Introduction to Quantum Mechanics I

Homework 2024

To be uploaded as a pdf on the Moodle by November 8, 2024, 23h59.

You can write in English or French. If you have reasonably well understood the lectures, this homework should not take more than a couple of hours. Also your pdf should not exceed 4 pages.

1 Some operator identities

Let A and B be some linear operators on a finite-dimensional Hilbert space, n a non-negative integer, and s a real number. A and B do not necessarily commute.

1-a) Show that A and A^n commute, recall the definition of e^A in terms of the power series and show that $\frac{\mathrm{d}}{\mathrm{d}s}e^{sA} = Ae^{sA} = e^{sA}A$, as well as $\frac{\mathrm{d}}{\mathrm{d}s}e^{s(A+B)} = (A+B)e^{s(A+B)} = e^{s(A+B)}(A+B)$ (even if A and B do not commute). Explain why there is no such simple answer for $\frac{\mathrm{d}}{\mathrm{d}s}e^{(sA+B)}$.

1-b) Consider $f(s) = e^{sA}Be^{-sA}$ and compute $f'(0)$ and $f''(0)$. Deduce (without detailed proof) $f^{(n)}(0)$. Use this result to show that

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A,[A[\ldots[A,B]\ldots]]}_{n \text{ commutators}} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots \tag{1.1}$$

Hint: You may expand f in a Taylor series around $s = 0$.

1-c) In this exercise assume that A and B commute with $[A,B]$. First, show that $g(s) = e^{sA}e^{sB}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}g(s) = \big(A + g(s)B(g(s))^{-1}\big)g(s) \ . \tag{1.2}$$

Use the result from 1-b) to compute $A + g(s)B(g(s))^{-1}$ and obtain that g satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}g(s) = \big(A + B + s[A,B]\big)g(s) \ , \tag{1.3}$$

with initial condition $g(0) = \mathbf{1}$. Explain why this is solved as

$$g(s) = \exp\left(s(A+B) + \frac{s^2}{2}[A,B]\right) \ . \tag{1.4}$$

Conclude that one has

$$\boxed{e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \ .} \tag{1.5}$$

This is a special case of the Campbell-Baker-Haussdorf formula. The latter applies even if $[A,B]$ does not commute with A and B and then the exponent on the right-hand side contains further multiple commutators of $[A,B]$ with A or B.

2 Rotations

Recall the infinitesimal rotation generators J_a as given in (3.38) and the matrices $\mathcal{R}_a(\alpha)$ of the finite rotations (3.36).

2-a) For $a = x, y, z$, compute J_a^n for all integer $n \geq 1$. Explicitly compute $\sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} J_z^n = e^{i\alpha J_z}$ and compare the result with $\mathcal{R}_z(\alpha)$. Argue that a similar result holds for J_x and J_y.

2-b) Explicitly compute the product of the three matrices $\mathcal{R}_z(\alpha)\, J_y\, \mathcal{R}_z(-\alpha)$ and express the result in the form $(...)J_y + (...)J_x$. Then compute the same expression as $e^{i\alpha J_z} J_y e^{-i\alpha J_z}$, using (1.1). You should find the same result.

2-c) Recall equations (3.35) that relates the coordinates x', y', z' rotated by $\mathcal{R}(\vec{u}, \alpha)$ to the unrotated ones x, y, z. Consider a function f of the coordinates and let

$$(\delta_{\vec{u},\alpha} f) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = f\left(\mathcal{R}(\vec{u}, \alpha)^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) - f\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{2.6}$$

Note the appearance of $\mathcal{R}(\vec{u}, \alpha)^{-1}$. Recall how $\mathcal{R}(\vec{u}, \alpha)^{-1}$ is related to $\mathcal{R}(\vec{u}, \alpha)$. For infinitesimal α, and $\vec{u} = \vec{e}_z$, express $\delta_{\vec{e}_z,\alpha} f$ as $i\alpha \mathcal{L}_z f$ where $\mathcal{L}_z$ is some first-order differential operator, acting on f, to be determined.

2-d) Deduce (or guess) similarly the differential operators $\mathcal{L}_x$ and $\mathcal{L}_y$. Compute the commutator of these differential operators as

$$\big[\mathcal{L}_x, \mathcal{L}_y\big] f \equiv \mathcal{L}_x\big(\mathcal{L}_y f\big) - \mathcal{L}_y\big(\mathcal{L}_x f\big) , \tag{2.7}$$

and try to express the result in terms of $\mathcal{L}_z f$. What do you observe ? Interpretation ?

3 Time evolution of a 2-state system

Consider an arbitrary 2-state system with orthonormal basis $\{|1\rangle, |2\rangle\}$ and Hamiltonian H defined by (a priori $a, b, c, d \in \mathbf{C}$ and we assume $a \neq b$)

$$H|1\rangle = a|1\rangle + c|2\rangle \quad , \quad H|2\rangle = d|1\rangle + b|2\rangle . \tag{3.8}$$

3-a) What can you say about a and b ? Can you express d in terms of $c = c_1 + ic_2$ (with $c_1, c_2 \in \mathbf{R}$) ? Write down the corresponding matrix $\widehat{H}$.

3-b) Express $\widehat{H}$ in terms of the 3 Pauli matrices and the identity matrix as $\widehat{H} = E_0 \mathbf{1} + \frac{\hbar\omega}{2} \vec{u} \cdot \vec{\sigma}$ where $\vec{u}$ is a unit vector. Identify E_0, $\hbar\omega$ and the components of $\vec{u}$ in terms of a, b, c_1, c_2. Use this result to state the eigenvalues of H without any further computation. Verify that their sum equals the trace of $\widehat{H}$ and their product equals its determinant. Give the eigenvectors of H in terms of the $|\pm\rangle_{\vec{u}}$ given in the lecture notes in eq. (3.32) as parametrised by θ and φ, and determine $\tan^2\theta$ and $\tan\varphi$

3-c) Let X be the observable $X = x_0\big(|1\rangle\langle 1| - |2\rangle\langle 2|\big)$. Suppose at $t = 0$ one measures X and finds x_0. What is the state just after the measurement, and what is the state at any later time t (if no further measurement is made in this time interval). What is the probability that a measurement of X at $t = T$ yields $-x_0$?

3-d) Compute the expectation value of X at any $t \in (0, T)$? Is this consistent with the result of the previous question ?