Introduction to Quantum Mechanics I

Homework - Solutions

1 A 3-state system : Permutation operator

We write the action of Π as

$$\Pi |i\rangle = |i+1\rangle \quad , \quad i \equiv i+3 \tag{1.1}$$

1-a) The matrix elements of Π in the basis of the $|i\rangle$ are

$$\Pi_{ij} = \langle i| \Pi |j\rangle \quad \Rightarrow \quad \Pi_{21} = \Pi_{32} = \Pi_{13} = 1 \ , \quad \text{all other elements vanish} \ . \tag{1.2}$$

1-b) On any of the basis states $|i\rangle$ one has $\Pi^3 |i\rangle = \Pi\,\Pi\,\Pi\,|i\rangle = \Pi\,\Pi\,|i+1\rangle = \Pi\,|i+2\rangle = |i+3\rangle \equiv |i\rangle$. Hence for all basis states one has $\Pi^3 |i\rangle = \mathbf{1}\,|i\rangle$ which is enough to conclude that $\Pi^3 = \mathbf{1}$. Then $\Pi^2\Pi = \Pi\,\Pi^2 = \Pi^3 = \mathbf{1}$ which means that $\Pi^2 = \Pi^{-1}$.

1-c) In terms of the matrix elements (1.2) one has $(\Pi^\dagger)_{ij} = \Pi^*_{ji}$. Since the Π_{ij} are real one simply has

$$(\Pi^\dagger)_{12} = (\Pi^\dagger)_{23} = (\Pi^\dagger)_{31} = 1 \ , \ \text{all other elements vanish} \quad \Leftrightarrow \quad \Pi^\dagger |i\rangle = |i-1\rangle \ . \tag{1.3}$$

Clearly, Π is not hermitian, but we see that $\Pi^\dagger\Pi = \Pi\,\Pi^\dagger = \mathbf{1}$ which shows that $\Pi^\dagger = \Pi^{-1}$ and this means that Π is unitary.

1-d) Assume that $|v_n\rangle$ is an eigenvector of Π with eigenvalue λ_n : $\Pi\,|v_n\rangle = \lambda_n\,|v_n\rangle$. Then $\Pi^3\,|v_n\rangle = \Pi^2\,\Pi\,|v_n\rangle = \Pi^2\lambda_n\,|v_n\rangle = \lambda_n\Pi^2\,|v_n\rangle = \lambda_n\Pi\,\Pi\,|v_n\rangle = \lambda_n\Pi\,\lambda_n\,|v_n\rangle = \lambda_n^2\Pi\,|v_n\rangle = \lambda_n^2\lambda_n\,|v_n\rangle = \lambda_n^3\,|v_n\rangle$. But we have also seen that $\Pi^3 = \mathbf{1}$ so that $\Pi^3\,|v_n\rangle = \mathbf{1}\,|v_n\rangle = |v_n\rangle$. Hence $\lambda_n^3 = 1$ and then λ_n is either $\lambda_1 = 1$ or $\lambda_2 = q$ or $\lambda_3 = q^2$ with $q = e^{2i\pi/3}$. The eigenvector $|v_1\rangle$ corresponding to $\lambda_1 = 1$ must be such that when acting with Π on it, it remains unchanged. Hence it must be invariant under cyclic permutation of the basis states:

$$|v_1\rangle = \frac{1}{\sqrt{3}}\big(\,|1\rangle + |2\rangle + |3\rangle\,\big) \ . \tag{1.4}$$

The prefactor ensures that this is normalised to 1. The other eigenvectors are easily guessed as

$$|v_2\rangle = \frac{1}{\sqrt{3}}\big(\,|1\rangle + q^2\,|2\rangle + q\,|3\rangle\,\big) \quad , \quad |v_3\rangle = \frac{1}{\sqrt{3}}\big(\,|1\rangle + q\,|2\rangle + q^2\,|3\rangle\,\big) \ . \tag{1.5}$$

Of course, they are only defined up to an overall phase, and one could equally well use e.g. $|\widetilde{v_2}\rangle = q\,|v_2\rangle = \frac{1}{\sqrt{3}}\big(q\,|1\rangle + |2\rangle + q^2\,|3\rangle\,\big)$, etc.

2 A 3-state system : Eigenvectors and eigenvalues of the Hamiltonian

The action of the Hamiltonian on the basis states can also be written as

$$H|i\rangle = (E_0 - a)|i\rangle + a\big(|1\rangle + |2\rangle + |3\rangle\big) = (E_0 - a)|i\rangle + a\sqrt{3}\,|v_1\rangle \ . \tag{2.1}$$

2-a) To show that H and Π commute, it is enough to show that on the basis states we have $H\Pi|i\rangle = \Pi H|i\rangle$:

$$\begin{aligned} H\Pi|i\rangle &= H|i+1\rangle = (E_0 - a)|i+1\rangle + a\sqrt{3}\,|v_1\rangle \ , \\ \Pi H|i\rangle &= \Pi\Big((E_0 - a)|i\rangle + a\sqrt{3}\,|v_1\rangle\Big) = (E_0 - a)\Pi|i\rangle + a\sqrt{3}\Pi|v_1\rangle = (E_0 - a)|i+1\rangle + a\sqrt{3}\,|v_1\rangle \ , \end{aligned} \tag{2.2}$$

where we used that $\Pi|v_1\rangle = |v_1\rangle$. Both lines give indeed the same result and, hence, $H\Pi = \Pi H$ or equivalently $[H, \Pi] = 0$. Indeed, the action of H does respect the cyclic symmetry between the 3 basis states and therefore it does not matter whether one first cyclically permutes the states and then applies H or one first applies H and then permutes the states.

2-b) One has

$$\begin{aligned} [A, BC] &= ABC - BCA = ABC - BAC + BAC - BCA = [A, B]C + B[A, C] \ , \\ [AB, C] &= ABC - CAB = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B \ . \end{aligned} \tag{2.3}$$

Then $[H, \Pi^2] = [H, \Pi\,\Pi] = \Pi[H, \Pi] + [H, \Pi]\Pi = 0 + 0 = 0$.

2-c) Let $H|\varphi_j\rangle = E_j|\varphi_j\rangle$. Then

$$H\big(\Pi|\varphi_j\rangle\big) = \Pi H|\varphi_j\rangle = \Pi E_j|\varphi_j\rangle = E_j\big(\Pi|\varphi_j\rangle\big) \ , \tag{2.4}$$

and $\Pi|\varphi_j\rangle$ (if non-zero) is an eigenvector of H with the same eigenvalue E_j. In exactly the same way this holds if one replaces Π by Π^2 since Π^2 also commutes with H and then $H\Pi^2 = \Pi^2 H$. We have shown before that Π is unitary, and then also $\Pi^2 = \Pi^{-1}$ is unitary, and a unitary operator preserves the norm. Then $\Pi|\varphi_j\rangle$ and $\Pi^2|\varphi_j\rangle$ have the same (non-vanishing) norm as $|\varphi_j\rangle$. A state with a non-vanishing norm cannot be vanishing.

2-d) The proposed eigenvector $|\varphi_1\rangle$ of H is actually the eigenvector $|v_1\rangle$ of Π with Π-eigenvalue $\lambda_1 = 1$. Then, if we sum (2.1) over $i = 1, 2, 3$ and divide by $\sqrt{3}$ we get

$$H|v_1\rangle = (E_0 - a)|v_1\rangle + 3a|v_1\rangle = (E_0 + 2a)|v_1\rangle \ , \tag{2.5}$$

so that $|\varphi_1\rangle = |v_1\rangle$ is indeed eigenvector of H with eigenvalue $E_1 = E_0 + 2a$. Of course, since $\Pi|v_1\rangle = |v_1\rangle$, applying Π just gives back the same eigenvector of H.

2-e) Clearly, the proposed $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ and $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$ are orthogonal to $|\varphi_1\rangle = |v_1\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle) + |3\rangle)$, e.g. $\langle\psi_2\,|\varphi_1\rangle = \frac{1}{\sqrt{2}\sqrt{3}}(\langle 1\,|1\rangle - \langle 2\,|2\rangle) = \frac{1}{\sqrt{6}}(1-1) = 0$. Since $|\psi_2\rangle$ and $|\psi_3\rangle$ are linearly independent (although not orthogonal) they span a subspace of dimension 2, orthogonal to $|\varphi_1\rangle$. Since our Hilbert space is 3-dimensional, they must span the orthogonal complement to $|\varphi_1\rangle$. Since the eigenvectors of H can be taken to form an orthonormal basis, the other two eigenvectors can be found in this orthogonal complement.

2-f) $\Pi\,|\psi_2\rangle = |\psi_3\rangle$ and $\Pi\,|\psi_3\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |1\rangle)$. Using (2.1) we see that

$$H\,|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big((E_0-a)\,|1\rangle + a\sqrt{3}\,|v_1\rangle - (E_0-a)\,|2\rangle - a\sqrt{3}\,|v_1\rangle\Big) = (E_0-a)\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) = (E_0-a)\,|\psi_2\rangle \tag{2.6}$$

and $|\psi_2\rangle$ is an eigenvector of H with eigenvalue $E_2 = E_0 - a$. But $\Pi\,|\psi_2\rangle = |\psi_3\rangle$ and since Π and H commute, we know that $|\psi_3\rangle$ must also be eigenvector of H with this same eigenvalue $E_2 = E_0 - a$. Hence this eigenvalue E_2 is two times degenerate. Similarly, $\Pi\,|\psi_3\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |1\rangle) = -\,|\psi_2\rangle - |\psi_2\rangle$ is a linear combination of the two eigenvectors corresponding to this same degenerate eigenvalue $E_0 - a$. We can then choose within this eigen-space spanned by $|\psi_2\rangle$ and $|\psi_3\rangle$ two orthonormal eigen-vectors. For the first, we can choose $|\psi_2\rangle$, and to obtain the second we take $|\psi_3\rangle$ and subtract its projection on $|\psi_2\rangle$: $|\psi_3\rangle - |\psi_2\rangle\,\langle\psi_2\,|\psi_3\rangle = |\psi_3\rangle + \frac{1}{2}\,|\psi_2\rangle = \frac{1}{2\sqrt{2}}\big(\,|1\rangle + |2\rangle - 2\,|3\rangle\,\big)$. This now is clearly orthogonal to $|\psi_2\rangle$ and, of course, orthogonal to $|\varphi_1\rangle$. But it is not yet normalized. The correctly normalized state is $|\varphi_3\rangle = \frac{1}{\sqrt{6}}\big(\,|1\rangle + |2\rangle - 2\,|3\rangle\,\big)$. To summarize,

$$\begin{aligned}
|\varphi_1\rangle &= |v_1\rangle = \frac{1}{\sqrt{3}}\big(\,|1\rangle + |2\rangle + |3\rangle\,\big) \quad , \quad E_1 = 2E_0 + 2a\ ,\\
|\varphi_2\rangle &= |\psi_2\rangle = \frac{1}{\sqrt{2}}\big(\,|1\rangle - |2\rangle\,\big) \quad , \quad E_2 = 2E_0 - a\ ,\\
|\varphi_3\rangle &= \frac{1}{\sqrt{6}}\big(\,|1\rangle + |2\rangle - 2\,|3\rangle\,\big) \quad , \quad E_3 = E_2 = 2E_0 - a\ .
\end{aligned} \tag{2.7}$$

2-g) We have

$$\mathtt{H} = \begin{pmatrix} E_0 & a & a \\ a & E_0 & a \\ a & a & E_0 \end{pmatrix} \quad , \quad \mathtt{H} - E_2\mathbf{1} = \begin{pmatrix} E_0 & a & a \\ a & E_0 & a \\ a & a & E_0 \end{pmatrix} - \begin{pmatrix} E_0-a & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix} = \begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix} \tag{2.8}$$

and obviously, $\mathtt{H} - E_2\mathbf{1}$ has a vanishing determinant.

2-h) We have $2\sqrt{3}\,|\varphi_1\rangle + \sqrt{6}\,|\varphi_3\rangle = 3(|1\rangle + |2\rangle)$, i.e. $|1\rangle + |2\rangle = \frac{2}{\sqrt{3}}\,|\varphi_1\rangle + \sqrt{\frac{2}{3}}\,|\varphi_3\rangle$. If we add $\sqrt{2}\,|\varphi_2\rangle$ we eliminate the $|2\rangle$ and get $2\,|1\rangle$. Then

$$|1\rangle = \frac{1}{\sqrt{3}}\,|\varphi_1\rangle + \frac{1}{\sqrt{2}}\,|\varphi_2\rangle + \frac{1}{\sqrt{6}}\,|\varphi_3\rangle\ . \tag{2.9}$$

We know that in general, if the $|\varphi_n\rangle$ are the eigenvectors of H with eigenvalues E_n, and if at time $t=0$ one has $|\Psi(0)\rangle = \sum_n c_n\,|\varphi_n\rangle$ then at time t we have $|\Psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}\,|\varphi_n\rangle$. Applying this to $|\Psi(0)\rangle = \frac{1+i}{\sqrt{2}}\,|1\rangle$ (note that $|\frac{1+i}{\sqrt{2}}|^2 = \frac{1+1}{2} = 1$) we get

$$\begin{aligned} |\Psi(t)\rangle &= \frac{1+i}{\sqrt{2}}\Big(\frac{e^{-iE_1t/\hbar}}{\sqrt{3}}\,|\varphi_1\rangle + \frac{e^{-iE_2t/\hbar}}{\sqrt{2}}\,|\varphi_2\rangle + \frac{e^{-iE_3t/\hbar}}{\sqrt{6}}\,|\varphi_3\rangle\Big) \\ &= \frac{1+i}{\sqrt{2}}e^{-iE_0t/\hbar}\Big(\frac{e^{-i2at/\hbar}}{\sqrt{3}}\,|\varphi_1\rangle + \frac{e^{iat/\hbar}}{\sqrt{2}}\,|\varphi_2\rangle + \frac{e^{iat/\hbar}}{\sqrt{6}}\,|\varphi_3\rangle\Big)\ . \end{aligned} \tag{2.10}$$

2-i) The probability to “find it” in the state $|2\rangle$ at time t is

$$\begin{aligned} P(1\to 2,t) &= |\langle 2\,|\Psi(t)\rangle\,|^2 = \Big|\,\langle 2|\,\Big(\frac{e^{-i2at/\hbar}}{\sqrt{3}}\,|\varphi_1\rangle + \frac{e^{iat/\hbar}}{\sqrt{2}}\,|\varphi_2\rangle + \frac{e^{iat/\hbar}}{\sqrt{6}}\,|\varphi_3\rangle\Big)\Big|^2 \\ &= \Big|\Big(\frac{e^{-i2at/\hbar}}{\sqrt{3}}\frac{1}{\sqrt{3}} + \frac{e^{iat/\hbar}}{\sqrt{2}}(-\frac{1}{\sqrt{2}}) + \frac{e^{iat/\hbar}}{\sqrt{6}}\frac{1}{\sqrt{6}}\Big)\Big|^2 = \Big|\Big(\frac{e^{-i2at/\hbar}}{3} - \frac{e^{iat/\hbar}}{3}\Big)\Big|^2 \\ &= \frac{1}{9}\Big|e^{-i2at/\hbar} - e^{iat/\hbar}\Big|^2 = \frac{1}{9}\Big|e^{-iat/(2\hbar)}\Big(e^{-i3at/(2\hbar)} - e^{3iat/(2\hbar)}\Big|^2 \\ &= \frac{4}{9}\sin^2\frac{3at}{2\hbar} = \frac{2}{9}(1-\cos\frac{3at}{\hbar})\ . \end{aligned} \tag{2.11}$$

We see the characteristic frequency $\omega = \frac{3a}{\hbar} = \frac{E_1-E_2}{\hbar}$

3 Time-dependences

The measurement of an observable $\mathcal{A}$ can only give any of the eigenvalues of the associated hermitian linear operator A. Distinct eigenvalues correspond to orthogonal eigenvectors. If there are four different possible measurements, i.e. four different eigenvalues, then there are at least four orthogonal eigenvectors. (If some eigenvalue is degenerate, there can be more than one different eigenvectors for this eigenvalue.) Thus, the dimension of the Hilbert space is at least four.

3-a) One has $A\,|j\rangle = a_j\,|j\rangle$. And since A is hermitian, also $\langle j|\,A = \langle j|\,a_j$. Then

$$a_j\langle j\,|i\rangle = \big(\,\langle j|\,a_j\big)\,|i\rangle = \big(\,\langle j|\,A\big)\,|i\rangle = \langle j|\,\big(A\,|i\rangle\,\big) = \langle j|\,a_i\,|i\rangle = a_i\langle j\,|i\rangle \quad\Rightarrow\quad (a_j-a_i)\langle j\,|i\rangle = 0\ . \tag{3.1}$$

Then, if $a_j \neq a_i$, we must have $\langle j\,|i\rangle = 0$.

3-b) (Note the misprint on the homework assignment : it should be $H\,|4\rangle = \widetilde{E}_0\,|4\rangle + \eta\,|3\rangle$). The matrix $\mathtt{H}$ associated with H in the basis of the $|i\rangle$ is

$$\mathtt{H} = \begin{pmatrix} E_0 & -i\epsilon & 0 & 0 \\ i\epsilon & E_0 & 0 & 0 \\ 0 & 0 & \widetilde{E}_0 & \eta \\ 0 & 0 & \eta & \widetilde{E}_0 \end{pmatrix} \tag{3.2}$$

Clearly, if we transpose and complex conjugate this matrix, we get back the same matrix. Hence, $\mathtt{H}$ is a hermitian matrix, and this is equivalent to H being a hermitian linear operator.

3-c) The matrix $\mathtt{H}$ is bloc-diagonal. It is then enough to determine the eigenvalues and eigenvectors separately in the subspace $\mathcal{E}_{1,2}$ spanned by $|1\rangle\,, |2\rangle$ and in the subspace $\mathcal{E}_{3,4}$ spanned by $|3\rangle\,, |4\rangle$. We have

$$\mathtt{H}\Big|_{\mathcal{E}_{1,2}} = \begin{pmatrix} E_0 & -i\epsilon \\ i\epsilon & E_0 \end{pmatrix} \quad , \quad \mathtt{H}\Big|_{\mathcal{E}_{3,4}} = \begin{pmatrix} \widetilde{E}_0 & \eta \\ \eta & \widetilde{E}_0 \end{pmatrix} \tag{3.3}$$

We have

$$0 = \det(\mathtt{H}\Big|_{\mathcal{E}_{1,2}} - \lambda_{2\times 2}) = \det \begin{pmatrix} E_0 - \lambda & -i\epsilon \\ i\epsilon & E_0 - \lambda \end{pmatrix} = (E_0 - \lambda)^2 - \epsilon^2 \quad \Rightarrow \quad \lambda = E_0 \pm \epsilon\ , \tag{3.4}$$

and

$$0 = \det(\mathtt{H}\Big|_{\mathcal{E}_{3,4}} - \lambda_{2\times 2}) = \det \begin{pmatrix} \widetilde{E}_0 - \lambda & \eta \\ \eta & \widetilde{E}_0 - \lambda \end{pmatrix} = (E_0 - \lambda)^2 - \eta^2 \quad \Rightarrow \quad \lambda = E_0 \pm \eta\ . \tag{3.5}$$

Then $E_1 = E_0 - \epsilon$, $E_2 = E_0 + \epsilon$, $E_3 = \widetilde{E}_0 - \eta$, $E_4 = \widetilde{E}_0 + \eta$. To obtain the corresponding eigenvectors we have to solve the eigenvector equation. For example, for $|\varphi_1\rangle = a\,|1\rangle + b\,|2\rangle$ we get the equation

$$0 = \begin{pmatrix} E_0 - E_1 & -i\epsilon \\ i\epsilon & E_0 - E_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \epsilon & -i\epsilon \\ i\epsilon & \epsilon \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}\ . \tag{3.6}$$

This leads to $a - ib = 0$ and $ia + b = 0$. Both equations are equivalent and the un-normalized solution is $a = 1$, $b = -i$. One proceeds similarly for the other eigenvectors. The rnormalized eigenvectors then are

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}\big(\,|1\rangle - i\,|2\rangle\,\big)\ , \quad |\varphi_2\rangle = \frac{1}{\sqrt{2}}\big(\,|1\rangle + i\,|2\rangle\,\big)\ , \quad |\varphi_3\rangle = \frac{1}{\sqrt{2}}\big(\,|3\rangle - |4\rangle\,\big)\ , \quad |\varphi_4\rangle = \frac{1}{\sqrt{2}}\big(\,|3\rangle + |4\rangle\,\big)\ . \tag{3.7}$$

3-d) If at $t = 0$ one has $|\Psi(0)\rangle = e^{i\alpha}\,|\varphi_j\rangle$, then at time t one has $|\Psi(t)\rangle = e^{-iE_j t/\hbar + i\alpha}\,|\varphi_j\rangle$. The expectation value of A is

$$\langle A\rangle(t) = \langle \Psi(t)|\,A\,|\Psi(t)\rangle = \langle \varphi_j|\,A\,|\varphi_j\rangle\ , \tag{3.8}$$

where the time-dependent phase has dropped out. This is always the case for any eigenstate of H, since it's time-dependence is only a phase which always drops out when taking the expectation value.

3-e) Suppose now that $|\Psi(t_0)\rangle = |2\rangle$. Decomposing on the eigenstates of H this is $|\Psi(t_0)\rangle = |2\rangle = \frac{i}{\sqrt{2}}\big(\,|\varphi_1\rangle - |\varphi_2\rangle\,\big)$. Then

$$|\Psi(t)\rangle = \frac{i}{\sqrt{2}}\big(e^{-iE_1(t-t_0)/\hbar}\,|\varphi_1\rangle - e^{-iE_2(t-t_0)/\hbar}\,|\varphi_2\rangle\,\big) = i\frac{e^{-iE_0(t-t_0)/\hbar}}{\sqrt{2}}\big(e^{i\epsilon(t-t_0)/\hbar}\,|\varphi_1\rangle - e^{-i\epsilon(t-t_0)/\hbar}\,|\varphi_2\rangle\,\big) \tag{3.9}$$

3-f) The probability $P_{2\to 1}(t)$ that a measurement of A at time t yields a_1 is

$$\begin{aligned} P_{2\to 1}(t) &= \big|\langle 1\,|\Psi(t)\rangle\,\big|^2 = \Big|\frac{e^{i\epsilon(t-t_0)/\hbar}}{\sqrt{2}}\langle 1\,|\varphi_1\rangle - \frac{e^{-i\epsilon(t-t_0)/\hbar}}{\sqrt{2}}\langle 1\,|\varphi_2\rangle\,\Big|^2 = \Big|\frac{1}{2}\big(e^{i\epsilon(t-t_0)/\hbar} - e^{-i\epsilon(t-t_0)/\hbar}\big)\Big|^2 \\ &= \sin^2\frac{\epsilon(t-t_0)}{\hbar} = \frac{1}{2}\big(1 - \cos\frac{2\epsilon(t-t_0)}{\hbar}\big)\ . \end{aligned} \tag{3.10}$$

To compute the expectation value $\langle A \rangle_{\Psi(t)}$ of A in this state, first note that $A = \sum_{j=1}^{4} a_j \left| j \right\rangle \left\langle j \right|$, and that $|\Psi(t)\rangle$ is a linear superposition of $|\varphi_1\rangle$ and $|\varphi_2\rangle$, i.e. of $|1\rangle$ and $|2\rangle$ only. Then

$$\langle A \rangle_{\Psi(t)} = \langle \Psi(t) | \, A \, | \Psi(t) \rangle = \sum_{j=1,2} \langle \Psi(t) | \, j \rangle \langle j \, | \Psi(t) \rangle = \sum_{j=1,2} |\langle j \, | \Psi(t) \rangle \, |^2 \tag{3.11}$$

But $|\langle 1 \, | \Psi(t) \rangle \, |^2$ was just computed for the probability, and one similarly finds

$$|\langle 2 \, | \Psi(t) \rangle \, |^2 = \Big| \frac{1}{2} \big(e^{i\epsilon(t-t_0)/\hbar} + e^{-i\epsilon(t-t_0)/\hbar} \big) \Big|^2 = \cos^2 \frac{\epsilon(t-t_0)}{\hbar} = \frac{1}{2} \big(1 + \cos \frac{2\epsilon(t-t_0)}{\hbar} \big) \ . \tag{3.12}$$

Then

$$\langle A \rangle_{\Psi(t)} = a_1 \, \frac{1}{2} \big(1 - \cos \frac{2\epsilon(t-t_0)}{\hbar} \big) + a_2 \, \frac{1}{2} \big(1 + \cos \frac{2\epsilon(t-t_0)}{\hbar} \big) = \frac{a_1 + a_2}{2} + \frac{a_2 - a_1}{2} \cos \frac{2\epsilon(t-t_0)}{\hbar} \ . \tag{3.13}$$

We see that at $t = t_0$ the mean value is a_2, consistent with the initial state being $|2\rangle$. At time t such that $\frac{2\epsilon(t-t_0)}{=}\pi$, i.e. $t - t_0 = \frac{\pi}{2\epsilon}$, the mean value is a_1. This being the other eigenvalue, it must be that at this time the state is the corresponding eigenvector $|1\rangle$. Indeed, for $t - t_0 = \frac{\pi}{2\epsilon}$, we have that $P_{2\to1}(t) = 1$, i.e. we are certain to measure a_1, i.e. the state must be $\sim |1\rangle$. This is perfectly consistent.

3-g) Since $\langle 3 \, | \Psi(t) \rangle = \langle 4 \, | \Psi(t) \rangle = 0$, the transition probabilities $P_{2\to3}(t)$ and $P_{2\to4}(t)$ vanish. This is because the Hamiltonian is bloc diagonal and has no matrix elements connecting $|2\rangle$ to $|3\rangle$ or $|4\rangle$.