Final Jam Session Problems

Long/Research Projects

1. Make a gravitational lens image distorter.
2. <u>2D Gravity</u>: Find the metric of the MTA subway map, and the distribution of energy density in New York City that could produce such a 2 dimensional geometry. Does this stress energy tensor correlate with anything? Property value? Density of Papaya Dogs?
3. Make a General Relativity calculator. Include things like
   (a) Christoffel symbol calculator
   (b) Riemann, Ricci, and Scalar Curvature symbolic calculator
   (c) Geodesic Equation Integrator

   Bonus points for using open source software (like SymPy instead of Mathematica)
4. When you first open a GR book, it says "From the equivalence principle, Einstein derived General Relativity." That seems like a bold claim, write an essay to back it up.
5. You're given an infinitely precise Michelson interferometer and want to look for gravitational waves. If a gravitational wave passes through your detector with some amplitude, frequency, direction of propagation, and phase, what would your data look like?
6. Consider a static, spherically symmetric star of mass M, radius R. Suppose we can model the stellar material as a perfect fluid of constant energy density e. Assuming the energy density of the star is constant, what is the pressure at the center of the star? Can a star of mass M have a minimum size?
7. Can a gravitational wave kill you? Can you estimate the required masses, periods and distances using using the binary star example from last class? Recall, we made a lot of Newtonian approximations in that derivation, so make sure you don't go outside the Newtonian regime. What would the body's response be to a such a gravitational wave traveling through you? Think hard.

8. Investigate the pericenter precession of orbits in the Schwarzschild metric by numerically solving the geodesic equation. Demonstrate the post-Newtonian prediction is valid for distant, nearly circular orbits. What is $\delta\phi_{prec}$ for orbits closer to the event horizon? How does it depend on ϵ and ℓ? When does the post-Newtonian prediction cease to be valid?

   Bonus points for doing the same analysis in the Kerr Metric.

9. Find the stress energy tensor for chromodynamics in flat spacetime via the Nöther prescription, and also by varying the generally covariant theory with respect to the metric. Show the symmetrization of the former is proportional to the latter.

10. As you recall, Hilbert guessed the form of general relativity very quickly from an action principle. His guess was the scalar curvature. Using his logic, we could've also guessed terms like R^2 or $R_{\mu\nu}R^{\mu\nu}$. Assuming the only dimensionful parameter you have is the Planck length, what would the coefficient of these higher order terms be, assuming the dimensionless contribution is order 1? Now the hard part...try and estimate in terms of this order 1 dimensionless parameter, how these higher order terms contributes to observables. Equivalence principle? Perihelion precession? Light bending? Black holes?

11. Invent your own! (but talk to one of us first...)

Medium Length Problems

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1. Consider the following metric
$$ds^2 = \frac{1}{z^2}\left(-f(z)^2dt^2 + dx^2 + dy^2 + f(z)^{-2}dz^2\right) \tag{1}$$
with $f(z) = \sqrt{1-\alpha z^3}$. Show that this metric satisfies the vacuum Einstein field equations with a cosmological constant Λ. What is Λ in terms of the parameters in the metric?

2. Check that Kerr satisfies the Einstein Field Equations

3. A perfect fluid has a stress-energy tensor $T^{\mu\nu} = (e+P)u^\mu u^\nu + Pg^{\mu\nu}$, where $e = \rho + \epsilon$ is the energy density, P is the pressure, ρ is the mass density, ϵ is the internal energy density, and u^μ is the fluid four-velocity. Conservation of energy-momentum tells us $\nabla_\mu T^{\mu\nu} = 0$. Write out this equation for the perfect fluid stress-energy above. Show it reduces to the Euler equations (ie. Navier-Stokes without viscosity) and Newtonian energy conservation in a gravitational field in the Newtonian static weak-field limit.

4. Derive the Schwarzschild metric from the Einstein field equations, assuming the space-time is static and spherically symmetric.

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6. A lightbulb emits light isotropically in its own rest frame. In your (inertial) frame the lightbulb moves with speed V, and due to relativity emits light anisotropically (ie. in a preferred direction). If the lightbulb emits (in its frame) at a power P, at what rate does it lose momentum in your frame? At what rate does it lose speed?

7. Find the stress energy tensor for electrodynamics in flat spacetime via the Nöther prescription, and also by varying the generally covariant theory with respect to the metric. Show the symmetrization of the former is proportional to the latter.

8. Show that the Weyl Tensor $C^\rho{}_{\sigma\mu\nu}$ is invariant with respect to conformal transformations, $g_{\mu\nu}(x^\mu) \to \omega(x^\mu)g_{\mu\nu}(x^\mu)$ with
$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - g_{\rho[\mu}R_{\nu]\sigma} + g_{\sigma[\mu}R_{\nu]\rho} + \frac{1}{3}g_{\rho[\mu}g_{\nu]\sigma}R \tag{2}$$

9. What action would lead to Einstein's original guess for the Field Equations $R_{\mu\nu} \propto T_{\mu\nu}$? What is wrong with this action?

10. Consider Einstein's equations in vacuum but with a cosmological constant, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.

(a) Solve for the most general spherically symmetric metric, in coordinates (t,r) that reduce to the ordinary Schwarzschild coordinates when $\Lambda = 0$.

(b) Write down the equation of motion for radial geodesics in terms of an effective potential. Sketch the potential for massive particles.

11. A particle is in a circular orbit around a black hole of mass M. You give the particle a small kick radially away from the black hole, what is the frequency of the resulting oscillations?

    Bonus: What if the kick was vertical, out of the orbital plane?

    Double Bonus: What if the black hole had angular momentum J?

12. What's the area of the event horizon of a Kerr black hole?

13. A neutron star has mass M, radius R, and negligible angular momentum. What is its angular diameter when viewed by a stationary observer a distance D away?

14. A black hole has mass M and negligible angular momentum. What is the angular diameter of the "shadow" when viewed by a stationary observer a distance D away? What if the observer were falling towards the black hole from rest at infinity?

Short Confidence Building Victories

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1. Show that $f^\mu = qF^\mu_\nu u^\nu$ yields the Lorentz force law that you know and love.

2. Show that the number of independent components of the Riemann tensor in dimension D is $D^2(D^2-1)/12$.

3. Show that $\sqrt{-g}d^4x$ is invariant with respect to coordinate transformations. (In class, we showed this was true for orthogonal coordinates, prove this in general here)

4. Find the metric for a non-rotating black hole of mass M boosted to a speed V. What is the shape of the event horizon? Homework 8 might help!

5. A lightbulb emits light isotropically in its own rest frame. In your (inertial) frame the lightbulb travels with speed V. What angle in front of the light bulb will contain 50% of the light?

6. Find the stress-energy tensor required to produce a wormhole with a 'mouth' of size A. What is the energy density? Can you make a wormhole with classical particles?

7. Given the power formula for a nearly Newtonian source of gravitational radiation
$$\frac{dE}{dt} = \frac{1}{5}\langle \dddot{I\!\!\!-}_{ij} \dddot{I\!\!\!-}^{ij} \rangle \tag{3}$$
Estimate the fraction of energy lost to gravitational radiation when fist pumping like a champ.

8. The muon has a mass of 0.106 Gev and a rest frame lifetime of 2.19×10^{-6} s. Imagine that such a muon is moving in the circular storage ring of a particle accelerator 1 kilometer in diameter, such that the muon's total energy is 1000 GeV. How long would it appear to live from the experimenter's point of view? How many radians would it travel around the ring?

9. If $T^{\mu\nu}$ is the stress energy tensor in special relativity, and $\partial_\mu T^{\mu\nu} = Q^\nu$, what physically does Q^i represent?

10. How do $\vec{E}$ and $\vec{B}$ transform with respect to a boost in the z-direction? Show that Maxwell's equations are invariant with respect to this transformation.

11. Show that $\star F$ reverses the role of the electric and magnetic field. Also, show that $\star(\star F) = -F$.

12. Show that Maxwell's equations don't obey Galilean transformations.
13. Find a wine glass and a laser pointer. Take a selfie with your face and an Einstein ring that forms through the glass, due to the curvature of glass causing lensing effects similar to general relativity. Send it to your parents in an email with subject line "Thanks for paying for me to go to college" and attach said photo. Bonus points if they aren't paying and you're actually taking out student loans, thus changing the mood of the email from gracious to sarcastic and snarky.
14. How do you know you can make the Christoffel symbols vanish with a coordinate transformation, but not the Riemann Curvature tensor? How many components of the Riemann curvature tensor can you set to zero with a coordinate transformation?
15. Find the matrix for a Lorentz transformation that represents a boost in the x-direction followed by a boost in the y-direction. Find the same for a boost in the y, followed by a boost in the x. Are they the same? What is their difference?
16. If two frames move with 3-velocities $\vec{v}_1$ and $\vec{v}_2$, show that their relative velocity is given by

$$v^2 = \frac{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}{(1 - \vec{v}_1 \cdot \vec{v}_2)^2} \tag{4}$$

17. Show that it is impossible for an isolated free electron to absorb or emit a photon.
18. Prove that the Einstein tensor, $G_{\mu\nu}$, is covariantly conserved.
19. Show that under a variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$:

$$\delta R_{\alpha\beta} = \nabla_\gamma \delta\Gamma^\gamma_{\alpha\beta} - \nabla_\alpha \delta\Gamma^\gamma_{\beta\gamma} \tag{5}$$

20. Recall in class we discussed briefly Nordstrøm's theory (1913) of gravity, which was an early attempt to make Newton's theory of gravity relativistic. As an obvious first guess, the gravitational potential, instead of a rank 2 tensor, was a scalar field ϕ. The field satisfied the following field equation

$$\eta^{ij}\phi_{,ij} = -4\pi\phi\eta^{ij}T_{ij} \tag{6}$$

The physical metric that determines geodesic motion is given by

$$g_{ij} = \phi^2\ \eta_{ij} \tag{7}$$

Let spacetime contain a pressureless fluid. Show that in the limit that time variations of the field ϕ are small compared to spatial variations, these equations reduce to the Poisson equation for Newtonian gravity.

Show also that trajectories $x^\alpha(t)$ of freely falling Newtonian trajectories correspond to geodesic motion in Nordstrøm's theory for non-relativistic velocities $||\dot{x}^\alpha(t)|| \ll 1$. Is this theory consistent with the observed light bending of the sun?

21. Just for appreciation of our beautiful notation, write down <u>one</u> of the Einstein field equations in terms of the metric tensor, no summation convention allowed. Title the top of the page "Formula Sheet" and give it to a freshman who you'd like to scare away from physics.