Tufts University
Department of Mathematics

Math 135 **Homework 6** [1] **Fall 2024**

Due date: 11:59 pm, Sunday, October 27, 2024 on Gradescope.

You are encouraged to work on problems with other Math 135 students and to talk with your professor and TA but your answers should be in your own words.

A proper subset of the problems will be selected for grading.

This homework covers sections 9.3, 9.4, 10.1 and 10.2 of Fitzpatrick.

Problems:

1. (20 points) You may use the following theorem for this problem.

   Theorem 1 (Weierstrass M-Test) *Let $\sum_{k=0}^{\infty} M_k$ be a convergent infinite series of real numbers and let $\sum_{k=0}^{\infty} f_k(x)$ be an infinite series of functions on the domain D. Assume that, for each $k \in \mathbb{N} \cup \{0\}$ and each $x \in D$, we have $|f_k(x)| \leq M_k$. Then $\sum_{k=0}^{\infty} f_k(x)$ converges absolutely and uniformly to a function $f\colon D \to \mathbb{R}$.*

   Let (a_n) be a bounded sequence of numbers, let $r \geq 0$ and consider the following infinite series of functions.

$$\sum_{k=0}^{\infty} \frac{a_k x^k}{k!} = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \frac{a_4 x^4}{4!} + \cdots$$

   Prove that this series of functions converges to a *continuous* function $f\colon [-r, r] \to \mathbb{R}$. (HINT: Prove that the series is uniformly convergent using the Weierstrass M-Test and then apply another important theorem.)

2. (15 points) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Recall the Cauchy-Schwarz Inequality: $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$.

   Prove that $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \, \|\mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. You may use the fact that $\mathbf{u}$ and $\mathbf{v}$ are *linearly dependent* if and only if $\mathbf{u} = 0$ or $\mathbf{v} = \alpha \mathbf{u}$ for some $\alpha \in \mathbb{R}$.

3. (20 points) Let $i \in \{1, 2, \ldots, n\}$ and let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the ith coordinate function

$$p_i(x_1, \ldots, x_i, \ldots, x_n) = x_i$$

   (see p. 279 of Fitzpatrick).

   (a) Prove that p_i is linear. That is, prove for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, that $p_i(\mathbf{x} + \mathbf{y}) = p_i(\mathbf{x}) + p_i(\mathbf{y})$ and $p_i(\alpha \mathbf{x}) = \alpha p_i(\mathbf{x})$.

   (b) Prove for every $\mathbf{u} \in \mathbb{R}^n$ that $|p_1(\mathbf{u})| + |p_2(\mathbf{u})| + \cdots + |p_n(\mathbf{u})| \geq \|\mathbf{u}\|$.

4. (20 points) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let $(\mathbf{u}_k)$ and $(\mathbf{v}_k)$ be sequences in $\mathbb{R}^n$ that converge to $\mathbf{u}$ and $\mathbf{v}$, respectively. Prove that $\lim_{k \to \infty} \langle \mathbf{u}_k, \mathbf{v}_k \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$.

   (NOTE: Once we define continuity in $\mathbb{R}^n$ you will know that the function $\Lambda\colon \mathbb{R}^{2n} \to \mathbb{R}$ given by $\Lambda\big((u_1, u_2, \ldots, u_{2n})\big) = \langle (u_1, u_2, \ldots, u_n), (u_{n+1}, u_{n+2}, \ldots, u_{2n}) \rangle$ is continuous.)

5. (25 points) A sequence $(\mathbf{u}_k)$ in $\mathbb{R}^n$ is *Cauchy* if, for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for any $k, \ell \geq K$ we have $\|\mathbf{u}_k - \mathbf{u}_\ell\| < \epsilon$. Prove that a sequence $(\mathbf{u}_k)$ in $\mathbb{R}^n$ converges if and only if it is Cauchy. (HINT: Prove that $(\mathbf{u}_k)$ is Cauchy if and only if each component sequence is Cauchy.)