Stückelberg Fields on the Effective p-brane

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We demonstrate the one-to-one correspondence between reparametrization invariant effective actions for relativistic p-branes in flat target space and effective actions for transverse brane perturbations with non-linearly realized Poincare symmetry. Starting with an action with non-linearly realized symmetry we construct the corresponding reparametrization invariant action by introducing Stückelberg fields. They combine with the transverse modes to form a Lorentz vector. The manifest Lorentz symmetry of the reparametrization invariant action follows directly from the non-linearly realized Lorentz symmetry of the initial action in terms of the physical modes.

I. INTRODUCTION

Confinement in quantum chromodynamics (QCD) arises due to the formation of gluonic flux tubes as is nicely visualized by lattice simulations (see, e.g., [1]). The dynamics of the transverse oscillations of a flux tube are described by the 2D effective field theory of a string. The leading term, known as the Nambu-Goto action, is simply the area of the dynamical, effectively two-dimensional surface of the tube, known as the worldsheet. In the QCD case, where the effective string has a finite width associated to it, there are higher order terms given by combinations of the first and second fundamental form associated with the induced metric of the worldsheet in an ambient spacetime,

$$S_{string} = -\int d^2\sigma\sqrt{-h}\left(l_s^{-2} + \frac{1}{\alpha_0}(K^i_{ab})^2 + \cdots\right). \quad (1)$$

Here the worldsheet metric h_{ab}, extrinsic curvature K^i_{ab} and all higher order terms are expressed as functions of the embedding coordinates X^μ,

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu \ ,$$

Where the bulk inner product is chosen to be $g_{\mu\nu} = \mathrm{diag}(-,+,\cdots,+)$. The worldsheet description is helpful in that we know all of the local geometric invariants of embedded 2D surfaces, thus we know the most general local action compatible with the Poincaré symmetry of the theory, as well as diffeomorphism invariance of the worldsheet. On the other hand, as with any gauge symmetry, the diff invariance of the theory leaves us with a huge redundancy in our description and obscures the counting of physical degrees of freedom. The natural language to describe the string dynamics directly in terms of propagating degrees freedom is that of Goldstone bosons (see, e.g., [2] for a recent discussion of effective strings from this viewpoint). A straight p-brane spontaneously breaks the target space Poincaré group $ISO(1, D-1)$ down to the direct product of Poincaré transformations along the p-brane and rotations in the transverse hyperplane, $ISO(1,p) \times SO(D-p-1)$. The Goldstone Lagrangian can then be written as a derivative expansion of the form

$$S = \int d^d\sigma (c_0 \partial_a X^i \partial^a X_i + c_2(\partial_a X^i \partial^a X_i)^2 + c_3(\partial_a X^i \partial_b X_i)(\partial^a X^j \partial^b X_j) + \cdots \ , \quad (2)$$

where X^i are the dynamical degrees of freedom corresponding to $(D-p-1)$ transverse oscillations of the brane. Both transverse translations and off-diagonal generators of the target space Lorentz group are realized non-linearly. Non-linearly realized translations imply the shift invariance of the action (2). Non-linearly realized rotations/boosts in the the (ai) plane, where a labels a hypersurface tangent to the p-brane and i is normal to this surface in the bulk, act as

$$\delta_{NL} X^j = -\epsilon_{aj}(\delta^{ij}\sigma^a + X^i\partial^a X^j) \equiv -\epsilon_{aj}\delta^{aj}_{NL} X^j \ , \quad (3)$$

This transformation law implies an infinite number of relations between coefficients c_i in front of the individual terms in the action (2). This can be deduced by noticing that actions of the form (2) can be obtained by fixing to the static gauge $X^a = \sigma^a$ in the reparametrization invariant action (1). Then the transformation (3) is a combination of a boost and a compensating diffeomorphism, required to satisfy the gauge condition.

In principle, one may consider actions of the form (2) invariant under (3) on its own without any reference to gauge fixing. It is natural to expect, however, that all of them can be obtained by gauge fixing some reparametrization invariant action invariant under the linearly realized Poincaré group.

This expectation was challenged recently in Ref. [3], where an inductive procedure was developed to construct actions of the form (2) invariant under (3) starting with an arbitrary monomial "seed" term with a minimal number of X^i's, invariant under $\delta^{ai}X^j = \delta^{ij}\sigma^a$. It is convenient to use the "scaling" ($d^n X^m \implies n-m$) of an operator to label operators that do not mix under (3). Initially, it was claimed that this way, already at scaling two, one may construct actions invariant under (3), which do not correspond to any local geometric invariant.

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This claim was later revoked in [4], however, the general question remains.

The purpose of this note is to show that the natural expectation is correct and there is a one-to-one correspondence between p-brane actions in the form (1) and (2). To achieve this we use the Stückelberg technique to (re)introduce reparametrization invariance. We find that in the presence of the symmetry defined by (3) Stückelberg fields automatically provide the proper degrees of freedom to restore the manifest D-dimensional Poincaré invariance. That is, the non-linear invariance of the initial Lagrangian translates into a linear Poincaré symmetry of the Stückelberg action. We follow the variation of the Stückelberg procedure described in [5], which is most convenient for our purposes. Namely, to introduce diff invariance in any theory, one replaces all the fields in the action with their image under a diff

$$\sigma^a \to \eta^a(\vec{\sigma}) \, ,$$

and adds η^a to the set of dynamical fields. The resulting action is equivalent to the initial one and invariant under coordinate transformations, $\sigma^a \to \sigma'^a(\vec{\sigma})$, provided the new fields transform as

$$\eta^a \to \left(\sigma' \circ \hat{\eta}\right)^a \, ,$$

with $\circ$ denoting the composition of the diffeomorphism σ' with $\hat{\eta}$. The Stückelberg fields η^a do not transform as scalars with respect to coordinate transformations, however the inverse components of the diffeomorphism induced by η^a do. That is, if we perform a field redefinition from η^a to ξ^a such that

$$\xi^a(\vec{\eta}(\vec{\sigma})) = \sigma^a \tag{4}$$

then

$$\xi^a(\vec{\sigma}) \to \left(\xi \circ \sigma'\right)^a = \xi^a(\vec{\sigma}'(\vec{\sigma})) \, ,$$

which is the transformation rule for a scalar. As we show, these fields, when packaged with the physical transverse oscillations of the worldsheet, form a D-dimensional Lorentz vector, $X^\mu(\vec{\sigma}) \equiv (\xi^a(\vec{\sigma}), X^i(\vec{\sigma}))$. This proves the one-to-one correspondence between actions (1) and (2).

II. THE GENERAL PROOF

We start with the action (2), depending only on the physical transverse degrees of freedom of the worldsheet ("Goldstone fields"). As explained above, it should be invariant under the non-linearly realized Lorentz symmetry, i.e.

$$\delta^{ai}_{NL} S\left[\vec{X}(\vec{\sigma})\right] \equiv \int d^p\sigma \frac{\delta S[\vec{X}(\vec{\sigma})]}{\delta X^j} \delta^{ai}_{NL} X^j(\vec{\sigma}) = 0 \, ,$$

where $\delta^{ai}_{NL} X^j(\vec{\sigma})$ is given by equation (3). Our goal is to check that this invariance translates into the invariance under the linear Lorentz transformations after the reparametrization invariance is (re)introduced through the Stückelberg trick. The Stückelberg prescription is to replace the action with a new one defined as

$$S[X^j(\vec{\sigma})] \to S[X^j(\vec{\eta}(\vec{\sigma}))] \, .$$

This new action is a functional depending on $(D-p-1)$ Goldstones fields X^i and $(p+1)$ Stückelberg fields η^a. Equivalently, we can make a field redefinition and treat this action as a functional depending on the Goldstone fields X^i and the inverse Stückelberg fields, $\xi^a(\vec{\sigma})$, defined above. By construction, this functional is reparametrization invariant with both X^i and $\xi^a(\vec{\sigma})$ transforming as scalars. Our claim is that in addition, as a consequence of the non-linear Lorentz symmetry (3), this action is also invariant under a *linearly* realized Lorentz symmetry with X^i and ξ^a transforming as components of a Lorentz vector $X^\mu = (\xi^a, X^i)$,

$$\delta_L X^\mu = -\epsilon_{ai}(-\xi^a g^{i\mu} + X^i g^{a\mu}) \equiv -\epsilon_{ai}\delta^{ai}_L X^\mu \, . \tag{5}$$

To prove this, lets us show that $\delta_{NL} S[X(\vec{\eta}(\vec{\sigma}))] = 0$ implies $\delta_L S[X(\vec{\sigma}), \xi(\vec{\sigma})] = 0$. The latter variation is proportional to

$$\frac{\delta S[X^i(\eta(\vec{\sigma}))]}{\delta X^i} \left(\delta^{ai}_L X^j(\vec{\eta}(\vec{\sigma})) + \frac{\partial X^j(\vec{\eta}(\vec{\sigma}))}{\partial \eta^b(\vec{\sigma})} \delta^{ai}_L \eta^b(\vec{\sigma}) \right) \, . \tag{6}$$

To evaluate $\delta^{ai}_L \eta^b(\vec{\sigma})$ let us take the variation of (4), which gives

$$-\frac{\partial \eta^b(\vec{\sigma})}{\partial \sigma^c} \delta\xi^c(\vec{\eta}(\vec{\sigma})) = \delta\eta^b(\vec{\sigma}) \, . \tag{7}$$

After plugging (7) and (5) into the variation (6) and making use of the chain rule the variation (6) takes the form

$$\frac{\delta S[X^i(\eta(\vec{\sigma}))]}{\delta X^i} \left(\delta^{ij}\sigma^a + X^i(\vec{\eta}(\vec{\sigma})) \, \frac{\partial X^j(\vec{\eta}(\vec{\sigma}))}{\partial \sigma_a} \right) \, ,$$

which is simply $\delta^{ai}_{NL} S\left[\vec{X}(\vec{\eta}(\vec{\sigma}))\right]$ and, as a result, vanishes as a consequence of the non-linearly realized Lorentz symmetry of the original action. Thus δ^{bi}_L is equivalent to δ^{bi}_{NL} when the linear transformation is seen as acting on the appropriate combination of Goldstone and inverse Stückelberg fields. This completes the proof that a *generic* effective Lagrangian invariant under non-linear Lorentz symmetry can be obtained as a result of gauge fixing to the static gauge the corresponding reparametrization invariant Lagrangian with linear Lorentz symmetry. After the Stückelberg procedure, the non-linear symmetry of the original Lagrangian translates directly into in the linear Poincaré invariance, transforming fields X^μ as a vector. Hence, as was natural to expect, by gauge fixing generic geometric actions of the form (1) one obtains an exhaustive list of actions invariant under the non-linearly realized Lorentz (3) and shift symmetry.

III. A CONCRETE EXAMPLE

Given a somewhat abstract nature of the general proof of the previous section, we feel it is instructve to follow in more details how the Stückelberg procedure works in a concrete example. For simplicity, let us consider the case when the action has scaling zero, *i.e.* the Lagrangian is a function of the first derivatives ∂X^i only. The first step of the Stückelberg procedure results in an action of the form

$$S[X^i(\vec{\eta}(\vec{\sigma}))] = \int d^p\sigma \mathcal{L}(\partial_c X^i(\vec{\eta}(\vec{\sigma}))) ,$$

where partial derivatives ∂_c will always refer to differentiation with respect to the variable of integration unless stated otherwise. To introduce the inverse Stückelberg fields, let us change the integration variable, $\sigma^a \to \eta^a(\vec{\sigma}) \equiv \alpha^a$, so that $\sigma^a = \xi^a(\vec{\alpha})$. To make sense of the argument of the Lagrangian in these variables, we introduce the inverse Jacobian of this transformation

$$J(\vec{\alpha})^c_b \equiv [(\partial\xi)^{-1}]^c_b , \tag{8}$$

where -1 is understood as inverting the matrix of first partials of the variables $\vec{\xi}$. This substitution leaves us with

$$S[X,\xi] = \int d^p\alpha \det\left(\partial_e\xi^d\right) \mathcal{L}(J(\vec{\alpha})^c_b \partial_c X^i(\vec{\alpha})) .$$

Let us check that this Lagrangian is invariant under linear Lorentz transformations on the vector $X^\mu(\vec{\alpha}) \equiv (\xi^a(\vec{\alpha}), X^i(\vec{\alpha}))$, provided the original action is invariant under non-linear Lorentz transformations. Under a rotation in the (aj) plane the action transforms as

$$\delta^{aj}_L S[X] = \int d^p\alpha \frac{\partial L(\partial_c\xi^b, \partial_c X^i)}{\partial(\partial_f X^k)} \delta^{aj}_L(\partial_f X^k) + \frac{\partial L(\partial_c\xi^b, \partial_c X^i)}{\partial(\partial_f\xi^h)} \delta^{aj}_L(\partial_f \xi^h) , \tag{9}$$

where

$$L(\partial_c\xi^b, \partial_c X^i) = \det(\partial_e\xi^d) \mathcal{L}(J(\vec{\alpha})^c_b \partial_c X^i(\vec{\alpha})) .$$

At this point it will be useful to introduce some notation to clean up our calculation. Let us define

$$\frac{\partial \mathcal{L}(J^c_b \partial_c X^i)}{\partial(J^c_g \partial_c X^l)} \equiv \mathcal{L}_{(gl)} .$$

Now we notice the following chain rules, making use of this notation to change the differential operators acting on the Lagrangian to be with respect to its argument

$$\frac{\partial}{\partial(\partial_f X^k)} \to \frac{\partial(J^c_g \partial_c X^l)}{\partial(\partial_f X^k)} \mathcal{L}_{(gl)} = = J^c_g \delta^f_c \delta^l_k \mathcal{L}_{(gl)} = J^f_g \mathcal{L}_{(gk)} , \tag{10}$$

as well as

$$\frac{\partial}{\partial(\partial_f\xi^h)} \to \frac{\partial(J^c_g \partial_c X^l)}{\partial(\partial_f\xi^h)} \mathcal{L}_{(gl)} = \partial_c X^l(-J^c_h J^f_g)\mathcal{L}_{(gl)} . \tag{11}$$

The integrand of the first term, using equation (5) for the linear variation of X^k, becomes

$$\det(\partial_e\xi^d) \frac{\partial\left(\mathcal{L}(J(\vec{\alpha})^c_b \partial_c X^i(\vec{\alpha}))\right)}{\partial\left(\partial_f X^k\right)} \partial_f\xi^a \delta^{jk} .$$

Since J doesn't depend on ∂X, using (10) we arrive at

$$\det(\partial_e\xi^d)\mathcal{L}_{(ak)}\delta^{jk} . \tag{12}$$

The $\delta(\partial\xi)$ term in the variation breaks off further into two terms since both the determinant and original Lagrangian depend on $\partial\xi$. The differentiation of the determinant along with the variation of the inverse Stückelberg field yields:

$$\det(\partial_e\xi^d) J^f_h \partial_f X^j \mathcal{L}\; g^{ah} . \tag{13}$$

The differentiation of the Lagrangian, after implementing (11) leaves us with

$$-\det(\partial_e\xi^d)(J^f_g \partial_f X^j)(J^c_h \partial_c X^l)\mathcal{L}_{(gl)} g^{ah} . \tag{14}$$

The variation of the action is now the sum of equations (12), (13) and (14). We perform one more change of variables to get this into a form that we can juxtapose against the gauge fixed action in order to make use of the fact that the original Lagrangian possessed non-linear symmetry. Since every term has $d^p\alpha \det\left(\partial\xi/\partial\alpha\right)$ we simply make ξ our variable of integration. This also implies $J^a_b \partial_a X^i \to \partial_b X^i$ when we change notation $\partial_a \to \partial/\partial\xi^a$. The variation becomes:

$$\int d^p\xi \left(\mathcal{L}_{(ak)}\delta^{jk} + \partial_h X^j \mathcal{L} g^{ah} - \partial_g X^j \partial_h X^l \mathcal{L}_{(gl)} g^{ah}\right) . \tag{15}$$

The second term we integrate by parts:

$$\partial_h X^j \mathcal{L} \to -X^j \partial_h \mathcal{L} = -X^i \frac{\partial(\partial_d X^k)}{\partial\xi^h} \mathcal{L}_{(dk)} = -X^j \partial_h \partial_d X^k \mathcal{L}_{(dk)} .$$

Now we raise the derivatives with the metric of the second two terms and rename the indices d and g of the second and third term respectively both to h. We also rename the index l of the last term k and use a Kronecker delta to rename the index a of the first term h, and alas we can factor out the common derivative of the Lagrangian leaving us with

$$-\int d^p\xi \mathcal{L}_{(hk)} \left(-\delta^a_h\delta^{jk} + X^j\partial^a\partial_h X^k + \partial^a X^j\partial_h X^k\right) \ .$$

which by equation (3) is proportional to

$$\int d^p\xi \mathcal{L}_{(hk)}\ \delta^{aj}_{NL}(\partial_h X^k) = \delta^{aj}_{NL} S[\vec{X}(\vec{\xi})] \ . \qquad (16)$$

This vanishes by our initial assumption. Thus any Lagrangian with this non-linear symmetry can be turned into a manifestly Poincaré and diffeomorphism invariant action. One simply restores diff invariance with Stückelberg fields and identifies the appropriate degrees of freedom which combine with the physical degrees of freedom to form the embedding coordinates of the worldsheet in spacetime. The original Lagrangian can always be interpreted as the static gauge of this procedure.

As an aside, if we knew that our initial Lagrangian had to be invariant with respect to linear Lorentz transformations then equation (15) provides us with a differential equation for $\mathcal{L}$. The integral would have to vanish meaning the integrand was proportional to a linear combination of $\partial^a X^j$ that satisfies the required symmetries of the Goldstone Lagrangian. Equivalently, by shifting $\mathcal{L}$ by a meaningless constant, we can cancel this total derivative by noticing that only the second term in (15) sees this correction and we pick up exactly the allowed form of the total derivative, which we can tune accordingly. To solve this differential equation, we can write the Lagrangian as

$$\mathcal{L}(\partial_c X^k) = \sum_{c,k}\sum_n a_n^{(c,k)}(\partial_c X^k)^n \ .$$

again, noting that the coefficients $a_n^{(ck)}$ aren't completely arbitrary in c and k, since the Lagrangian still must respect the manifest $ISO(1,p)\times SO(D-p-1)$ symmetry. The derivatives and products of ∂X then provide us with recurrences relations between the powers of $\partial_c X^k$ in the argument of the Lagrangian. The resummation of these terms is precisely the procedure in [3] to show that $\mathcal{L}$ is the invariant area of the p-brane. As a trivial example, for the 0-brane, better known as the relativistic point particle, this differential equation has the form:

$$-\mathcal{L}'(\dot{x}(t)) - \dot{x}(t)\mathcal{L}(\dot{x}(t)) + \dot{x}(t)^2\mathcal{L}'(\dot{x}(t)) = 0 \ .$$

It can be easily shown that the Nambu Goto Lagrangian solves this differential equation

$$\mathcal{L}(\dot{x}(t)) = \sqrt{-1+\dot{x}(t)^2} \ .$$

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