# How Hard Are Verifiable Delay Functions?

Souvik Sur

Indian Institute of Technology Kharagpur, Kharagpur, West Bengal, India souviksur@iitkgp.ac.in

**Abstract.** Verifiable delay functions (VDF) are functions that enable a verifier to verify if a prover has spent a specified number of sequential steps to execute the function. VDFs are useful in several applications ranging from non-interactive time-stamping to randomness beacons. A close resemble of VDFs are interactive proofs, however have the following difference. VDFs stand sequential against a prover using even poly(T) parallelism to evaluate the function for T sequential steps. We know that the class of all interactive proofs  $\mathbf{IP} = \mathbf{PSPACE}$ . Then it seems natural to ask this question that how hard are VDFs? Equivalently does this sequentiality against parallel provers add more power to a Turing machine deciding larger class of languages than  $\mathbf{IP}$ ?

In this paper, we show that the class of all the VDFs,  $\mathbf{VDF} \nsubseteq \mathbf{IP}$ . In particular, we construct a VDF from an  $\mathbf{EXP}$ -complete language and reduce the  $\mathbf{EXP}$ -complete language to the derived VDF. Thus if  $\mathbf{VDF} \subseteq \mathbf{PSPACE} = \mathbf{IP}$  then  $\mathbf{EXP} \subseteq \mathbf{PSPACE} = \mathbf{IP}$  which has no proof yet. So  $\mathbf{VDF} \nsubseteq \mathbf{IP}$ .

**Keywords:** Verifiable delay functions  $\cdot$  Sequentiality  $\cdot$  Turing machine  $\cdot$  Space-time hierarchy

## 1 Introduction

In 1992, Dwork and Naor [3] introduced the very first notion of VDF under a different nomenclature "pricing function". It is a computationally hard puzzle that needs to be solved to send a mail, whereas the solution of the puzzle can be verified efficiently. Later, the concept of verifiable delay functions was formalized in [2]. A verifiable delay function is a function that takes a specified number of sequential steps T to be evaluated (even using poly(T) parallelism) and can be verified efficiently (even without parallelism) and publicly.

At a first glance VDFs looks like another well-known class of languages interactive proofs. However the key difference is that VDFs must achieve sequentiality even if poly(T) parallelism is used by the prover.

Then an interesting question would be does this sequentiality add more power to a Turing machine that decides the VDFs than that decides interactive proofs? More precisely we investigate if  $\mathbf{VDF} \subseteq \mathbf{IP}$  where  $\mathbf{VDF}$  and  $\mathbf{IP}$  are the classes of all the VDFs and interactive proofs respectively?

#### 1.1 Our Contribution

We show that there exists a VDF  $\notin$  **PSPACE** thus **VDF**  $\nsubseteq$  **PSPACE**. In particular, we derive the VDF from an **EXP**-complete language EXPHALT and prove that EXPHALT is polynomial time reducible to the derived VDF. Therefore, if the derived VDF  $\in$  **PSPACE** then EXPHALT  $\in$  **PSPACE** thus **EXP**  $\subseteq$  **PSPACE** which has not been proved yet. Thus **VDF**  $\not\subset$  **PSPACE** = **IP**.

The language EXPHALT decides if a deterministic Turing machine M halts on input x in time T. It is an **EXP**-complete language as any language  $\mathbf{L} \in \mathbf{EXP}$  can be reduced to EXPHALT in polynomial-time. For any  $x \in \mathbf{L}$ , the reduction  $\mathbf{L} \leq_p \mathbf{EXPHALT}$  is nothing but  $f(x) = (M, x, 2^{\varnothing(|x|)})$ . Moreover, EXPHALT is a inherently sequential language in a sense that EXPHALT can not be parallelized. If it could be then all the languages in  $\mathbf{EXP}$  could be parallelized by the definition of completeness of a language for a class. But it is known that  $\mathbf{EXP} \neq \mathbf{NC}$  the class of parallelizable languages.

Given a security parameter  $\lambda$  and the time parameter T, VDFs consist of three algorithms,

- 1. Setup generates the public parameters **pp**.
- 2. Eval sequentially maps an input statement x to an output  $\phi$  in time T. Also it gives a proof  $\pi$ , if required, using randomness.
- 3. Verify checks if  $\phi$  is the correct mapping of x using the proof  $\pi$ , if any.

The fundamental idea of deriving VDFs from EXPHALT is to choose a Turing machine M and a random oracle H as the public parameter  $\mathbf{pp}$ . Then Eval computes the state  $q_T$  of M after T number of transitions starting from its initial state. The proof  $\pi$  is the state  $q_{T-t}$  of M after T-t transitions from the initial state. Here t=H(x||y). The Verify checks if M reaches at  $q_T$  in time t starting from  $q_{T-t}$ . We show that the derived VDF is correct, sound and sequential.

#### 2 Related Work

In this section, we mention some well-known schemes qualified as VDFs. The pricing function by Dwork–Naor scheme [3] asks a prover, given a prime  $p \equiv 3 \pmod{4}$  and a quadratic residue  $x \pmod{p}$ , to find a y such

that  $y^2 \equiv x \pmod{p}$ . The prover has no other choice other than using the identity  $y \equiv x^{\frac{(p+1)}{4}} \pmod{p}$ , but the verifier verifies the correctness using  $y^2 \equiv x \pmod{p}$ . Evidently, it is difficult to generate difficult instances of this VDF without using larger primes p. Further the massive parallelism with the prover violates its sequentiality.

In 2018, Boneh et al. [2] propose a VDF based on injective rational maps of degree T, where the fastest possible inversion is to compute the polynomial GCD of degree-T polynomials. They conjecture that it achieves  $(t^2, \emptyset(t))$  sequentiality using permutation polynomials as the candidate map. However, it is a weak form of VDF as it needs  $\emptyset(T)$  processors for the inversion in parallel time T.

Rivest, Shamir, and Wagner [7] introduced an another discipline of VDFs known as time-lock puzzle. These puzzles enables an encryption that can be decrypted only sequentially. Starting with N=pq such that p,q are large primes, the key y is enumerated as  $y\equiv x^{2^T}\pmod{N}$ . Then the verifier, uses the value of  $\phi(N)$  to reduce the exponent to  $e\equiv 2^T\pmod{\phi(N)}$  and finds out  $y\equiv x^e\pmod{N}$ . On the contrary, without the knowledge of  $\phi(N)$ , the only option available to the prover is to raise x to the power  $2^T$  sequentially. As the verification stands upon a secret, the knowledge of  $\phi(N)$ , it is not a VDF as verification should depend only on public parameters.

Wesolowski [8] and Pietrzak [6] give two different VDFs based on the time-lock puzzle [7]. The first one asks the prover to compute  $y = x^{2^T}$  and  $w = x^{\lfloor 2^T/l \rfloor}$ , where l is a prime chosen by the verifier. The verifier checks whether  $y = w^l x^{(2^T \mod l)}$ . Two candidate groups suits well in this scheme – an RSA group  $(\mathbb{Z}/N\mathbb{Z})^*$ , and the class group of an imaginary quadratic number field. On the other hand, Pietrzak's VDF asks the prover to compute  $2^u$  elements  $x^{2^{iT/2^u}}$  for  $i = 0, 1, 2, \ldots, 2^u - 1$ , where  $u = \frac{1}{2} \log_2 T$ . The verifier checks them in  $O(\log T)$  time, whereas the prover needs  $O(\sqrt{T})$  time to generate them. It uses the RSA group and the class groups of imaginary quadratic number fields.

Feo et al. [4] presents two VDFs based on isogenies of super-singular elliptic curves. They start with five groups  $\langle G_1, G_2, G_3, G_4, G_5 \rangle$  of prime order N with two non-degenerate bilinear pairing maps  $e_{12}: G_1 \times G_2 \to G_5$  and  $e_{34}: G_3 \times G_4 \to G_5$ . Also there are two group isomorphisms  $\phi: G_1 \to G_3$  and  $\overline{\phi}: G_4 \to G_2$ . Given all the above descriptions as the public parameters along with a generator  $P \in G_1$ , the prover needs to find  $\overline{\phi}(Q)$ , where  $Q \in G_4$ , using T sequential steps. The verifier checks

if  $e_{12}(P, \overline{\phi}(Q)) = e_{34}(\phi(P), Q)$  in poly(log T) time. However it requires a trusted setup, and the setup phase may take same time as the evaluation.

Mahmoody et al. [5] have recently ruled out the possibility of having perfectly unique VDFs using random oracles only.

## 3 Preliminaries

We start with the notations.

#### 3.1 Notations

We denote the security parameter with  $\lambda \in \mathbb{Z}^+$ . The term  $\operatorname{poly}(\lambda)$  refers to some polynomial of  $\lambda$ , and  $\operatorname{negl}(\lambda)$  represents some function  $\lambda^{-\omega(1)}$ . If any randomized algorithm  $\mathcal{A}$  outputs y on an input x, we write  $y \overset{\mathcal{R}}{\leftarrow} \mathcal{A}(x)$ . By  $x \overset{\$}{\leftarrow} \mathcal{X}$ , we mean that x is sampled uniformly at random from  $\mathcal{X}$ . For a string x, |x| denotes the bit-length of x, whereas for any set  $\mathcal{X}$ ,  $|\mathcal{X}|$  denotes the cardinality of the set  $\mathcal{X}$ . If x is a string then  $x[i \dots j]$  denotes the substring starting from the literal x[i] ending at the literal x[j]. We consider an algorithm  $\mathcal{A}$  as efficient if it runs in probabilistic polynomial time (PPT).

## 3.2 Verifiable Delay Function

We borrow this formalization from [2].

**Definition 1.** (Verifiable Delay Function). A VDF V = (Setup, Eval, Verify) that implements a function  $\mathcal{X} \to \mathcal{Y}$  is specified by three algorithms.

- Setup $(1^{\lambda}, T) \to \mathbf{pp}$  is a randomized algorithm that takes as input a security parameter  $\lambda$  and a targeted time bound T, and produces the public parameters  $\mathbf{pp}$ . We require Setup to run in  $\mathrm{poly}(\lambda, \log T)$  time.
- Eval( $\mathbf{pp}, x$ )  $\to (y, \pi)$  takes an input  $x \in \mathcal{X}$ , and produces an output  $y \in \mathcal{Y}$  and a (possibly empty) proof  $\pi$ . Eval may use random bits to generate the proof  $\pi$ . For all  $\mathbf{pp}$  generated by  $\mathsf{Setup}(\lambda, T)$  and all  $x \in \mathcal{X}$ , the algorithm  $\mathsf{Eval}(\mathbf{pp}, x)$  must run in parallel time T with  $\mathsf{poly}(\lambda, \log T)$  processors.
- Verify( $\mathbf{pp}, x, y, \pi$ )  $\to \{0, 1\}$  is a deterministic algorithm that takes an input  $x \in \mathcal{X}$ , an output  $y \in \mathcal{Y}$ , and a proof  $\pi$  (if any), and either accepts (1) or rejects (0). The algorithm must run in  $\operatorname{poly}(\lambda, \log T)$  time.

**Subexponentiality of** T As Verify is efficient, we need  $T \leq 2^{o(\lambda)}$ , otherwise a prover with poly(T) processors will always be able to brute-force  $2^{\lambda}$  possible solutions.  $T \leq 2^{o(\lambda)}$  enforces complexity of this brute-force approach to be  $2^{\lambda}/2^{o(\lambda)} = 2^{\Omega(\lambda)}$ .

Interactive VDFs Def. 3.2 uses Fiat-Shamir heuristic to generate the proof  $\pi$ . It is the non-interactive version of VDFs where the verifier does not interact with the prover. Fiat-Shamir heuristic replaces this interaction in *any* public coin interactive proof using a random oracle. The interactive versions of VDFs asks the prover to compute the proof  $\pi$  on challenges chosen by the verifier.

The three desirable properties of a VDF are now introduced.

**Definition 2.** (Correctness) A VDF is correct with some error probability  $\varepsilon$ , if for all  $\lambda, T$ , parameters **pp**, and  $x \in \mathcal{X}$ , we have

$$\Pr\left[ \mathsf{Verify}(\mathbf{pp}, x, y, \pi) = 1 \, \left| \begin{array}{c} \mathbf{pp} \leftarrow \mathsf{Setup}(1^{\lambda}, T) \\ x \xleftarrow{\$} \mathcal{X} \\ (y, \pi) = \mathsf{Eval}(\mathbf{pp}, x) \end{array} \right] = 1.$$

**Definition 3.** (Soundness) A VDF is sound if for all non-uniform algorithms  $\mathcal{A}$  that run in time poly $(T, \lambda)$ , we have

$$\Pr\left[ \begin{array}{l} y \neq \mathsf{Eval}(\mathbf{pp}, x) \\ \mathsf{Verify}(\mathbf{pp}, x, y, \pi) = 1 \end{array} \middle| \begin{array}{l} \mathbf{pp} \leftarrow \mathsf{Setup}(1^{\lambda}, T) \\ (x, y, \pi) \leftarrow \mathcal{A}(1^{\lambda}, T, \mathbf{pp}) \end{array} \right] \leq \operatorname{negl}(\lambda).$$

We call the VDF *perfectly* sound if this probability is 0.

**Definition 4.** (Sequentiality) A VDF is  $(\Delta, \sigma)$ -sequential if there exists no pair of randomized algorithms  $\mathcal{A}_0$  with total running time poly $(T, \lambda)$  and  $\mathcal{A}_1$  which runs in parallel time  $\sigma$  on at most  $\Delta$  processors, such that

$$\Pr\left[ y = \mathsf{Eval}(\mathbf{pp}, x) \; \middle| \; \begin{array}{l} \mathbf{pp} \leftarrow \mathsf{Setup}(1^{\lambda}, T) \\ state \leftarrow \mathcal{A}_0(1^{\lambda}, T, \mathbf{pp}) \\ x \overset{\$}{\leftarrow} \mathcal{X}; y \leftarrow \mathcal{A}_1(state, x) \end{array} \right] = \operatorname{negl}(\lambda).$$

Here,  $\mathcal{A}_0$  is a preprocessing algorithm that precomputes some *state* based only on the public parameters, and  $\mathcal{A}_1$  exploits this additional knowledge in Eval $(x, \mathbf{pp})$  in parallel running time  $\sigma$  on  $\Delta$  processors. An almost-perfect VDF would achieve a sequentiality  $\sigma = T - o(T)$ . Even a sequentiality  $\sigma = T - \epsilon T$  for small  $\epsilon$  is sufficient for most applications.

## 3.3 The Complexity Classes

We start with the definition of Turing machine in order to discuss complexity classes.

**Definition 5.** (Turing machine). A Turing Machine is a tuple  $TM = \langle Q, \Gamma, q_0, F, \delta \rangle$  with the following meaning,

- 1. Q is the finite and nonempty set of states.
- 2.  $\Gamma$  is the finite and non-empty set of tape alphabet symbols including the input alphabet  $\Sigma$ .
- 3.  $q_0 \in Q$  is the initial state.
- 4.  $F \subseteq Q$  is the set of halting states.
- 5.  $\delta: \{Q \setminus F\} \times \Gamma \to Q \times \Gamma \times D$  is the transition functions where  $D = \{-1, 0, +1\}$  is the set of directions along the tape.

Throughout the paper we assume that the initial state  $q_0$ , one of the final states  $q_F$  and the tape alphabet  $\Gamma = \{0,1,\vdash\}$  are implicit to the description of a TM. Here  $\Sigma = \{0,1\}$  and  $\vdash$  marks the left-end of the tape. Thus  $\langle Q, F, \delta \rangle$  suffices to describe any TM.

**Definition 6.** (Configuration). A configuration of a TM is a triple (q, z, n) where, at present,

- 1.  $q \in Q$  is the state of TM.
- 2.  $z \in \Gamma^*$  is the content of the tape.
- 3.  $n \in \mathbb{Z}$  is the position of the head at the tape.

 $(q_0, x, 0)$  denotes the starting configuration for an input string x instead of  $(q_0, \vdash x, 0)$  (w.l.o.g).

**Definition 7.**  $(\tau$ -th Configuration  $\stackrel{\tau}{\rightarrow}$ ). The relation  $\stackrel{\tau}{\rightarrow}$  is defined as,

- 1.  $(q, z, n) \stackrel{\tau}{\to} (q', z', n + d)$  where  $z' = \dots z[n-1] ||b|| z[n+1] \dots$  if  $\delta(q, z[n]) = (q', b, d)$ .
- 2.  $\alpha \stackrel{\tau+1}{\rightarrow} \gamma$  if there exists a  $\beta$  such that  $\alpha \stackrel{\tau}{\rightarrow} \beta \stackrel{1}{\rightarrow} \gamma$ .

We denote  $\alpha \xrightarrow{0} \alpha$ . We say that the TM halts on a string x if  $(q_0, x, 0) \xrightarrow{\tau} (q_F, x_F, n)$  where  $q_F \in F$  for some  $\tau \geq 0$ . We say that a TM computes a function  $f: \Sigma^* \to \Sigma^*$  in time  $\tau$  if  $\forall x \in \Sigma^*$ ,  $(q_0, x, 0) \xrightarrow{\tau} (q_F, f(x), n)$  in time at most  $\tau \geq 0$ .

We call a language L is decidable by a TM if and only if there exists a TM that accepts all the strings belong to  $\mathcal{L}$  and rejects all the strings belong to  $\overline{L} = \mathcal{L}^* \setminus \mathcal{L}$ . We say that a language L is reducible to another

language L' if and only if there exists a function f such that  $f(x) \in L'$  if and only if  $x \in L$ . If the function f is computable in poly(|x|)-time then we call it as a polynomial time reduction  $L \leq_p L'$ .

In order to discuss the complexity classes we follow the definitions provided in [1].

**Definition 8.** (DTIME). Suppose  $f : \mathbb{N} \to \mathbb{N}$  be some function. A language L is in  $\mathbf{DTIME}(f(n))$  if and only if there is a TM that decides L in time  $\mathcal{O}(f(n))$ .

**Definition 9.** (**DSPACE**). Suppose  $f : \mathbb{N} \to \mathbb{N}$  be some function. A language L is in **DSPACE**(f(n)) if and only if there is a TM that decides L in space  $\emptyset(f(n))$ .

Definition 10. (The Class P).

$$\mathbf{P} = \bigcup_{k \ge 1} \mathbf{DTIME}(n^k).$$

Definition 11. (The Class PSPACE).

$$\mathbf{PSPACE} = \bigcup_{k \geq 1} \mathbf{DSPACE}(n^k).$$

Definition 12. (The Class EXP).

$$\mathbf{EXP} = \bigcup_{k>1} \mathbf{DTIME}(2^{n^k}).$$

Definition 13. (The Class NC).

$$\mathbf{NC} = \bigcup_{k \ge 1} \mathbf{NC}^k$$

For every k,  $\mathbf{NC}^k$  is the set of languages that can be decided by a family of boolean circuits  $\{C_n\}$  where  $C_n$  has  $\mathrm{poly}(n)$  size and depth  $\emptyset(\log^k n)$ .

NC denotes the class of efficiently parallelizable languages.

Definition 14. (The Class IP).

$$\mathbf{IP} = \bigcup_{k \geq 1} \mathbf{IP}^{n^k}.$$

For every k',  $\mathbf{IP}^{k'}$  is the set of languages L such that there exist a probabilistic polynomial time  $\mathsf{TM}\ \mathcal{V}$  that can have a k'-round interaction with a prover  $\mathcal{P}: \{0,1\}^* \to \{0,1\}^*$  having these two following properties

(Correctness). 
$$(x \in L) \implies \exists \mathcal{P}, \Pr[\mathcal{V}(\mathcal{P}(x)) = 1] \ge \frac{2}{3}$$
. (Soundness).  $(x \notin L) \implies \forall \mathcal{P}, \Pr[\mathcal{V}(\mathcal{P}(x)) = 1] < \frac{1}{3}$ .

Using Chernoff bounds it can be shown that if  $\mathcal{V}$  repeats this experiment for m times then these two probabilities become  $(1-2^{-\Omega(m)})$  and  $2^{-\Omega(m)}$  causing no harm to the definition. The class **IP** is the set of interactive proofs with probabilistic polynomial time verifier allowing polynomial number of rounds of interaction with the prover.

We define,

Definition 15. (The Class VDF).

$$\mathbf{VDF} = \bigcup_{\lambda \geq 1, T \geq 1} \mathbf{VDF}^{\lambda, T}.$$

For every  $\lambda, T$ ,  $\mathbf{VDF}^{\lambda,T}$  is the set of languages L such that there exists a probabilistic polynomial time TM  $\mathcal V$  that can have a  $\operatorname{poly}(\lambda, \log T)$ -round interaction with a  $\operatorname{poly}(\lambda, T)$  parallel TM  $\mathcal P: \{0,1\}^* \to \{0,1\}^*$  that runs in time  $\geq T$  having these three following properties

(Correctness). 
$$(x \in L) \implies \exists \mathcal{P}, \Pr[\mathcal{V}(\mathcal{P}(x)) = 1] = 1.$$
  
(Soundness).  $(x \notin L) \implies \forall \mathcal{P}, \Pr[\mathcal{V}(\mathcal{P}(x)) = 1] \leq \operatorname{negl}(\lambda).$   
(Sequentiality).  $(x \in L) \implies \forall \mathcal{A}, \Pr[\mathcal{V}(\mathcal{A}(x)) = 1] \leq \operatorname{negl}(\lambda)$  where  $\mathcal{A}$  is a  $\operatorname{poly}(\lambda, T)$  parallel TM that runs in time  $< T$ .

We know that  $\mathbf{NC} \subseteq \mathbf{P} \subsetneq \mathbf{IP} = \mathbf{PSPACE} \subseteq \mathbf{EXP}$ . If a class  $\mathbb{S} \supsetneq \mathbf{NC}$  then the set  $\{\mathbb{S} \setminus \mathbf{NC}\}$  represent *some* of the languages that are non-parallelizable or inherently sequential. Informally, a complete language for any class is hardest to decide. Therefore it is always safe to assume that  $\mathbb{S}$ -complete languages are sequential if  $\mathbb{S} \supsetneq \mathbf{NC}$ . In our approach, we need  $\mathbf{EXP}$  to be the class  $\mathbb{S}$ .

**Definition 16.** (EXP-complete). A language is EXP-complete if it is in EXP and every language in EXP is reducible to it in polynomial time.

Here is an example of **EXP**-complete languages.

**Definition 17.** (EXPHALT). Suppose M is a Turing machine,  $x \in \Sigma^*$  is an input string and  $T \in \mathbb{N}$ . The language EXPHALT is the set of all the tuples  $(\langle M \rangle, x, T)$  such that M halts on input x in time T. Formally,

$$\mathsf{EXPHALT} = \{ (\langle Q, F, \delta \rangle, x, T) \mid (q_0, x, 0) \overset{T}{\to} (q_F, y, n) \}.$$

**Lemma 1.** EXPHALT is **EXP**-complete.

*Proof.* We will show that any language  $L \in \mathbf{EXP}$  is reducible to EXPHALT in polynomial-time. Suppose  $L \in \mathbf{EXP} = \mathbf{DTIME}(2^{\emptyset(n^k)})$  is decided by a TM M. Then the function  $f(x) = (\langle M \rangle, x, 2^{\emptyset(|x|)})$  is a polynomial-time reduction from L to EXPHALT.

## Lemma 2. EXPHALT is inherently sequential.

*Proof.* We prove this by contradiction. Suppose  $(\langle M \rangle, x, T) \in \mathsf{EXPHALT}$  and we parallelize the simulation of M with another  $\mathsf{TM}$   $\widehat{M}$ . Then any  $\mathsf{TM}$  that decides a language  $\mathsf{L} \in \mathsf{EXP}$  must be parallelizable using the  $\mathsf{TM}$   $\widehat{M}$ . The it means  $\mathsf{NC} = \mathsf{EXP}$  which is a contradiction.

# 4 $VDF \nsubseteq IP$

We derive a VDF from the language EXPHALT such that EXPHALT is polynomial time reducible to the derived VDF.

## Theorem 1. VDF $\nsubseteq$ IP = PSPACE.

*Proof.* We prove this by contradiction. Suppose **VDF**  $\subseteq$  **PSPACE**. Then EXPHALT  $\in$  **PSPACE** as EXPHALT is polynomial time reducible to the derived VDF (proved in Theorem 3). Then **EXP**  $\subseteq$  **PSPACE** as EXPHALT is an **EXP**-complete language (See lemma 1). But we do not know yet if **EXP**  $\subseteq$  **PSPACE**. Thus **VDF**  $\nsubseteq$  **IP** = **PSPACE**.  $\square$ 

#### 4.1 VDF From EXPHALT

Here we specify the algorithms of the derived VDF.

The Setup $(1^{\lambda}, T) \to pp$  Algorithm The chosen public parameters are  $pp = (\langle M \rangle, H)$  with the following meaning,

- 1.  $M = (Q, F, \delta)$  is a TM.
- 2.  $\mathsf{H}:\{0,1\}^* \to \mathbb{Z}_{\lambda}$  is a random oracle.

The Eval $(x, pp) \to (\phi, \pi)$  Algorithm The prover  $\mathcal{P}$  computes,

- 1.  $(q_0, x, 0) \xrightarrow{T} (q_T, y, n)$ , i.e., M reaches at state  $q_T$  starting from the state  $q_0$  after T transitions on input x.
- 2.  $t = \mathsf{H}_x(q_T)$  where  $\mathsf{H}_x(\cdot) \stackrel{def}{=} \mathsf{H}(x||\cdot)$ .
- 3.  $(q_0, x, 0) \stackrel{T-t}{\to} (q_{T-t}, y_{T-t}, n_{T-t})$  i.e.,  $q_{T-t}$  is the state starting from which M reaches the state  $q_T$  at time t.

- 4. Initialize a string  $z = y_{T-t}[n_{T-t}]$ .
- 5. Repeat  $1 \le i \le t$ ,
  - (a)  $(q_{T-t}, y_{T-t}, n_{T-t}) \xrightarrow{i} (q_{T-t+i}, y_{T-t+i}, n_{T-t+i}).$
  - (b)  $z = z || y_{T-t+i} [n_{T-t+i}].$

The output  $\phi = q_T$  and the proof  $\pi = (q_{T-t}, z)$ . Observe that  $(q_{T-t}, z, 0) \xrightarrow{t} (q_T, y, n)$  as z is the tape content that M reads in t moves to reach state  $q_T$  from  $q_{T-t}$ . Theorem 3 shows that  $\mathsf{EXPHALT} \leq_p \mathsf{Eval}$ .

The Verify $(x, \operatorname{pp}, \phi, \pi) \to \{0, 1\}$  Algorithm  $\mathcal V$  executes,

- 1.  $t = \mathsf{H}_x(\phi)$  where  $\mathsf{H}_x(\cdot) \stackrel{def}{=} \mathsf{H}(x||\cdot)$ .
- 2.  $(q_{T-t}, z, 0) \xrightarrow{t} (q_T, y, n)$ .
- 3. checks if  $\phi = q_T$ . If yes then (s)he accepts the tuple  $(x, T, \phi, \pi)$ , rejects otherwise.

## 5 Security

Now we claim that the derived VDF is correct, sound and sequential.

**Theorem 2.** The constructed VDF is correct.

*Proof.* If  $\mathcal{P}$  has run Eval honestly then  $(q_0, x, 0) \xrightarrow{T} (q_T, y, n)$ . The integer t is solely determined by the input statement x and the output  $q_T$ . Thus  $\mathcal{V}$  will always find  $\phi = q_T$  by computing  $(q_t, z, 0) \xrightarrow{t} (q_T, y, n)$ .

**Theorem 3.** If there is an adversary  $\mathcal{A}$  breaking the soundness of this VDF with the probability p then there is a Turing machine  $\mathcal{A}'$  that decides the language EXPHALT with the same probability.

*Proof.* Given a string  $(M, x, T) \in \mathsf{EXPHALT}$ ,  $\mathcal{A}'$  passes it to  $\mathcal{A}$ . When  $\mathcal{A}$  outputs  $q, \mathcal{A}'$  checks if  $q \in F$  or not and decides the string (M, x, T). As  $\mathcal{A}$  breaks the soundness of this VDF with the probability p so  $\Pr[q \in F] = p$ . Hence  $\mathcal{A}'$  decides EXPHALT with the probability p.

**Theorem 4.** If there is an adversary  $\mathcal{A}$  breaking the sequentiality of this VDF with the probability p then there is a Turing machine  $\mathcal{A}'$  breaking the sequentiality of the language EXPHALT with the same probability.

*Proof.* Given a string  $(M, x, T) \in \mathsf{EXPHALT}$ ,  $\mathcal{A}'$  passes it to  $\mathcal{A}$  that runs in time < T. When  $\mathcal{A}$  outputs q,  $\mathcal{A}'$  checks if  $q \in F$  or not and decides the string (M, x, T). As  $\mathcal{A}$  breaks the sequentiality of this VDF with the probability p so  $\Pr[q \in F] = p$ . Hence  $\mathcal{A}'$  decides EXPHALT in time < T with the probability p violating the sequentiality of EXPHALT.  $\square$ 

#### 5.1 Efficiency

Here we discuss the time and the memory required by the prover and the verifier.

**Proof Size** The output  $\phi = q_T$  needs *n*-bits where  $\lambda \leq n \leq \text{poly}(\lambda)$ . The proof  $\pi = (q_{T-t}, z)$  requires n + t-bits as |z| = t.

 $\mathcal{P}$ 's Effort  $\mathcal{P}$  needs T time to find  $q_T$ , then T-t time to find  $q_{T-t}$  and finally t time to find z. By the Theorem 4 it requires 2T time in total.  $\mathcal{V}$ 's Effort  $\mathcal{V}$  needs only t time to find  $q_T$  from  $q_{T-t}$  using z. Deciding  $q \in F$  is already shown to be efficient.

## 6 Conclusions

This paper presents an idea of constructing a verifiable delay function from **EXP**-complete problems. It shows that  $\mathbf{VDF} \nsubseteq \mathbf{IP} = \mathbf{PSPACE}$ .

#### References

- Arora, S., Barak, B.: Computational Complexity A Modern Approach. Cambridge University Press (2009), http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521424264
- Boneh, D., Bonneau, J., Bünz, B., Fisch, B.: Verifiable delay functions. In: Shacham, H., Boldyreva, A. (eds.) Advances in Cryptology CRYPTO 2018 38th Annual International Cryptology Conference, Santa Barbara, CA, USA, August 19-23, 2018, Proceedings, Part I. Lecture Notes in Computer Science, vol. 10991, pp. 757-788. Springer (2018). https://doi.org/10.1007/978-3-319-96884-1\_25
- 3. Dwork, C., Naor, M.: Pricing via processing or combatting junk mail. In: Brickell, E.F. (ed.) Advances in Cryptology CRYPTO '92, 12th Annual International Cryptology Conference, Santa Barbara, California, USA, August 16-20, 1992, Proceedings. Lecture Notes in Computer Science, vol. 740, pp. 139–147. Springer (1992). https://doi.org/10.1007/3-540-48071-4\_10
- 4. Feo, L.D., Masson, S., Petit, C., Sanso, A.: Verifiable delay functions from super-singular isogenies and pairings. In: Galbraith, S.D., Moriai, S. (eds.) Advances in Cryptology ASIACRYPT 2019 25th International Conference on the Theory and Application of Cryptology and Information Security, Kobe, Japan, December 8-12, 2019, Proceedings, Part I. Lecture Notes in Computer Science, vol. 11921, pp. 248–277. Springer (2019). https://doi.org/10.1007/978-3-030-34578-5\_10
- Mahmoody, M., Smith, C., Wu, D.J.: Can verifiable delay functions be based on random oracles? In: Czumaj, A., Dawar, A., Merelli, E. (eds.) 47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference). LIPIcs, vol. 168, pp. 83:1–83:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020). https://doi.org/10.4230/LIPIcs.ICALP.2020.83
- 6. Pietrzak, K.: Simple verifiable delay functions. In: Blum, A. (ed.) 10th Innovations in Theoretical Computer Science Conference, ITCS 2019, January 10-12, 2019, San Diego, California, USA. LIPIcs, vol. 124, pp. 60:1–60:15. Schloss Dagstuhl Leibniz-Zentrum für Informatik (2019). https://doi.org/10.4230/LIPIcs.ITCS.2019.60

- 7. Rivest, R.L., Shamir, A., Wagner, D.A.: Time-lock puzzles and timed-release crypto. Tech. rep., USA (1996)
- 8. Wesolowski, B.: Efficient verifiable delay functions. In: Ishai, Y., Rijmen, V. (eds.) Advances in Cryptology EUROCRYPT 2019 38th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Darmstadt, Germany, May 19-23, 2019, Proceedings, Part III. Lecture Notes in Computer Science, vol. 11478, pp. 379–407. Springer (2019). https://doi.org/10.1007/978-3-030-17659-4\_13