Question 1: Chapter 5 #3

Let $X_1, \ldots, X_n$ be i.i.d. and let $\mu = \mathbb{E}(X_1)$. Suppose that the variance is finite. Show that $\bar{X}_n \xrightarrow{qm} \mu$.

We need to show:

$$\lim_{n\to\infty} \mathbb{E}[(\bar{X}_n - \mu)^2] = 0.$$

Proof:

Let $Y_i = X_i - \mu$, $\mu_{Y_i} = 0$ and Let $\sigma_{Y_i} = \sigma < \infty$

$$\mathbb{E}[(\bar{X}_n - \mu)^2] = \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)\right)^2\right]. \tag{1}$$

$$= \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n} Y_i\right)^2\right] \tag{2}$$

$$\textbf{expanding the square} \tag{3}$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n}\mathbb{E}[Y_i]^2 - \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}[Y_iY_j]\right] \tag{4}$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n}(Var(Y_i) + (\mathbb{E}[Y_i])^2 - \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}[Y_i]\mathbb{E}[Y_j]\right] \tag{5}$$

$$= \frac{1}{n^2}(n(\sigma^2 + 0) + (n)(n-1)(0)) \tag{6}$$

$$= \frac{\sigma^2}{n} \tag{7}$$

Since $\sigma^2 < \infty$, we conclude that:

$$\lim_{n\to\infty} \mathbb{E}[(\bar{X}_n - \mu)^2] = \lim_{n\to\infty}\frac{\sigma^2}{n} = 0 \tag{8}$$

$\square$

Question 2: Chapter 5 #4

Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$P(X_n = 1) = 1 - \frac{1}{n^2} \quad \text{and} \quad P(X_n = n) = \frac{1}{n^2}.$$

Does X_n converge in probability? Does X_n converge in quadratic mean?

Proof Convergence in Probability

Let $\epsilon > 0$

$\mathbb{P}(|X_n - X| > \epsilon) =$

$\mathbb{P}(|X_n - X| > \epsilon \mid X_n = \frac{1}{n})\mathbb{P}(X_n = \frac{1}{n}) + \mathbb{P}(|X_n - X| > \epsilon \mid X_n = n)\mathbb{P}(X_n = n)$

By Law of Conditional Probability (9)

$$= \mathbb{P}(|\frac{1}{n} - X| > \epsilon)(1 - \frac{1}{n^2}) + \mathbb{P}(|n - X| > \epsilon)(\frac{1}{n^2}) \tag{10}$$

Using the law of large numbers, lets examine $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon)$. We have established that:

$$\mathbb{P}(|X_n - X| > \epsilon) = \mathbb{P}(|\frac{1}{n} - X| > \epsilon)(1 - \frac{1}{n^2}) + \mathbb{P}(|n - X| > \epsilon)(\frac{1}{n^2}) \tag{11}$$

Lets examine the limit as $n \to \infty$ of the first term,

$$\lim_{n\to\infty} \mathbb{P}(|\frac{1}{n} - X| > \epsilon)(1 - \frac{1}{n^2}) \tag{12}$$

$$= \mathbb{P}(| - X| > \epsilon)(1) \tag{13}$$

$$= \mathbb{P}(|X| > \epsilon) \tag{14}$$

Now, lets examine the limit of the second term:

$$\lim_{n\to\infty} \mathbb{P}(|n - X| > \epsilon)(\frac{1}{n^2}) = 1 * 0 = 0 \tag{15}$$

Because $\lim_{n\to\infty} \mathbb{P}(|n - X| > \epsilon) = 1$

Combining these we get:

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = \lim_{n\to\infty} \mathbb{P}(|X| > \epsilon) \tag{16}$$

If we set $X = 0$, then because $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}(0 > \epsilon) = 0$$

Therefore, X_n converges in probability to 0

Proof Convergence in QM

$$E[(X_n - X)^2] \tag{17}$$
$$= E[(X_n - X)^2 | X_n = \frac{1}{n}]P(X_n = \frac{1}{n}) + E[(X_n - X)^2 | X_n = n]P(X_n = n) \tag{18}$$
$$= E[(\frac{1}{n} - X)^2](1 - \frac{1}{n^2}) + E[(n - X)^2](\frac{1}{n^2}) \tag{19}$$
$$= E[n^{-2} - 2n^{-1}X + X^2](1 - n^{-2}) + E[n^2 - 2nX + X^2](n^{-2}) \tag{20}$$

After foiling and simplifying this, we get:

$$E[(X_n - X)^2] = E(X^2) - E(X)2n^{-1}(2 + n^{-2}) + n^{-2}(1 - n^{-2}) + 1 \tag{21}$$

Taking the limit of both sides we get:

$$\lim_{n\to\infty} E[(X_n - X)^2] = \tag{22}$$
$$\lim_{n\to\infty} [E(X^2) - E(X)2n^{-1}(2 + n^{-2}) + n^{-2}(1 - n^{-2}) + 1] \tag{23}$$
$$= E(X^2) + 1 \tag{24}$$

There is no value of X such that:

$$E(X^2) + 1 = 0$$

Therefore X_n does not converge in QM

Question 3: Chapter 5 #14

Let $X_1, \ldots, X_n$ be i.i.d. random variables from Uniform$(0,1)$. Let $Y_n = \bar{X_n}^2$. Find the limiting distribution of Y_n.

Answer

If $X_1, \ldots, X_n$ be i.i.d. random variables from Uniform$(0,1)$, then $\bar{X_n}$ has limiting distribution $1/2$ by the strong law of large numbers.

What is the limiting dist. of Y_n?

$$P(Y_n > x) = P(\bar{X_n}^2 > x) = P(\bar{X_n} > \sqrt{x})$$

This is can be expressed in terms of the CDF of Y_n

$$F_{Y_n}(x) = P(Y_n > x) = P(\bar{X_n} > \sqrt{x}) = F_{\bar{X}_n}(\sqrt{x})$$

Now let's examine the limit:

$$\lim_{n\to\infty} F_{\bar{X}_n}(\sqrt{x})$$

Becuase we know $\bar{X_n}$ has limiting distribution $1/2$, we can replace $F_{\bar{X}_n}$ in the limit with $F_{1/2}$ as follows:

$$\lim_{n\to\infty} F_{1/2}(\sqrt{x})$$

Now there is nothing inside the limit that depends on n, so this becomes:

$$F_{1/2}(\sqrt{x})$$

Recall that:

$$F_{1/2}(x) = P(1/2 > x)$$

Therefore:

$$F_{1/2}(\sqrt{x}) = P(1/2 > \sqrt{x})$$

Question 4: Chapter 6 #1

Let $X_1, \dots, X_n$ be i.i.d. random variables with $X_i \sim \text{Poisson}(\lambda)$, and let $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the bias, standard error (SE), and mean squared error (MSE) of this estimator.

4.1 Bias

$$Bias(\hat{\lambda}) = \mathbb{E}[\hat{\lambda}] - \lambda = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} X_i] - \lambda \tag{25}$$

$$= \frac{1}{n}\mathbb{E}[\sum_{i=1}^{n} X_i] - \lambda \tag{26}$$

By linearity of expectation

$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] - \lambda \tag{27}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\lambda - \lambda \tag{28}$$

$$= n\lambda/n - \lambda = \lambda - \lambda = 0 \tag{29}$$

Therefore, $\hat{\lambda}$ is an unbiased estimator

4.2 Standard Error

The standard error is given as:

$$SE(\hat{\lambda}) = \sqrt{Var(\hat{\lambda})}$$

To get the standard error of $\hat{\lambda}$, first compute $Var(\hat{\lambda})$:

$$
\begin{aligned}
Var(\hat{\lambda}) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i) && (30)\\
&\text{By variance laws}\\
&= \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) && (31)\\
&\text{Because each } X_i \text{ are iid}\\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) && (32)\\
&\text{Because each } X_i \sim \text{Poisson}(\lambda) && (33)\\
&= \frac{1}{n^2}\sum_{i=1}^{n} \lambda && (34)\\
&= \frac{1}{n^2} n\lambda && (35)\\
&= \frac{\lambda}{n} && (36)
\end{aligned}
$$

Therefore,

$$SE(\hat{\lambda}) = \sqrt{\frac{\lambda}{n}}$$

4.3 Mean Squared Error(MSE)

The Mean Squared Error(MSE) of an estimator is given as:

$$MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2]$$

Or, simplified:

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias(\hat{\theta})^2$$

Applying that to this case:

$$
\begin{aligned}
MSE(\hat{\lambda}) &= Var(\hat{\lambda}) + 0 && (37)\\
&\text{Because } Bias(\hat{\lambda}) = 0\\
&= \sqrt{\frac{\lambda}{n}} && (38)
\end{aligned}
$$

Therefore, the MSE is $\sqrt{\frac{\lambda}{n}}$

Question 5: Supplemental Question (Convergence in L^p, $p \neq 2$)

For $p > 1$, we say that a sequence $\{X_n\}_{n=1}^{\infty}$ of random variables converges to a random variable X in L^p if

$$\lim_{n\to\infty} \mathbb{E}|X - X_n|^p = 0.$$

Show that if $\{X_n\}_{n=1}^{\infty}$ converges to X in L^2, then $\{X_n\}_{n=1}^{\infty}$ converges to X in L^1. We sometimes call the former convergence in mean square and the latter convergence in mean.

Want to prove:

$$\lim_{n\to\infty} \mathbb{E}|X - X_n|^2 = 0.$$

Then,

$$\lim_{n\to\infty} \mathbb{E}|X - X_n| = 0.$$

Proof:

Jensen's inequality gives us, because x^2 is convex:

$$E[|X_n - X|] \leq (E[|X_n - X|^2])^{\frac{1}{2}}$$

We know that:

$$\lim_{n\to\infty} \mathbb{E}|X - X_n|^2 = 0.$$

Lets use that to talk about $\lim_{n\to\infty} \mathbb{E}|X - X_n|$. If we take the limit of both sizes of the inequality above we get.

$$\lim_{n\to\infty} E[|X_n - X|] \leq \lim_{n\to\infty} (E[|X_n - X|^2])^{\frac{1}{2}} = 0$$

Therefore,

$$\lim_{n\to\infty} E|X_n - X| = 0$$

□