Introduction to Quantum Mechanics I

Solutions to the Homework 2024

Here is a proposition for the solutions to the homework.

1 Some operator identities

Since A and B are linear operators on a *finite-dimensional Hilbert space* there are no subtleties associated with defining power series of these operators.

1-a) We have $AA^n = A^{n+1} = A^n A$, so A and A^n obviously commute. $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. One has $\frac{\mathrm{d}}{\mathrm{d}s}(sA)^n = ns^{n-1}A^n = n(sA)^{n-1}A = nA(sA)^{n-1}$ and it follows that $\frac{\mathrm{d}}{\mathrm{d}s}e^{sA} = Ae^{sA} = e^{sA}A$. Replacing A by $C = A + B$ then also $\frac{\mathrm{d}}{\mathrm{d}s}e^{s(A+B)} = (A+B)e^{s(A+B)} = e^{s(A+B)}(A+B)$. This is true even if A and B do not commute, since the only combination ever appearing is $C = A + B$ and its powers. On the other hand, $\frac{\mathrm{d}}{\mathrm{d}s}(sA+B) = A$ and then A does not commute with $(sA+B)$ if A and B do not commute. Then e.g. $\frac{\mathrm{d}}{\mathrm{d}s}(sA+B)^3 = A(sA+B)^2 + (sA+B)A(sA+B) + (sA+B)^2 A \neq 3A(sA+B)^2$. Hence, there is no simple form for $\frac{\mathrm{d}}{\mathrm{d}s}e^{(sA+B)}$.

1-b) Consider $f(s) = e^{sA}Be^{-sA}$. We have $f(0) = B$, $f'(s) = e^{sA}(AB - BA)e^{-sA}$ and $f'(0) = [A, B]$. Similarly, $f''(s) = e^{sA}(A^2B - 2ABA + BA^2)e^{-sA}$ and then $f''(0) = A^2B - 2ABA + BA^2 = A(AB - BA) - (AB - BA)A = [A, [A, B]]$. One can similarly compute $f'''(0$ and find $[A, [A, [A, B]]]$ and then guess the general result. Writing $f(s)$ as the Taylor series $f(s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}s^n$ and setting $s = 1$ yields the desired result :

$$e^A Be^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A[\ldots[A, B]\ldots]]}_{n \text{ commutators}} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \ldots \tag{1.1}$$

1-c) In this exercise we assume that $[A, [A, B]] = [B, [A, B]] = 0$. Let $g(s) = e^{sA}e^{sB}$. Then $g(s)^{-1} = e^{-sB}e^{-sA}$, and $g(s)B(g(s))^{-1} = e^{sA}Be^{-sA}$. It follows

$$\frac{\mathrm{d}}{\mathrm{d}s}g(s) = e^{sA}(A+B)e^{sB} = Ag(s) + e^{sA}Be^{-sA}g(s) = \big(A + g(s)B(g(s))^{-1}\big)g(s) \ . \tag{1.2}$$

Using (1.1) one has $g(s)B(g(s))^{-1} = e^{sA}Be^{-sA} = B + s[A, B] + 0$ since all higher commutators vanish, due to our assumption. Thus we have the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}g(s) = \big(A + B + s[A, B]\big)g(s) \ , \tag{1.3}$$

with obvious initial condition $g(0) = \mathbf{1}$. Since $\frac{\mathrm{d}}{\mathrm{d}s}\Big(s(A+B) + \frac{s^2}{2}[A, B]\Big) = A + B + s[A, B]$, and since $A + B + s[A, B]$ and $s(A+B) + \frac{s^2}{2}[A, B]$ commute, we have, as in 1-a that

$$\frac{\mathrm{d}}{\mathrm{d}s}\exp\Big(s(A+B) + \frac{s^2}{2}[A, B]\Big) = \big(A + B + s[A, B]\big)\exp\Big(s(A+B) + \frac{s^2}{2}[A, B]\Big) \ , \tag{1.4}$$

so this exponential solves the differential equation and satisfies the initial condition. Since the solution is unique we have $g(s) = \exp\Big(s(A+B) + \frac{s^2}{2}[A, B]\Big)$. Setting $s = 1$ gives the desired result. $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$.

2 Rotations

The infinitesimal rotation generators J_a as given in (3.38) and the matrices $\mathcal{R}_a(\alpha)$ of the finite rotations (3.36).

2-a) Consider e.g. J_z. Then

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad J_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad J_z^3 = J_z \ , \tag{2.5}$$

etc, i.e. $J_z^{2n} = J_z^2$ and $J_z^{2n-1} = J_z$, $n = 1, 2, 3, \ldots$. Then, separating the odd and even powers in the exponential series,

$$e^{i\alpha J_z} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha)^{2n-1}}{(2n-1)!} J_z + \sum_{n=1}^{\infty} \frac{(i\alpha)^{2n}}{(2n)!} J_z^2 = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathcal{R}_z(\alpha) \ . \tag{2.6}$$

For J_x and J_y one similarly has that J_x^2 and J_y^2 are diagonal with entries $(0, 1, 1)$ and $(1, 0, 1)$ and then $J_x^3 = J_x$ and $J_y^3 = J_y$. The rest of the computation proceeds in the same way.

2-b) For the product of the three matrices $\mathcal{R}_z(\alpha)\, J_y\, \mathcal{R}_z(-\alpha)$ we have

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{pmatrix}$$

$$= \cos\alpha\ J_y + \sin\alpha\ J_x \tag{2.7}$$

On the other hand, we can use (1.1) to compute

$$e^{i\alpha J_z} J_y e^{-i\alpha J_z} = J_y + i\alpha [J_z, J_y] + \frac{(i\alpha)^2}{2!} [J_z, [J_z, J_y]] + \frac{(i\alpha)^3}{3!} [J_z, [J_z, [J_z, J_y]]] + \ldots \tag{2.8}$$

Now, $[J_z, J_y] = -iJ_x$, $[J_z, [J_z, J_y]] = [J_z, -iJ_x] = i(-i)J_y = J_y$ and $[J_z, [J_z, [J_z, J_y]]] = [J_z, J_y] = -iJ_x$, etc. Then

$$e^{i\alpha J_z} J_y e^{-i\alpha J_z} = J_y + \alpha J_x + \frac{(i\alpha)^2}{2!} J_y - i \frac{(i\alpha)^3}{3!} J_x + \ldots = \cos\alpha\ J_y + \sin\alpha\ J_x \ , \tag{2.9}$$

in agreement with the previous result.

2-c) First, although not part of the expected answer, note that it must be the inverse (or transposed) matrices that appear, so that when doing two successive transformations one correctly obtains a representation. More precisely, if we define $\left(\widehat{R}(\vec{u}, \alpha) f\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = f\left(\mathcal{R}(\vec{u}, \alpha)^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$, we correctly have that the $\widehat{R}(\vec{u}, \alpha)$, as operators acting on some appropriate space of functions, form a (possibly infinite-dimensional) representation of the rotation group.

Of course, $\mathcal{R}(\vec{u}, \alpha)^{-1} = \mathcal{R}(\vec{u}, -\alpha)$, and for infinitesimal α one has

$$f\left(\mathcal{R}(\vec{e}_z, \alpha)^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = f\begin{pmatrix} x - \alpha y \\ y + \alpha x \\ z \end{pmatrix} = f\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \alpha y \frac{\partial}{\partial x} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \alpha x \frac{\partial}{\partial y} f\begin{pmatrix} x \\ y \\ z \end{pmatrix} \ , \tag{2.10}$$

so that

$$\delta_{\vec{e}_z,\alpha} f = -\alpha y \frac{\partial}{\partial x} f + \alpha x \frac{\partial}{\partial y} f = i\alpha \Big(-ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x}\Big) f \tag{2.11}$$

and we identify

$$\mathcal{L}_z = -ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} \ . \tag{2.12}$$

2-d) By cyclicity one guesses the other two differential operators as

$$\mathcal{L}_x = -iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y} \quad , \quad \mathcal{L}_y = -iz\frac{\partial}{\partial x} + ix\frac{\partial}{\partial z} \ . \tag{2.13}$$

One has

$$\begin{aligned}
\mathcal{L}_x\big(\mathcal{L}_y f\big) &= \big(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y}\big)\big(-iz\frac{\partial f}{\partial x} + ix\frac{\partial f}{\partial z}\big) \\
&= -yz\frac{\partial^2 f}{\partial z \partial x} - y\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial z^2} + z^2\frac{\partial^2 f}{\partial y \partial x} - xz\frac{\partial^2 f}{\partial y \partial z} \ , \\
\mathcal{L}_y\big(\mathcal{L}_x f\big) &= \big(-iz\frac{\partial}{\partial x} + ix\frac{\partial}{\partial z}\big)\big(-iy\frac{\partial f}{\partial z} + iz\frac{\partial f}{\partial y}\big) \\
&= -yz\frac{\partial^2 f}{\partial x \partial z} + z^2\frac{\partial^2 f}{\partial x \partial y} + xy\frac{\partial^2 f}{\partial z^2} - xz\frac{\partial^2 f}{\partial z \partial y} - x\frac{\partial f}{\partial y} \ .
\end{aligned} \tag{2.14}$$

All second-order derivatives cancel in the difference and

$$\mathcal{L}_x\big(\mathcal{L}_y f\big) - \mathcal{L}_y\big(\mathcal{L}_x f\big) = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = i\big(-ix\frac{\partial f}{\partial y} + iy\frac{\partial f}{\partial x}\big) = i\mathcal{L}_z f \ . \tag{2.15}$$

Hence,

$$[\mathcal{L}_x, \mathcal{L}_y] = i\mathcal{L}_z \ , \tag{2.16}$$

as well as the cyclically permuted relations. We see that the $\mathcal{L}_a$ satisfy the same commutator algebra as the J_a. Indeed, they are the infinitesimal generators of rotations acting on functions of the space coordinates and, hence, must satisfy the same Lie algebra commutators.

3 Time evolution of a 2-state system

3-a) The corresponding matrix is

$$\widehat{H} = \begin{pmatrix} a & d \\ c & b \end{pmatrix} \quad , \quad \widehat{H}^\dagger = \begin{pmatrix} a^* & c^* \\ d^* & b^* \end{pmatrix} \ , \tag{3.17}$$

which is hermitian if a and b are real and $d = c^*$. Hence $\widehat{H} = \begin{pmatrix} a & c_1 - ic_2 \\ c_1 + ic_2 & b \end{pmatrix}$.

3-b) One has

$$\widehat{H} = \frac{a+b}{2}\mathbf{1} + \frac{a-b}{2}\sigma_z + c_1\sigma_x + c_2\sigma_y \tag{3.18}$$

We identify $E_0 = \frac{a+b}{2}$ and $\hbar\omega = \sqrt{(a-b)^2 + 4(c_1^2 + c_2^2)} = \sqrt{(a-b)^2 + 4|c|^2}$. Furthermore, in terms of the standard parametrization of $\vec{u}$ in terms of the spherical angles θ and φ one has $\tan^2\theta = \frac{u_x^2+u_y^2}{u_z^2} = \frac{4|c|^2}{(a-b)^2}$ and $\tan\varphi = \frac{u_y}{u_x} = \frac{c_2}{c_1}$.

The eigenvectors of H are the eigenvectors of $\vec{u}\cdot\vec{\sigma}$ and are those given in the lecture notes, eq. (3.32), with the obvious replacements $|+\rangle_z \to |1\rangle$ and $|-\rangle_z \to |2\rangle$:

$$|+\rangle_{\vec{u}} = \cos\frac{\theta}{2}\, e^{-i\varphi/2}\, |1\rangle + \sin\frac{\theta}{2}\, e^{i\varphi/2}\, |2\rangle \quad , \quad |-\rangle_{\vec{u}} = -\sin\frac{\theta}{2}\, e^{-i\varphi/2}\, |1\rangle + \cos\frac{\theta}{2}\, e^{i\varphi/2}\, |2\rangle \ , \tag{3.19}$$

and the eigenvalues are

$$E_\pm = E_0 \pm \frac{\hbar\omega}{2} \ . \tag{3.20}$$

3-c) The inital measurement of X "prepares" our state to be $|1\rangle$ at $t=0$, i.e. $|\psi(0)\rangle = |1\rangle$. We can decompose this on the eigenbasis of H as

$$|\psi(0)\rangle = |+\rangle_{\vec{u}}\, \langle +|_{\vec{u}}\, |1\rangle + |-\rangle_{\vec{u}}\, \langle -|_{\vec{u}}\, |1\rangle \ . \tag{3.21}$$

Now, $\langle +|_{\vec{u}}\, |1\rangle = (\langle 1\, |+\rangle_{\vec{u}})^* = \cos\frac{\theta}{2}\, e^{i\varphi/2}$ and $\langle -|_{\vec{u}}\, |1\rangle = (\langle 1\, |-\rangle_{\vec{u}})^* = -\sin\frac{\theta}{2}\, e^{i\varphi/2}$, so that

$$|\psi(0)\rangle = e^{i\varphi/2}\Big(\cos\frac{\theta}{2}\, |+\rangle_{\vec{u}} - \sin\frac{\theta}{2}\, |-\rangle_{\vec{u}} \Big) \ . \tag{3.22}$$

Then at time t we have

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar} e^{i\varphi/2}\Big(e^{-i\omega t/2} \cos\frac{\theta}{2}\, |+\rangle_{\vec{u}} - e^{i\omega t/2} \sin\frac{\theta}{2}\, |-\rangle_{\vec{u}} \Big) \ . \tag{3.23}$$

The probability that a measurement of X at $t=T$ yields $-x_0$ is

$$\begin{aligned} P(t,2) &= |\langle 2\, |\psi(t)\rangle\, |^2 = \Big| e^{-i\omega t/2} \cos\frac{\theta}{2} \langle 2\, |+\rangle_{\vec{u}} - e^{i\omega t/2} \sin\frac{\theta}{2} \langle 2\, |-\rangle_{\vec{u}} \Big|^2 \\ &= \Big| e^{-i\omega t/2} \cos\frac{\theta}{2} \sin\frac{\theta}{2} - e^{i\omega t/2} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \Big|^2 = \frac{1}{4} \sin^2\theta \Big| e^{-i\omega t/2} - e^{i\omega t/2} \Big|^2 \\ &= \sin^2\theta\ \sin^2\frac{\omega t}{2} = \frac{1}{2} \sin^2\theta\, \big(1 - \cos\omega t\big) \ . \end{aligned} \tag{3.24}$$

It remains to express this in terms of a, b and c. One has $\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta} = \frac{4|c|^2}{(a-b)^2+4|c|^2}$, so that we can also write

$$P(t,2) = \frac{4|c|^2}{(a-b)^2 + 4|c|^2}\, \sin^2\frac{\omega t}{2} = \frac{2|c|^2}{(a-b)^2 + 4|c|^2} \big(1 - \cos\omega t\big) \ . \tag{3.25}$$

3-d) The expectation value of X at time t is

$$\begin{aligned} \langle X\rangle_{\psi(t)} &= \langle\psi(t)|\, X\, |\psi(t)\rangle = x_0\Big(\langle\psi(t)\, |1\rangle\, \langle 1\, |\psi(t)\rangle - \langle\psi(t)\, |2\rangle\, \langle 2\, |\psi(t)\rangle \Big) = x_0\Big(P(t,1) - P(t,2) \Big) \\ &= x_0\Big(1 - 2P(t,2)\Big) = x_0\Big(1 - \frac{4|c|^2}{(a-b)^2 + 4|c|^2} \big(1 - \cos\omega t\big)\Big) \ . \end{aligned} \tag{3.26}$$

We see that this expectation value oscillates between x_0 and $x_0\big(1 - \frac{8|c|^2}{(a-b)^2+4|c|^2}\big) \geq -x_0$.