Homework 6

Due March 7th

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1. *Rotating Frames* - The line element of flat spacetime in a frame (t, x, y, z) that is rotating with an angular velocity Ω about the $z-$axis of an inertial frame is

$$ds^2 = -[1 - \Omega^2(x^2 + y^2)]dt^2 + 2\Omega(ydx - xdy)dt + dx^2 + dy^2 + dz^2 \ .$$

   (a) Verify that this is flat spacetime by transforming to polar coordinates and making the substitution $\phi \to \phi - \Omega t$

   (b) Find the geodesic equations for x, y, and z in the rotating frame.

   (c) Show that in the the nonrelativistic limit these reduce to the usual equations of Newtonian mechanics for a free particle in a rotating frame exhibiting the centrifugal force and the Coriolis force.

2. Show by direct calculation from the geodesic equation that the norm of the four-velocity $u \cdot u$ is a constant along a geodesic.

3. *Fermat's Principle of Least Time* - Consider a medium with an index of refraction $n(x^i)$ that is a function of position. The velocity of light in the medium varies with position and is $c/n(x^i)$. Fermat's principle states that light rays follow paths between two points in *space* (not spacetime) that take the least travel time.

   (a) Show that the paths of least time are geodesics in three-dimensional space with the line element

$$dS^2_{\mathrm{fermat}} = n^2(x^i)dS^2 \ ,$$

   where dS^2 is the usual line element for flat three-dimensional space, e.g.,

$$dS^2 = dx^2 + dy^2 + dz^2 \ .$$

   (b) Write out the geodesic equations for the extremal paths in (x, y, z) rectangular coordinates.

4. *Generators of transformations* Define the follow matrix

$$i \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

   What is $e^{i\theta}$?