Cooper Golemme

Question 1: Chapter 1 #4

1.1 Part 1

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$$

Proof:

To show that the LHS and RHS are equal we need to show set equivalence. I will do this by showing first that given an arbitrary event $w \in LHS$, it is necessarily true that $w \in RHS$. I will then show the opposite to prove these sets are equivalent.

1.1.1 LHS $\subseteq$ RHS

Let

$$w \in \left(\bigcup_{i \in I} A_i\right)^c$$

It follows, by definition of complement, that:

$$w \notin \bigcup_{i \in I} A_i$$

Thus, by the definition of a union:

$$\forall i \in I \; w \notin A_i$$

By the definition of complement:

$$\forall i \in I \; w \in A_i^c$$

Thus, because $\forall i \in I \; w \in A_i^c$:

$$w \in \bigcap_{i \in I} A_i^c$$

1.1.2 RHS $\subseteq$ LHS

Let

$$w \in \bigcap_{i \in I} A_i^c$$

By definition of intersection:

$$\forall i \in I \; w \in A_i^c$$

By definition of complement:

$$\forall i \in I \; w \notin A_i$$

By definition of union:

$$w \notin \bigcup_{i \in I} A_i$$

By definition of complement:

$$w \in \left(\bigcup_{i \in I} A_i \right)^c$$

1.1.3 RHS = LHS

Because RHS $\subseteq$ LHS and LHS $\subseteq$ RHS, RHS = LHS, thus

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

1.2 Part 2

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$$

Proof:

Following similar steps to above, I will prove set equivalence given an arbitrary event w.

Let $w \in \left(\bigcap_{i\in I} A_i\right)^c$

$$\begin{aligned} w &\notin \bigcap_{i\in I} A_i \\ \exists i \in I\ w &\notin A_i \\ \exists i \in I\ w &\in A_i^c \\ w &\in \bigcup_{i\in I} A_i^c \end{aligned}$$

Let $w \in \bigcup_{i\in I} A_i^c$

$$\begin{aligned} \exists i \in I\ w &\in A_i^c \\ \exists i \in I\ w &\notin A_i \\ w &\notin \bigcap_{i\in I} A_i \\ w &\in \left(\bigcap_{i\in I} A_i\right)^c \end{aligned}$$

Because RHS $\subseteq$ LHS and LHS $\subseteq$ RHS, RHS $=$ LHS, thus

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$$

Question 2: Chapter 1 #8

Prove: $P(\bigcap_{i=1}^{\infty} A_i) = 1$ given that $\forall i \; P(A_i) = 1$

$$P(\bigcap_{i=1}^{\infty} A_i) = 1 - P(\bigcap_{i=1}^{\infty} A_i)^c = 1 - P(\bigcup_{i=1}^{\infty} A_i^c) \text{ by } Question 1$$

$$P(\bigcup_{i=1}^{\infty} A_i^c) \leq \sum_{i=1}^{\infty} P(A_i^c) \text{ by union bound theorem}$$

$$P(A_i^c) = 1 - P(A_i) = 1 - 1 = 0$$

$$P(\bigcup_{i=1}^{\infty} A_i^c) \leq 0$$

$$P(\bigcup_{i=1}^{\infty} A_i^c) \geq 0 \text{ by non-negativity constraint of prob. distribution}$$

$$P(\bigcup_{i=1}^{\infty} A_i^c) = 0$$

Substitution of line 1 expression

$$P(\bigcap_{i=1}^{\infty} A_i) = 1 - P(\bigcap_{i=1}^{\infty} A_i)^c = 1 - P(\bigcup_{i=1}^{\infty} A_i^c) = 1 - 0 = 1$$

$$P(\bigcap_{i=1}^{\infty} A_i) = 1$$

Question 3: Chapter 2 #14

Let (X, Y) be uniformly distributed on the unit disk $(x, y) : x^2 + y^2 \leq 1$. Let $R = \sqrt{X^2 + Y^2}$. Find the CDF and PDF of R.

3.1 CDF

$$F_R(r) = P(r \leq R) = \frac{area\, of\, r}{area\, of\, R} = \frac{\pi r^2}{\pi} = r^2$$

$F_R(r) = 0$ for r < 0 and $F_R(r) = 1$ for r > 1, so

$$F_R(r) = \begin{cases} 0 & \text{if } r < 0 \\ r^2 & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r > 1 \end{cases}$$

3.2 PDF

The PDF: $f_R(r)$ is the derivative of the CDF: $F_R(r)$:

$$f_R(r) = \frac{d}{dr} F_R(r)$$

$$f_R(r) \begin{cases} 0 & \text{if } r < 0 \\ 2r & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r > 1 \end{cases}$$

Question 4: Chapter 3 #7

Let X be a continuous random variable with CDF F. Suppose that $P(X > 0) = 1$ and that $E(X)$ exists. We want to show that:

$$E(X) = \int_0^\infty P(X > x)\, dx$$

Proof:
1. The expected value of X is given by:

$$E(X) = \int_0^\infty x f_X(x)\, dx$$

where $f_X(x)$ is the PDF of X.
2. We can integrate by parts:

$$f_X(x) = \frac{d}{dx} F_X(x) \Rightarrow F_X(x) = \int f_X(x)\, dx$$

Let $u = x$, so $du = dx$, - $dv = f_X(x)\, dx$, so $v = 1 - F(x) = P(X > x)$.

$$\mathbb{E}(X) = \int_0^\infty x f_X(x)\, dx = [x(1 - F(x))]_0^\infty + \int_0^\infty (1 - F(x))\, dx.$$

Given $E(X)$ exists:

$$\lim_{x\to\infty} x[1 - F(x)] = 0$$

$$[x(1 - F(x))]_0^\infty = \lim_{x\to\infty} x[1 - F(x)] - \lim_{x\to 0} x[1 - F(x)] = 0 - \lim_{x\to 0} x[1 - F(x)]$$

$$\lim_{x\to 0} x[1 - F(x)] = \lim_{x\to 0} x - \lim_{x\to 0} xF(x) = 0 - 0 = 0 \qquad \text{(propterty of CDF)}$$

Thus,

$$[x(1 - F(x))]_0^\infty = 0$$

So,

$$\mathbb{E}(X) = \int_0^\infty (1 - F(x))\, dx = \int_0^\infty P(X > x)\, dx$$

Question 5: Chapter 4 #3

Let $X_1, X_2, \ldots, X_n \sim \text{Bernoulli}(p)$ and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Bound $P(|\bar{X}_n - p| > \epsilon)$ using Chebyshev's and Hoeffding's inequalities and show that the Hoeffding's bound is smaller when n is large.

1. **Chebyshev's Inequality:**
Chebyshev's inequality states:

$$P\left(|Y - \mathbb{E}[Y]| \geq \epsilon\right) \leq \frac{\text{Var}(Y)}{\epsilon^2}$$

For $\bar{X}_n$:

$$\mathbb{E}[\bar{X}_n] = p$$

$$\text{Var}(\bar{X}_n) = \frac{p(1-p)}{n}$$

Thus:

$$P\left(|\bar{X}_n - p| \geq \epsilon\right) \leq \frac{p(1-p)}{n\epsilon^2}$$

2. **Hoeffding's Inequality:**

$$P\left(|\bar{X}_n - p| > \epsilon\right) \leq 2\exp\left(-2n\epsilon^2\right)$$

This is true for $X_1, X_2, \ldots, X_n \sim \text{Bernoulli}(p)$ and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.
3. **Comparison:**
- **Chebyshev's Bound:**

$$P\left(|\bar{X}_n - p| \geq \epsilon\right) \leq \frac{p(1-p)}{n\epsilon^2}$$

- **Hoeffding's Bound:**

$$P\left(|\bar{X}_n - p| \geq \epsilon\right) \leq 2\exp(-2\epsilon^2)$$

For large n, Hoeffding's bound is smaller than Chebyshev's bound because Hoeffding's bound decays exponentially while Chebyshev's bound decays polynomial, which is always slower than exponential decay for large n.

Question 6: Supplemental Question

6.1 Part A

For $n = 10, 20, 30, \ldots, 10000$, sample n i.i.d. samples from $N(0,1)$, i.e., the random variable X with density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Let $\bar{X}_n$ be the corresponding sample average. Plot $\bar{X}_n$ as a function of n. Describe the behavior as n increases. What does the Law of Large Numbers suggest will happen as $n \to \infty$?

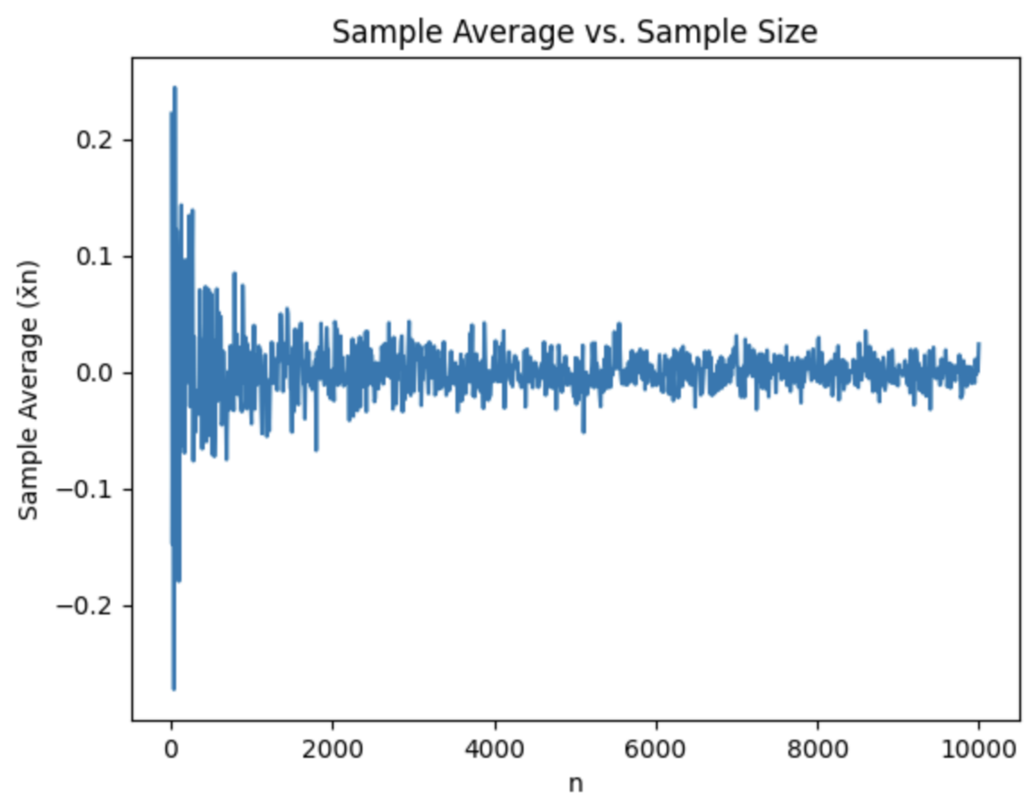

Figure 1: $\bar{X}_n$ as a function of n

As n increases, the sample average $(\bar{X}_n)$ tends to stabilize around the mean, 0 in this case. The Law of Large Numbers applies to this problem and states that as n increases, the average of the sum of the random Gaussian variables converges to the mean which in this case is 0.

6.2 Part B

For $n = 10, 20, 30, \ldots, 10000$, sample n i.i.d. samples from the Cauchy distribution, i.e., the random variable X with density

$$f_X(x) = \frac{1}{\pi(1 + x^2)}.$$

Let $\bar{X}_n$ be the corresponding sample average. Plot $\bar{X}_n$ as a function of n. Describe the behavior as n increases. What does the Law of Large Numbers suggest will happen as $n \to \infty$?

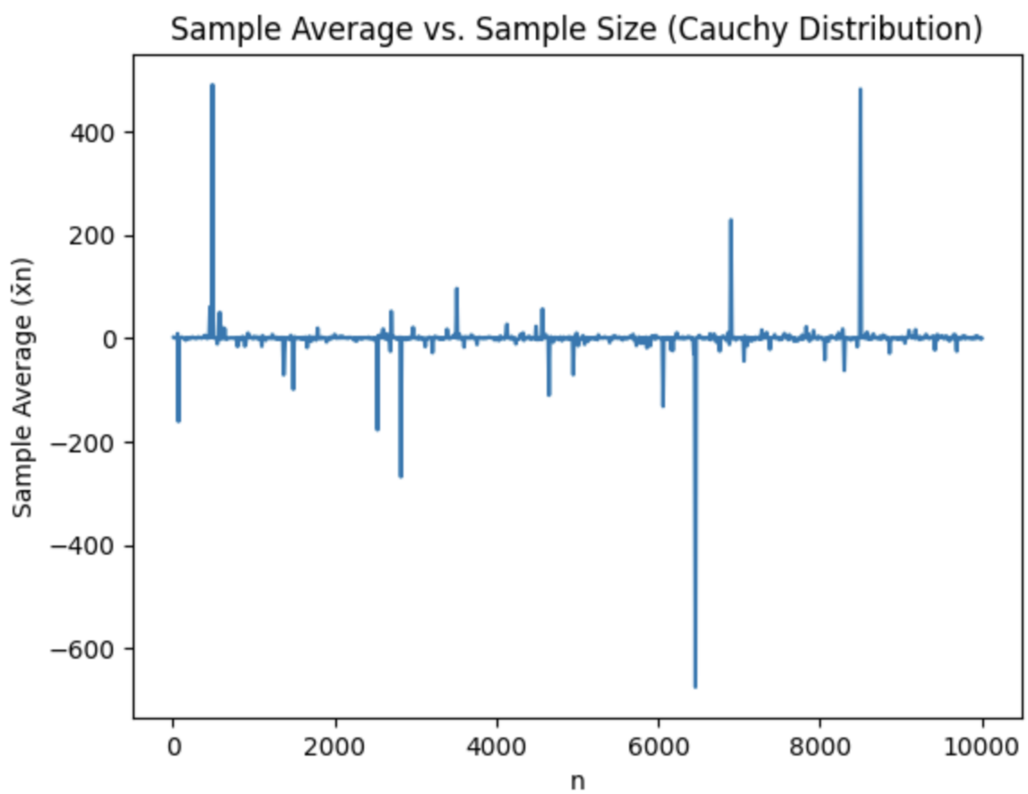

Figure 2: $\bar{X}_n$ as a function of n

The Law of Large Numbers does not apply to this distribution because it is Cauchy and therefore does not have a finite mean. This can be seen in the graph which fluctuates randomly and does not tend to standardize toward a particular value.

6.3 Code for Supplemental

6.3.1 A

```
import numpy as np
import matplotlib.pyplot as plt

n_values = np.arange(10, 10001, 10)

sample_averages = []

for n in n_values:
    samples = np.random.normal(0, 1, n)
    sample_average = np.mean(samples)
    sample_averages.append(sample_average)

plt.plot(n_values, sample_averages)
plt.xlabel("n")
plt.ylabel("Sample Average (xn)")
plt.title("Sample Average vs. Sample Size")
plt.show()
```

6.3.2 B

```
import numpy as np
import matplotlib.pyplot as plt

n_values = np.arange(10, 10001, 10)

sample_averages = []

for n in n_values:
    samples = np.random.standard_cauchy(n)
    sample_average = np.mean(samples)
```

```
    sample_averages.append(sample_average)

plt.plot(n_values, sample_averages)
plt.xlabel("n")
plt.ylabel("Sample Average (xbn)")
plt.title("Sample Average vs. Sample Size (Cauchy Distribution)")
plt.show()
```