**Bounded Operators & Closed Subspaces**
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November 2014[1]

## 1 Bounded operators & examples

Let $V$ and $W$ be Banach spaces. We say that a linear transformation $L : V \to W$ is *bounded* if and only if there is a constant $K$ such that $\|Lv\|_W \le K\|v\|_V$ for all $v \in V$. Equivalently, $L$ is bounded whenever

$$\|L\|_{op} := \sup_{v\neq 0} \frac{\|Lv\|_W}{\|v\|_V} \tag{1.1}$$

is finite. $\|L\|_{op}$ is called the norm of $L$. Frequently, the same operator may map another space $\widetilde{V} \to \widetilde{W}$, rather than $V \to W$. When this happens, we will need to note which spaces are involved. For instance, if $V$ and $W$ are the spaces involved, we will use the notation $\|L\|_{V\to W}$ for the operator norm. In addition to the expression given in (1.1), it is easy to show that $\|L\|_{op}$ is also given by

$$\|L\|_{op} := \min\{K > 0 \colon \|Lv\|_W \le K\|v\|_V \ \forall v \in V\}. \tag{1.2}$$

As usual, we say $L : V \to W$ is continuous at $v \in V$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Lu - Lv\|_W < \varepsilon$ whenever $\|u - v\|_V < \delta$. Of course, this is just the standard definition of continuity. Be aware that it holds whether or not $L$ is linear. When $L$ is linear, the distinction between $u, v$ becomes irrelevant, because $\|Lu - Lv\|_W = \|L(u - v)\|_W$. From this it immediately follows that $L$ will be continuous at every $v \in V$ whenever it is continuous at $v = 0$. The proposition below connects boundedness and continuity for linear transformations. The proof is left as an exercise.

**Proposition 1.** *A linear transformation $L : V \to W$ is continuous if and only if it is bounded.*

We will now provide a number of examples of bounded operators and unbounded operators.

[1] Revised October 2019

**Example 1.** Let $L : C[0,1] \to C[0,1]$ be given by $Lu(x) = \int_0^1 k(x,y)u(y)dy$, where $k \in C(R)$, $R = [0,1]\times[0,1]$. We have that $|Lu(x)| \le \int_0^1 |k(x,y)|\,|u(y)|dy$, so $|Lu(x)| \le \|k\|_{C(R)}\|u\|_{C([0,1])}$. Consequently, $\|L\|_{C\to C} \le \|k\|_{C(R)}\|u\|_{C([0,1])}$

**Example 2.** Hilbert-Schmidt operators.

**Definition 1.** Let $R = [0,1] \times [0,1]$ and let $k : R \to \mathbb{R}$. If $k \in L^2(R)$, then $k$ is called a *Hilbert-Schmidt kernel.*

**Proposition 2.** *Let $k$ be a Hilbert-Schmidt kernel. The linear operator $Lu(x) = \int_0^1 k(x,y)u(y)dy$ maps $L^2[0,1] \to L^2[0,1]$ and is bounded. Moreover, $\|L\|_{L^2\to L^2} \le \|k\|_{L^2(R)}$.*

*Proof.* Since $k(x,y) \in L^2(R)$, $\int_R |k(x,y)|^2 dxdy < \infty$, we have that $|k(x,y)|^2 \in L^1(R)$. Fubini's theorem then implies that $\int_0^1 |k(x,y)|^2 dy$ exists for almost every $x$ and, in $x$, is in $L^1[0,1]$. But this also implies that for almost every $x$, $|k(x,y)|^2$ is $L^2$ in $y$. Hence, by Schwarz's inequality,

$$|Lu(x)|^2 = \left|\int_0^1 k(x,y)u(y)dy\right|^2 \le \int_0^1 |k(x,y)|^2 dy \underbrace{\int_0^1 |u(y)|^2 dy}_{\|u\|_{L^2}^2}.$$

Integrating both sides in $x$ then yields $\|Lu\|^2_{L^2[0,1]} \le \|k\|^2_{L^2(R)}\|u\|^2_{L^2[0,1]}$, so $\|Lu\|_{L^2[0,1]} \le \|k\|_{L^2(R)}\|u\|_{L^2[0,1]}$. Then by (1.2), we see that $\|L\|_{L^2\to L^2} \le \|k\|_{L^2(R)}$, which completes the proof. □

**Example 3.** Consider $L^2[0,1]$. The differentiation operator $D = \frac{d}{dx}$ is defined on all $f \in C^1[0,1]$, which is dense in $L^2$ because it contains the set of polynomials. The question is whether $D$ is bounded, or at least can be extended to a bounded operator on $L^2$. The answer is no. Let $u_n(x) := \sqrt{2}\sin(n\pi x)$. These functions are in $C^1[0,1]$ and they satisfy $\|u_n\|_{L^2} = 1$. Since $Du_n = n\pi\sqrt{2}\cos(n\pi x)$, $\|Du_n\|_{L^2} = n\pi$. Consequently,

$$\frac{\|Du_n\|_{L^2}}{\|u_n\|_{L^2}} = n\pi \to \infty, \text{ as } n \to \infty.$$

Thus $D$ is an *unbounded* operator on $L^2[0,1]$.

The situation changes if we use a different space. Consider the Sobolev space $H^1[0,1]$, which has the inner product

$$\langle f, g\rangle_{H^1} = \int_0^1 f(x)\overline{g(x)} + f'(x)\overline{g'(x)}dx.$$

The operator $D : H^1 \to L^2$ turns out to be bounded. In fact, one can show that $\|D\|_{H^1 \to L^2} = 1$. (It's easy to show that $\|D\|_{H^1 \to L^2}$ is at *most* 1. Showing that it's exactly one requires more work.)

## 2 Closed subspaces

The usual definition of subspace holds for Banach spaces and for Hilbert spaces. Such subspaces inherit norms and/or inner products from the larger spaces. They are said to be *closed* if they contain all of their limit points.

Finite dimensional subspaces are always closed. Earlier, when we discussed completeness of an orthonormal set $U = \{u_n\}_{n=1}^\infty$ in a Hilbert space $\mathcal{H}$, we saw that the space $\mathcal{H}_U = \{f \in \mathcal{H} \colon f = \sum_n \langle f, u_n \rangle u_n\}$ is closed in $\mathcal{H}$. When $C[0,1]$ is considered to be a subspace of $L^2[0,1]$, it is not closed. However, $C[0,1]$ *is* a closed subspace of $L_\infty[0,1]$.

Given a subspace $V$ of a Hilbert space $\mathcal{H}$, we define the *orthogonal complement* of $V$ to be

$$V^\perp := \{f \in \mathcal{H} \colon \langle f, g \rangle = 0 \ \forall g \in V\}.$$

**Proposition 3.** *$V^\perp$ is a closed subspace of $\mathcal{H}$.*

*Proof.* Let $\{f_n\}_{n=1}^\infty$ be a sequence in $V^\perp$ that converges to a function $f \in \mathcal{H}$. Since each $f_n$ is in $V^\perp$, $\langle f_n, g \rangle = 0$ for every $g \in V$. Also, because the inner product is continuous, $\lim_{n\to\infty} \langle f_n, g \rangle = \langle f, g \rangle$. It immediately follows that $\langle f, g \rangle = 0$. so $f \in V^\perp$. Consequently, $V^\perp$ is closed in $\mathcal{H}$. $\square$

Bounded linear operators mapping $V \to W$, where $V$ and $W$ are Banach spaces, have all of the usual subspaces associated with them. Let $L : V \to W$ be bounded and linear. The domain of $L$ is $D(L) = V$. The range of $L$ is defined as $R(L) := \{w \in W \colon \exists v \in W \text{ for which } Lv = W\}$. Finally, the null space (or kernel) of $L$ is $N(L) := \{v \in V \colon Lv = 0\}$.

**Proposition 4.** *If $L : V \to W$ is bounded and linear, then the null space $N(L)$ is a closed subspace of $V$.*

*Proof.* The proof again relies on the continuity of $L$. If $\{f_n\}_{n=1}^\infty$ is a sequence in $N(L)$ that converges to $f \in V$. By Proposition 1, $L$ is continuous, so $\lim_{n\to\infty} Lf_n = Lf$. But, because $f_n \in N(L)$, $Lf_n = 0$. Combining this with $\lim_{n\to\infty} Lf_n = Lf$, we see that $Lf = 0$ and so $f \in N(L)$. Thus, $N(L)$ is a closed subspace of $V$. $\square$

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