Introduction to Quantum Mechanics I

Final exam, January 24, 2025

4 hours

Please write as neatly as possible. Any unreadable text will be ignored.

This exam is composed of three independent problems, and a bonus problem.
The number of points for each problem is only an approximative indication.

Use separate sets of paper to deal with the first two and the last two problems.

You can write in English or French.

Problem 1 : Alice and Bob (4 points)

We consider 3 particles, called A, B and C, each of spin $\frac{1}{2}$. Recall that for any two of them the Bell basis is defined as

$$
\begin{aligned}
\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}\big(\,|{+}{+}\rangle + |{-}{-}\rangle\,\big) \quad , \quad \left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}}\big(\,|{+}{+}\rangle - |{-}{-}\rangle\,\big)\ , \\
\left|\Psi^+\right\rangle &= \frac{1}{\sqrt{2}}\big(\,|{+}{-}\rangle + |{-}{+}\rangle\,\big) \quad , \quad \left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\big(\,|{+}{-}\rangle - |{-}{+}\rangle\,\big)\ ,
\end{aligned} \tag{1.1}
$$

where here the $|\pm\rangle$ are meant to be the eigenstates of the individual S_z (called the z-basis, for short).

To begin, Alice has only particle A which she prepares in the eigenstate of S_x (*not* of S_z) with eigenvalue $+\frac{\hbar}{2}$. Bob has two particles, B and C, which he prepares in the Bell state $|\Phi^+\rangle$ which we could denote as $\left|\Phi^+_{\rm BC}\right\rangle$.

1) Write Alice's initial state in the z-basis. Write the total "initial" state of the 3 particles A, B and C as the superposition of the 8 possible states $|\pm\pm\pm\rangle$ where the first $\pm$ refers to particle A, the second to particle B, and the third to particle C, all in the z-basis.

2) Bob sends his particle B to Alice. Now Alice has particles A and B. She has some observable

$$O = \vec{S}_{\rm A} \cdot \vec{S}_{\rm B}\ , \tag{1.2}$$

(e.g. related to the energy of the hyperfine structure). Alice performs a measurement of this observable O which yields $-\frac{3}{4}\hbar^2$. What is the state of Alice's particles A and B after this measurement, and what is the state of the 3 particles A, B and C after this measurement ?

3) What are the possible outcomes and corresponding probabilities if Bob now measures S_x, S_y or S_z on his particle C ?

Problem 2 : Electron in an infinite well and electric field (10 points)

In this problem one wants to study an electron (in one spatial dimension), of mass m and electric charge $q = -e$, confined between two infinite potential walls at $x = \pm\frac{L}{2}$ (i.e. $V(x) = 0$ for $x \in [-\frac{L}{2}, \frac{L}{2}]$, and $V(x) = \infty$ for $x \notin [-\frac{L}{2}, \frac{L}{2}]$) and interacting with an electric field $\mathcal{E}_x = \mathcal{E}_0 \cos\frac{\pi x}{L}$.

1) First, recall the eigenfunctions $\varphi_n(x)$ of the Hamiltonian H_0 without electric field, and associated eigenvalues (energies, called $E_n^{(0)}$). Explicitly write out the integrals that express that these eigenfunctions φ_n form an orthonormal set.

2) One now adds the electric field. Determine the additional term H_1, to be added to H_0. (Reminder : the expression for the potential energy is $q\Phi$ where $\mathcal{E}_x = -\frac{\mathrm{d}\Phi}{\mathrm{d}x}$.) As the notation suggests, we will treat H_1 as a perturbation. Give a preliminary discussion, by comparing orders of magnitude, of when you expect this perturbative treatment to work well. It will be useful to define the two quantities ϵ and η as follows

$$\epsilon = \frac{\hbar^2\pi^2}{2mL^2} \quad , \quad \eta\,\epsilon = \frac{eL\mathcal{E}_0}{2\pi} \ . \tag{1.3}$$

What are the units of these quantities ?

3) Recall the formula in perturbation theory for the energies E_n up to second order, and of the eigenfunctions $\varphi_n(x)$, up to first order. Explain how the matrix elements in these formula are related to integrals. Do not compute the integrals yet.

4) Using the trigonometric relations

$$\begin{aligned} \sin\alpha\cos\beta &= \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big) \ , \\ \sin\alpha\sin\beta &= \frac{1}{2}\big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\big) \ , \end{aligned} \tag{1.4}$$

together with the orthonormality relations, explicitly compute the integrals that give the relevant matrix elements.

5) Using these results, compute the energies and eigenfunctions from question 3.

6) Suppose at $t = 0$ the electron is in the *unperturbed* ground state of energy $E_1^{(0)}$. In *first* order perturbation theory, what is the wave function at some later time $t > 0$? What is the probability, up to and including terms of order η^2, that the electron is found at $t > 0$ in the unperturbed first excited state ? (This can be phrased as “At $t = t_1 > 0$ one turns off the perturbation. What is the probability that a measurement of the energy immediately afterwards yields $E_2^{(0)}$? ”). Compare your result with eq. (9.34) of the lecture notes.

Problem 3 : A diatomic molecule (8 points + 1 bonus)

Questions 4-7 are independent of questions 1-3.

In this problem one considers a diatomic molecule composed of atom 1 and atom 2 of equal masses m. We suppose that they are confined to one spatial direction (which coincides with the axis of the molecule, with coordinate called x). We further suppose that these atoms have spin $\frac{1}{2}$ each. There are some complicated interactions between these atoms that we suppose only depend on their distance. (Later-on we also introduce a dependence on the spins.) Classically, the potential energy of the atoms is minimum when they are separated by a distance d_0.

1) Briefly recall what is the appropriate Hilbert space $\mathcal{H}_{(1,x)}$ to describe the one-dimensional motion of the first atom and what is the appropriate Hilbert space $\mathcal{H}_{(1,s)}$ to describe its spin ? Same question for the second atom. The Hilbert space to describe the full system then is the tensor product of all four Hilbert spaces $\mathcal{H} = \mathcal{H}_{(1,x)} \otimes \mathcal{H}_{(2,x)} \otimes \mathcal{H}_{(1,s)} \otimes \mathcal{H}_{(2,s)}$.

2) On $\mathcal{H}_{(1,x)}$ one has, as usual, position and momentum operators X_1 and P_1, and similarly X_2 and P_2 on $\mathcal{H}_{(2,x)}$. On the full Hilbert space one should write $X_1 \otimes 1 \otimes 1 \otimes 1$, etc, but for simplicity we simply continue writing X_1, etc. What are the (six) commutation relations between all four operators $X_1,\ X_2,\ P_1,\ P_2$?

3) We assume that the Hamiltonian is

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + V(X_1 - X_2 - d_0\mathbf{1}) \tag{1.5}$$

One introduces "total" and "relative" position and momentum operators as

$$X_{\rm tot} = \frac{1}{2}(X_1 + X_2)\ ,\ P_{\rm tot} = P_1 + P_2\ ,\ X \equiv X_{\rm rel} = X_1 - X_2 - d_0\mathbf{1}\ ,\ P \equiv P_{\rm rel} = \frac{1}{2}(P_1 - P_2)\ . \tag{1.6}$$

a) Show that all relative operators commute with all total operators and that $[X, P] = i\hbar\mathbf{1} = [X_{\rm tot}, P_{\rm tot}]$.

b) Write the Hamiltonian in terms of total and relative operators and show that it separates as $H = H_{\rm tot} + H_{\rm rel}$. Explicitly give $H_{\rm tot}$ and $H_{\rm rel}$ and show that they commute.

c) Conclude that the eigenstates of H can be chosen to be simultaneous (non-normalisable) eigenstates of $P_{\rm tot}$ and eigenstates of $H_{\rm rel}$. In the sequel we concentrate on the Hilbert space $\widehat{\mathcal{H}} = \mathcal{H}_{\rm rel,x} \otimes \mathcal{H}_{1,s} \otimes \mathcal{H}_{2,s}$, where $\mathcal{H}_{\rm rel,x}$ is the Hilbert space on which $X_{\rm rel}$ and $P_{\rm rel}$ act.

4) One now supposes that the potential also depends on the spins, i.e.

$$V(X) \quad \longrightarrow \quad V(X, \vec{S}_{(1)}, \vec{S}_{(2)}) = \frac{\mu}{2}\omega^2\Big(\alpha + \frac{\beta}{\hbar^2}\vec{S}_{(1)} \cdot \vec{S}_{(2)}\Big)X^2\ , \tag{1.7}$$

with $0 \leq \beta \leq \alpha$. Obviously, this can be viewed as a harmonic oscillator potential with a frequency that depends on the spin state of the two atoms. Determine the eigenvalues and eigenstates of $H_{\rm rel} = \frac{P^2}{2\mu} + V(X, \vec{S}_{(1)}, \vec{S}_{(2)})$.

5) Assume that at $t = 0$ the system is in its ground state.

a) What is the probability that a measurement at time $t > 0$ of $S_{(1),x}$ gives $+\frac{\hbar}{2}$?

b) Assuming that the measurement of $S_{(1),x}$ at t yielded $+\frac{\hbar}{2}$, what are the probabilities that a measurement of $S_{(2),x}$ immediately afterwards gives $\pm\frac{\hbar}{2}$?

c) Assuming that the measurement of $S_{(1),x}$ at t yielded $+\frac{\hbar}{2}$, what is the probabilities that a measurement of $S_{(2),y}$ immediately afterwards gives $\pm\frac{\hbar}{2}$?

6) Assume now that $\beta \ll \alpha$. How does the spectrum of $H_{\rm rel}$ look like ? Discuss under which conditions it is a good approximation to consider only the 4 states of lowest energy. In the sequel we only consider these 4 states. Assume that a measurement at $t = 0$ of the spins of the atoms in the z-direction yields $+\frac{\hbar}{2}$ for both of them. What is the state immediately after the measurement ? What is the state at any later time t ? Give the probability that a measurement at $t > 0$ of the spins of the atoms in the z-direction yields again $+\frac{\hbar}{2}$ for both of them.

7) (*Bonus*) Same as the previous question 6, but with the z-direction replaced by the x-direction throughout.

Problem 4 : A simple quantum field in 1+1 dimensions (bonus : 5 points)

Consider an *infinite* set of harmonic oscillators, one for each index $n \in \mathbb{Z}$, with operators a_n and $a_n^\dagger$, such that

$$[a_n, a_m^\dagger] = \delta_{nm} \ . \tag{1.8}$$

Our Hilbert space is an infinite tensor product of Hilbert spaces, one for each harmonic oscillator.

Define the 1+1 dimensional quantum field $\Phi(t, \sigma)$ where $\sigma \in S^1$ is the (dimensionless) spatial coordinate on the unit circle, i.e. $\sigma \in [0, 2\pi]$ with 2π and 0 identified, and t is a (dimensionless) time coordinate, as

$$\Phi(t, \sigma) = \frac{1}{\sqrt{4\pi}} \Big(a_0 + a_0^\dagger - it(a_0 - a_0^\dagger) \Big) + \sum_{n \neq 0} \frac{1}{\sqrt{4\pi |n|}} \Big(e^{-i|n|t + in\sigma} a_n + e^{i|n|t - in\sigma} a_n^\dagger \Big) \ . \tag{1.9}$$

(Note that the spatial coordinate σ has nothing to do with any "position operators" X_n one might want to define from $a_n + a_n^\dagger$.)

1) Compute the time-derivative $\dot{\Phi}(t, \sigma) \equiv \frac{\partial}{\partial t} \Phi(t, \sigma)$ and show that one has the following "equal time" commutators

$$[\Phi(t, \sigma), \Phi(t, \sigma')] = 0 \quad , \quad [\Phi(t, \sigma), \dot{\Phi}(t, \sigma')] = \delta(\sigma - \sigma') \ . \tag{1.10}$$

2) The so-called vacuum state $|\text{vac}\rangle$ is defined by $a_n |\text{vac}\rangle = 0$, $\forall n \in \mathbb{Z}$. It can be formally constructed from the individual ground states $|0\rangle_n$ of the n^{th} harmonic oscillator as an infinite tensor product $|\text{vac}\rangle = \ldots \otimes |0\rangle_{-2} \otimes |0\rangle_{-1} \otimes |0\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2 \otimes \ldots$. The Hamiltonian is taken to be

$$H = \epsilon \sum_{n \in \mathbb{Z}} |n| a_n^\dagger a_n \ , \tag{1.11}$$

where we have "removed" the $+\frac{1}{2}$. Why ? A so-called one-particle state is $|n_1\rangle = a_{n_1}^\dagger |\text{vac}\rangle$, and a two-particle state is $|n_1, n_2\rangle = a_{n_2}^\dagger a_{n_1}^\dagger |\text{vac}\rangle$. Show that they are eigenstates of H and determine the corresponding eigenvalues.

Solutions

Problem 1 : Alice and Bob (4 points)

1) (1 point) Alice's particle A is in the state $|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, and Bob's particles B and C are in $|\Phi^+\rangle \equiv |\Phi^+_{BC}\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$ so that the total state is

$$|\chi_{ABC}\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle) = \frac{1}{2}\Big(|{+}{+}{+}\rangle + |{+}{-}{-}\rangle + |{-}{+}{+}\rangle + |{-}{-}{-}\rangle \Big) . \tag{1.12}$$

2) (2 points) Alice's two-particle observable is

$$O = \vec{S}_A \cdot \vec{S}_B = \frac{1}{2}\Big((\vec{S}_A + \vec{S}_B)^2 - \vec{S}_A^2 - \vec{S}_B^2 \Big) = \frac{1}{2}\vec{S}^2_{\rm tot} - \frac{3}{4}\hbar^2 , \tag{1.13}$$

as discussed at length in sect. 5.4 of the lecture notes, see eq. (5.28). Since Alice measures $-\frac{3}{4}\hbar^2$ this means that she projects on the eigenstate of $\vec{S}^2_{\rm tot}$ with eigenvalue 0. This is the singlet state $|\Psi^-\rangle \equiv |\Psi^-_{AB}\rangle$. Hence, up to normalisation the state is now

$$\begin{aligned} &|\Psi^-_{AB}\rangle \langle \Psi^-_{AB} | \chi_{ABC}\rangle \\ &\quad = |\Psi^-_{AB}\rangle \frac{1}{\sqrt{2}}\big(\langle{+}{-}|_{AB} - \langle{-}{+}|_{AB} \big) \frac{1}{2}\Big(|{+}{+}{+}\rangle + |{+}{-}{-}\rangle + |{-}{+}{+}\rangle + |{-}{-}{-}\rangle \Big) \\ &\quad = -\frac{1}{2\sqrt{2}} |\Psi^-_{AB}\rangle \Big(|+\rangle_C - |-\rangle_C \Big) . \end{aligned} \tag{1.14}$$

3) (1 point) This means that Bob's particle C is in the state $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = |-\rangle_x$ which is the eigenstate of S_x with eigenvalue $-\frac{\hbar}{2}$. Hence, a measurement of S_x gives $-\frac{\hbar}{2}$ with probability 1, while the measurements of S_y or of S_z both have equal probabilities $\frac{1}{2}$ to yield either $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$.

Problem 2 : Electron in an infnite well and electric field (10 points)

1) (1,5 points) In terms of the quantity ϵ, the unperturbed energies are simply

$$E^{(0)}_n = \epsilon\, n^2 , \tag{1.15}$$

so that ϵ actually is the ground state energy $E^{(0)}_1$. One has to be careful, since the ground state has $n = 1$ and not $n = 0$. The unperturbed eigenfunctions are ($L = 2a$ as compared to the lecture notes)

$$\text{odd } n \ : \varphi^{(0)}_n(x) = \sqrt{\frac{2}{L}} \cos\frac{n\pi x}{L} \quad , \quad \text{even } n \ : \varphi^{(0)}_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} . \tag{1.16}$$

The orthonormality relations are

$$
\begin{aligned}
\frac{2}{L}\int_{-L/2}^{L/2} \mathrm{d}x \, \sin\frac{n\pi x}{L} \, \sin\frac{m\pi x}{L} &= \delta_{n,m} \ , \\
\frac{2}{L}\int_{-L/2}^{L/2} \mathrm{d}x \, \cos\frac{n\pi x}{L} \, \cos\frac{m\pi x}{L} &= \delta_{n,m} \ , \\
\frac{2}{L}\int_{-L/2}^{L/2} \mathrm{d}x \, \sin\frac{n\pi x}{L} \, \cos\frac{m\pi x}{L} &= 0 \ .
\end{aligned}
\tag{1.17}
$$

2) (1 point) The electric field is $\mathcal{E}_x = \mathcal{E}_0 \cos\frac{\pi x}{L}$. It derives from an electrostatic potential

$$
\Phi(x) = -\frac{L}{\pi}\mathcal{E}_0 \sin\frac{\pi x}{L} \tag{1.18}
$$

and thus gives rise to an additional potential energy

$$
V(x) = q\Phi(x) = -e\Phi(x) = e\frac{L}{\pi}\mathcal{E}_0 \sin\frac{\pi x}{L} = 2\eta\,\epsilon \sin\frac{\pi x}{L} \ . \tag{1.19}
$$

Thus the perturbation Hamiltonian is

$$
\widehat{H}_1 = 2\eta\,\epsilon \sin\frac{\pi x}{L} \ . \tag{1.20}
$$

Since ϵ is an energy, the parameter η is dimensionless, and we expect perturbation theory to work well if $\eta\epsilon \ll \epsilon$, i.e. if $\eta \ll 1$. Thus η will be our small parameter, by assumption.

3) (1 point) The unperturbed spectrum is non-degenerate. The rest is copying from the notes: We let $E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \ldots$ and $|\varphi_n\rangle = |\varphi_n^{(0)}\rangle + |\varphi_n^{(1)}\rangle + \ldots$ where a term $E_n^{(p)}$ or $|\varphi_n^{(p)}\rangle$ is of order p in η, i.e. $\sim \eta^p$. Then

$$
E_n^{(1)} = \langle\varphi_n^{(0)}|H_1|\varphi_n^{(0)}\rangle \quad , \quad E_n^{(2)} = -\sum_{m\neq n} \frac{|\langle\varphi_m^{(0)}|H_1|\varphi_n^{(0)}\rangle|^2}{E_m^{(0)} - E_n^{(0)}} \ , \tag{1.21}
$$

as well as

$$
|\varphi_n^{(1)}\rangle = -\sum_{m\neq n} |\varphi_m^{(0)}\rangle \frac{\langle\varphi_m^{(0)}|H_1|\varphi_n^{(0)}\rangle}{E_m^{(0)} - E_n^{(0)}} \ . \tag{1.22}
$$

At present the matrix element is

$$
\langle\varphi_m^{(0)}|H_1|\varphi_n^{(0)}\rangle = 2\eta\,\epsilon \int_{-L/2}^{L/2} \mathrm{d}x \, \varphi_m^{(0)}(x) \, \sin\frac{\pi x}{L} \, \varphi_n^{(0)}(x) \ . \tag{1.23}
$$

4) (2 points) We need to decompose $\sin\frac{\pi x}{L}\ \varphi_n^{(0)}(x)$ on the basis of the $\varphi_k^{(0)}(x)$.
<u>If n is odd</u>, this is $\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\ \cos\frac{n\pi x}{L}$, and using the first trigonometric identity given this equals

$$
\sin\frac{\pi x}{L}\ \varphi_n^{(0)}(x) = \sqrt{\frac{2}{L}}\frac{1}{2}\Big(\sin\frac{(n+1)\pi x}{L} - \sin\frac{(n-1)\pi x}{L}\Big) = \frac{1}{2}\Big(\varphi_{n+1}^{(0)}(x) - \varphi_{n-1}^{(0)}(x)\Big) \ . \tag{1.24}
$$

Of course, for $n=1$ the last term vanishes. One can now multiply this with $\varphi_m^{(0)}(x)$ and integrate and use the orthonormality to get

$$\langle\varphi_m^{(0)}|H_1|\varphi_n^{(0)}\rangle = \eta\,\epsilon\left(\delta_{m,n+1}-\delta_{m,n-1}\right)\ , \tag{1.25}$$

again with the last $\delta_{m,n-1}$ vanishing if $n=1$ since m cannot be 0.
<u>If n is even</u>, $\sin\frac{\pi x}{L}\ \varphi_n^{(0)}(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\ \sin\frac{n\pi x}{L}$, and using the second trigonometric identity given this equals

$$\sin\frac{\pi x}{L}\ \varphi_n^{(0)}(x)=\sqrt{\frac{2}{L}}\frac{1}{2}\Big(\cos\frac{(n-1)\pi x}{L}-\cos\frac{(n+1)\pi x}{L}\Big)=\frac{1}{2}\Big(\varphi_{n-1}^{(0)}(x)-\varphi_{n+1}^{(0)}(x)\Big)\ . \tag{1.26}$$

One again multiplies this with $\varphi_m^{(0)}(x)$ and integrates to get

$$\langle\varphi_m^{(0)}|H_1|\varphi_n^{(0)}\rangle = -\eta\,\epsilon\left(\delta_{m,n+1}-\delta_{m,n-1}\right)\ . \tag{1.27}$$

Note the overall change of sign compared to the case where n was odd.

The formula for both cases can be combined as

$$\langle\varphi_m^{(0)}|H_1|\varphi_n^{(0)}\rangle = (-)^n\eta\,\epsilon\left(\delta_{m,n-1}-\delta_{m,n+1}\right)\ . \tag{1.28}$$

5) (2 points) First we see that $E_n^{(1)}=0$. Next, for $n\geq 2$,

$$\begin{aligned} E_n^{(2)} &= -\sum_{m=n\pm1}\frac{|\eta\,\epsilon|^2}{\epsilon(m^2-n^2)}=-\eta^2\epsilon\left(\frac{1}{(n+1)^2-n^2}+\frac{1}{(n-1)^2-n^2}\right)\\ &= -\eta^2\epsilon\left(\frac{1}{2n+1}-\frac{1}{2n-1}\right)=\eta^2\epsilon\ \frac{2}{4n^2-1}\ ,\quad (n\neq 1)\ , \end{aligned} \tag{1.29}$$

while for $n=1$ the term $m=n-1$ is absent and then

$$E_1^{(2)}=-\eta^2\epsilon\ \frac{1}{(1+1)^2-1^2}=-\frac{\eta^2\epsilon}{3}\ . \tag{1.30}$$

For the eigenstate we get the correction

$$\begin{aligned} |\varphi_n^{(1)}\rangle &= -|\varphi_{n+1}^{(0)}\rangle\frac{(-)^{n+1}\eta\,\epsilon}{\epsilon((n+1)^2-n^2)}-|\varphi_{n-1}^{(0)}\rangle\frac{(-)^n\eta\,\epsilon}{\epsilon((n-1)^2-n^2)}\\ &= (-)^n\eta\left(\frac{1}{2n+1}|\varphi_{n+1}^{(0)}\rangle+\frac{1}{2n-1}|\varphi_{n-1}^{(0)}\rangle\right)\ . \end{aligned} \tag{1.31}$$

Again, if $n=1$, the last term is simply absent.

6) (2,5 points) Instead of following the method of the lecture notes, here we give an alternative "pedestrian" approach. To get the time evolution, we must express the initial state $|\psi(0)\rangle=|\varphi_1^{(0)}\rangle$ in terms of the eigenstates of the perturbed hamiltonian, namely the $|\varphi_n\rangle=|\varphi_n^{(0)}\rangle+|\varphi_n^{(1)}\rangle$, where we can drop the higher order terms. Indeed, we want the transition probability to order η^2 which means that we need the amplitude to order η, i.e. $|\varphi_1^{(1)}\rangle$ is enough. Then we have

$$\begin{aligned}
|\varphi_1\rangle &= |\varphi_1^{(0)}\rangle + |\varphi_1^{(1)}\rangle = |\varphi_1^{(0)}\rangle - \frac{\eta}{3}|\varphi_2^{(0)}\rangle \\
|\varphi_2\rangle &= |\varphi_2^{(0)}\rangle + |\varphi_2^{(1)}\rangle = |\varphi_2^{(0)}\rangle + \frac{\eta}{3}|\varphi_1^{(0)}\rangle + \frac{\eta}{5}|\varphi_3^{(0)}\rangle \ .
\end{aligned} \tag{1.32}$$

The first equation can be rewritten as $|\varphi_1^{(0)}\rangle = |\varphi_1\rangle + \frac{\eta}{3}|\varphi_2^{(0)}\rangle$, and the second equation as $|\varphi_2^{(0)}\rangle = |\varphi_2\rangle + \mathcal{O}(\eta)$. Inserting this into the previous equation gives

$$|\psi(0)\rangle = |\varphi_1^{(0)}\rangle = |\varphi_1\rangle + \frac{\eta}{3}|\varphi_2\rangle + \mathcal{O}(\eta^2) \ , \tag{1.33}$$

which is the desired decomposition of the initial state on the eigenstates of the full Hamiltonian. Then at time t the state is

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar}|\varphi_1\rangle + \frac{\eta}{3}e^{-iE_2 t/\hbar}|\varphi_2\rangle + \mathcal{O}(\eta^2) \ . \tag{1.34}$$

The probability to find the state in the eigenstate of H_0 with eigenvalue $E_2^{(0)}$ is

$$\begin{aligned}
\mathcal{P}(E_2^{(0)}, t) &= |\langle\varphi_2^{(0)}|\psi(t)\rangle|^2 = \left| e^{-iE_1 t/\hbar}\langle\varphi_2^{(0)}|\varphi_1\rangle + \frac{\eta}{3}e^{-iE_2 t/\hbar}\langle\varphi_2^{(0)}|\varphi_2\rangle + \mathcal{O}(\eta^2)\right|^2 \\
&= \left| -\frac{\eta}{3}e^{-iE_1 t/\hbar} + \frac{\eta}{3}e^{-iE_2 t/\hbar} + \mathcal{O}(\eta^2)\right|^2 = \left| -\frac{\eta}{3}e^{-iE_1 t/\hbar} + \frac{\eta}{3}e^{-iE_2 t/\hbar}\right|^2 + \mathcal{O}(\eta^3) \\
&= \frac{\eta^2}{9}\left| - e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar}\right|^2 + \mathcal{O}(\eta^3) = \frac{\eta^2}{9} 4 \sin^2 \frac{E_2 - E_1}{2\hbar} t + \mathcal{O}(\eta^3) \ .
\end{aligned} \tag{1.35}$$

Now, $E_2 - E_1 = E_2^{(0)} - E_1^{(0)} + \mathcal{O}(\eta) = 3\epsilon + \mathcal{O}(\eta)$, so that

$$\mathcal{P}(E_2^{(0)}, t) = \frac{4}{9}\,\eta^2 \sin^2\big(\frac{3\epsilon\, t}{2\hbar}\big) + \mathcal{O}(\eta^3) \ . \tag{1.36}$$

Equation (9.34) of the lecture notes is

$$\mathcal{P}(t) = \frac{t^2}{\hbar^2}|\langle\varphi_2^{(0)}|H_1|\varphi_1^{(0)}\rangle|^2 \, \frac{\sin^2(\frac{3\epsilon t}{2\hbar})}{(\frac{3\epsilon t}{2\hbar})^2} = \frac{4}{9\epsilon^2}|\langle\varphi_2^{(0)}|H_1|\varphi_1^{(0)}\rangle|^2 \sin^2(\frac{3\epsilon t}{2\hbar}) \tag{1.37}$$

The matrix element was computed above in (1.28), for $n = 1$, $m = 2$ and equals $\eta\epsilon$, so that

$$\mathcal{P}(t) = \frac{4}{9}\eta^2 \sin^2(\frac{3\epsilon t}{2\hbar}) \ , \tag{1.38}$$

which coincides with the present result (1.36).

Problem 3 : A diatomic molecule (8 + 1 points)

1) (0,5 point) The Hilbert spaces are $\mathcal{H}_{(1,x)} \simeq \mathcal{H}_{(2,x)} \simeq L^2(\mathbb{R})$ and $\mathcal{H}_{(1,s)} \simeq \mathcal{H}_{(2,s)} \simeq \mathbb{C}^2$.

2) (0,5 point) The commutation relations are

$$[X_1, X_2] = [X_1, P_2] = [X_2, P_1] = [P_1, P_2] = 0 \quad , \quad [X_1, P_1] = [X_2, P_2] = i\hbar \mathbf{1} \; . \tag{1.39}$$

3a) (0,5 point) Obviously, $[X_{\rm tot}, X] = [P_{\rm tot}, P] = 0$. Next

$$\begin{aligned}
[X, P_{\rm tot}] &= [X_1 - X_2, P_1 + P_2] = [X_1, P_1] - [X_2, P_2] = i\hbar\mathbf{1} - i\hbar\mathbf{1} = 0 \; , \\
[X_{\rm tot}, P] &= \frac{1}{4}[X_1 + X_2, P_1 - P_2] = \frac{1}{4}\Big([X_1, P_1] - [X_2, P_2]\Big) = i\hbar\mathbf{1} - i\hbar\mathbf{1} = 0 \; , \\
[X, P] &= \frac{1}{2}[X_1 - X_2, P_1 - P_2] = \frac{1}{2}\Big([X_1, P_1] + [X_2, P_2]\Big) = \frac{1}{2}\big(i\hbar\mathbf{1} + i\hbar\mathbf{1}\big) = i\hbar\mathbf{1} \; , \\
[X_{\rm tot}, P_{\rm tot}] &= \frac{1}{2}[X_1 + X_2, P_1 + P_2] = \frac{1}{2}\Big([X_1, P_1] + [X_2, P_2]\Big) = \frac{1}{2}\big(i\hbar\mathbf{1} + i\hbar\mathbf{1}\big) = i\hbar\mathbf{1} \, .
\end{aligned} \tag{1.40}$$

3b) (1 point) We have $P_1 = \frac{1}{2}P_{\rm tot} + P$ and $P_2 = \frac{1}{2}P_{\rm tot} - P$. Then

$$\frac{P_1^2}{2m} + \frac{P_2^2}{2m} = \frac{1}{2m}\Big(\big(\frac{1}{2}P_{\rm tot} + P\big)^2 + \big(\frac{1}{2}P_{\rm tot} - P\big)^2\Big) = \frac{1}{2m}\big(\frac{1}{2}P_{\rm tot}^2 + 2P^2\big) = \frac{P_{\rm tot}^2}{4m} + \frac{P^2}{m} = \frac{P_{\rm tot}^2}{2M} + \frac{P^2}{2\mu} \; , \tag{1.41}$$

where $M = 2m$ is the total mass and $\mu = \frac{m}{2}$ the reduced mass. Then

$$H = H_{\rm tot} + H_{\rm rel} \quad , \quad H_{\rm tot} = \frac{P_{\rm tot}^2}{2M} \quad , \quad H_{\rm rel} = \frac{P^2}{2\mu} + V(X) \; . \tag{1.42}$$

It is then obvious from (1.40) that $H_{\rm tot}$ and $H_{\rm rel}$ commute.

3c) (0,5 point) Because these are two commuting observables, there is a common basis of eigenstates. Actually, although not totally obvious mathematically, we may view the tensor product Hilbert space $\mathcal{H}_{1,x} \otimes \mathcal{H}_{2,x}$ also as the tensor product $\mathcal{H}_{{\rm tot},x} \otimes \mathcal{H}_{{\rm rel},x}$. Then $H_{\rm tot}$ acts on $\mathcal{H}_{{\rm tot},x}$ and $H_{\rm rel}$ acts on $\mathcal{H}_{{\rm rel},x}$. Then the eigenstates of H are of the form $|p_{\rm tot}\rangle \otimes |\varphi_n\rangle$ with $P_{\rm tot}\,|p_{\rm tot}\rangle = p_{\rm tot}\,|p_{\rm tot}\rangle$ and $H_{\rm rel}\,|\varphi_n\rangle = E_n\,|\varphi_n\rangle$. The eigenvalues of H then are $\frac{p_{\rm tot}^2}{2M} + E_n$.

4) (2 points) We now concentrate on $H_{\rm rel}$ and assume that the potential $V(X)$ also involves the spin-operators

$$V(X) \equiv V(X, \vec{S}_{(1)}, \vec{S}_{(2)}) = \frac{\mu}{2}\omega^2\Big(\alpha + \frac{\beta}{\hbar^2}\vec{S}_{(1)} \cdot \vec{S}_{(2)}\Big)X^2 \; , \tag{1.43}$$

As before, we let $\vec{S}_{\rm tot} = \vec{S}_{(1)} + \vec{S}_{(2)}$ so that $\vec{S}_{\rm tot}^2 = \vec{S}_{(1)}^2 + \vec{S}_{(2)}^2 + 2\vec{S}_{(1)} \cdot \vec{S}_{(2)}$ and $\vec{S}_{(1)} \cdot \vec{S}_{(2)} = \frac{1}{2}\vec{S}_{\rm tot}^2 - \frac{3}{4}\hbar^2$. The eigenstates of $\vec{S}_{\rm tot}^2$ are the triplet states of total spin $S = 1$, and eigenvalue $S(S+1)\hbar^2 = 2\hbar^2$ and the singlet state with total spin $s = 0$ and eigenvalue $s(s+1)\hbar^2 = 0$. Thus

$$\vec{S}_{(1)} \cdot \vec{S}_{(2)}\,|S, M\rangle = \Big(\frac{S(S+1)}{2} - \frac{3}{4}\Big)\hbar^2\,|S, M\rangle \; . \tag{1.44}$$

Then

$$V(X,\vec{S}_{(1)},\vec{S}_{(2)})\ |\varphi\rangle\otimes|S,M\rangle=\frac{\mu}{2}\omega^2\Big(\alpha+\beta\big(\frac{S(S+1)}{2}-\frac{3}{4}\big)\Big)X^2\ \ |\varphi\rangle\otimes|S,M\rangle \tag{1.45}$$

Define

$$\omega_S^2=\omega^2\Big(\alpha+\beta\big(\frac{S(S+1)}{2}-\frac{3}{4}\big)\Big)\ , \tag{1.46}$$

Then, the eigenvalue problem of $H_{\rm rel}$ is

$$H_{\rm rel}\ \ |\varphi\rangle\otimes|S,M\rangle=\Big(\frac{P^2}{2\mu}+\frac{\mu}{2}\omega_S^2X^2\Big)\ \ |\varphi\rangle\otimes|S,M\rangle=E_{n,S}\ \ |\varphi\rangle\otimes|S,M\rangle\ \ . \tag{1.47}$$

One then needs to solve

$$\Big(\frac{P^2}{2\mu}+\frac{\mu}{2}\omega_S^2X^2\Big)\ \ |\varphi\rangle=E_{n,S}\ \ |\varphi\rangle \tag{1.48}$$

But we know that the eigenstates are just those of the harmonic oscillator with frequency ω_S and the eigenvalues are

$$E_{n,S}=\hbar\omega_S(n+\frac{1}{2})\quad,\quad n=0,1,2,\ldots,S=0,1\ . \tag{1.49}$$

The $E_{n,0}$ are non-degenerate (singlet), while the $E_{n,1}$ are 3 times degenerate (triplet). The smallest energy (ground state) is the one for $n=0$ and the smallest ω_S which is the one with $S=0$ (singlet) : $\omega_0^2=\omega^2(\alpha-\frac{3}{4}\beta)$. Thus the ground state and its energy are

$$|0\rangle_{\omega_0}\otimes|\text{singlet}\rangle\quad,\quad E_{0,0}=\frac{1}{2}\hbar\omega\sqrt{\alpha-\frac{3}{4}\beta}\ . \tag{1.50}$$

5a) (0,5 point) Being in the ground state (which is non-degenerate) at $t=0$, the system will still be in the ground state at any later time (up to some irrelevant phase $e^{-iE_{0,0}t/\hbar}$). Thus the spin state is still the singlet state

$$|\text{singlet}\rangle=\frac{1}{\sqrt{2}}\big(\,|+-\rangle-|-+\rangle\,\big)\ . \tag{1.51}$$

This is written in the z-basis, but (by rotational invariance of the singlet state), it can also be written in any of the basis's $|\pm\rangle_x$ or $|\pm\rangle_y$:

$$|\text{singlet}\rangle=\frac{1}{\sqrt{2}}\big(\,|+-\rangle_x-|-+\rangle_x\,\big)=\frac{1}{\sqrt{2}}\big(\,|+-\rangle_y-|-+\rangle_y\,\big)\ . \tag{1.52}$$

Thus the probability that a measurement of $S^x_{(1)}$ gives $+\frac{\hbar}{2}$ is $|\frac{1}{\sqrt{2}}|^2=\frac{1}{2}$.

5b) (0,5 point) If this measurement has indeed resulted in $+\frac{\hbar}{2}$, the state gets projected with $|+\rangle_x\,\langle+|_x\otimes\mathbf{1}$ which gives (after normalising) $|+-\rangle_x$. Thus a measurement of $S^x_{(2)}$ yields $-\frac{\hbar}{2}$ with probability one.

5c) (0,5 point) Now, on this same state $|+-\rangle_x = |+\rangle_x \otimes |-\rangle_x$ one measures $S^y_{(2)}$. Since $|-\rangle_x$ decomposes on $|+\rangle_y$ and $|-\rangle_y$ with amplitudes of equal modulus $|\frac{1}{\sqrt{2}}|$, the probabilities to find $\pm\frac{\hbar}{2}$ are both $\frac{1}{2}$.

6) (1,5 points) If $0 < \beta \ll \alpha$, the splitting of the harmonic oscillator levels $\hbar\omega\sqrt{\alpha}(n+\frac{1}{2})$ is very small. More precisely

$$\begin{aligned} \omega_S &= \omega\sqrt{\alpha}\sqrt{1+\frac{\beta}{\alpha}\left(\frac{S(S+1)}{2}-\frac{3}{4}\right)} \simeq \omega\sqrt{\alpha}\Big(1+\frac{\beta}{2\alpha}\left(\frac{S(S+1)}{2}-\frac{3}{4}\right)\Big) \\ &= \omega\sqrt{\alpha}\big(1-\frac{3\beta}{8\alpha}\big)+\omega\frac{\beta}{4\sqrt{\alpha}}S(S+1)\ , \end{aligned} \tag{1.53}$$

so that

$$E_{n,S} = \hbar\omega\sqrt{\alpha}\big(1-\frac{3\beta}{8\alpha}\big)(n+\frac{1}{2})+\hbar\omega\frac{\beta}{4\sqrt{\alpha}}S(S+1)(n+\frac{1}{2})\ , \tag{1.54}$$

which is of the form

$$E_{n,S} = \epsilon(n+\frac{1}{2})+\eta S(S+1)(n+\frac{1}{2}) \quad , \quad 0<\eta\ll\epsilon\ . \tag{1.55}$$

Then the energies with $n \geq 1$ are all "much" larger that the 4 energy levels with $n = 0$. For example, if one considers a gas of these molecules at temperatures T such that $k_BT \ll \epsilon \simeq \hbar\omega\sqrt{\alpha}$ then almost all molecules will be in the four lowest lying states with $n = 0$ and $S = 0$ (one state) or $S = 1$ (3 states). If a measurement of both spins in the z-direction yields $+\frac{\hbar}{2}$ for both of them, the spin state is $|++\rangle$ which is part of the triplet ($S = 1$). Then, after this measurement, the state is $|0\rangle \otimes |++\rangle$ with energy $E_{0,1} = \frac{\epsilon}{2} + \eta$. At a later time the state is $e^{-iE_{0,1}t/\hbar}\,|0\rangle \otimes |++\rangle$. Hence, the spins stay both eigenstates of $S^z_{(1)}$ and $S^z_{(2)}$ with eigenvalue $+\frac{\hbar}{2}$, and a measurement at any later time of the z-components will yield $+\frac{\hbar}{2}$ for both with probability one.

7) (1 point) Of course, when discussing the triplet and singlet states, the choice of the z-basis was arbitrary. We have already seen that the singlet keeps its same form with any of the x or y or z-basis. Similarly, an orthonormal basis of the $S = 1$ eigenspace is equivalently be given by

$$|++\rangle_x \ , \quad \frac{1}{2}\big(|+-\rangle_x+|-+\rangle_x\big)\ , \quad |--\rangle_x\ . \tag{1.56}$$

Of course, none of these states equals the three states of the triplet written in the z-basis, but the claim is that any of these three states can be written as a linear combination of the three triplet states in the z-basis and vice versa. Hence, we can repeat, the results of the previous question word by word, only replacing $S^z_{(1)}$ and $S^z_{(2)}$ by $S^x_{(1)}$ and $S^x_{(2)}$, and $|++\rangle \equiv |++\rangle_z$ by $|++\rangle_x$.

Problem 4 : A simple quantum field in 1+1 dimensions (5 bonus points)

1) (3 points) We have

$$
\begin{aligned}
\Phi(t,\sigma) &= \frac{1}{\sqrt{4\pi}}\Big(a_0+a_0^\dagger - it(a_0-a_0^\dagger)\Big) + \sum_{n\neq 0}\frac{1}{\sqrt{4\pi|n|}}\Big(e^{-i|n|t+in\sigma}a_n + e^{i|n|t-in\sigma}a_n^\dagger\Big)\ , \\
\dot\Phi(t,\sigma) &= \frac{-i}{\sqrt{4\pi}}(a_0-a_0^\dagger) - i\sum_{n\neq 0}\sqrt{\frac{|n|}{4\pi}}\Big(e^{-i|n|t+in\sigma}a_n - e^{i|n|t-in\sigma}a_n^\dagger\Big)\ .
\end{aligned}
\tag{1.57}
$$

Then,

$$
\begin{aligned}
&[\Phi(t,\sigma),\Phi(t,\sigma')] \\
&\quad= \Big[\sum_{n\neq 0}\frac{1}{\sqrt{4\pi|n|}}\Big(e^{-i|n|t+in\sigma}a_n + e^{i|n|t-in\sigma}a_n^\dagger\Big), \sum_{m\neq 0}\frac{1}{\sqrt{4\pi|m|}}\Big(e^{-i|m|t+im\sigma'}a_m + e^{i|m|t-im\sigma'}a_m^\dagger\Big)\Big] \\
&\quad= \frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{\sqrt{|n|}}e^{-i|n|t+in\sigma}\sum_{m\neq 0}\frac{1}{\sqrt{|m|}}e^{i|m|t-im\sigma'}\big[a_n,a_m^\dagger\big] \\
&\qquad\qquad+\frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{\sqrt{|n|}}e^{i|n|t-in\sigma}\sum_{m\neq 0}\frac{1}{\sqrt{|m|}}e^{-i|m|t+im\sigma'}\big[a_n^\dagger,a_m^\dagger\big]\ .
\end{aligned}
\tag{1.58}
$$

Now, $\big[a_n,a_m^\dagger\big]=\delta_{nm}$ and $\big[a_n^\dagger,a_m^\dagger\big]=-\delta_{nm}$. Then

$$
[\Phi(t,\sigma),\Phi(t,\sigma')] = \frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{|n|}e^{in(\sigma-\sigma')} - \frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{|n|}e^{-in(\sigma-\sigma')}\ . \tag{1.59}
$$

Upon renaming n to $-n$ in the second sum, it equals the first one, and they cancel each other. Hence

$$
[\Phi(t,\sigma),\Phi(t,\sigma')] = 0\ . \tag{1.60}
$$

Note that this cancellation only occurs for equal times, i.e. $t'=t$.

A similar computation for $[\Phi(t,\sigma),\dot\Phi(t,\sigma')]$ gives

$$
\begin{aligned}
&[\Phi(t,\sigma),\dot\Phi(t,\sigma')] = -\frac{i}{4\pi}\big[a_0+a_0^\dagger, a_0-a_0^\dagger\big] \\
&\quad -\frac{i}{4\pi}\Big[\sum_{n\neq 0}\frac{1}{\sqrt{|n|}}\Big(e^{-i|n|t+in\sigma}a_n + e^{i|n|t-in\sigma}a_n^\dagger\Big), \sum_{m\neq 0}\sqrt{|m|}\Big(e^{-i|m|t+im\sigma'}a_m - e^{i|m|t-im\sigma'}a_m^\dagger\Big)\Big] \\
&\quad = \frac{i}{2\pi} + \frac{i}{4\pi}\sum_{n\neq 0}e^{in\sigma-in\sigma'} + \frac{i}{4\pi}\sum_{n\neq 0}e^{-in\sigma+in\sigma'}\ .
\end{aligned}
\tag{1.61}
$$

Again, upon renaming n to $-n$ in the second sum, it equals the first one, but htis time they add up :

$$
[\Phi(t,\sigma),\dot\Phi(t,\sigma')] = \frac{i}{2\pi} + \frac{i}{2\pi}\sum_{n\neq 0}e^{in(\sigma-\sigma')} = \frac{i}{2\pi}\sum_{n\in\mathbb{Z}}e^{in(\sigma-\sigma')} = i\delta(\sigma-\sigma')\ . \tag{1.62}
$$

2) (2 points) We have an infinity of harmonic oscillators, and we want to assign them some frequencies ω_n. Then the natural Hamiltonian would be $H = \sum n\hbar\omega_n\big(a_n^\dagger a_n + \frac{1}{2}\big)$ with a ground state energy that would be $\sum n\hbar\omega_n/2$ which would almost certainly diverge, unless the ω_n are chosen to decrease sufficiently fast with n. Instead one decides to shift the energy such that $H = \sum n\hbar\omega_n a_n^\dagger a_n$ without the (infinite) constant piece. Furthermore, we choose $\hbar\omega_n = \epsilon|n|$ so that

$$H = \sum_{n\in\mathbb{Z}} \epsilon|n| a_n^\dagger a_n \ , \tag{1.63}$$

and a ground state (vacuum state) such that $a_n\,|\text{vac}\rangle = 0$ for all n. Then

$$H\,|\text{vac}\rangle = 0 \ . \tag{1.64}$$

Now

$$[H, a_k^\dagger] = \sum_{n\in\mathbb{Z}} \epsilon|n|\,[a_n^\dagger a_n, a_k^\dagger] = \sum_{n\in\mathbb{Z}} \epsilon|n|\,a_n^\dagger [a_n, a_k^\dagger] = \sum_{n\in\mathbb{Z}} \epsilon|n|\,a_n^\dagger \delta_{n,k} = \epsilon|k| a_k^\dagger \ , \tag{1.65}$$

so that

$$H a_k^\dagger\,|\text{vac}\rangle = \Big(a_k^\dagger H + [H, a_k^\dagger]\Big)\,|\text{vac}\rangle = 0 + \epsilon|k| a_k^\dagger\,|\text{vac}\rangle \ . \tag{1.66}$$

Hence, $a_k^\dagger\,|\text{vac}\rangle$ is an eigenstate of H with eigenvalue (energy) $\epsilon|k|$. Similarly,

$$\begin{aligned} [H, a_k^\dagger a_l^\dagger] &= \sum_{n\in\mathbb{Z}} \epsilon|n|\,[a_n^\dagger a_n, a_k^\dagger a_l^\dagger] = \sum_{n\in\mathbb{Z}} \epsilon|n|\,\Big(a_k^\dagger [a_n^\dagger a_n, a_l^\dagger] + [a_n^\dagger a_n, a_k^\dagger] a_l^\dagger\Big) \\ &= \sum_{n\in\mathbb{Z}} \epsilon|n|\,\Big(a_k^\dagger \delta_{nl} a_l^\dagger + \delta_{nk} a_k^\dagger a_l^\dagger\Big) = \epsilon\big(|l| + |k|\big) a_k^\dagger a_l^\dagger \ , \end{aligned} \tag{1.67}$$

and then

$$H a_k^\dagger a_l^\dagger\,|\text{vac}\rangle = \Big(a_k^\dagger a_l^\dagger H + [H, a_k^\dagger a_l^\dagger]\Big)\,|\text{vac}\rangle = 0 + \epsilon\big(|k| + |l|\big) a_k^\dagger a_l^\dagger\,|\text{vac}\rangle \ . \tag{1.68}$$

Hence, $a_k^\dagger a_l^\dagger\,|\text{vac}\rangle$ is an eigenstate of H with eigenvalue (energy) $\epsilon|k| + \epsilon|l|$.

The interpretation is that each $a_k^\dagger$ creates a particle of energy $\epsilon|k|$ from the vacuum. Since the energies simply add up, there are no interactions between these particles (with the Hamiltonian we have chosen).