambda \to \infty$, in which case there would be market failure.

Problem 4. (The Roll model)

In the Roll model, trade prices p_t are modeled as

$$p_t = m_t + c\,\epsilon_t$$

where m_t is the efficient price, c is the market maker's cost to trade and $\epsilon_t = \pm 1$ is a trade sign indicator. Trade signs are assumed to be serially independent with equal probabilities of buys and sells.

(a) Compute the variance and first order autocovariance of the trade price process in terms of the mid-price volatility per trade σ and the half-spread c.

(b) Deduce a formula for the half-spread in terms of the first order autocovariance.

(c) Why is the first order autocovariance negative?

(d) The Zhou realized variance estimator is given by

$$\sigma^2_{Zhou} := \sum_t^T (p_t - p_{t-1})^2 + \sum_t^T (p_t - p_{t-1})(p_{t-1} - p_{t-2}) + \sum_t^T (p_t - p_{t-1})(p_{t+1} - p_t)$$

Show that in the Roll model,

$$\mathbb{E}\left[\sigma^2_{Zhou}\right] = \sum_t^T (m_t - m_{t-1})^2,$$

the realized variance of the efficient price process.

Solution

(a) We have

$$\begin{aligned}
\gamma_0 := \text{Var}[\Delta p_t] &= \mathbb{E}[\Delta p_t^2] \\
&= \mathbb{E}[\Delta m_t^2] + c^2\,\text{Var}[\epsilon_t - \epsilon_{t-1}] \\
&= \sigma^2 + 2\,c^2
\end{aligned}$$

Also,

$$\begin{aligned}
\gamma_1 &:= \text{Cov}[\Delta p_{t-1}, \Delta p_t] \\
&= \mathbb{E}[\Delta p_{t-1}\,\Delta p_t] \\
&= \mathbb{E}[\{\Delta m_{t-1} + c\,(\epsilon_{t-1} - \epsilon_{t-2})\}\,\{\Delta m_t + c\,(\epsilon_t - \epsilon_{t-1})\}] \\
&= -c^2
\end{aligned}$$

(b) We conclude that the effective half-spread is given by

$$c = \sqrt{-\gamma_1}.$$

(c) The first order autocovariance is negative because of bid-ask bounce.

(d) In the Roll model,

$$p_t - p_{t-1} = m_t - m_{t-1} + \epsilon_t - \epsilon_{t-1}.$$

Then

$$\begin{aligned}
\mathbb{E}[(p_t - p_{t-1})^2] &= \mathbb{E}\left[(m_t - m_{t-1})^2\right] + 2\,c^2 \\
\mathbb{E}[(p_t - p_{t-1})\,(p_{t-1} p_{t-2})] &= -c^2 \\
\mathbb{E}[(p_t - p_{t-1})\,(p_{t+1} p_t)] &= -c^2.
\end{aligned}$$

Substituting into the definition of the Zhou estimator gives the result.

Problem 5. (The Madhavan-Richardson-Roomans (MRR) model)
In the MRR model, innovations in the fair price V_t are modeled as

$$V_t = V_{t-1} + \lambda\,(\epsilon_t - \mathbf{E}[\epsilon_t|\mathcal{F}_{t-1}]) + \eta_t$$

where ϵ_t is the sign of the order at time t and η_t is a (zero expectation) noise that represents for example news.

(a) Explain the concept of *perfect competition* as it relates to the determination of the prices at which trades occur.

(b) Assuming perfect competition, where will the market maker set bid and ask prices given V_{t-1} and $\hat{\epsilon}_t := \mathbf{E}[\epsilon_t|\mathcal{F}_{t-1}]$? What is the bid-offer spread s in this simplified version of the MRR model?

(c) From the above, deduce an expression for the trade price p_t in terms of V_{t-1} and the realized trade sign ϵ_t.

(d) Now model the order sign process as AR(1) so that

$$\epsilon_t = \rho\,\epsilon_{t-1} + \xi_t$$

where ξ_t is independent noise. Compute $\hat{\epsilon}_t = \mathbf{E}[\epsilon_t|\epsilon_{t-1}]$ with this assumption.

(e) Deduce an expression for innovations in trade prices of the form

$$\Delta p_t = \alpha\,\epsilon_t + \beta\,\epsilon_{t-1} + \text{noise}$$

providing explicit expressions for α, β and the noise term.

(f) How would you estimate λ and ρ from TAQ data?

(g) How realistic is it to model the order sign process ϵ_t as AR(1)? Suggest a better model for the order sign process.

Solution

(a) Under perfect competition, the bid(offer) price is set to the expectation of the price conditional on a market sell(buy). The market maker makes no profit.

(b) Under perfect competition, the expected profit if hit or taken should be zero. So

$$\begin{aligned} p_A &= \mathbf{E}[V_t|\epsilon_t = +1] = V_{t-1} + \lambda\,(1 - \hat{\epsilon}_t) \\ p_B &= \mathbf{E}[V_t|\epsilon_t = -1] = V_{t-1} + \lambda\,(-1 - \hat{\epsilon}_t) \end{aligned}$$

Thus $s = 2\,\lambda$.

(c)

$$p_t = V_{t-1} + \lambda\,(\epsilon_t - \hat{\epsilon}_t).$$

(d) With

$$\epsilon_t = \rho\,\epsilon_{t-1} + \xi_t,$$

we have

$$\hat{\epsilon}_t = \rho\,\epsilon_{t-1}.$$

(e) From above,

$$\begin{aligned}
\Delta p_t &= \Delta V_{t-1} + \lambda\,(\epsilon_t - \hat{\epsilon}_t) - \lambda\,(\epsilon_{t-1} - \hat{\epsilon}_{t-1}) \\
&= \lambda\,(\epsilon_{t-1} - \hat{\epsilon}_{t-1}) + \eta_{t-1} + \lambda\,(\epsilon_t - \hat{\epsilon}_t) - \lambda\,(\epsilon_{t-1} - \hat{\epsilon}_{t-1}) \\
&= \lambda\,(\epsilon_t - \rho\,\epsilon_{t-1}) + \eta_{t-1}.
\end{aligned}$$

(f) First assign order signs ϵ using for example the Lee-Ready rule. Compute the first order autocorrelation coefficient ρ. Then estimate λ by regression.

(g) The order sign process is found to be long-memory in practice whereas AR(1) is a very short memory process. One could model order signs using an AR(n) process with n large or using a long memory model such as ARFIMA.

Problem 6. (Obizhaeva and Wang)

Denote the position at time t by x_t with $x_0 = X$ and $x_T = 0$. In the Obizhaeva-Wang model, the stock price S_t evolves as

$$S_t = S_0 + \eta \int_0^t \dot{x}_s\, e^{-\rho(t-s)}\, ds + \int_0^t \sigma\, dZ_s \tag{1}$$

where $\dot{x}_t$ is the (signed) rate of trading.

(a) The (random) cost of execution (implementation shortfall) is given by

$$\mathcal{C}[x] = \int_0^T (S_t - S_0)\, dx_t = S_0\, X + \int_0^T S_t\, dx_t$$

Assuming that x is continuous, show that

$$\mathcal{C}[x] = \eta \int_0^T \dot{x}_t\, dt \int_0^t \dot{x}_s\, e^{-\rho(t-s)}\, ds - \sigma \int_0^T x_t\, dZ_t.$$

(b) The Obizhaeva-Wang optimal trading strategy minimizes the expected cost $\mathbb{E}\,[\mathcal{C}[x]]$. Does the optimal strategy depend on the stock price S_t?

(c) Recall from variational calculus that if

$$\mathcal{C}[x] = \int_0^T L(t, x_t, \dot{x}_t)\, dt$$

with boundary conditions $x_0 = X$, $x_T = 0$, we have the Euler-Lagrange equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Show that the statically optimal strategy x satisfies the Fredholm integral equation

$$\int_0^T \dot{x}_s\, e^{-\rho|t-s|}\, ds = C \tag{2}$$

where C is a constant.

(d) Verify by substitution into (2) that the optimal trading rate $\{\dot{x}_s\}$ is given by

$$\dot{x}_s = a\,\{\delta(s) + \rho + \delta(T-s)\}$$

for some constant a where δ denotes the Dirac delta function. Explain what the three terms mean (how would you execute this strategy in practice?).

(e) Can you lower the expected cost by allowing the trading rate $\dot{x}_t$ to depend on the stock price S_t?

Solution

(a)

$$\begin{aligned}
\mathcal{C}[x] &= \int_0^T (S_t - S_0)\, dx_t \\
&= \int_0^T \dot{x}_t\, dt \left\{\eta \int_0^t \dot{x}_s\, e^{-\rho(t-s)}\, ds + \int_0^t \sigma\, dZ_s\right\} \\
&= \eta \int_0^T \dot{x}_t\, dt \int_0^t \dot{x}_s\, e^{-\rho(t-s)}\, ds + \int_0^T \dot{x}_t\, dt \int_0^t \sigma\, dZ_s \\
&= \eta \int_0^T \dot{x}_t\, dt \int_0^t \dot{x}_s\, e^{-\rho(t-s)}\, ds + \int_0^T \sigma\, dZ_s \int_s^T \dot{x}_t\, dt \\
&= \eta \int_0^T \dot{x}_t\, dt \int_0^t \dot{x}_s\, e^{-\rho(t-s)}\, ds - \int_0^T \sigma\, x_s\, dZ_s.
\end{aligned}$$

(b) $\mathcal{C}[x]$ does not depend on the stock price so the optimal strategy does not depend on the stock price.

(c) In this case, $L(t, x, \dot{x})$ does not depend on x, only on $\dot{x}$, so the E-L equation becomes

$$\frac{\partial L}{\partial \dot{x}} = C.$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{x}_t} &= \eta \int_0^t \dot{x}_s \, e^{-\rho(t-s)} \, ds + \eta \int_t^T \dot{x}_s \, e^{-\rho(s-t)} \, ds \\
&= \eta \int_0^T \dot{x}_s \, e^{-\rho|t-s|} \, ds
\end{aligned}
$$

as required.

(d) Considering each of the terms in the proposed solution in turn:

$$
\begin{aligned}
I_1 &= \int_0^T \delta(s) \, e^{-\rho|t-s|} \, ds = e^{-\rho t} \\
I_2 &= \int_0^T \delta(T-s) \, e^{-\rho|t-s|} \, ds = e^{-\rho(T-t)} \\
I_3 &= \int_0^T \rho \, e^{-\rho|t-s|} \, ds \\
&= \int_0^t \rho \, e^{-\rho(t-s)} \, ds + \int_t^T \rho \, e^{-\rho(s-t)} \, ds \\
&= 1 - e^{-\rho t} + 1 - e^{-\rho(T-t)}.
\end{aligned}
$$

Adding these gives $I_1 + I_2 + I_3 = 2$ which is constant as required.

(e) No, the optimal strategy does not depend on the stock price.