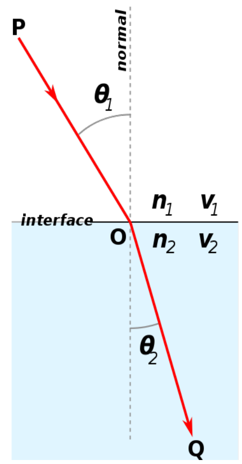
Homework 2

Due February 8th

1. Consider two slabs of optically transparent media with indices of refraction n_1 and n_2 respectively, shown in the image below[1]



   Snell's law states that the relationship between the angle of incidence, θ_1, and the angle of refraction, θ_2, is related by the following:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \ .$$

   Use Fermat's principle and the fact that light travels at an effective speed, $v = c/n$, through a medium with index of refraction n, to derive Snell's law.

2. (a) The Einstein velocity-addition law in one dimension is given as follows,

$$W' = \frac{W + v}{1 + Wv} \ ,$$

   where W is the original velocity and W' is the velocity as observed by an observer comoving with velocity v. This has a simpler form by introducing the *rapidity* parameter ψ defined by:

$$v = \tanh \psi \ .$$

[1]Image Source: https://en.wikipedia.org/

show that if in addition,

$$W = \tanh \Psi \ ,$$

that the rapidity adds linearly, namely

$$W' = \tanh(\psi + \Psi) \ .$$

(b) Use this to solve the following problem. A star measures a second star to be moving away at speed $v = 0.9c$. The second star measures a third to be receding in the same direction at $0.9c$. Similarly, the third measures a fourth, and so on, up to some large number N of stars. What is the velocity of the Nth star relative to the first? Give an exact answer and an approximation useful for large N.

(c) Write the Lorentz Transformation matrix, M_L, for a boost in the x-direction,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = M_L \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

in terms of the rapidity parameter (do not just replace v with tanh ψ and call it a day. It should simplify).

3. Consider two bodies of mass m_1 and m_2 located at points (x_1, y_1) and (x_2, y_2) respectively in empty space only effected by their mutual gravitational attraction. Such motion is confined to a plane so for simplicity we'll set that equal to the $x - y$ plane.

(a) Write down the Lagrangian for such a system and derive the Euler-Lagrange equations.

(b) Perform the following simplfication of the Lagrangian. First, go to the center of mass frame by the transformation

$$X = \tfrac{x_1 m_1 + x_2 m_2}{m_1 + m_2} \ ; \qquad Y = \frac{y_1 m_1 + y_2 m_2}{m_1 + m_2}$$
$$x = x_2 - x_1 \ ; \qquad y = y_2 - y_1$$

Then go into polar coordinates, and make use of the total mass, M and reduced mass μ:

$$r^2 = x^2 + y^2 \ ; \qquad \phi = \arctan \frac{y}{x} \ ; \qquad M = m_1 + m_2 \ ; \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Write down the Lagrangian, $L(X, Y, r, \phi)$, and the new Euler-Lagrange equations. Show that three of them are equivalent to conservation of linear and angular momentum.