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SURVEY

Jigsaw puzzle solving techniques and applications: a survey

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Abstract

A jigsaw puzzle is a recreational activity that involves assembling a certain number of pieces into a combined and well-fitting unit without creating gaps between adjacent pieces. Two-dimensional puzzles are divided into two main categories, the "apictorial" in which the only information available is the shape of the pieces and the "pictorial" which may take into account not only the shape of the pieces, but also their content. Jigsaw puzzles are considered as one of the most popular category of puzzles. The majority of them are accompanied by a guiding image and there is only one "counterpart" for each side of each piece (pictorial jigsaw puzzles), although some more difficult variants have blank pieces, the so-called apictorial jigsaw puzzles. In this paper, we will examine the open problem of solving pictorial and apictorial jigsaw puzzles, and their various applications, such as the reconstruction of two-dimensional fragmented objects, the restoration of fragmented wall-paintings and the repair of shredded documents. We will also present an evaluation of the state-of-the- art jigsaw puzzle reassembly techniques in pictorial and apictorial puzzles.

Keywords Jigsaw puzzles · Pictorial puzzles · Apictorial puzzles · Shape matching · Fresco fragments · Image reconstruction

1 Introduction

In a puzzle, the pieces must be arranged in a logical way to obtain the correct solution. Thus, the main goal of a two-dimensional puzzle is to put together a number of particular pieces into a combined and well-fitting arrangement without any gaps between the adjacent pieces [1]. Two-dimensional puzzles are divided into two main categories, pictorial and apictorial, as shown in Fig. 1.

Apictorial puzzles are called those puzzles in which the only information available is the shape of the pieces, while the color of the pieces is not considered. Apictorial jigsaw puzzles and polyform packing belong to this category. As an application of this type of puzzles we can consider the reconstruction of two-dimensional fragmented objects.

On the other hand, a pictorial puzzle can take into account not only the shape of the pieces, but also their content, such as color information. The pictorial jigsaw puzzles and the

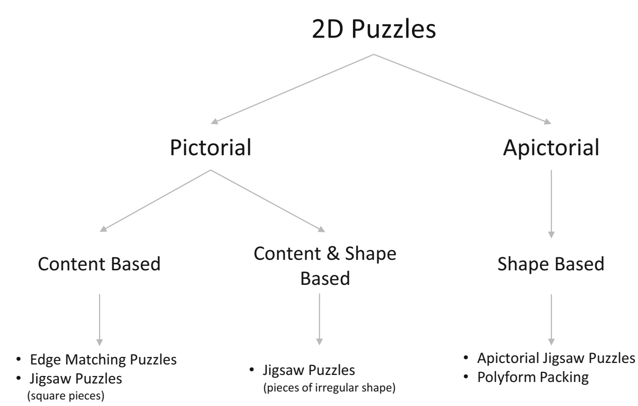

Fig. 1 Main categories of two-dimensional puzzles

reconstruction of fragmented wall paintings belong to the category of content and shape based pictorial puzzles. Edge-matching puzzles also belong to this category since they take into account the content of the pieces.

Even today, jigsaw puzzles are considered as one of the most widespread categories of puzzles. The first jigsaw puzzles were created in 1760 from wooden boards by a hand tool called "jigsaw", from which the puzzles are named after. They were used as educational tools with pictures of European maps on them. A handmade wooden version for adults

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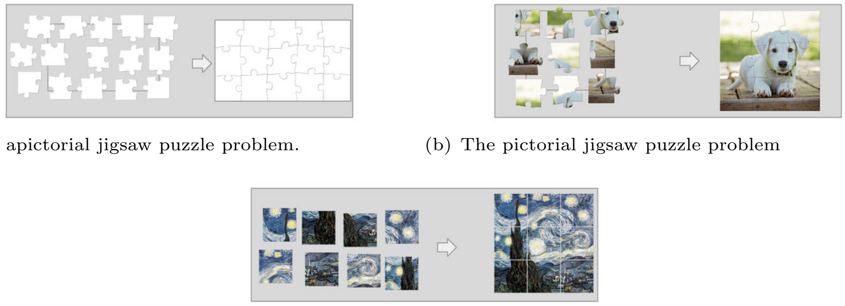


Fig. 2 Jigsaw puzzle reassembly problem categories

(a) The apictorial jigsaw puzzle problem.

(b) The pictorial jigsaw puzzle problem

(c) The square piece jigsaw puzzle problem

came on the market around 1900. Nowadays, jigsaw puzzles are made of cardboard. However, there are still artisans who make original wooden ones by hand. The majority of them have a guiding image and there is only one "counterpart" for each side of each piece (pictorial jigsaw puzzles), although some more difficult variants have blank pieces, the so-called apictorial jigsaw puzzles. [2] Fig. 2 presents the categories of the jigsaw puzzle reassembly problem.

Other popular categories of puzzles that work similarly to jigsaw puzzles are the so-called edge-matching puzzles and the polyform packing puzzles. The first ones are aiming to put together a given set of several pieces of the same shape (usually squares), but with different patterns on them, in order to match the patterns along the boundaries of the neighboring pieces [2]. Polyform packing puzzles [3,4], on the other hand, are aiming to pack a given set of multiple polyforms into exactly one target shape, such as a larger rectangle, a rhombus, or a dodecagon (as in Eternity) [2].

In this paper, the problem of solving pictorial and apictorial jigsaw puzzles is studied, since this is an open problem available for further research and in which the scientific community shows great interest. Jigsaw puzzle solving is considered as the problem of assembling a set of non-overlapping, disordered pieces into a coherent whole. This paper also describes various applications of jigsaw puzzles, such as the reconstruction of two-dimensional fragmented objects [5], the restoration of fragmented wall paintings and the repair of shredded documents. Finally, an evaluation of the state-of-the-art jigsaw puzzle reassembly techniques in pictorial and apictorial puzzles is also presented.

The main contribution and goal of this survey is to thoroughly review the literature on the advances of 2d jigsaw puzzle reassembly techniques and applications. Additionally, to the best of our knowledge, there is no other survey on this particular topic in the last decade, the last one being from 2009 [6]. Moreover, there is a great development in this field during the last decade while "Deep Learning Techniques" play an important role among the Jigsaw Puzzle solving methods that are worth mentioning.

The remainder of the paper is organized as follows: Sect. 2 describes the apictorial jigsaw puzzle problem, applications and solution techniques through an overview of their development over time. Section 3 discusses the pictorial jigsaw puzzles, focusing on the puzzles with pieces of irregular shape and their applications in real world problems. Section 4 examines the square jigsaw puzzle solving techniques as well as the deep neural network techniques. Section 5 presents the evaluation of the state-of-the-art jigsaw puzzle reassembly methods. Section 6 presents our proposals for future research which is followed by the summary and conclusion of the paper.

2 Apictorial Jigsaw puzzles

2.1 Solving apictorial Jigsaw puzzles

The first solving technique of an apictorial jigsaw puzzle was performed in 1964 by Freeman and Garder [1], whose method consists of three basic steps: encoding and classifying the pieces, rotating and matching them, and evaluating the "correct fit." Freeman and Garder, also introduced a method for chain coding a boundary line. Then followed several similar techniques for assembling jigsaw puzzles, [7–9] based on critical points extraction from local boundary information. An algorithm for assembling large apictorial jigsaw puzzles using curve matching and combinatorial optimization techniques was presented by Wolfson in 1988 [10]. Another interesting method was presented by Webster [11], based on isthmus critical points. This method helps to control the amount of matches to be evaluated. The class of jigsaw puzzles that can be solved using this method are regular interlocking puzzles, where each match segment contains a unique isthmus [11].

The temporal development of the apictorial jigsaw puzzles reassembling techniques is described in the following subsections and summarized in Table 1.

2.1.1 Studies from 2000 until 2010

In 2002, Goldberg et al. [13] presented a robust algorithm that was able to handle puzzles with pieces that must be

Table 1 The temporal development of the apictorial jigsaw puzzle solving techniques

Method	Description	Puzzle size
Freeman and Garder [1]	Freeman chain encoding	Up to 9 pieces
	Partial curve matching	
	Based on slop discontinuities	
Wolfson et al. [10]	Schwartz–Sharir curve-to-sub curve matching algorithm [12]	Up to 104 pieces
		Many nearly identical pieces
	Partial Curve Matching Combined with an optimal search algorithm	Mixed Puzzles
Goldberg et al. [13]	Robust reference points	Up to 200 pieces
	Assembling the edge pieces using a heuristic for the TSP	Pieces adjacent to more than 4 neighbors
Hoff and Olver [14]	Based on the extended Euclidean signature method for object detection & Curve Matching [15]	Up to 103 pieces
		Puzzles with no constraints
Harel and Ben-Shahar [16]	Inspired by the Lazy Catererís Sequence	Up to 1806 pieces
		Convex polygon pieces

adjacent to more than four neighbors, and puzzles with up to 200 pieces, larger puzzles than ever before. The novelty of this method was based on its robust reference points, its "highest-confidence-first" search, and the frequent global re-optimization of partial solutions. This algorithm first assembles the edge pieces using a heuristic for the traveling salesman problem and then fills up the interior pockets, same approach as in [10]. The algorithm of Goldberg et al. [13], on the other hand, makes more use of global geometry, e.g., by maintaining a geometric embedding of the best partial solution at all times, while Wolfson et al. [10] only use local geometry, i.e., pairwise matching of the sides of the pieces. In De Bock et al. [17], the topological solution of the jigsaw puzzle was presented. A topological solution involves a single position for each puzzle piece in a rectangular grid. This algorithm was able to solve large puzzles, consisting of up to 300 pieces.

2.1.2 Studies from 2010 until 2020

In 2014, Hoff and Olver [14] proposed a new technique based on an extension of the differential invariant signature. This technique was based on the extended Euclidean signature method for object detection and curve matching developed by [15]. This method is capable of solving challenging puzzles without constraints on the shape of the pieces or their internal arrangement.

In 2020, Harel and Ben-Shahar [16] introduced a new category of puzzles consisting of convex polygon pieces generated by cutting through a global polygon shape with a random number of straight cuts, a generation model inspired by the well-known lazy caterer's sequence.

2.2 Applications of the apictorial Jigsaw puzzles

Apictorial jigsaw puzzle's solution, in addition to its theoretical significance, is of great interest when applied in practice, especially in archaeology and art restoration. In such applications, a large amount of irregular fragments coming from more than one broken objects or shredded documents must be automatically reassembled in order to avoid the difficult and time-consuming manual work.

2.2.1 Shredded documents

In 2001, Kong and Kimia [18] presented an automatic method for solving jigsaw puzzles from a practical point of view, using a partial curve fitting technique. The problem is approached in two stages: First, local shape matching aims to find likely candidate pairs for adjacent pieces, using only local shape information. The second stage, finding a global solution, involves resolving ambiguities arising from local shape matching and reassembling the pieces. This method was applied to real fragmented two-dimensional objects such as a torn map and several ceramic fragments. Another similar method for reassembling document fragments was introduced by Zhu et al. [19]. In this method, the contours of the fragments were represented by their turning functions. Lalitha et al. [20] proposes an efficient iterative framework for solving apictorial jigsaw puzzles of hand-shredded blank pages, using a modified graph-based for agglomerative clustering framework to reconstruct a hand shredded content-less page by its shape information alone. The iterative framework presented by [20] for reassembling content-less torn documents is shown in Fig. 3.

2.2.2 Fragmented objects and wall paintings

Reassembling two-dimensional jigsaw puzzles is a common problem in archeology when reconstructing fragmented objects and wall paintings. In [21] a multiscale method for reassembling two-dimensional fragmented objects is described. The method compares the curvature-encoded fragment outlines, at progressively increasing resolution scales using an incremental dynamic programming sequence matching algorithm. This method has been validated on small quantities of real ceramic fragments such as tiles and wall paintings and can be applied to other objects with curved surfaces such as pottery.

A novel general methodology is presented by Panagopoulos et al. [22] for the computer-aided reconstruction of the outstanding wall paintings of the Greek island Thera, painted in the middle of the second millennium BC. According to this methodology, each fragment is photographed, its image is entered into the computer, its contour is obtained and all fragment contours are compared accordingly. Another interesting methodology for archaeological fragment reconstruction is introduced by McBride and Kimia [23]. According to this methodology each pair of fragments is compared using a partial curve matching technique to identify similar sections of their respective boundaries.

A statistical model analyzing fracture patterns in Theran wall paintings is presented by Shin et al. [24]. This model could be used for predicting other wall paintings fragmentation patterns and for assisting computer-aided reconstruction algorithms in the future.

In Funkhouser et al. [25], a machine learning approach is used to reconstruct fragmented objects and detect matches. The innovation of this approach is the idea of using machine learning to combine a variety of match features into a probability estimation.

Sizikova and Funkhouser [27] present a novel method for reconstructing wall paintings using a genetic algorithm. This method uses a robust optimization technique that iteratively rejects outlier matches while optimizing fragment transformations, allowing the algorithm to handle large amounts of noise and avoid converging to local minima.

In Naiman et al. [26] a method of reassembling broken or torn pieces of material, known as "physical match" is introduced. In this method, the algorithm finds the pixels of the boundary, arranges them, and uses their positions to characterize the edge slope. The boundary curves are then matched using a modified version of the longest common subsequence algorithm. Finally, the different parts are translated, rotated and mirrored (if necessary), and then merged for further fitting to other parts. Figure 4 summarizes the procedure followed by the "Physical Match" [26].

A linguistic method is introduced by Montusiewicz and Skulimowski [28] to reassemble the 2D elements from archaeological excavations. This method, first, generates the edge description of an object and assigns them a label in the form of letters, which leads to the formation of abstract words in the form of a string. Then, the words belonging to two different objects are compared with a defined length of strings.

3 Solving pictorial Jigsaw puzzles

Including the available chromatic information into puzzle solving techniques is closer to real life problems, as people always use all available means to achieve their goals. For this reason, the pictorial jigsaw puzzle reassembling problem has attracted the interest of the scientific community very early. Figure 2b depicts the pictorial jigsaw puzzle reassembling problem utilizing not only shape but also color information.

The first method that takes into account the color information of the pieces was initially proposed by Kosiba et al. [29]. An automatic puzzle solver based on shape and color features was also presented in [30].

3.1 Studies from 2000 until 2010

In 2003, Yao and Shao [31] presented a new method for solving puzzles that combines shape and image matching. According to this method, puzzle pieces are categorized into 18 patterns, in order to reduce the computation time. The procedure includes puzzle piece extraction, corner point detection, piece type recognition, boundary shape matching and image merging. The effectiveness of the algorithm was tested using real-world images with numerous puzzle pieces.

In 2009, Makridis and Papamarkos [32] introduced another technique for solving pictorial jigsaw puzzles that use boundary and color matching features. The novelty of this method is that it provides a solution without any initial constraint on the shape of the pieces, the number of neighboring pieces, etc. A recurrent process is used that recovers the original image by comparing and merging puzzle pieces in pairs. Geometric and color information is extracted at points of interest. The features associated with color are provided by applying color reduction technique using the Kohonen self-organized feature map. This method has been successfully tested using various artificial puzzles with an original image of a landscape or a document divided into parts.

On the other hand, in 2008 Nielsen et al. [33] presented a jigsaw puzzle solving technique using only image features.

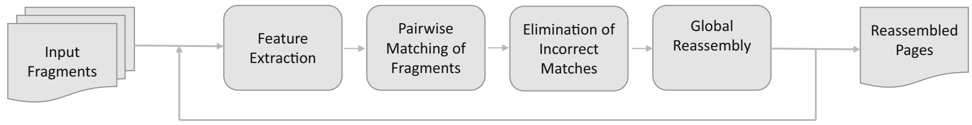


Fig. 3 Iterative framework for content-less shredded document reassembly [20]

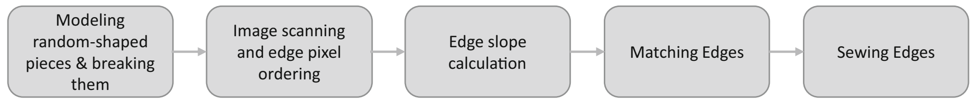


Fig. 4 The procedure of the "Physical Match" [26]

This method can be considered similar the edge-matching puzzle solving methods, which are not studied in this paper, since this method is based on examining the color/pattern similarity between the edges of the pieces. However, edge-matching puzzles by definition consist of pieces with the same and specific shape, commonly square without any guiding image on them. Consequently, this algorithm can be considered as the first successful attempt to reassemble puzzles without geometric information and it is able to reassemble puzzles with up to 320 pieces.

3.2 Studies from 2010 until 2020

In 2018, Shih and Lu [34] presented a method for reconstructing 2d pictorial jigsaw puzzles using a robotic arm. The proposed method is able to solve the reconstruction problem accurately by using the contexts in an image as matching features. It is a robust method unless the puzzle image contains many homogeneous areas. In this method, a patch matching algorithm is used to determine feature point pairs among puzzle piece images. The output of the algorithm is a transformation matrix (TM) containing the rotation angles between original images and transformed pieces. By performing the scale invariant feature transformation (SIFT), the feature points of the pieces as well as those from the original whole image are identified. Finally, the mismatched pairs are removed and then the image is completed by executing the random sample consensus (RANSAC) algorithm.

In the same year, Shen et al. [35] introduced an approach using stigmergy-inspired Internet-based human collective intelligence. The idea of this approach is a continuously executing loop consisting of three asynchronously connected activities: exploration, integration, and feedback. The key artifact generated by the loop is a continuously-updated collective opinion graph, which integrates every human players' opinions in a structured way and in real time, and also serves as the input to the feedback activity.

3.3 Applications of pictorial Jigsaw puzzles

3.3.1 Fragmented objects and wall paintings

The reassembling of fragmented objects and images is the most common application of the pictorial jigsaw puzzle problem. Automating this task is crucial leading to robust and fast solutions that reduce the human effort needed. According to Fornasier and Toniolo [36] any feasible computer-based solution of reassembling fresco fragments must be:

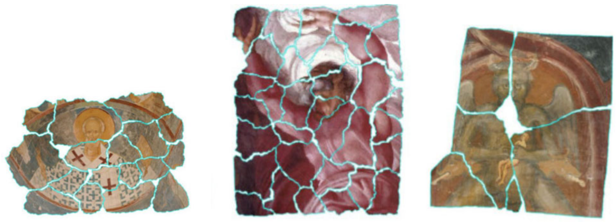
- Fast, due to the amount of pieces
- Robust, because of the strong noise presence
- Accurate, due to the small size of the pieces
- Translation-rotation invariant, due to the unknown initial position and orientation of the pieces

Papaodysseus et al. [37] presented a method based on color image segmentation and pattern analysis for the reconstruction of the wall paintings at Akrotiri of the Greek island Thera. Thousands of fragments were excavated suffering from severe damage. Tsamoura and Pitas [38] also presented another method for automatic color-based 2-D image fragment reassembly, including four steps:

1. Finding neighboring image fragments.
2. Finding matching boundary segments of neighboring image fragments.
3. Image fragments boundary alignment.
4. Image reassembly.

In Toler et al. [39] an approach to determining matches between small fragments of archeological artifacts is presented that is based not only on color and shape, but also on normal maps. The normal maps show conspicuous surface features such as string impressions, brush strokes, surface roughness, and fine cracks. The same paper also shows for the first time how machine learning techniques can be used to effectively combine the new features with the traditional ones.

Derech et al. [40] presented a new method for solving archaeological puzzles as well. More specifically, the reassembly of an archaeological artifact is considered as the



(a) 17 pieces of fragments from the Ferapontov monastery, Russia (b) 32 pieces of fragments from the Sistine Chapel, Vatican City, 1512 (c) 6 pieces of fragments from the church of Christ Antiphonitis in Cyprus

Fig. 5 Fresco fragments re-assembled by [40]

(a) Flower image fragments (b) final reassembly

Fig. 6 Image re-assembled by [44]

puzzle problem, given images of its fragments which are usually characterized by abrasion, color fading, and continuity. The proposed algorithm handles these challenges efficiently as shown in Fig. 5.

Another interesting approach, called Javastylosis, was introduced by Cantoni et al. [41]. Javastylosis allows the user to apply chromatic and semantic principles by shifting, grouping, rotating and merging the virtual fragments, and finally discarding the wrong ones to reassemble a digital solution of the puzzle from the images of the broken fragments. In this way, the restorers can use an automatically generated table with the coordinates of each fragment to correctly position each genuine fragment. It is worth noting that the graphical user interface, designed to make the fragments easy and intuitive to use, has been developed for and tested by people on the autism spectrum, with highly developed visual skills, to provide inclusion opportunities for this group of people.

3.3.2 Shredded documents

The reassembling of shredded pictures or documents is a very common problem of real life and another equally important application of the pictorial jigsaw puzzle problem. Putting the pieces of a shredded document back together manually is a tedious job, especially when the number of fragments is huge or they belong to different pictures or documents. Thus it is necessary to automate this process as well.

Liu et al. [42] introduced a method for reassembling shredded pieces from different photographs. The proposed method uses geometric and content information. The robustness of this method to material loss and missing pieces is proven by the experimental results.

A two-step algorithm assembles shredded documents is presented in Richter et al. [43]. In the first step, an SVM classifier is used to detect pairs of support points between pieces that are appropriate for the alignment of each piece. SVM enabled the distinction between matching connectives and false matches based on feature differences that used shape and content-based information. In the second step, all fragments are iteratively aligned into groups. By greedily selecting the best alignment in each iteration, groups of fragments are combined until the document is completely reassembled.

Zhang and Li [44] introduced another method for reassembling fragmented images, consisting of three main steps: pairwise matching between two image fragments, graph-based global reassembly of fragments, and refinement of the reassembly by graph optimization. The proposed algorithm has been evaluated in real images and its robustness is shown in Fig. 6.

4 Solving square Jigsaw puzzles

Square piece jigsaw puzzle reassembly problem is aiming to recreate the initial image from a given number of non-overlapping, square pieces. The square piece jigsaw puzzle reassembling problem is presented in Fig. 2c.

This puzzle category lacks information about the shape of the individual pieces, which is important for evaluating whether the pieces belong in pairs. Therefore, puzzles with square pieces are even more difficult to be solved than traditional puzzles. Puzzle solving also requires concentration on the image content, which makes it a good framework to analyze structural regularities in images.

4.1 Studies from 2000 until 2010

In Toyama et al. [45], a technique for the square piece jigsaw puzzle reassembly problem is introduced. In this technique, the information of pixel values (1 or 0) at the boundary of the pieces is used only for reassembling the puzzle, which are processed as binary images. This is considered as the problem of finding the appropriate arrangement of the pieces so that the total difference of pixel values on the boundary becomes smallest. This problem requires a global matching since there are many similar pieces and hence it is not able to be resolved by simply matching the pieces locally. This technique uses a genetic algorithm (GA) for finding the best arrangement of the pieces since this type of algorithms is capable of finding a global solution in a large optimization space. The proposed method was successfully applied in puzzles of 8×8 pieces.

Alajlan [46] proposed a method based on the simultaneous assignment of puzzle pieces using the Hungarian method. This technique uses the grayscale profiles of the edge pixels for local matching of the puzzle pieces. The proposed algorithm achieved a high precision rate for puzzles with four known images and up to 8×8 pieces.

On the other hand, Cho et al. [47] introduced a probabilistic approach. A graphical model represents the puzzle where each node corresponds to the location of a piece and each label corresponds to a piece. To find the most probable configuration of the pieces on the graph, loopy belief propagation is used.

4.2 Studies from 2010 until 2020

Pomeranz et al. [48] suggest a computational framework based on a greedy algorithm that can handle very large square jigsaw puzzles in reasonable time excluding any prior knowledge of the initial image. The proposed method attempted accurate solutions to bigger puzzles than ever before.

Gallagher [49] was the first to introduce the problem of assembling puzzles with square pieces of unknown orientation. His method introduces a tree-based reassembly that greedily merges components and succeeds even with mixed pieces from different puzzles. The proposed method was also successful on puzzles with up to 9600 pieces, the largest automatically reassembled puzzle to date.

Yu et al. [50] have proposed a puzzle solving formulation based on linear program (LP). Unlike existing greedy methods, the LP-solver uses all pairwise matches simultaneously and computes the position of each piece globally.

Paikin and Tal [51] presented a method that not only includes all of the above-mentioned constraints, but also additional challenges. More specifically, the proposed method assumes no prior knowledge about the initial image, handles puzzles with unknown size, puzzles with missing pieces, puzzles with pieces of unknown orientation, and even mixed puzzles. The proposed greedy algorithm uses a compatibility function between pieces, taking special care in selecting the first piece.

The first robust puzzle solver based on the genetic algorithm (GA) was developed by Solomon et al. [52]. This method merges two "parent" solutions into an improved "child" configuration by detecting, extracting, and combining correctly assembled puzzle segments. It shows high accuracy and efficiency and can solve puzzles with up to 30,745 pieces.

Andalo et al. [53] have proposed a fully automatic Quadratic Programming method capable of solving puzzles with pieces of arbitrary rectangular shape. The proposed method is based on a global compatibility function, which is maximized to find the best permutation between pieces, and a local compatibility function, which provides a compatibility measure for matching pieces to adjacent locations.

In contrast to previous methods for solving puzzles, which aim to maximize the compatibility measures between all pairs of pieces and therefore depend greatly on the pairwise compatibility measures used, Son et al. [54] suggest a new technique that reduces the dependence on pairwise compatibility measures, which become increasingly uninformative at small scales and instead uses geometric correspondence between pieces. It also includes a developed pairwise compatibility measure that uses the derived directional information along the adjacent boundaries of the pieces. This method aims to find the maximum geometric correspondence between pieces, especially for hierarchical piece loops. The proposed algorithm searches for loops of four pieces and groups the smaller loops into higher order "loops of loops" in a bottom-up method.

Huroyan et al. [55] proposed a mathematical framework for large puzzles with square pieces that are randomly rotated and shuffled. The proposed method is based on estimating the graph connection Laplacian.

4.3 Deep neural network techniques

Deep neural networks (DNN) are structures with many layers created to extract significant features in data to solve a specific task. A special category of DNNs, the convolutional neural networks (CNNs), usually uses images as inputs. They are composed of a sequence of convolutional layers followed by pooling layers. Then, the computed feature maps are passed to a classifier, which consists of fully linked layers called dense layers. In a classification task, the CNN results are the odds of belonging to each of the predefined categories. Finally, the category with the highest probability is the predicted class.

DNNs play an important role in the progress of modern computer vision systems and have been incorporated into jigsaw puzzle-solving techniques with great success in the last decade. While these networks are very efficient, they still rely heavily on labeled image data making the process expensive and time consuming. Therefore, efforts have been made to develop unsupervised [56] or weakly supervised techniques [57] that learn visual knowledge from unlabeled data, or semi-supervised learning algorithms that aim to combine a limited amount of labeled data with a large corpus of unlabeled data to achieve better performance.

Siamese neural networks [58], which are suitable for predicting pairwise patch matching, contain two identical branches that have the same weights during initialization and all training stages. Then, two inputs are compared using a distance or similarity function, and based on this, the network is trained to distinguish similar pairs from dissimilar pairs.

Noroozi and Favaro [56] introduced the context-free network (CFN) shown in Fig. 7. The CFN is a Siamese-ennead convolutional neural network whose properties can be easily transferred between recognition, classification, and puzzle reassembly tasks. The network is trained in an unsupervised manner using the puzzle as the input. The basic idea is that the CFN learns to recognize each tile as part of an object and how to reassemble the parts into an object by solving jigsaw puzzles. Figure 7 depicts the procedure of puzzle generation and solution by a context-free network.

Dery et al. [59] proposed a neural network to reassemble jigsaw puzzles of arbitrary size by combining a pre-trained VGG model as a feature extractor and a pointer network to correct the order.

Paumard et al. [60] have introduced a deep learning method for classifying the position of two neighboring fragments from images of artworks applied to puzzles with 9 pieces. More specifically, square pieces of equal size are extracted from a given image. The visual features are then extracted using a convolutional neural network (CNN). These features are then combined and passed to a classifier to predict the correct relative position. "Deepzzle" [61] is proposed to be an extension of [60] that addresses the problem of reassembling images with large spacing between fragments. The distance mimics the erosion that archaeological fragments suffer, rendering the continuity of patterns and colors useless. Then, a neural network is required to predict the positions of the fragments without considering the gaps between them. This method also requires a graph that gives the best compositions from these predictions. "Deepzzle" [61] is able to solve more difficult puzzles than those studied in [60], such as missing pieces, an unknown central fragment, additional unrelated fragments, or fragments from various images of the same object.

"JigsawGAN" [62] proposes a GAN-based auxiliary learning method for solving jigsaw puzzles without prior knowledge of the initial images. The architecture of the proposed method includes the classification branch and the GAN branch, which restores the features of the images. The two branches are connected by an encoder and a flow-based warp module as shown in Fig. 8. The encoder extracts the features of the input mixed parts. The mixed features are then combined, called Fshuffle, and fed to the classification module to predict the permutation of the puzzle. The classification results provide information to generate flow fields that can be used to combine the feature Fshuffle to a new combined feature Fwarp. The distorted feature Fwarp is then sent to the residual blocks and the decoder to restore the initial image. GAN loss and a novel edge loss are used to force the network to focus on the semantic information and the edge information, respectively.

4.4 Applications of square Jigsaw puzzles

An application of square jigsaw puzzles is detected in internet security by Jigsaw Puzzle CAPTCHAs. CAPTCHA is the acronym for "Completely Automated Public Turing Test to Tell Computers and Humans Apart" and it is a security measurement used by websites to distinguish human users from bots. The main three categories of CAPTCHAs are: the text-based [63,64], the image-based [65,66], and the sound-based [67] CAPTCHAs.

Text-based CAPTCHAs rely on the distortion of digits or letters and other visual effects, and the user is prompted to identify and enter the distorted characters. If the input matches, the user is accepted. Sound-based CAPTCHAs are

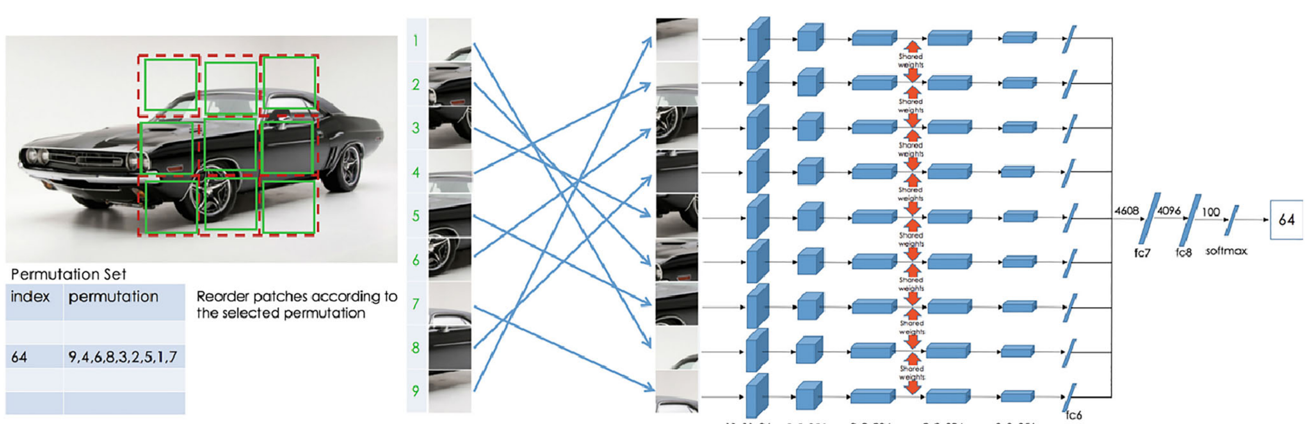


Fig. 7 Puzzle re-assembling using the context-free network [56]

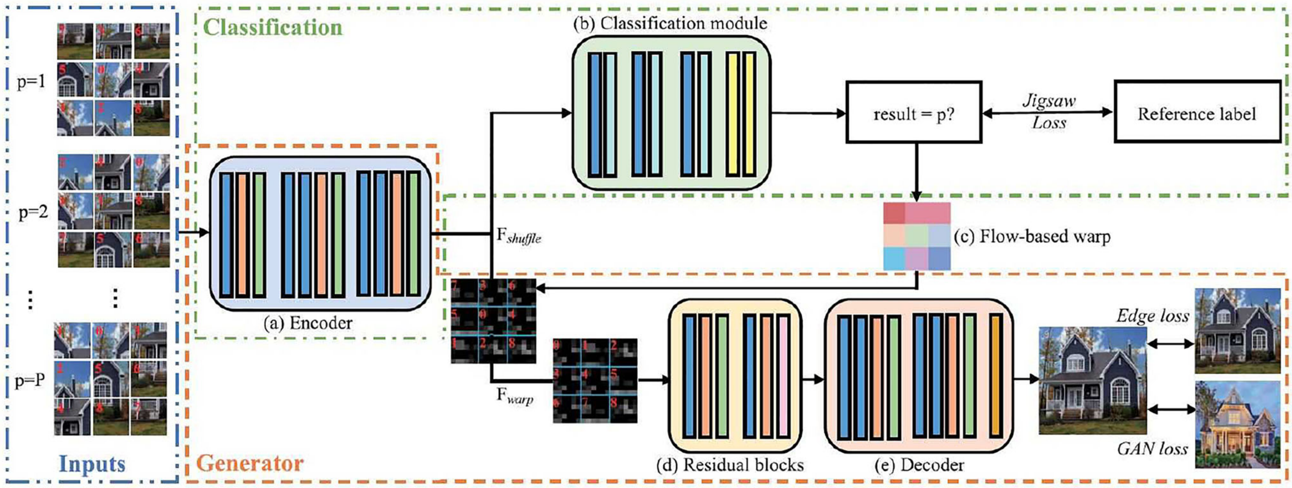


Fig. 8 The architecture of the "JigsawGAN" method [62]

Table 2 A summary of the main Jigsaw puzzle reassembly problem techniques

Method	Puzzle type	Pieces shape	Puzzle size	Technique
Freeman and Garder [1]	Apictorial	Irregular	9 pieces	Partial curve matching
Wolfson et al. [10]	Apictorial	Regular	104 pieces	Partial curve matching
Chung et al. [30]	Pictorial	Regular	54 pieces	Boundary and color matching
Goldberg et al. [13]	Apictorial	Regular	200 pieces	A heuristic for the TSP
Nielsen et al. [33]	Pictorial	Regular	320 pieces	Color matching
Cho et al. [47]	Pictorial	Square	400 pieces	Graphical model
Gallagher [49]	Pictorial	Square	9600 pieces	Tree-based assembly
Noroozi [56]	Pictorial	Square	9 pieces	Context-free network
Ostertag and Beurton [58]	Pictorial	Irregular pottery pieces	7000 pieces	Siamese neural network

based on the auditory perception of the human user. In traditional image-based CAPTCHAs, users are prompted to enter labels that can correctly describe the image, or to rotate the image in the correct direction.

Jigsaw Puzzle CAPTCHAs belong to the category of image-based CAPTCHAs and they were first introduced by [68]. Jigsaw Puzzle CAPTCHAs require users to solve a puzzle with a few misplaced pieces in order to be identified [69]. This type of CAPTCHA has the advantage of being language-independent, since no text input is required. Humans can perform the verification quickly and accurately, unlike computers, and users find it entertaining [68].

5 Evaluation

Freeman and Garder [1] introduced the first solving technique of an apictorial jigsaw puzzle in 1964. In 2002, Golberg's algorithm [13] was able to handle puzzles with pieces that must be adjacent to more than four neighbors, and puzzles with up to 200 pieces, larger puzzles than ever before. Kosiba et al. [29] introduced the first method that takes into account the color information of the pieces and Gallagher [49] was the first who introduced the square jigsaw puzzle reassembly problem with pieces of unknown orientation. His method succeeded in reassembling puzzles with up to 9600 pieces, the largest puzzle up to then. A summary of the main jigsaw puzzle reassembly problem techniques is presented in Table 2.

Deep learning techniques [70–72] were also applied to solve the problem of reassembling pictorial jigsaw puzzles and showed great experimental results. The Siamese model proposed in [58] shows robustness for pairwise assemblies, but achieves poor results for an entire image reconstruction. Solving puzzles with considerable erosion of the pieces are a demanding task; however, the "deepzzle" method [61] is able to reassemble most fragments in their correct position.

5.1 Comparison of apictorial Jigsaw puzzle reassembling techniques

The evaluation of three state-of-the-art wall painting reconstruction methods, the GA approach [27], the hierarchical clustering and the dense cluster growth method is presented in [27].

Dense cluster growth [13,18,23] is a widespread technique based on the idea that given a starting cluster, the right fragment can be iteratively added and then the starting cluster is replaced by the new cluster. This technique is equivalent to the configuration of a GA algorithm [27], where only one suitable cluster is selected in each iteration and clusters can be formed by merging with it. The disadvantage of this method is the impossibility to go back to the seed cluster after adding many false matches.

Hierarchical clustering [18,32,37,73] increases the search space by maintaining a queue of several existing clusters, iteratively merging the appropriate pair of clusters and deleting the initial ones.

The three techniques are evaluated using the F-score (harmonic mean of accuracy and recall) of correct matches in the biggest output cluster. The percentage of correct matches in the cluster is the accuracy, and the percentage of correct matches retrieved from that cluster is the recall. For example, small but mostly correct clusters are considered to have high accuracy but low recall.

Table 3 presents the comparison between the quantity of fragments at the largest reassembly and the best F-score for the above methods. According to this comparison, the GA approach seems to achieve the highest F-score with the greatest amount of reassembled pieces compared to the other methods. The comparison results based on Table 3 are also displayed in Fig. 9, where correct matches are depicted with green lines and incorrect matches with red lines [27].

Table 3 Evaluation of three wall painting reconstruction methods (GA approach, hierarchical clustering, and dense cluster growth method) [27]

Method	Number of fragments	F-Score
GA [27]	90	0.823
HC [18,32,37,73]	42	0.411
DCG [13,18,23]	7	0.082

(a) DCG [13, 18, 23]

(b) HC [18, 32, 37, 73]

(c) GA [27]

Fig. 9 Evaluation of reconstruction quality and size of the best runs of dense cluster growth (DCG) [13,18,23], hierarchical clustering (HC) [18,32,37,73] , and (GA) method [27], corresponding to numerical results in Table 3. [27]

5.2 Comparison of pictorial Jigsaw puzzle reassembling techniques

5.2.1 Traditional techniques

Datasets and Metrics

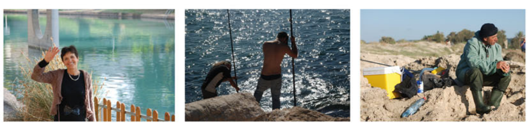
The datasets used to evaluate the top traditional jigsaw puzzle reassembling techniques according to [55] are the following:

- The MIT dataset by Cho et al. [47], which consists of 20 images, each with 432 patches of size 28x28.
- The McGill Calibrated Colour Image Database [48] which consists of 20 images with 540 patches of size 28x28.

(a) A sample from the MIT dataset by Cho et al. [47]

(b) A sample from the McGill Dataset [48]

(c) A sample from the BGU-805 Dataset [48]



(d) A sample from the BGU-2360 Dataset [48]

(e) A sample from the BGU-3300 Dataset [48]

Fig. 10 Samples of puzzle datasets

- The BGU-805 Dataset by Pomeranz et al. [48] which consists of 20 images with 805 patches of size 28x28.
- The BGU-2360 Dataset by Pomeranz et al. [48] which consists of 3 images with 2360 patches of size 28x28.
- The BGU-3300 Dataset by Pomeranz et al. [48] which consists of 3 images with 3300 patches of size 28x28.

In Fig. 10 samples from the above-mentioned puzzle datasets are presented.

The traditional jigsaw puzzle reassembling techniques were evaluated using the measures according to [49]:

- Direct comparison: indicates the percentage of pieces in the reassembled puzzle that are placed in the absolutely correct position.
- Neighbor comparison: defines the percentage of paired neighboring pieces that are correct. For a $m \times n$ jigsaw puzzle, $2 \times m \times n - m - n$ adjacencies are possible.
- Largest Component: indicates the percentage of pieces in the largest reassembled component that are correctly contiguous in pairs within the component. This essentially measures the size of the largest correct portion of the reassembled puzzle.
- Perfect Reconstruction: the number of perfectly reassembled puzzles.

Comparison

In Huroyan et al. [55] an assessment of the top traditional jigsaw puzzle reassembling methods is presented. For each puzzle each algorithm was executed 20 times and the averaged result was reported.

Table 4 presents the evaluation of three current puzzle reconstruction methods [49,50,55] for puzzles with pieces of unknown location and orientation. For the first three metrics (Direct Comparison, Neighbor Comparison and Largest Component), the means and standard deviations are reported across all images in each dataset and across the 20 instances of solving each puzzle. For the fourth metric (Perfect Reconstruction), the total number of perfectly solved images (which was identical across all 20 instances per puzzle) is reported. This metric indicates the number of perfectly reassembled images.

Figure 11 depicts the four metrics for each dataset and method. The first metric, direct comparison indicates the percentage of each puzzle's pieces placed in the absolutely correct position. The second metric, neighbor comparison, shows the percentage of neighboring pieces in pairs placed correctly. The third metric, largest component, indicates the percentage of pieces in the largest reassembled component that are correctly contiguous in pairs within the component. And the fourth metric, perfect reconstruction, is expressed as a percentage of the perfectly reassembled images related to the total number of images in each dataset.

As shown in Fig. 11 Yu et al. method [50] shows better results in all datasets for the three first metrics while for the fourth metric seems to achieve same results with the Huroyan et al. method [55] in all datasets. Thus, Yu et al. method [50] and Huroyan et al. method [55] present exactly the same percentage of perfectly reassembled puzzles in all datasets; however, Yu et al. method [50] seems to achieve better percentages of correctly positioned pieces in puzzles that are not totally reassembled.

5.2.2 Traditional and deep learning techniques

Dataset

For evaluating the top traditional [49,50,55] and the top DL-based [61,62] puzzle reassembling methods, Li et al. [62] used the PACS dataset [74], which includes 4 object categories (house, person, elephant, and guitar), and each of

Table 4 Reconstruction performance on Type 2 puzzles [55]

Dataset	Method	Direct Comp.		Neighbor Comp.		Largest Comp.		Perfect Rec.
		Mean	Std	Mean	Std	Mean	Std	
MIT dataset [47]	Gallagher [49]	84.2	19.7	89.1	12.4	87.2	14.3	9
	Yu et al. [50]	95.5	13.0	95.4	8.7	95.4	13.2	13
	Huroyan et al. [55]	94.8	11.3	95.2	9.2	95.4	9.1	13
McGill dataset [48]	Gallagher [49]	77.2	35.3	85.8	19.8	84.6	21.3	7
	Yu et al. [50]	92.9	24.6	93.5	14.8	93.1	15.4	13
	Huroyan et al. [55]	88.3	25.6	92.2	15.2	91.4	17.2	13
BGU-805 dataset [48]	Gallagher [49]	77.5	27.8	85.3	15.5	79.3	22.6	5
	Yu et al. [50]	91.8	14.2	92.7	13.0	91.7	14.2	9
	Huroyan et al. [55]	86.8	21.4	90.0	14.2	89.3	15.4	9
BGU-3300 dataset [48]	Gallagher [49]	82.9	15.6	84.2	14.2	82.8	15.7	1
	Yu et al. [50]	89.7	12.3	90.2	11.0	89.7	12.3	1
	Huroyan et al. [55]	86.4	14.0	88.1	11.7	86.4	14.0	1

Table 5 Evaluation of the top traditional [49,50,55] and the top DL-based [61,62] jigsaw puzzle reassembling methods by [62]

	Traditional techniques			DL-based techniques	
	Gallagher [49]	Yu [50]	Huroyan [55]	Paumard 2020“Deepzzle” [61]	Li 2021“JigsawGAN” [62]
House	64.5%	73.8%	76.4%	61.2%	80.2%
Person	63.8%	70.6%	73.7%	63.6%	79.9%
Guitar	59.4%	68.3%	71.1%	60.4%	77.4%
Elephant	60.5%	71.4%	73.1%	58.2%	78.5%
Mean	62.1%	71.0%	74.6%	60.9%	79.0%
Computational time	0.274	0.525	0.496	3.054	0.293

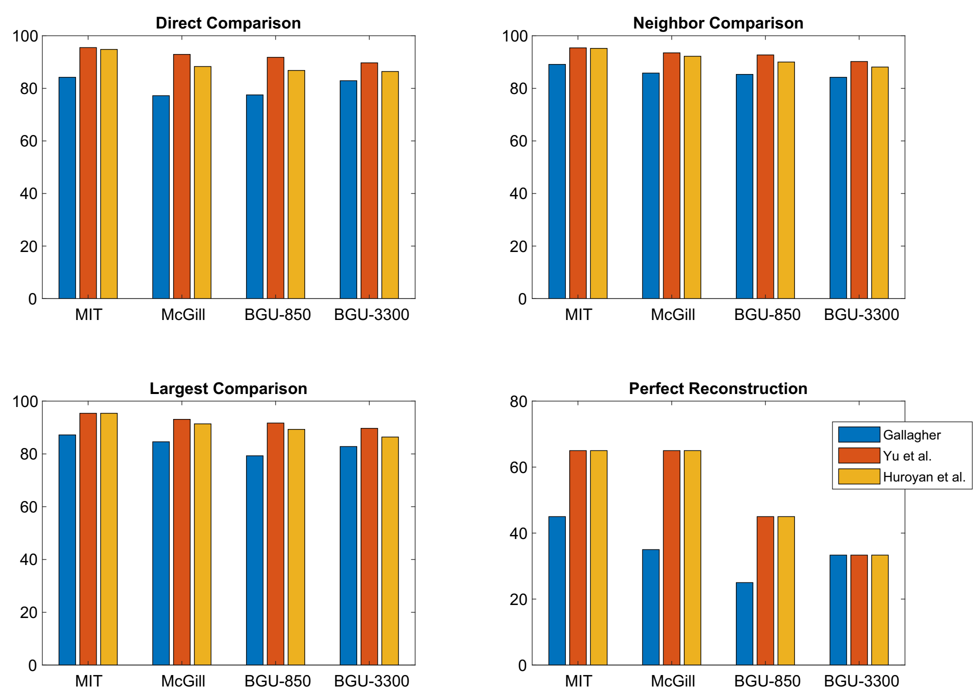


Fig. 11 Reconstruction performance for each method (Gallagher [49],Yu et al. [50], Huroyan et al. [55]) in each dataset (MIT Dataset [47], McGill Dataset [48], BGU-805 Dataset [48], BGU-3300 Dataset [48]) according to Table 4 results

them can be divided into 4 domains (photograph, painting, cartoon, and sketch).

Comparison Li et al. [62] present an evaluation of the top traditional [49,50,55] and the top DL-based [61,62] jigsaw puzzle reassembling methods.

Table 5 presents the reassembling accuracy of the above-mentioned methods on 3×3 square jigsaw puzzles as resulted by the evaluation conducted by [62]. The accuracy describes the proportion of correctly reassembled images in all test images, and the values given in Table 5 are the average results over the test sets. According to the results presented in Table 5, the "JigsawGAN" method [62] seems to provide the best results compared to all the other methods. The "Deepzzle" method [61], although a dl-based method, shows the worst results even compared to the traditional methods. Evaluating the accuracy of the traditional methods, the method of Huroyan et al. [55] achieves the best results among them, confirming the results presented in Section 5.2.1. as well.

Table 5 also lists the computation times of the above-mentioned methods for 3×3 square jigsaw puzzles. Gallagher et al. method [49] is the fastest, while the "JigsawGAN" method [62] is second. On the other hand, the "Deepzzle" method [61] is the slowest due to its complicated network architecture.

In Fig. 12 a visual evaluation of the above- mentioned methods [49,50,55,61,62] for puzzles with 9 (upper 2 lines) and 16 pieces (lower 2 lines) is presented by [62]. In the first house example, the results of Gallagher et al. [49] and Yu et al. [50] cannot solve the tree and roof parts, while the result of "Deepzzle" method [61] cannot distinguish two street parts. In the example of the elephant, all methods are unable to determine the leg parts since puzzles that have large portions of pieces with the same uniform texture and color have low percentages of retrieval by these algorithms. The global semantic information and the rigorous boundary detection are essential to achieve better results in the reassembly process. The failure of the last two house examples of [49,50,55,61] on puzzles with 16 pieces is caused by the similar texture and non-obvious boundary. Note that the results of the "Deepzzle" method [61] are the same as the inputs when the puzzle consists of 16 pieces, since their method can only handle puzzles with 9 pieces. In comparison, the "JigsawGAN" method [62] has the puzzle pieces perfectly assembled in all cases.

6 Future research directions and open issues

There is no doubt that puzzle solving techniques have been extensively explored by the scientific community, but there are still some challenges ahead. This section presents some promising research directions that will hopefully contribute to further advances in puzzle solving techniques and applications.

6.1 Restoration of frescoes in the absence of the initial image

Restoration of damaged frescoes is an important cultural heritage preservation project and digital solutions are precious tools for archaeologists [37–41]. Usually, the original image of the fresco is available and very crucial for the reconstruction process. However, in many cases there is no evidence of the initial condition of the frescoes which makes the situation very difficult. This can allow for the restoration of murals without any prior knowledge of the initial and complete condition. This challenging task can help to solve very difficult and time-consuming situations.

6.2 Development of deep learning techniques

Solving the jigsaw puzzle using deep learning methods [70–72] is an evolving field of research. The Siamese model proposed in [58] shows robustness for pairwise assemblies, however, for the reconstruction of an entire image obtains poor results. Deep learning techniques have undoubtedly made great progress. However, we believe that monitoring the deep learning techniques and, in particular, unsupervised learning is still far from the end. Most deep learning methods are capable of solving puzzles with 3×3 pieces [56,61], whereas the "JigsawGAN" method [62] can handle a maximum of 16 pieces. To solve puzzles with more pieces, a deeper classification network with more convolutional blocks is required to gain a larger corresponding field. Moreover, solving non-square puzzles with deep learning techniques is still a challenging research topic.

6.3 Coastline matching

Coastline matching is very important in various scientific disciplines that study coastal evolution at different time scales (in the present, in history and in geology) [75]. The idea of geometrical coastline matching is based on the work of Alfred Wegener who suggested the hypothesis of continental drift (the basis for today's model of plate tectonics) in 1912 [76]. Coastline matching with emphasis on computing the maximum and average discrepancy between sinuous lines was studied by [77]. Coastline matching can be viewed as an extension of the apictorial jigsaw puzzle reassembling problem and to our knowledge it has not been studied before. The study of geometric adjustment of coastlines as an application of the apictorial jigsaw puzzle problem can effectively contribute to the study of many areas such as sea level rise and coastal erosion.

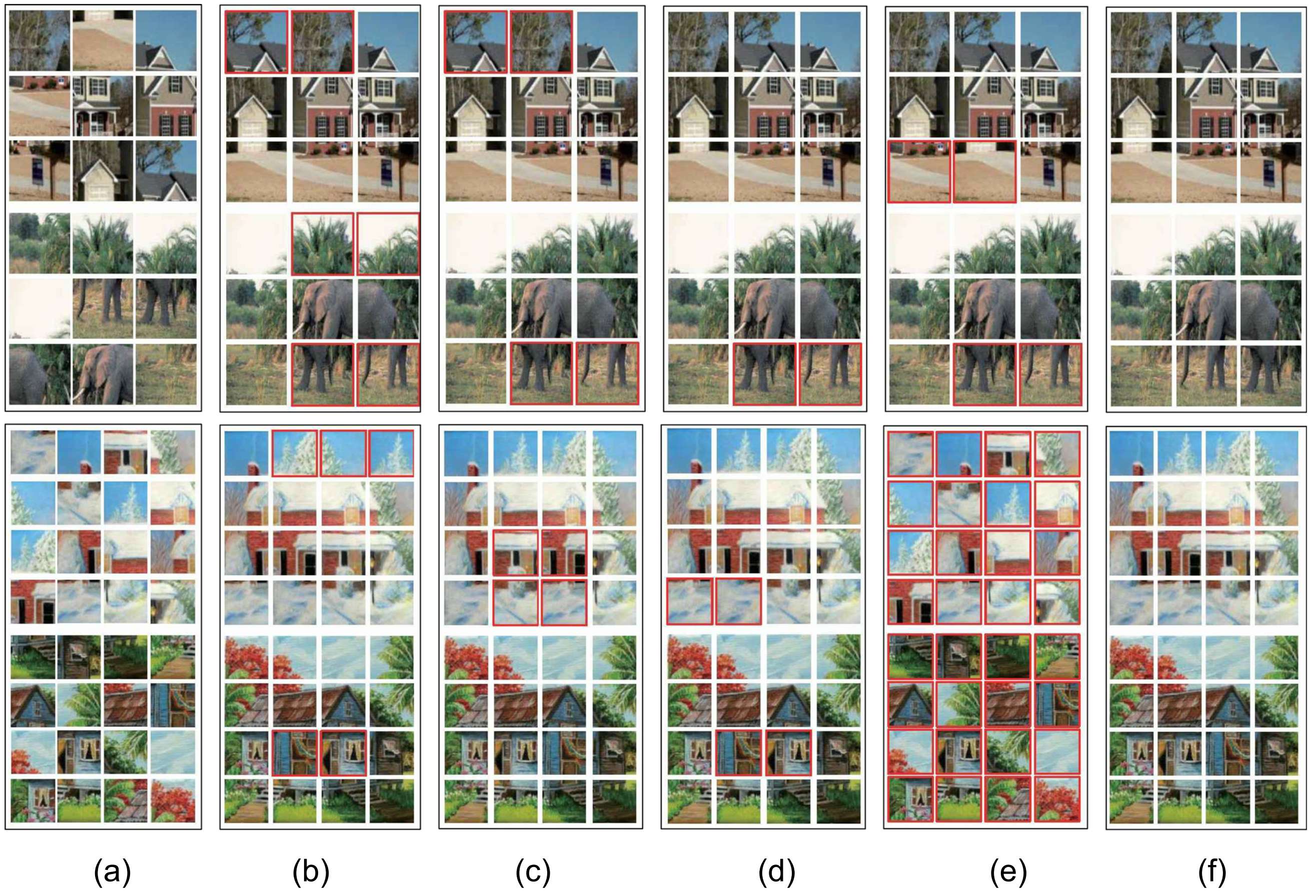


Fig. 12 A visual comparison of the top traditional and the top DL-based methods for puzzles with 9 pieces (upper 2 lines) and 16 pieces (lower 2 lines) by [62]. **a** inputs, **b** Gallagher et al. [49], **c** Yu et al. [50], **d** Huroyan et al. [55], **e** "Deepzzle" [61], **f** "JigsawGAN" [62]

7 Summary and conclusions

This article explores the open problem of solving pictorial and apictorial jigsaw puzzles by reviewing their evolution over time. It also examines the various real-world applications of the jigsaw puzzles, such as the reconstruction of two-dimensional fragmented objects, the restoration of fragmented wall-paintings and the repair of shredded documents. An evaluation of the state-of-the-art jigsaw puzzle reassembling techniques in pictorial and apictorial puzzles is also presented.

The traditional methods [49,50,55] obey the basic rules for solving puzzles, which include boundary detection to determine the relationship between the pieces and the application of various optimization methods to reassemble the images. These methods are limited by the boundary detection step, which leads to the randomness of their results. On the other hand, more DL-based techniques can handle puzzles of a maximum of 9 pieces while the "JigsawGAN" method [62] can handle a maximum of 16 pieces. For larger puzzles, a deeper classification network is required with more convolutional blocks to gain a larger respective field.

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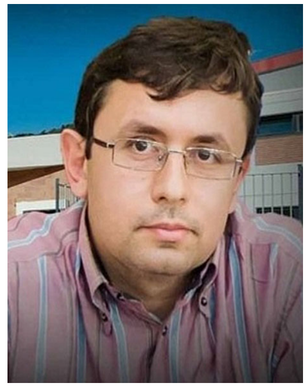

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