S M T W T F

DATE

4.1

(a) $H(b)-H(a) = -\sum_j b_j \log b_j + \sum_i a_i \log a_i$

$$= \sum_j\sum_i a_i P_{ij} \log (\sum a_k P_{kj}) + \sum_i a_i \log a_i$$

$$= \sum_i\sum_j a_i P_{ij} \log \frac{a_i}{\sum_k a_k P_{kj}} \geq (\sum_{ij} a_i P_{ij}) \log \frac{\sum_{ij} a_{ij}}{\sum_{ij} b_j} = 1 \log \frac{m}{m} = 0$$

(b) If the matrix is doubly stochastic, the substituting $\mu_i = \frac{1}{m}$,
Therefore, it satisfies $\mu = \mu P$.

(c) If the uniform is a stationary distribution, then

$$\frac{1}{m} = \sum_j \mu_j P_{ji} = \frac{1}{m} \sum_j P_{ji},$$

Therefore, P is doubly stochastic.

4.3. $H(TX) \geq H(TX|T)$ — It follows the fact that conditions reduce entro

$= H(T^{-1}TX|T)$ } $T^{-1}T = I$

$= H(X|T)$

$= H(X)$ → T and X are independent.

4.6 (a) By the chain rule,

$$\frac{1}{n} H(X_1, X_2, \dots, X_n) = \frac{1}{n}\sum_{i=1}^{n} H(X_i | X^{i-1})$$

$$= \frac{1}{n}[H(X_n|X^{n-1}) + \sum_{i=1}^{n-1} H(X_i|X^{i-1})]$$

$$= \frac{1}{n}[H(X_n|X^{n-1}) + H(X_1, \dots, X_{n-1})]$$

$\because H(X_n|X^{n-1}) \leq H(X_i|X^{i-1})$ for $1 \leq i \leq n$

$\therefore H(X_n|X^{n-1}) \leq \frac{1}{n-1}\sum_{i=1}^{n-1} H(X_i|X^{i-1})$

$$= \frac{1}{n-1} H(X_1, X_2, \dots, X_{n-1})$$

$\therefore \frac{1}{n} H(X_1, X_2, \dots, X_n) \leq \frac{1}{n}[\frac{1}{n-1} H(X_1, X_2, \dots, X_{n-1}) + H(X_1, \dots, X_{n-1})]$

$$= \frac{1}{n-1} H(X_1, X_2, \dots, X_{n-1})$$

(b) $H(X_n|X^{n-1}) \leq H(X_i|X^{i-1})$ for $i \leq n$

$$\frac{1}{n}\sum_{i=1}^{n} H(X_n|X^{n-1}) \leq \frac{1}{n}\sum_{i=1}^{n} H(X_i|X^{i-1})$$

$$H(X_n|X^{n-1}) \leq \frac{1}{n} H(X_1, \dots, X_n).$$

S M T W T F S

DATE

4.7

(a) $\mu_0 = \frac{p_{10}}{p_{01}+p_{10}}$, $\mu_1 = \frac{p_{01}}{p_{01}+p_{10}}$

$\therefore H(X_2|X_1) = \mu_0 H(p_{01}) + \mu_1 H(p_{10}) = \frac{1}{p_{01}+p_{10}}[p_{10} H(p_{01}) + p_{01} H(p_{10})]$

(b) $\because$ the process has only two states

$\therefore H(X_2|X_1) \leq 1$

The maximum can be achieved iff $p_{01} = p_{10} = \frac{1}{2}$.

(c) $H(X_2|X_1) = \mu_0 H(p) + \mu_1 H(1) = \frac{H(p)}{p+1}$.

(d) The maximum value of $H(X)$ can be achieved when $p = \frac{3-\sqrt{5}}{2}$.

$\therefore H_m(p) = H_m(1-p) = H(\frac{-1+\sqrt{5}}{2}) = 0.694$ bits

4.10.

(a) Let $S_k = \sum_{i=1}^{k} X_i$

$P(S_k \text{ odd}) = P(S_{k-1} \text{ odd}) P(X_k=0) + P(S_{k-1} \text{ even}) P(X_k=1)$

$= (\frac{1}{2}\cdot\frac{1}{2}) \times 2 = \frac{1}{2}$

$\therefore P(X_n=1) = P(X_n=0) = \frac{1}{2}$

$P(X_1=1, X_n=1) = P(X_1=1, \sum_{i=2}^{n-1} X_i \text{ even})$

$= P(X_1=1) P(\sum_{i=2}^{n-1} X_i \text{ even})$

$= \frac{1}{2}\cdot\frac{1}{2} = P(X_1=1)\ P(X_n=1)$.

$\therefore$ X_1 and X_n are independent.

(b) $H(X_i, X_j) = H(X_i) + H(X_j) = 1+1 = 2$ bits

(c). $H(X_1, X_2, \cdots, H_n) = H(X_1, X_2, \cdots, X_{n-1}) + H(X_n | X_{n-1}, \cdots, X_1)$

$= \sum_{i=1}^{n-1} H(X_i) + 0 = n-1 \neq n H(X_1)$

4.12 (a) $H(X_0, X_1, \cdots, X_n) = \sum_{i=0}^{n} H(X_i|X^{i-1}) = H(X_0) + H(X_1|X_0) + \sum_{i=2}^{n} H(X_i|X_{i-1}, X_{i-2})$

$H(X_i | X_{i-1}, X_{i-2}) = H(.1, .9)$

$\therefore H(X_0, \cdots, X_n) = 1 + (n-1) H(.1, .9)$

(b) $\frac{1}{n+1} H(X_0, \cdots, X_n) = \frac{1}{n+1}(1 + (n-1)H(.1, .9)) \rightarrow H(.1, .9)$

(c) $E(S) = \sum_{s=1}^{\infty} s(.9)^{s-1}(.1) = 10$

the expected number of steps to the first ~~reversal~~ is 11.

S M T W T F S

4.7

(a) $\mu_0 = \frac{p_{10}}{p_{01}+p_{10}}$, $\mu_1 = \frac{p_{01}}{p_{01}+p_{10}}$

$\therefore H(X_2|X_1) = \mu_0 H(p_{01}) + \mu_1 H(p_{10}) = \frac{1}{p_{01}+p_{10}}[p_{10} H(p_{01}) + p_{01} H(p_{10})]$

(b) $\because$ the process has only two states

$\therefore H(X_2|X_1) \le 1$

The maximum can be achieved iff $p_{01} = p_{10} = \frac{1}{2}$

(c) $H(X_2|X_1) = \mu_0 H(p) + \mu_1 H(1) = \frac{H(p)}{p+1}$

(d) $H(p) = \frac{1}{p+1}[-p\log p - (1-p)\log(1-p)]$

$\frac{dH}{dp} = (p\log p - \log(1-p)(p-1))/(p+1)^2 + (\log(1-p) - \log p)/(p+1) = 0$

So, $p = \frac{3}{2} - \frac{\sqrt{5}}{2}$ (solved by matlab)

$H_{max}(p) = H_m(1-p) = 0.694$ bits

4.10

(a) Let $S_k = \sum_{i=1}^{k} X_i$

$P(S_k \text{ odd}) = P(S_{k-1} \text{ odd}) P(X_k=0) + P(S_{k-1} \text{ even}) P(X_k=1)$

$= (\frac{1}{2}\cdot\frac{1}{2}) \times 2 = \frac{1}{2}$

$\therefore P(X_n=1) = P(X_n=0) = \frac{1}{2}$

$P(X_1=1, X_n=1) = P(X_1=1, \sum_{i=2}^{n-1} X_i \text{ even})$

$= P(X_1=1) P(\sum_{i=2}^{n-1} X_i \text{ even})$

$= \frac{1}{2}\cdot\frac{1}{2} = P(X_1=1) P(X_n=1)$

$\therefore$ X_1 and X_n are independent

(b) $H(X_i, X_j) = H(X_i) + H(X_j) = 1+1 = 2$ bits

(c) $H(X_1, X_2, \cdots, H_n) = H(X_1, X_2, \cdots, X_{n-1}) + H(X_n | X_{n-1}, \cdots, X_1)$

$= \sum_{i=1}^{n-1} H(X_i) + 0 = n-1 \ne n H(X_1)$

4.12 (a) $H(X_0, X_1, \cdots, X_n) = \sum_{i=0}^{n} H(X_i | X^{i-1}) = H(X_0) + H(X_1|X_0) + \sum_{i=2}^{n} H(X_i|X_{i-1}, \ldots$

$H(X_i | X_{i-1}, X_{i-2}) = H(0.1, 0.9)$

$\therefore H(X_0, \cdots, X_n) = 1 + (n-1) H(0.1, 0.9)$

(b) $\frac{1}{n+1} H(X_0, \cdots, X_n) = \frac{1}{n+1}(1 + (n-1)H(0.1, 0.9)) \to H(0.1, 0.9)$

(c) $E(S) = \sum_{s=1}^{\infty} s(0.9)^{s-1}(0.1) = 10$

the expected number of steps to the first ...

4.34.

$I(X;Y) + I(Z;W) - I(X;Z) - I(X;W)$

$\geq I(\cancel{X}\,X;Z,W) + I(Z;W) - I(X;Z) - I(X;W)$

$= H(Z,W) + H(X) + [-H(X,W,Z)] + H(W) + H(Z) - H(W,Z)$

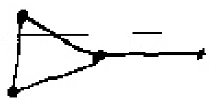
$\quad - H(Z) - H(X) + H(X,Z) - H(W) - H(X) + H(W,X)$

$= -H(X,W,Z) + H(X,Z) + H(X,W) - H(X)$

$= H(W|X) - H(W|X,Z)$

$= I(W;Z|X) \geq 0$

4.21

(a)  $H = 1/2 + 3/8 \log 3 \approx 1.094$

(b)  $H = 0.75$