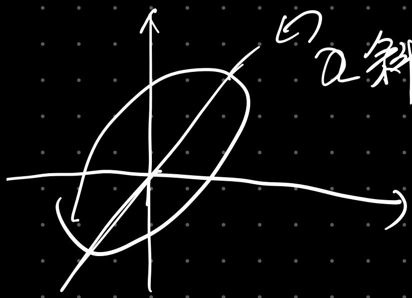
$Y = aX \Rightarrow E(Y - aX)^2$ (mean sequare Error)



$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & (x,y) \in D \\ 0 & \text{others} \end{cases}$

$\frac{\partial}{\partial x} = 2(\cdot)x)a \Rightarrow a = \frac{Y}{X} = \frac{XY}{X^2}$

两者相差一个参数

$\Rightarrow \min_a (Y - aX)^2 \Rightarrow a_{opt} = \frac{E(XY)}{E(X^2)} \longrightarrow E(XY)$ Correlation $\mathbb{R}$

$E(X - EX)(Y - EY) = E(XY) - E(X \cdot EY) - E(Y \cdot EX) + EX\,EY$

$= E(XY) - EY \cdot EX - EX\,EY + EX \cdot EY = E(XY) - EX \cdot EY$

不区分两者相关的表达(只差一个常数而已)

uncorrelated: $E(XY) = 0$

$\Uparrow$

Independence:

$\theta \sim U(0, 2\pi)$

$X = \cos(\theta)$, $Y = \sin\theta$

$X^2 + Y^2 = 1$

$\Rightarrow EX = \int_{-\infty}^{+\infty} \cos\theta \cdot f_\theta(\theta) d\theta = \int_0^{2\pi} \sin\theta \cdot \frac{1}{2\pi} d\theta = 0$

$EY = 0$, $E(XY) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta \cdot \cos\theta \, d\theta = 0$

$E(XY) = EX \cdot EY$ 故 X 与 Y 不相关

Geometric View

$E(XY) = \langle X, Y \rangle$ Inner Product 内积

$\langle x, y \rangle : H \times H \to \mathbb{R}$ ① $\langle X, Y \rangle = \langle Y, X \rangle$ ② $\langle X, X \rangle \geq 0$

$\langle X, X \rangle = 0 \Rightarrow X = 0$

③ Bilinear: $\langle X, ay + \beta Z \rangle = a\langle X, Y \rangle + \beta \langle X, Z \rangle$

$\langle ax + \beta Y, Z \rangle = a\langle X, Z \rangle + \beta \langle Y, Z \rangle$

$E|X|^2 = 0 \Rightarrow X = 0$

$\Downarrow$

$P(X = 0) = 1$

概率为0并不是不发生，反例：数轴上取一个点，取到有理数的P为0，但不能说有理数取不到

把相关看成内积

$\cos\angle(x,y) = \dfrac{\langle x,y\rangle}{(\langle x,x\rangle\langle y,y\rangle)^{1/2}}$ Random Variable $\Longleftrightarrow$ Vector

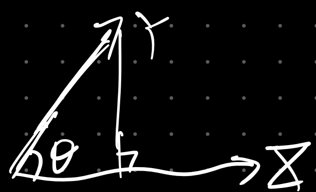
Cauchy–Schwarz

$\cos\angle(X,Y) = \dfrac{E(XY)}{[EX^2\cdot EY^2]^{1/2}}$ 保证了 $\cos\angle$ 在 $[-1,1]$ 之间 $|\langle x,y\rangle| \le (\langle x,x\rangle\langle y,y\rangle)^{1/2}$

$0 \le g(\lambda) = \langle \lambda x+y, \lambda x+y\rangle = \lambda^2\underbrace{\langle x,x\rangle}_{>0} + 2\lambda\langle x,y\rangle + \langle y,y\rangle \Rightarrow \Delta \le 0$

$|\sum_k x_k y_k| \le (\sum_k x_k^2)^{1/2}(\sum_k y_k^2)^{1/2}$, $\int f(x)g(x)dx \le (\int f^2(x)dx \int g^2(x)dx)^{1/2}$

Cauchy-Schwarz 其他形式



$\|Y\|\cdot\cos\theta\dfrac{X}{\|X\|} = \left(\dfrac{\|Y\|}{\|X\|}\cdot\cos\theta\right)\cdot X$

$\dfrac{\|Y\|}{\|X\|}\cdot\dfrac{E(XY)}{\|X\|\cdot\|Y\|} = \dfrac{E(XY)}{EX^2} = a$

Correlation Function 自相关 Auto

$X(t)$：随机过程是一组随机变量，t: Index | 两维：Random Field. 随机场

$R_X(t,s) = E(X(t)\cdot X(s))$. Binary $\longrightarrow$ Univariate Assumption

① $R_X(t,s) = R_X(s,t)$

② $|R_X(t,s)| \le (R_X(t,t)R_X(s,s))^{1/2}$ ← 相关运算带来的

令 D 满足 $X(t)-m(t)$

Stationary 平稳性 ⇐ Invariance 不变性，某一类统计性质随着时间变化保持不变的特性

Wide-Sense 宽平稳

① 均值 $E(X(t)) = m(t) \equiv m$（常数）

② $R_X(t,s) = R_X(t+D, s+D)$, $\forall D \in \mathbb{R}$ 两个变元上同时平移 D

$\hookrightarrow R_X(t,s) = R_X(t-s) = R_X(\tau)$

相关函数的简化 $\tau = t-s$

二元 → 一元

R_X 函数只与 $t-s$ 有关 $\tau = t-s$

例子：

① Modulated Signal.

$X(t) = A(t)\cdot\cos(2\pi f_0 t+\theta)$, $A(t)$, r.v. $\theta \sim U(0,2\pi)$ Independent

一阶矩

$$E(X(t)) = E(A(t))\cdot E(\cos(2\pi f_0 t+\theta))$$
$$= E(A(t))\int_0^{2\pi}\frac{1}{2\pi}\cos(2\pi f_0+\theta)\,d\theta = 0$$

$$R_X(t,s) = E(A(t)\cdot A(s))\,E(\cos(2\pi f_0 t+\theta)\cdot\cos(2\pi f_0 s+\theta))$$
$$= E(A(t)\cdot A(s))\left[\frac{1}{2}\left(E\cos(2\pi f_0(t+s)+2\theta) + E(\cos(2\pi f_0(t-s))\right)\right]$$
$$E\cos(2\pi f_0(t+s)+2\theta) = 0$$
$$= E(A(t)A(s))\,E(\cos(2\pi f_0(t-s)))$$
$$= \frac{1}{2}R_A(t-s)\cdot\cos(2\pi f_0(t-s))$$

振幅调制是宽平稳，且随机噪声与振幅独立，那么总的就是宽平稳的

② Random Telegraph Signal

任意时段内，随机跳

$[s,t]$ 内，K，切为 1 或 -1 具有随机性

$$P = \frac{\lambda(t-s)^k}{k!}\exp(-\lambda(t-s))$$ Poisson Distribution

$$E(X(t)\cdot X(s)) = R_X(t,s) = 1\cdot P_1 + (-1)\cdot P_{-1}$$

$X(t)$ 与 $X(s)$ 两者乘积只能是 1 或 -1

用翻转次数来描述

$$\begin{cases} P_1 = P([s,t].\,\text{even}) = \sum\limits_{k\ \text{even}}\frac{(\lambda(t-s))^k}{k!}\exp(-\lambda(t-s)) \\ P_{-1} = P([s,t],\,\text{odd}) = \sum\limits_{k\ \text{odd}}\frac{(\lambda(t-s))^k}{k!}\exp(-\lambda(t-s)) \end{cases}$$

$$\sum_{k=0}^{\infty}\frac{\lambda(t-s)^k}{k!} = \exp(\lambda(t-s)),\quad \sum_{k\ \text{even}}\frac{(\lambda(t-s))^k}{k!} = \frac{1}{2}(\exp(\lambda(t-s))+\exp(-\lambda(t-s)))$$

$$P_1 = \frac{1}{2}(1+\exp(-2\lambda(t-s))),\quad P_{-1} = \frac{1}{2}(1-\exp(-2\lambda(t-s)))$$

$$R_X(t,s) = \frac{1}{2}(1+\exp(-2\lambda(t-s))) - \frac{1}{2}(1-\exp(-2\lambda(t-s)))$$
$$= \exp(-2\lambda(t-s))$$