1. $\forall x, y \in M$.
   By the definition of M, $f(x) = f(y)$.
   Because S is a convex set, $\forall\, \theta \in [0,1]$, $\theta x + \bar{\theta} y \in S = dom\, f$.
   Because f is convex, $f(\theta x + \bar{\theta} y) \le \theta f(x) + \bar{\theta} f(y) = f(x)$.
   By the definition of M, $f(x)$ is the global minima of f.
   Thus, $f(\theta x + \bar{\theta} y) \ge f(x)$.
   So $f(\theta x + \bar{\theta} y) = f(x)$.
   Namely, $\theta x + \bar{\theta} y \in M. M$ is a convex set.
2. Suppose that $\exists \alpha \in [\,0, \theta_0]$ such that $f(\alpha x + \bar{\alpha} y) < \alpha f(x) + \bar{\alpha} f(y)$.
   Apparently, there exists $\beta \in [0,1]$, such that $\theta_0 x + \overline{\theta_0} y = \beta(\alpha x + \bar{\alpha} y) + \bar{\beta} y$.
   Because f is a convex function, $f(\theta_0 x + \overline{\theta_0} y) \le \beta f(\alpha x + \bar{\alpha} y) + \bar{\beta} f(y)$.
   But $f(\theta_0 x + \overline{\theta_0} y) = \theta_0 f(x) + \overline{\theta_0} f(y) > \beta f(\alpha x + \bar{\alpha} y) + \bar{\beta} f(y)$.
   It's a contradiction.
   So the supposition is false. There doesn't exist such α.
   Namely, the conclusion holds for all $\theta \in [0,1]$.
3. (a) convex
   (b) convex
   (c) neither
   (d) neither
   (e) convex, when $\alpha = 1$
       Neither, when $\alpha < 1$
4. $\forall x_i, y_i, i = 1,2$ and $\theta \in [0,1]$.
   Because f_1 and f_2 are strictly convex,
   $f_1(\theta x_1 + \bar{\theta} x_2) < \theta f_1(x_1) + \bar{\theta} f_1(x_2)$ and $f_2(\theta y_1 + \bar{\theta} y_2) < \theta f_2(y_1) + \bar{\theta} f_2(y_2)$.
   So $f_1(\theta x_1 + \bar{\theta} x_2) + f_2(\theta y_1 + \bar{\theta} y_2) < \theta f_1(x_1) + \bar{\theta} f_1(x_2) + \theta f_2(y_1) + \bar{\theta} f_2(y_2)$
   Namely, $f(\theta x_1 + \bar{\theta} x_2, \theta y_1 + \bar{\theta} y_2) < \theta f(x_1, y_1) + \bar{\theta} f(x_2, y_2)$.
   So, f is convex.
   Specially, $H(x_1{}^2 + x_2{}^4) = \begin{matrix} 2 & 0 \\ 0 & 12x_2{}^2 \end{matrix}$ is a definite matrix.
   So $f(x_1, x_2) = x_1{}^2 + x_2{}^4$ is strictly convex.
5. ←:

   $\forall\, x < y$ and $\theta \in [0, 1]$.
   $\theta[f(x) - f(\theta x + \bar{\theta} y)] = \theta \nabla f(\delta_1)(\theta x + \bar{\theta} y - x)$…*1, where $\delta_1 \in [x, \theta x + \bar{\theta} y\,]$
   And $\bar{\theta}[\,f(\theta x + \bar{\theta} y) - f(y)] = \bar{\theta} \nabla f(\delta_2)(y - \theta x - \bar{\theta} y)$…*2, where $\delta_2 \in [\theta x + \bar{\theta} y, y\,]$.
   Let *1-*2.
   $$\theta f(x) + \bar{\theta} f(y) - f(\theta x + \bar{\theta} y) = \theta \nabla f(\delta_1)(\theta x + \bar{\theta} y - x) - \bar{\theta} \nabla f(\delta_2)(\theta x + \bar{\theta} y - y)$$
   $$= 2\theta\bar{\theta} < \big(\nabla f(\delta_1) - \nabla f(\delta_2)\big), (x - y) >$$
   Because $< \big(\nabla f(\delta_1) - \nabla f(\delta_2)\big), (\delta_1 - \delta_2) > \ge 0$, $< \big(\nabla f(\delta_1) - \nabla f(\delta_2)\big), (x - y) > \ge 0$.
   Thus, $\theta f(x) + \bar{\theta} f(y) - f(\theta x + \bar{\theta} y) \ge 0$. f is convex.

   →:

   The above process is reversible.