Q1: Suppose we roll a red die and a green die

(i) What is the sample space for this experiment?

{1,1} {1,2} {1,3} {1,4} {1,5} {1,6}
{2,1} {2,2} {2,3} {2,4} {2,5} {2,6}
{3,1} {3,2} {3,3} {3,4} {3,5} {3,6}
{4,1} {4,2} {4,3} {4,4} {4,5} {4,6}
{5,1} {5,2} {5,3} {5,4} {5,5} {5,6}
{6,1} {6,2} {6,3} {6,4} {6,5} {6,6}

(ii) What is the probability that the number on the green die is larger than the number on the red die?

5/12

(iii)Define what it means for two events E and F to be independent.

Two events E and F are independent if the order in which they occur doesn't matter. Alternatively, if observing one doesn't affect the other.

$P(E \cap F) = P(E)P(F)$

(iv) Let event E be that the sum of the dice equals 2 or 3 and event F be that the sum equals 3. Are E and F independent? Explain with reference to the definition given above

No, E and F are not independent. Observing E affects the outcome of F. Using the mathematical definition $P(E \cap F) = P(E)P(F)$:

$P(E \cap F) = 1/36$, $P(E)P(F) = 1/12 * 1/18 = 1/216$

Thus, E and F are dependent as $P(E \cap F) != P(E)P(F)$

Q2:

(i) State Bayes Rule

$P(E|F) = (P(F|E)P(E))/P(F)$

(ii) Suppose 1% of computers are infected with a virus.There is an imperfect test for detecting the virus.When applied to a computer with the virus the test gives a positive result 90% of the time.
When applied to a computer which does not have the virus, the test gives a negative result 99% of the time.Suppose that the test is positive for a computer.What is the probability that the computer has the virus?

E = actually has virus, F = test positive for virus

$P(F|E) = 0.9$, $P(E)=0.01$, $P(F) = (0.9*0.01 + 0.01*(1-0.01)) = 0.0189$

$P(E|F) = (0.9*0.01)/0.0189 = 0.4761904761904762$

Q3: You invent a game where the player bets €1, and rolls two dice. If the sum is 7, the player wins €k, and otherwise loses their bet.

(i)Define the expectation and variance of a discrete random variable.

For random variable X taking values x1, …, xn the expected value is E[X]=x1*P(X=x1)+… xn*P(X=xn).The variance is Var(X)=E[(X-E[X])^2] = (x1-E[X])^2*P(X=x1)+… (xn-E[X])^2*P(X=xn).

(ii) What is the expected reward in this game?

Probability of winning = 1/6 -> expected reward = k/6 - 5/6

(iii)What value of k makes the game fair (i.e. makes the expected reward zero)?What is the variance of the reward in this case ?

To make the expected reward zero, k=5. Variance = (5-0)^2 * 1/6 + (-1-0)^2 * 5/6 = 5^2/6 + 5/6 = 5.

(iv)For two independent random variables X and Y show that Var(X+Y)=Var(X)+Var(Y). Hint: Recall that E[X+Y]=E[X]+E[Y] and that when X and Y are independent then E[XY]=E[X]E[Y]

Var(X+Y) = E[(X+Y)^2] - E[X+Y]^2
= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2
= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2
= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2
= Var(X) + Var(Y)

(v)Suppose that you play the game 2 times in a row with k=5.What is the expected value of the reward (i.e. of the aggregate winnings after playing 2 times)?What is its variance ? What is the expectation and variance of the reward after 100 plays ?

Expected value = 0 for both. Variance = 10 for 2 plays, 500 for 100 plays