HW4

1. a) Suppose the first k components are nonzero.

   Because $\sum_{i=1}^{k} x_i = 1$ and $\log x$ is a concave function,

   $-H(x) \geq \log\left(\frac{\sum_{i=1}^{k} x_i}{k}\right) = \log\left(\frac{1}{k}\right) = -\log(k).$

   Because $k \leq n$, $H(x) \leq \log(k) \leq \log(n)$.

   b) $H(\bar{x}) = -\sum_{i=1}^{n} \frac{1}{n} \log\left(\frac{1}{n}\right) = -\log\left(\frac{1}{n}\right) = \log(n).$

   Because $H(x) \leq \log(n)$, $H(\bar{x})$ sis the maximum of $H(x)$.

   $$Hessian(H(x)) = \begin{bmatrix} (-1)^{n-1}(n-2)!\frac{1}{{x_1}^{n-1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (-1)^{n-1}(n-2)!\frac{1}{{x_n}^{n-1}} \end{bmatrix}.$$

   Apparently, $Hessian(H(x))$ is positive-definite.

   $H(x)$ is strictly concave.

   So $\bar{x}$ is the unique maximum.

2. a) Let $\mu =$ EX, for $a < u < \mu < s < b$.

   Apparently, there exists $\theta \in [0,1]$ such that $\theta u + \bar{\theta} s = \mu$.

   Because, f is convex. $\frac{f(\mu)-f(s)}{\mu-s} = \frac{f(\theta u+\bar{\theta} s)-f(s)}{\theta u+\bar{\theta} s-s} = \frac{f(\theta u+\bar{\theta} s)-f(s)}{\theta(u-s)} \leq \frac{f(u)-f(s)}{(u-s)}$

   For the same reason, $\frac{f(u)-f(\mu)}{u-\mu} \geq \frac{f(u)-f(s)}{(u-s)}$.

   So, $\frac{f(u)-f(\mu)}{u-\mu} \geq \frac{f(\mu)-f(s)}{\mu-s}$.

   b) Take $\beta = sup\frac{f(\mu)-f(s)}{\mu-s}$. Because $\frac{f(u)-f(\mu)}{u-\mu} \geq \frac{f(\mu)-f(s)}{\mu-s}$, $\beta < +\infty$.

   Obviously, $\beta > -\infty$.

   So, there exists β such that $f(x) \geq f(\mu) + \beta(x - \mu), \forall x \in (a, b)$

   c) therefore, $f(X) \geq f(\mu) + \beta(X - \mu)$

   Namely, $f(X) \geq f(EX) + \beta(X - EX)$.

   $Ef(X) \geq f(EX)$

3. $Hessian(\log(1 + e^{3x_1+2x_2})) = \begin{bmatrix} \frac{9e^{3x_1+2x_2}}{(1+e^{3x_1+2x_2})^2} & \frac{6e^{3x_1+2x_2}}{(1+e^{3x_1+2x_2})^2} \\ \frac{6e^{3x_1+2x_2}}{(1+e^{3x_1+2x_2})^2} & \frac{4e^{3x_1+2x_2}}{(1+e^{3x_1+2x_2})^2} \end{bmatrix}.$

   $D_1 = \frac{9e^{3x_1+2x_2}}{(1+e^{3x_1+2x_2})^2}$, $D_2 = 0$.

   So $\log(1 + e^{3x_1+2x_2})$ is convex.

   Therefore, $\{\boldsymbol{x}: \log(1 + e^{3x_1+2x_2}) \leq 2\}$ is convex.

   Because $A\boldsymbol{x} + b$ is an affine function, so S is convex.

4. a) It's a convex optimization problem.

   b) It's not a convex optimization problem.