CS2601 Linear and Convex Optimization

Homework 9

Due: 2021.12.2

1. Consider the equality constrained quadratic program

$$\begin{aligned} \min_{x_1,x_2} \quad & f(x_1,x_2) = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 - x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 = 1 \end{aligned}$$

(a). Find the optimal solution x_1^*, x_2^* by reduction to an unconstrained problem.

(b). Find the optimal solution and the corresponding Lagrange multiplier λ^* using the Lagragian multipliers method.

2. Consider the equality constrained quadratic program

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g}^T\boldsymbol{x} + c \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

where $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, $\boldsymbol{Q} \succ \boldsymbol{O}$, $\boldsymbol{g} \in \mathbb{R}^n$, $c \in \mathbb{R}$, $\boldsymbol{A} \in \mathbb{R}^{k\times n}$ with rank $\boldsymbol{A} = k$, and $\boldsymbol{b} \in \mathbb{R}^k$.

(a). Write down the Lagrange condition for this problem.

(b). Find a closed form solution for the optimal solution $\boldsymbol{x}^*$ and the corresponding Lagrange multiplier $\boldsymbol{\lambda}^*$. Hint: Show $\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T \succ \boldsymbol{O}$ and hence is invertible.

(c). Use part (b) to find the projection $\mathcal{P}_S(\boldsymbol{x}_0)$ of a point $\boldsymbol{x}_0$ onto the the affine space $S = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$, i.e. solve

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|_2^2 \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

When $\boldsymbol{x}_0 = \boldsymbol{0}$, you should recover the result on slide 11 of §9.

(d). Consider a hyperplane $P = \{\boldsymbol{x} : \boldsymbol{w}^T\boldsymbol{x} = b\}$. Use the result in (c) to find the distance $\mathrm{dist}(\boldsymbol{x}_0, P)$ between $\boldsymbol{x}_0$ and P. You should recover the result on slide 12 of §1.

3. Solve the following problem,

$$\begin{aligned} \min_{x_1,x_2} \quad & f(x_1,x_2) = x_1 x_2 \\ \text{s.t.} \quad & x_1^2 + 4x_2^2 = 1 \end{aligned}$$

4. In this problem, we characterize the eigenvalues of a symmetric matrix $\boldsymbol{A}$ as the optimal values of certain optimization problems. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $\boldsymbol{A}$.

(a). Consider the following optimization problem,

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \\ \text{s.t.} \quad & \|\boldsymbol{x}\|_2^2 = 1 \end{aligned} \tag{1}$$

Use the Lagrange multipliers method to show that a solution $\boldsymbol{x}^*$ to problem (1) is an eigenvector of $\boldsymbol{A}$ associated to λ_1, and the optimal value is λ_1.

(b). Let $\boldsymbol{v}_1$ be an eigenvector of $\boldsymbol{A}$ associated to the eigenvalue λ_1. Consider the following optimization problem,

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \\ \text{s.t.} \quad & \|\boldsymbol{x}\|_2^2 = 1 \\ & \boldsymbol{v}_1^T \boldsymbol{x} = 0 \end{aligned} \tag{2}$$

Let $\boldsymbol{x}^*$ be an optimal solution to (2).

i) Use the Lagrange multipliers method to show that there exist some $c_0, c_1 \in \mathbb{R}$ s.t.

$$\boldsymbol{A}\boldsymbol{x}^* = c_0 \boldsymbol{x}^* + c_1 \boldsymbol{v}_1$$

ii) Show $c_1 = 0$ in i).

iii) Conclude $\boldsymbol{x}^*$ is an eigenvector of $\boldsymbol{A}$ associated to λ_2, and the optimal value of (2) is λ_2. You can assume the fact that an eigenvector orthogonal to $\boldsymbol{v}_1$ must be associated to one of the eigenvalues $\lambda_2, \ldots, \lambda_n$.

Remark. The same argument can be used to show the following generalization. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}$ be linearly independent eigenvectors of $\boldsymbol{A}$ associated to $\lambda_1, \ldots, \lambda_{k-1}$, respectively. Then λ_k is the optimal value of the following problem,

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \\ \text{s.t.} \quad & \|\boldsymbol{x}\|_2^2 = 1 \\ & \boldsymbol{v}_i^T \boldsymbol{x} = 0, \quad i = 1, 2, \ldots, k-1 \end{aligned}$$

and the optimal solution is an eigenvector associated to λ_k.