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Linear Algebra

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Chapter 0

Sets and Proofs

0.1 Sets

We will begin by exploring the concept of a set through what is sometimes called intuitive or naive set theory. A more rigorous approach, axiomatic set theory, is outside the scope of this course. For the purposes of this course, our intuitive treatment of sets will suffice.

Definition 0.1.1. A set is a well-defined collection of objects.

0.2 Propositional logic

0.3 Proofs

Chapter 1

Vectors

1.1 Vector spaces

Definition 1.1.1. Let F be a set and let $+$ and $\cdot$ be two operations defined for elements of F. F, together with the operations $+$ and $\cdot$, is called a field if all of the following axioms are satisfied:

1. $+$ and $\cdot$ are associative, i.e. for all $a, b, c \in F$, we have

$$(a+b)+c = a+(b+c) \quad \text{and} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

2. $+$ and $\cdot$ are commutative, i.e. for all $a, b \in F$, we have

$$a+b = b+a \quad \text{and} \quad a \cdot b = b \cdot a.$$

3. There exists an element $0_F \in F$, called the additive identity, such that for all $a \in F$, we have

$$a + 0_F = a.$$

4. There exists an element $1_F \in F$, called the multiplicative identity, such that for all $a \in F$, we have

$$a \cdot 1_F = a.$$

5. For every $a \in F$, there exists an element $-a \in F$, called the additive inverse of a, such that

$$a + (-a) = 0_F.$$

6. For every $a \in F$ other than 0_F, there exists an element $a^{-1} \in F$, called the multiplicative inverse of a, such that

$$a \cdot a^{-1} = 1_F.$$

7. $\cdot$ is distributive over $+$, i.e. for all $a, b, c \in F$, we have

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c).$$

Example 1.1.2. Show that the set of real numbers $\mathbb{R}$, together with standard addition and multiplication, is a field.

Solution. We will prove this result by examining each of the axioms one-by-one:

1. We already know that standard addition and multiplication are associative.
2. We also know that standard addition and multiplication are commutative.
3. The additive identity is the number 0.
4. The multiplicative identity is the number 1.
5. For any $x \in \mathbb{R}$, the additive inverse is the number $-x$.
6. For any $x \in \mathbb{R}$ other than 0, the multiplicative inverse is the number $1/x$.
7. We already know that standard multiplication is distributive over standard addition.

Hence, $\mathbb{R}$ is a field. □

Example 1.1.3. Show that the set of integers $\mathbb{Z}$, together with standard addition and multiplication, is not a field.

Solution. The multiplicative identity is the number 1. Consider the number $2 \in \mathbb{Z}$. There does not exist a number $n \in \mathbb{Z}$ such that $2n = 1$; that is, 2 does not have a multiplicative inverse in $\mathbb{Z}$. Hence, $\mathbb{Z}$ is not a field. □