Introduction to Quantum Mechanics I

Homework

To be handed in on November 10, 2023, in old-fashioned paper-style.

This homework is decomposed into many short sub-problems. If you have well understood the lectures so far it should not take more than a few hours. If it takes much longer it is probably because you still need to assimilate the lectures, and then this homework will have fulfilled its purpose ....

You can write in English or French.

A 3-state system

A hypothetical ion is formed by three identical atoms disposed on the corners of an equilateral triangle and one excess electron which can be localised on either of the 3 atoms. The corresponding states are called $|1\rangle$, $|2\rangle$, $|3\rangle$ and are supposed to form an orthonormal basis of the relevant Hilbert space.

1) Define the permutation operator Π as

$$\Pi\,|1\rangle = |2\rangle \quad , \quad \Pi\,|2\rangle = |3\rangle \quad , \quad \Pi\,|3\rangle = |1\rangle \ . \tag{0.1}$$

1-a) Give the matrix elements of Π in this basis.

1-b) Show that $\Pi^3 = \mathbf{1}$ and conclude that the inverse is $\Pi^{-1} = \Pi^2$.

1-c) Determine $\Pi^\dagger$. Is Π a hermitian operator ? Is it unitary ?

1-d) Show that the eigenvalues λ_n of Π must satisfy $\lambda_n^3 = 1$ and conclude that they are $\lambda_1 = 1,\ \lambda_2 = e^{2\pi i/3}$ and $\lambda_3 = e^{-2\pi i/3}$. Determine the eigenvector corresponding to λ_1.

2) Suppose that the Hamiltonian is given by

$$H\,|1\rangle = E_0\,|1\rangle + a\,|2\rangle + a\,|3\rangle \ , \quad H\,|2\rangle = E_0\,|2\rangle + a\,|1\rangle + a\,|3\rangle \ , \quad H\,|3\rangle = E_0\,|3\rangle + a\,|1\rangle + a\,|2\rangle \ . \tag{0.2}$$

2-a) Show that $[H, \Pi] = 0$. Interpret this result in terms of the symmetries of H and the action of Π as a permutation operator.

2-b) By writing out the commutators, show that in general $[A, BC] = B[A, C] + [A, B]C$ and similarly $[AB, C] = A[B, C] + [A, C]B$. Use this to deduce that H also commutes with $\Pi^2 = \Pi^{-1}$.

2-c) Conclude that if $H\,|\varphi_j\rangle = E_j\,|\varphi_j\rangle$ then $\Pi\,|\varphi_j\rangle$ and $\Pi^2\,|\varphi_j\rangle$ are also eigenvectors of H. Show that $\Pi\,|\varphi_j\rangle$ and $\Pi^2\,|\varphi_j\rangle$ cannot vanish.

2-d) Check that $|\varphi_1\rangle \sim (|1\rangle + |2\rangle + |3\rangle)$ is an eigenvector of H. Determine the corresponding eigenvalue (energy) E_1 and correctly normalise this eigenvector. Does $\Pi\,|\varphi_1\rangle$ yield another eigenvector ? If not, explain why.

2-e) Show that the subspace which is orthogonal to $|\varphi_1\rangle$ is spanned by $|\psi_2\rangle = \frac{1}{\sqrt{2}}\big(|1\rangle - |2\rangle\big)$ and $|\psi_3\rangle = \frac{1}{\sqrt{2}}\big(|2\rangle - |3\rangle\big)$. Why is it possible to search for the two other eigenvectors of H in this orthogonal subspace (consisting of all $\alpha\,|\psi_2\rangle + \beta\,|\psi_3\rangle$) ?

2-f) How does Π act on $|\psi_2\rangle$ and $|\psi_3\rangle$? Compute $H\,|\psi_2\rangle$. Deduce from this what are the other two eigenvalues and eigenvectors of H.

2-g) Explicitly write the matrix $\mathtt{H}$ of H in the basis $\{|1\rangle\,,|2\rangle\,,|3\rangle\}$. Verify that the matrix $\mathtt{H} - E_2\mathbf{1}$ has a vanishing determinant.

2-h) Express the state $|1\rangle$ in the basis of the eigenstates of H. Suppose at time $t = 0$ the ion is in the state $|\Psi(0)\rangle = \frac{1+i}{\sqrt{2}}\,|1\rangle$. Check that this is correctly normalised. What is the state at some later time t ?

2-i) What is the probability to "find it" in the state $|2\rangle$ at time t ?

Time-dependences

Consider a quantum mechanical system such that when measuring a certain observable A one can obtain any of the 4 different results a_1, a_2, a_3 or a_4. Explain why this implies that the underlying Hilbert space is of dimension at least 4 ? In the sequel we will assume that the dimension is 4 and we denote the basis of eigenvectors of A by $\{|1\rangle\,,|2\rangle\,,|3\rangle\,,|4\rangle\}$.

3-a) Repeat the proof of the lectures that these basis states of eigenvectors must be orthogonal, i.e. $\langle i\,|j\rangle = 0$ if $i \neq j$. We may furthermore assume that they are normalised and then $\langle i\,|j\rangle = \delta_{ij}$.

3-b) Assume that these basis states are "almost" eigenstates of the Hamiltonian H, more precisely we assume (with $\epsilon, \eta \ll E_0, \widetilde{E}_0$)

$$\begin{aligned} H\,|1\rangle &= E_0\,|1\rangle + i\epsilon\,|2\rangle \quad , \quad H\,|2\rangle = E_0\,|2\rangle - i\epsilon\,|1\rangle \\ H\,|3\rangle &= \widetilde{E}_0\,|3\rangle + \eta\,|4\rangle \quad , \quad H\,|4\rangle = \widetilde{E}_0\,|4\rangle + \eta\,|1\rangle\ . \end{aligned} \tag{0.3}$$

Write the matrix associated with H in the basis of the $|j\rangle$. Show that H is hermitian if $E_0, \widetilde{E}_0, \epsilon$ and η are all real.

3-c) Determine the four eigenvalues E_a and eigenvectors $|\varphi_a\rangle$ of H. The indexing a should be chosen such that for $\epsilon = \eta = 0$ the eigenvectors $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are linear combinations of $|1\rangle$ and $|2\rangle$, and $|\varphi_3\rangle$ and $|\varphi_4\rangle$ are linear combinations of $|3\rangle$ and $|4\rangle$.

3-d) Suppose the system is initially at time t_0 in any of the eigenstates of H. Determine the state vector $|\Psi(t)\rangle$ (up to a phase) at an arbitrary later time. Compute the expectation value of the observable A in this state and observe that it does not depend on time. Explain why this is always the case for expectation values in an eigenstate of H.

3-e) Suppose now that $|\Psi(t_0)\rangle = |2\rangle$. Determine $|\Psi(t)\rangle$.

3-f) Compute the probability $P_{2\to1}(t)$ that a measurement of A at time t yields a_1. Also compute the expectation value $\langle A\rangle_{\Psi(t)}$ of A in this state as a function of t. What is the characteristic period associated ? Are the results for $P_{2\to1}(t)$ and $\langle A\rangle_{\Psi(t)}$ consistent ?

3-g) Compute similarly $P_{2\to3}(t)$ and $P_{2\to4}(t)$. Comment on the result.