DATE

4.1

(a) $H(b) - H(a) = -\sum_j b_j \log b_j + \sum_i a_i \log a_i$

$= \sum_j \sum_i a_i P_{ij} \log (\sum_k a_k P_{kj}) + \sum_i a_i \log a_i$

$= \sum_i \sum_j a_i P_{ij} \log \frac{a_i}{\sum_k a_k P_{kj}} \geq (\sum_{ij} a_i P_{ij}) \log \frac{\sum_{ij} a_{ij}}{\sum_{ij} b_j} = 1 \log \frac{m}{m} = 0$

(b) If the matrix is doubly stochastic, the substituting $\mu_i = \frac{1}{m}$,
Therefore, it satisfies $\mu = \mu P$.

(c) If the uniform is a stationary distribution, then

$$\frac{1}{m} = \sum_j \mu_j P_{ji} = \frac{1}{m} \sum_j P_{ji},$$

Therefore, P is doubly stochastic.

4.3. $H(TX) \geq H(TX|T)$ It follows the fact that conditions reduce entro

$= H(T^{-1}TX|T)$ } $T^{-1}T = I$

$= H(X|T)$

$= H(X)$ → T and X are independent.

4.6 (a) By the chain rule,

$$\frac{1}{n} H(X_1, X_2, \cdots, X_n) = \frac{1}{n} \sum_{i=1}^{n} H(X_i | X^{i-1})$$

$$= \frac{1}{n} \left[H(X_n | X^{n-1}) + \sum_{i=1}^{n-1} H(X_i | X^{i-1}) \right]$$

$$= \frac{1}{n} \left[H(X_n | X^{n-1}) + H(X_1, \cdots, X_{n-1}) \right]$$

$\because H(X_n | X^{n-1}) \leq H(X_i | X^{i-1})$ for $1 \leq i \leq n$

$\therefore H(X_n | X^{n-1}) \leq \frac{1}{n-1} \sum_{i=1}^{n-1} H(X_i | X^{i-1})$

$$= \frac{1}{n-1} H(X_1, X_2, \cdots, X_{n-1})$$

$\therefore \frac{1}{n} H(X_1, X_2, \cdots X_n) \leq \frac{1}{n} \left[\frac{1}{n-1} H(X_1, X_2, \cdots, X_{n-1}) + H(X_1, \cdots X_{n-1}) \right]$

$$= \frac{1}{n-1} H(X_1, X_2, \cdots, X_{n-1})$$

(b) $H(X_n | X^{n-1}) \leq H(X_i | X^{i-1})$ for $i \leq n$

$\frac{1}{n} \sum_{i=1}^{n} H(X_n | X^{n-1}) \leq \frac{1}{n} \sum_{i=1}^{n} H(X_i | X^{i-1})$

$H(X_n | X^{n-1}) \leq \frac{1}{n} H(X_1, \cdots, X_n)$.

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DATE

4.7

(a) $\mu_0 = \frac{P_{10}}{P_{01}+P_{10}}$, $\mu_1 = \frac{P_{01}}{P_{01}+P_{10}}$

$\therefore H(X_2|X_1) = \mu_0 H(P_{01}) + \mu_1 H(P_{10}) = \frac{1}{P_{01}+P_{10}}[P_{10} H(P_{01}) + P_{01} H(P_{10})]$

(b) $\because$ the process has only two states

$\therefore H(X_2|X_1) \leq 1$

The maximum can be achieved iff $P_{01} = P_{10} = \frac{1}{2}$

(c) $H(X_2|X_1) = \mu_0 H(p) + \mu_1 H(1) = \frac{H(p)}{p+1}$

(d) The maximum value of $H(X)$ can be achieved when $p = \frac{\sqrt{5}-1}{2}$

$\therefore H_{max}(P) = H_{max}(1-P) = H(\frac{\sqrt{5}-1}{2}) = 0.694$ bits

4.10

(a) Let $S_k = \sum_{i=1}^{k} X_i$

$P(S_k \text{ odd}) = P(S_{k-1} \text{ odd}) P(X_k = 0) + P(S_{k-1} \text{ even}) P(X_k = 1)$

$= (\frac{1}{2} \cdot \frac{1}{2}) \times 2 = \frac{1}{2}$

$\therefore P(X_n = 1) = P(X_n = 0) = \frac{1}{2}$

$P(X_1 = 1, X_n = 1) = P(X_1 = 1, \sum_{i=2}^{n-1} X_i \text{ even})$

$= P(X_1 = 1) P(\sum_{i=2}^{n-1} X_i \text{ even})$

$= \frac{1}{2} \cdot \frac{1}{2} = P(X_1 = 1) P(X_n = 1)$

$\therefore X_1$ and X_n are independent

(b) $H(X_i, X_j) = H(X_i) + H(X_j) = 1 + 1 = 2$ bits

(c) $H(X_1, X_2, \cdots, X_n) = H(X_1, X_2, \cdots, X_{n-1}) + H(X_n | X_{n-1}, \cdots, X_1)$

$= \sum_{i=1}^{n-1} H(X_i) + 0 = n-1 \neq n H(X_i)$

4.12 (a) $H(X_0, X_1, \cdots, X_n) = \sum_{i=0}^{n} H(X_i|X^{i-1}) = H(X_0) + H(X_1|X_0) + \sum_{i=2}^{n} H(X_i|X_{i-1}, X_{i-2})$

$H(X_i | X_{i-1}, X_{i-2}) = H(.1, .9)$

$\therefore H(X_0, \cdots, X_n) = 1 + (n-1) H(.1, .9)$

(b) $\frac{1}{n+1} H(X_0, \cdots, X_n) = \frac{1}{n+1}(1 + (n-1) H(.1, .9)) \to H(.1, .9)$

(c) $E(S) = \sum_{s=1}^{\infty} s(.9)^{s-1}(.1) = 10$

the expected number of steps to the first event is 10

DATE

4.13 $I(X_1, X_2, \cdots, X_n; X_{n+1}, \cdots, X_{2n})$

$= H(X_1, X_2, \cdots, X_n) + H(X_{n+1}, \cdots, X_{2n}) - H(X_1, X_2, \cdots, X_n, X_{n+1}, X_{n+2}, \cdots, X_{2n})$

$= 2H(X_1, X_2, \cdots, X_n) - H(X_1, X_2, \cdots, X_n, X_{n+1}, X_{n+2}, \cdots, X_{2n})$

$\because H(X_1, X_2, \cdots, X_n) = H(X_{n+1}, \cdots, X_{2n})$.

$\therefore \lim_{n\to\infty} \frac{1}{2n} I(X_1, X_2, \cdots, X_n; X_{n+1}, \cdots, X_{2n})$.

$= \lim_{n\to\infty} \frac{1}{2n} 2H(X_1, X_2, \cdots, X_n) - \lim_{n\to\infty} \frac{1}{2n} H(X_1, X_2, \cdots, X_{2n})$.

$= \lim_{n\to\infty} \frac{1}{n} H(X_1, \cdots, X_n) - \lim_{n\to\infty} \frac{1}{2n} H(X_1, \cdots, X_{2n})$.

$\because \lim_{n\to\infty} \frac{1}{n} H(X_1, \cdots, X_n) = \lim_{n\to\infty} \frac{1}{n+1} (X_1, \cdots, X_{n+1}) = \cdots = \lim_{n\to\infty} \frac{1}{2n} H(X_1, \cdots, X_{2n})$.

$\therefore \lim_{n\to\infty} I(X_1, X_2, \cdots, X_n; X_{n+1}, \cdots, X_{2n}) \cdot \frac{1}{2n} = 0$

4.20 $\mu_i = E_i / E$

$E_1 = E_3 = E_7 = E_9 = 3$

$E_2 = E_4 = E_6 = E_8 = 5$

$E_5 = 8$. $E = \sum E_i = 40$.

$\therefore \mu_1 = \mu_3 = \mu_7 = \mu_9 = \frac{3}{40}$

$\mu_2 = \mu_4 = \mu_6 = \mu_8 = \frac{5}{40}$

$\mu_5 = \frac{8}{40}$

$H(X_2 | X_1 = i) = \log 3$ bits $i = 1, 3, 7, 9$.

$H(X_2 | X_1 = i) = \log 5$ bits $i = 2, 4, 6, 8$

$H(X_2 | X_1 = i) = \log 8$ bits. $i = 5$.

$\therefore H = \sum_{i=1}^{9} \mu_i H(X_2 | X_1 = i)$

$= 0.3 \log 3 + 0.5 \log 5 + 0.2 \log 8$

$= 2.24$ bits.

4.33 $I(X_1; X_4) + I(X_2; X_3) - I(X_1; X_3) - I(X_2; X_4)$

$= H(X_1) - H(X_1 | X_4) + H(X_2) - H(X_2 | X_3) - H(X_1) + H(X_1 | X_3) - H(X_2) + H(X_2 | X_4)$

$= H(X_1 | X_3) - H(X_1 | X_4) + H(X_2 | X_4) - H(X_2 | X_3)$

$= H(X_1, X_2 | X_3) - H(X_2 | X_1, X_3) - H(X_1, X_2 | X_4) + H(X_2 | X_1, X_4) + H(X_1, X_2 | X_4) - H(X_1$

$- H(X_1, X_2 | X_3) + H(X_1 | X_2, X_3)$

$= - H(X_2 | X_1, X_3) + H(X_2 | X_1, X_4) - H(X_2 | X_1, X_4) - H(X_2 | X_1, X_3, X_4)$

$= I(X_2; X_3 | X_1, X_4) \geq 0$.

4.34.

$I(X;Y) + I(Z;W) - I(X;Z) = I(X;W)$

$\geq I(X;Z,W) + I(Z;W) - I(X;Z) - I(X;W)$

$= H(Z,W) + H(X) + (-H(X,W,Z)) + H(W) + H(Z) - H(W,Z)$

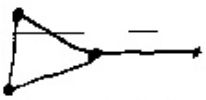
$- H(Z) - H(X) + H(X,Z) - H(W) - H(X) + H(W,X)$

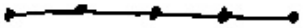
$= -H(X,W,Z) + H(X,Z) + H(X,W) - H(X)$

$= H(W|X) - H(W|X,Z)$

$= I(W;Z|X) \geq 0$

4.21

(a)  $H = 1/2 + 3/8 \log 3 \approx 1.094$

(b)  $H = 0.75$