Properties of the duality map

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In this work we derive properties of the duality map on a normed vector space E. In particular, we will consider spaces such that $E^\star$ is uniformly convex. The problem and the preliminary exercises are taken from *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, by Brezis, H. The two main results we are heading for are the following:

Proposition 1. *$E^\star$ is uniformely convex if and only if the duality map is uniquely defined and uniformely continuous on bounded sets,*

and

Proposition 2. *Let $\phi : E \to \mathbb{R}, x \mapsto \frac{1}{2}\|x\|^2$. If $E^\star$ is strictly convex then ϕ is Gâteaux differentiable, and if $E^\star$ is uniformely convex then ϕ is Fréchet differentiable. In both cases the derivative is the duality map.*

The two types of derivatives in proposition 2 are both natural generalizations of the "standard" derivative; the Gâteaux derivative is the generalization of the directional derivative while the Fréchet derivative is the derivative in the sense of the linear approximation of a map.

The proof of proposition 1 consists of Part B2 and Part C, while the proof of proposition 2 consists of Part A1 and part B3.

From here on, E is a normed vector space. Recall the definition of the *duality map*, here denoted by F. For $x \in E$, the duality map of x is the subset $F(x) \subset E^\star$ defined as

$$F(x) = \left\{ f \in E^\star : \|f\| = \|x\| \text{ and } \langle f, x \rangle = \|x\|^2 \right\}. \tag{1}$$

In case $F(x)$ consists of precisely one point, we will denote it as Fx.

1 Preliminary exercises

In this section, the solutions to Exercise 1.1 and parts of Exercise 1.25 are presented.

1.1 Exercise 1.1

The goal of this exercise is to show various elementary properties of F. We will find two equivalent ways of defining F (equations 2 and 3). We will also find a sufficient condition for the dual map to be uniquely defined, namely that $E^\star$ is strictly convex.

1.1.1 Part 1

Claim. *The following equality holds*

$$F(x) = \left\{ f \in E^\star : \|f\| \leq \|x\| \textit{ and } \langle f, x \rangle = \|x\|^2 \right\}. \tag{2}$$

Furthermore, $F(x)$ is nonempty, closed and convex for all $x \in E$.

Proof. Denote the right-hand side of equation 2 by M. There are two inclusions to be shown. The inclusion $F(x) \subset M$ is obvious. To show the inclusion $M \subset F(x)$, take some $f \in M$. We must show that $\|f\| = \|x\|$.

We can assume $x \neq 0$, otherwise the statement is trivial. Then by linearity $\left\langle f, \frac{1}{\|x\|}x \right\rangle = \|x\|$. But $\left\| \frac{1}{\|x\|}x \right\| = 1$, so

$$\|f\| = \sup_{\|y\| \leq 1} \langle f, y \rangle \geq \left\langle f, \frac{1}{\|x\|}x \right\rangle = \|x\|,$$

i.e. $\|f\| \geq \|x\|$. But since $f \in M$ we have that $\|f\| \leq \|x\|$. Consequently $\|f\| = \|x\|$, so $f \in F(x)$ and hence $F(x) = M$.

Now, $F(x)$ is nonempty as an immediate consequence of the Hahn-Banach theorem. To show that $F(x)$ is closed, observe that $F(x) = A \cap B$, where

$$\begin{aligned} A &= \{f \in E^\star : \|f\| = \|x\|\} \\ B &= \left\{f \in E^\star : \langle f, x\rangle = \|x\|^2\right\}. \end{aligned}$$

Both mappings $\| \ . \ \| : E^\star \to \mathbb{R}$ and $\langle \ . \ , x\rangle : E^\star \to \mathbb{R}$ are continuous, and both A and B are preimages of a point (which is a closed set in $\mathbb{R}$), hence A and B are closed. It follows that $F(x)$ is closed. To show that $F(x)$ is convex, pick $f, g \in F(x)$ and set $h = tf + (1-t)g$ for $0 \leq t \leq 1$. Then

$$\|h\| \leq t\|f\| + (1-t)\|g\| = t\|x\| + (1-t)\|x\| = \|x\|$$

and

$$\langle h, x\rangle = t\langle f, x\rangle + (1-t)\langle g, x\rangle = t\|x\|^2 + (1-t)\|x\|^2 = \|x\|^2,$$

so $h \in M$. But by the first part, $M = F(x)$, so $h \in F(x)$. Hence $F(x)$ is convex. □

1.1.2 Part 2

Claim. *If $E^\star$ is strictly convex, then $F(x)$ contains a single point.*

Proof. Recall the definition of strict convexity: If $f, g \in E^\star$ such that $f \neq g$ and $\|f\| = \|g\| = 1$, then for $0 < t < 1$ we have $\|tf + (1-t)g\| < 1$.

Now, pick two $f, g \in F(x)$ and assume $f \neq g$. Since $F(x)$ is convex, also $h = tf + (1-t)g$ is contained in $F(x)$ for all $0 < t < 1$. Hence $\|f\| = \|g\| = \|x\|$ and $\|tf + (1-t)g\| = \|x\|$. This contradicts the strict convexity, so the assumption was incorrect. Hence $f = g$, i.e. any two points in $F(x)$ are equal. Because $F(x)$ is nonempty, $F(x)$ consists of precisely one point. □

1.1.3 Part 3

Claim. *The following equality holds*

$$F(x) = \left\{f \in E^\star : \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \geq \langle f, y - x\rangle \ \forall y \in E\right\}. \tag{3}$$

Proof. Denote the right-hand side by N. Then we need to show the two inclusions $N \subset F(x)$ and $F(x) \subset N$. For the first inclusion, pick $f \in N$, and let $y = tx$ for $t \in \mathbb{R}$. Then

$$(t-1)\langle f, x\rangle \leq \frac{1}{2}(t^2 - 1)\|x\|^2 \qquad \forall t \in \mathbb{R}.$$

Rewriting we find that

$$(t-1)\left(\frac{t+1}{2}\|x\|^2 - \langle f, x, \rangle\right) \geq 0. \tag{4}$$

The left-hand side is a parabola with one zero $t = 1$, so for the parabola to be non-negative $t = 1$ must be a double zero. From this we find that $\|x\|^2 = \langle f, x\rangle$. Plugging this into the defining property of N we find that

$$\langle f, y\rangle \leq \frac{1}{2}\|y\|^2 + \frac{1}{2}\|x\|^2 \qquad \forall y \in E$$

Hence for all y with $\|y\| = \|x\|$, i.e. for all y on the sphere of radius $\|x\|$ we have that

$$\langle f, y\rangle \leq \|x\|^2.$$

Rescaling y it follows that $\|f\| \leq \|x\|$. The conclusion is that $\langle f, x\rangle = \|x\|^2$ and that $\|f\| \leq \|x\|$. Hence $f \in F(x)$.

To show the second inclusion, pick $f \in F(x)$. Observe that for any real numbers a, b we have $2ab \leq a^2 + b^2$. Consequently,

$$\begin{aligned} \langle f, y - x\rangle &= \langle f, y\rangle - \langle f, x\rangle \\ &\leq \|f\|\|y\| - \|x\|^2 \\ &= \|x\|\|y\| - \|x\|^2 \\ &\leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2. \end{aligned}$$

Hence $f \in N$. □

1.1.4 Part 4

Claim. *For all $x, y \in E$ and all $f \in F(x), g \in F(y)$, the following holds*

$$\langle f - g, x - y \rangle \geq (\|x\| - \|y\|)^2 . \tag{5}$$

Proof. We estimate the left-hand side:

$$\begin{aligned}\langle f - g, x - y \rangle &= \langle f, x \rangle + \langle g, y \rangle - \langle f, y \rangle - \langle g, x \rangle \\ &\geq \|x\|^2 + \|y\|^2 - \|f\|\|x\| - \|g\|\|y\| \\ &= \|x\|^2 + \|y\|^2 - \|y\|\|x\| - \|x\|\|y\| \\ &= (\|x\| - \|y\|)^2 .\end{aligned}$$

The claim follows. □

1.1.5 Part 5

Claim. *Assume $E^\star$ is strictly convex. Then $\langle Fx - Fy, x - y \rangle = 0$ implies $Fx = Fy$.*

Proof. Observe first that $\|x\| = \|y\|$ by part 4. Furthermore, $\langle Fx, y \rangle \leq \|x\|\|y\| = \|y\|^2$ and $\langle Fy, x \rangle \leq \|x\|^2$. Thus

$$\begin{aligned}\langle Fx - Fy, x - y \rangle &= \langle Fx, x \rangle + \langle Fy, y \rangle - \langle Fx, y \rangle - \langle Fy, x \rangle \\ &\geq \|x\|^2 + \|y\|^2 - \|y\|^2 - \|x\|^2 \\ &= 0.\end{aligned}$$

Hence the inequalities must actually be equalities, so $\langle Fx, y \rangle = \|y\|^2$ and $\langle Fy, x \rangle = \|x\|^2$. The conclusion is that Fx is a dual element of y and Fy is a dual element of x. According to Exersice 1.1.2, these are unique. It follows that $Fx = Fy$. □

1.2 Exercise 1.25

The goal of this exercise is to find an expression for the dual map in terms of a one-sided limit of a Newton quotient of the norm function. This will be an important ingredient in showing that the norm is Gâteaux differentiable. Parts of this exercise are not relevant for the discussion, and are hence omitted.

We define the semiscalar product, for $x, y \in E$,

$$[x, y] = \inf_{t>0} \frac{1}{2t} \left(\|x + ty\|^2 - \|x\|^2 \right) . \tag{6}$$

By Exercise 1.25.1 below, $[x, y]$ will always exist in $[-\infty, \infty)$ and

$$[x, y] = \lim_{t \to 0^+} \frac{1}{2t} \left(\|x + ty\|^2 - \|x\|^2 \right) . \tag{7}$$

1.2.1 Part 1

Claim. *Let $\phi : E \to \mathbb{R}$ be any convex function, and define, for fixed $x, y \in E$ the map h,*

$$h : (0, \infty) \to \mathbb{R}, \qquad h(t) = \frac{\phi(x + ty) - \phi(x)}{t} \tag{8}$$

Then h is non-decreasing.

Proof. Pick $0 < t_1 < t_2$. Then, using the fact that ϕ is convex, we find

$$\begin{aligned}h(t_2) - h(t_1) &= \frac{\phi(x + t_2 y) - \phi(x)}{t_2} - \frac{\phi(x + t_1 y) - \phi(x)}{t_1} \\ &= \frac{1}{t_2} \left(\frac{t_2}{t_1} \phi(x + t_1 y) + \frac{t_1 - t_2}{t_1} \phi(x) \right) - \frac{1}{t_2} \phi(x + t_2 y) \\ &\geq \frac{1}{t_2} \phi(x + t_2 y) - \frac{1}{t_2} \phi(x + t_2 y) \\ &= 0.\end{aligned}$$

The conclusion is that if $t_1 < t_2$ then $h(t_1) \leq h(t_2)$, so h is non-decreasing. □

1.2.2 Part 5

Claim. *For every $x, y \in E$,*

$$[x,y] = \max_{f \in F(x)} \langle f, y \rangle.$$

Proof. Set $\alpha = [x,y]$. We begin by observing that for any $f \in F(x)$, Exercise 1.1.3 implies that

$$t\langle f, y \rangle = \langle f, ty \rangle \leq \frac{1}{2}\left(\|x+ty\|^2 - \|x\|^2\right).$$

It follows that $\langle f, y \rangle \leq \alpha$ for all $f \in F(x)$. We will now show that there is some $f_0 \in E$ such that $\langle f_0, y \rangle = \alpha$. The claim then follows.

We will apply Theorem 1.12, the Fenchel-Rockafellar theorem. Define $\phi : E \to \mathbb{R}$ and $\psi : E \to (-\infty, +\infty]$ as

$$\begin{aligned}
\phi(z) &= \frac{1}{2}\left(\|x+z\|^2 - \|x\|^2\right) \\
\psi(z) &= \begin{cases} -\alpha t, & \text{if } z = ty,\ t > 0 \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Both ϕ and ψ are convex, and ϕ is continuous, so these satisfy the hypotheses of the Fenchel-Rockafellar theorem. We compute the corresponding conjugate functions. Estimating using the dual norm and the maximum of a quadratic function we obtain

$$\begin{aligned}
\phi^\star(f) &= \sup_{z \in E}\left\{\langle f, z \rangle - \phi(z)\right\} \\
&= \sup_{z \in E}\left\{\langle f, z \rangle - \frac{1}{2}\left(\|x+z\|^2 - \|x\|^2\right)\right\} \\
&= \sup_{z \in E}\left\{\langle f, z+x \rangle - \frac{1}{2}\|x+z\|^2\right\} - \langle f, x \rangle + \frac{1}{2}\|x\|^2 \\
&= \sup_{z \in E}\left\{\langle f, z \rangle - \frac{1}{2}\|z\|^2\right\} - \langle f, x \rangle + \frac{1}{2}\|x\|^2 \\
&= \sup_{z \in E}\left\{\|f\|\|z\| - \frac{1}{2}\|z\|^2\right\} - \langle f, x \rangle + \frac{1}{2}\|x\|^2 \\
&= \frac{1}{2}\|f\|^2 - \langle f, x \rangle + \frac{1}{2}\|x\|^2
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\psi^\star(f) &= \sup_{z \in E}\left\{\langle f, z \rangle - \psi(z)\right\} \\
&= \sup_{t > 0}\left\{\langle f, ty \rangle + t\alpha\right\} \\
&= \begin{cases} 0 & \text{if } \langle f, y \rangle + \alpha \leq 0 \\ \infty & \text{otherwise} \end{cases}
\end{aligned}$$

Now, the Fenchel-Rockafellar theorem states that

$$\inf_{z \in E}\left\{\phi(z) + \psi(z)\right\} = \max_{f \in E^\star}\left\{-\phi^\star(-f) - \psi^\star(f)\right\}.$$

The left-hand side can easily be computed:

$$\begin{aligned}
\inf_{z \in E}\left\{\phi(z) + \psi(z)\right\} &= \inf_{t > 0}\left\{\frac{1}{2}\left(\|x+ty\|^2 - \|x\|^2\right) - \alpha t\right\} \\
&= \inf_{t > 0}\left\{t\underbrace{\left(\frac{1}{2t}\left(\|x+ty\|^2 - \|x\|^2\right) - \alpha\right)}_{0 \leq \quad < \infty}\right\} \\
&= 0
\end{aligned}$$

Plugging in we find

$$
\begin{aligned}
0 = \max_{f\in E^\star}\big\{-\phi^\star(-f)-\psi^\star(f)\big\} &= \max_{f\in E^\star}\big\{-\phi^\star(f)-\psi^\star(-f)\big\}\\
&= \max_{-\langle f,y\rangle+\alpha\le 0}\left\{-\left(\frac{1}{2}\|f\|^2-\langle f,x\rangle+\frac{1}{2}\|x\|^2\right)\right\}.
\end{aligned}
$$

This max is attained, say by $f_0\in E^\star$. Then

$$
0=\|f_0\|^2-2\langle f_0,x\rangle+\|x\|^2\ge\|f_0\|^2-2\|f_0\|\|x\|+\|x\|^2=(\|f_0\|-\|x\|)^2
$$

i.e. $\|f_0\|=\|x\|$ and $\langle f_0,x\rangle=\|x\|^2$. Hence $f_0\in F(x)$. For this f_0 we also have $\langle f_0,y\rangle\ge\alpha$. But since $f_0\in F(x)$ we showed in the beginning of the proof that $\langle f_0,y\rangle\le\alpha$. Hence $\langle f_0,y\rangle=\alpha$. The conclusion is that

$$
[x,y]=\alpha=\max_{f\in F(x)}\langle f,y\rangle.
$$

□

2 Part A

In this part we show various properties of the dual map in case $E^\star$ is strictly convex.

2.1 Problem A1

Claim. *The map $x\mapsto\frac{1}{2}\|x\|^2$ is Gâteaux differentiable with derivative F, i.e. for every $x,y\in E$,*

$$
\lim_{\lambda\to 0}\frac{1}{2\lambda}\left(\|x+\lambda y\|^2-\|x\|^2\right)=\langle Fx,y\rangle. \tag{9}
$$

Proof. Because $E^\star$ is reflexive, Exercise 1.1.2 shows that the dual element is uniquely defined. By Exercise 1.25.5, it follows that

$$
[x,y]=\langle Fx,y\rangle
$$

We know by equation 7 that

$$
\langle Fx,y\rangle=[x,y]=\lim_{\lambda\to 0+}\frac{1}{2\lambda}\left(\|x+\lambda y\|^2-\|x\|\right).
$$

To show equality for the other side-limit, let $\lambda\to 0-$ and set $\tau=-\lambda$. Then

$$
\begin{aligned}
\lim_{\lambda\to 0-}\frac{1}{2\lambda}\left(\|x+\lambda y\|^2-\|x\|\right) &= -\lim_{\tau\to 0+}\frac{1}{2\tau}\left(\|x-\tau y\|^2-\|x\|\right)\\
&= -[x,-y]\\
&= -\langle Fx,-y\rangle\\
&= \langle Fx,y\rangle
\end{aligned}
$$

Combining the two side-limits, we obtain the desired result. □

2.2 Problem A2

Claim. *For any fixed $x,y\in E$, the map $\Psi:\mathbb{R}\to\mathbb{R}, t\mapsto\langle F(x+ty),y\rangle$ is continuous at $t=0$.*

Proof. First observe that $\Psi(0)=\langle Fx,y\rangle$. Next, by Exercise 1.1.3 we know that $\frac{1}{2}(\|v\|^2-\|u\|^2)\ge\langle Fu,v-u\rangle$ for all $u,v\in E$. Specifically, for any $\lambda,t\in\mathbb{R}$ set $u=x+ty$ and $v=x+\lambda y$. Then

$$
\langle F(x+ty),(\lambda-t)y\rangle\le\frac{1}{2}(\|x+\lambda y\|^2-\|x+ty\|^2).
$$

Taking the limit as $t\to 0$ gives

$$
\lambda\lim_{t\to 0}\langle F(x+ty),y\rangle\le\frac{1}{2}(\|x+\lambda y\|^2-\|x\|^2).
$$

This inequality holds true for both positive and negative λ. For $\lambda > 0$ we have

$$\lim_{t\to 0}\langle F(x+ty), y\rangle \le \frac{1}{2\lambda}(\|x+\lambda y\|^2 - \|x\|^2) \xrightarrow{\lambda\to 0+} \langle Fx, y\rangle,$$

and for $\lambda < 0$ we have

$$\lim_{t\to 0}\langle F(x+ty), y\rangle \ge \frac{1}{2\lambda}(\|x+\lambda y\|^2 - \|x\|^2) \xrightarrow{\lambda\to 0-} \langle Fx, y\rangle,$$

The conclusion is that $\lim_{t\to 0}\langle F(x+ty), y\rangle = \langle Fx, y\rangle$, i.e. Ψ is continuous at $t = 0$. □

2.3 Problem A3

Claim. $F : E \to E^\star$ *is continuous from* E *with the strong topology to* $E^\star$ *with the weak** *topology.*

Proof. We show this using two methods, one slightly less general than the other.

The general case follows directly from Exercise 3.11. This exercise shows that a monotone map satisfying the conclusion of A2 is continuous from E strong to $E^\star$ weak*. Observe that F is a monotone map, by Exercise 1.1.4.

For a different method, when E is separable or reflexive, it is enough to show that if $x_n \to x$ strongly in E then $Fx_n \rightharpoonup Fx$ weakly in $\sigma(E^\star, E)$. In this case, any bounded sequence in $E^\star$ will have a weak*-convergent subsequence. Clearly $\|Fx_n\|$ is bounded (because $\|Fx_n\| = \|x_n\|$), so there is a weakly convergent subsequence $Fx_{n_k} \rightharpoonup f$, where $f \in E^\star$ is the limit. We claim that $f = Fx$. Indeed, using Proposition 3.13 part (4) we find that

$$\|x_{n_k}\|^2 = \langle Fx_{n_k}, x_{n_k}\rangle \xrightarrow{\text{Prop 2.13}} \langle f, x\rangle = \|x\|^2.$$

Furthermore, by Proposition 3.13 part (3) we have that $\|f\| \le \liminf_{k\to\infty} \|Fx_{n_k}\| = \|x\|$. The conclusion is that $\langle f, x\rangle = \|x\|^2$ and $\|f\| \le \|x\|$, so by Exercise 1.1.1 $f = Fx$. It remains to show that the original sequence Fx_n also converge weakly to Fx. For this end, suppose not. Then there is a point $y \in E$ such that $\langle Fx_n, y\rangle \nrightarrow \langle Fx, y\rangle$, i.e. for some ε there is a subsequence Fx_{n_j} such that $|\langle Fx_{n_j}, y\rangle| > \varepsilon$ for all j. But clearly $x_{n_j} \to x$ strongly, so the above discussion shows that Fx_{n_j} has subsequence which converges weakly to Fx. This contradicts the fact that $|\langle Fx_{n_j}, y\rangle| > \varepsilon$ for all j. The conclusion is that $Fx_n \rightharpoonup Fx$, so F is continuous from E strong to $E^\star$ weak*. □

2.4 Problem A4

Claim. *For all* $x, y \in E$ *with* $\|x\| = \|y\| = 1$*, the following holds:*

$$\|Fx + Fy\| + \|x - y\| \ge 2. \tag{10}$$

Proof. We begin by proving that for all $x, y \in E$ the equality

$$\langle Fx + Fy, x + y\rangle + \langle Fx - Fy, x - y\rangle = 2(\|x\| + \|y\|) \tag{11}$$

holds. Expand the left-hand side:

$$\begin{gathered}\langle Fx + Fy, x + y\rangle + \langle Fx - Fy, x - y\rangle = \\ = \big(\langle Fx, x\rangle + \langle Fx, y\rangle + \langle Fy, x\rangle + \langle Fy, y\rangle\big) + \big(\langle Fx, x\rangle - \langle Fx, y\rangle - \langle Fy, x\rangle + \langle Fy, y\rangle\big)\end{gathered}$$

Canceling factors and using $\langle Fu, u\rangle = \|u\|^2$ for any $u \in E$ we obtain equation 11. Now, choose $x, y \in E$ with $\|x\| = \|y\| = 1$. Then, using equation 11 and the triangle inequality,

$$\begin{aligned}4 &= \langle Fx + Fy, x + y\rangle + \langle Fx - Fy, x - y\rangle \\ &\le \|Fx + Fy\|\|x + y\| + \|Fx - Fy\|\|x - y\| \\ &\le 2\|Fx + Fy\| + 2\|x - y\|.\end{aligned}$$

The claim follows. □

2.5 Problem A5

Claim. *Assume, in addition, that E is reflexive and strictly convex. Then F is bijective, with the inverse being the dual map on $E^\star$.*

Proof. Let $G : E^\star \to E^{\star\star} = E$ be the dual map on $E^\star$. It is enough to show that G is indeed the inverse of F, bijectivity then follows. For this we must show that for any $x \in E$ and $f \in E^\star$ we have $GFx = x$ and $FGf = f$.

Observe first that the dual element Gf of an element $f \in E^\star$ is characterized by $\langle f, Gf\rangle = \|f\|^2$ and $\|Gf\| = \|f\|$ (this is clear from the way we identify $E = E^{\star\star}$). Now, $\langle Fx, x\rangle = \|Fx\|^2$ and $\|x\| = \|Fx\|$ so x is the dual element of Fx, i.e. $x = GFx$. Furthermore, $\langle f, Gf\rangle = \|Gf\|^2$ and $\|f\| = \|Gf\|$, so f is the dual element of Gf, i.e. $f = FGf$. The conclusion is that $G = F^{-1}$. □

3 Part B

In this part, we further assume that $E^\star$ is uniformly convex. The goals will be to show that F is uniformly continuous on bounded sets, and that the norm-function is Fréchet-differentiable with derivative F.

3.1 Problem B1

Claim. *$F : E \to E^\star$ is continuous in the strong topologies.*

Proof. Pick a sequence $x_n \to x$ strongly in E. We need to show that $Fx_n \to Fx$ strongly in $E^\star$, and do so by applying Proposition 3.32. By Problem A3 we know that $Fx_n \rightharpoonup Fx$ weakly in $\sigma(E^\star, E)$. Since $E^\star$ is uniformly convex, $E^\star$ is reflexive and hence also E is reflexive. Consequently, $Fx_n \rightharpoonup Fx$ weakly in $\sigma(E^\star, E^{\star\star})$. Furthermore, $\limsup_{n\to\infty} \|Fx_n\| = \|x\| = \|Fx\|$, so all hypotheses of the proposition are valid. The conclusion is that $Fx_n \to Fx$ strongly. It follows that F is continuous in the strong topologies. □

3.2 Problem B2

Claim. *F is uniformly continuous on bounded sets of E.*

Proof. To reach a contradiction, we assume not. Let $B \subset E$ be a bounded set. Then there is an $\varepsilon > 0$ and two sequences $\{x_n\}, \{y_n\} \subset B$ such that $\|x_n - y_n\| \to 0$ as $n \to \infty$ but

$$\|Fx_n - Fy_n\| > \varepsilon \text{ for all } n. \tag{12}$$

Observe that $\|x_n\|$ is a bounded sequence in $\mathbb{R}$, so we can find a convergent subsequence $\|x_{n_k}\|$, converging to some limit l. Clearly, $\|y_{n_k}\|$ converges to the same limit l. We can assume $l \neq 0$, because if not, then the sequences x_{n_k}, y_{n_k} converges to 0 in E, so by continuity Fx_{n_k} and Fy_{n_k} converges to 0 in $E^\star$. This contradicts equation 12.

Now, set $a_k = \frac{x_{n_k}}{\|x_{n_k}\|}$ and $b_k = \frac{y_{n_k}}{\|y_{n_k}\|}$. Because $\|x_{n_k}\|$ and $\|y_{n_k}\|$ converge to the same limit, $\|a_k - b_k\| \to 0$ as $k \to \infty$. Observe that $Fa_k = \frac{Fx_{n_k}}{\|x_{n_k}\|}$ (and similarly for b). Then it follows from equation 12 that for some $\tilde{\varepsilon}$, $\|Fa_k - Fb_k\| > \tilde{\varepsilon}$ for all k large enough. Now, if we apply Problem A4 (recall that $\|a_k\| = \|b_k\| = 1$), we have

$$\left\| \frac{Fa_k + Fb_k}{2} \right\| \geq 1 - \frac{\|a_k - b_k\|}{2}.$$

The conclusion is that there is an $\tilde{\varepsilon}$ such that regardless of δ, we can find a k such that $\left\|\frac{Fa_k + Fb_k}{2}\right\| \geq 1 - \delta$. This contradicts the fact that $E^\star$ is uniformly convex. We have thus reached a contradiction, which shows that F is uniformly continuous on bounded subsets of E. □

3.3 Problem B3

Claim. *The function $\phi(x) = \frac{1}{2}\|x\|^2$ is Fréchet differentiable with derivative F, i.e. for every $x \in E$,*

$$\lim_{x \to x_0} \frac{\phi(x) - \phi(x_0) - \langle Fx_0, x - x_0\rangle}{\|x - x_0\|} = 0 \tag{13}$$

Proof. We know by equation 3 that $\langle Fx_0, x - x_0\rangle \leq \phi(x) - \phi(x_0)$ and $\langle Fx, x_0 - x\rangle \leq \phi(x_0) - \phi(x)$. Rearranging it follows that for all $x \in E$

$$0 \leq \phi(x) - \phi(x_0) - \langle Fx_0, x - x_0\rangle \leq \langle Fx - Fx_0, x - x_0\rangle \leq \|Fx - Fx_0\|\|x - x_0\|$$

Hence for $x \neq x_0$ we have

$$0 \leq \frac{\phi(x) - \phi(x_0) - \langle Fx_0, x - x_0\rangle}{\|x - x_0\|} \leq \|Fx - Fx_0\|.$$

Now, for a given ε, because F is continuous we can find a δ such that $\|x - x_0\| < \delta$ implies $\|Fx - Fx_0\| < \varepsilon$. Then $0 \leq \frac{\phi(x)-\phi(x_0)-\langle Fx_0,x-x_0\rangle}{\|x-x_0\|} < \varepsilon$ which shows the claim.

□

4 Part C

The aim of this part is to show the converse of Part B.

4.1 Problem C

Claim. *If F is uniquely defined, and uniformly continuous on bounded sets of E then $E^\star$ is uniformly convex.*

Proof. We begin by establishing the following inequality. For all $f, g \in E^\star$ and all $y \in E$ we have

$$\|f + g\| \leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2 - \langle f - g, y\rangle + \sup_{\substack{x\in E\\ \|x\|\leq 1}} \big\{\phi(x + y) + \phi(x - y)\big\}.$$

Indeed, we have:

$$\begin{aligned}
\|f + g\| &= \sup_{\substack{x\in E\\ \|x\|\leq 1}} \big\{\langle f + g, x\rangle\big\} \\
&= \sup_{\substack{x\in E\\ \|x\|\leq 1}} \big\{\langle f, x + y\rangle + \langle g, x - y\rangle - \langle f - g, y\rangle\big\} \\
&\leq \sup_{\substack{x\in E\\ \|x\|\leq 1}} \big\{\|f\|\|x + y\| + \|g\|\|x - y\|\big\} - \langle f - g, y\rangle \\
&\leq \sup_{\substack{x\in E\\ \|x\|\leq 1}} \left\{\frac{1}{2}\|f\|^2 + \frac{1}{2}\|x + y\|^2 + \frac{1}{2}\|g\|^2 + \frac{1}{2}\|x - y\|^2\right\} - \langle f - g, y\rangle \\
&= \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2 - \langle f - g, y\rangle + \sup_{\substack{x\in E\\ \|x\|\leq 1}} \big\{\phi(x + y) + \phi(x - y)\big\}.
\end{aligned}$$

Now, we are ready to show that $E^\star$ is uniformly convex. Choose some $\varepsilon > 0$, pick $f, g \in E^\star$ such that $\|f\| \leq 1$, $\|g\| \leq 1$ and $\|f - g\| > \varepsilon$. Then

$$\|f + g\| \leq 1 - \langle f - g, y\rangle + \sup_{\substack{x\in E\\ \|x\|\leq 1}} \big\{\phi(x + y) + \phi(x - y)\big\}. \tag{$\star$}$$

We wish to approximate the term $\phi(x + y) + \phi(x - y)$ in equation ($\star$), and do so using the computations from B3. By uniform continuity of F, we can find a δ such that $\|F(x + y) - Fx\| < \frac{\varepsilon}{4}$ for all x with $\|x\| \leq 1$ and all y with $\|y\| < \delta$. Now, from B3 (setting x and x_0 from B3 as $x + y$ and x, respectively) we have

$$\phi(x + y) - \phi(x) - \langle Fx, y\rangle \leq \delta\frac{\varepsilon}{4}$$

Similarly we find, setting x from B3 as $x - y$ instead, that

$$\phi(x - y) - \phi(x) + \langle Fx, y\rangle \leq \delta\frac{\varepsilon}{4}.$$

We conclude that

$$\phi(x + y) + \phi(x - y) \leq \delta\frac{\varepsilon}{2} + 2\phi(x),$$

so that

$$\sup_{\substack{x \in E \\ \|x\| \leq 1}} \left\{ \phi(x+y) + \phi(x-y) \right\} \leq \delta \frac{\varepsilon}{2} + 1$$

Now, to approximate the term $\langle f - g, y \rangle$ in equation $(\star)$ we can choose $y \in E$ with $\|y\| < \delta$ such that $\langle f - g, y \rangle > \|f - g\| \delta - \frac{\varepsilon}{4} \delta > \varepsilon \delta - \frac{\varepsilon}{4} \delta$. (Intuitively, here we are saying that the direction of y can be chosen to make the value of $f - g$ arbitrarily close to its operator norm.) Combining everything into equation $(\star)$ we obtain

$$\|f + g\| < 2 - \delta \varepsilon \left(1 - \frac{1}{2} - \frac{1}{4} \right),$$

i.e. for $\tilde{\delta} = \frac{\delta \varepsilon}{8}$ we have

$$\left\| \frac{f+g}{2} \right\| < 1 - \tilde{\delta}.$$

This shows that $E^\star$ is uniformly convex. $\square$