On Haar Measures

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1 Introduction and Motivation

The concept of a *measure* in mathematics is one that in many context codifies an intuitive idea mathematically. On the real number line, for example, the Lebesgue measure extends the idea of *length* or *size* in English to a large family of sets, called the Borel sets, which are combinations of intersections, complements, and unions applied to open and closed intervals. In particular, the Lebesgue measure matches the intuitive sense of *Length* for any unit interval, assigning the intervals $[a, b]$ and (a, b) both $|b - a|$ length. One difficulty with measure, though, is that it requires a lot of artifice to construct in a manner that acts in accordance with many properties one would want in a mathematical understanding of size. Some of those difficulties will be discussed in the first few sections of this paper, but suffice it to say for now that even when there is a somewhat natural expectation of what a measure *should* look like on a space, it often takes some pretty complex and idiosyncratic mathematics to write down what the measure actually is and does.

This difficulty is what makes the Haar measure such a powerful tool. It equips us with a measure on a fairly large class of mathematical spaces, namely compact topological groups.[1] Even more importantly, Haar measures are translation-invariant, meaning that if we have the set S and then define the sets ${}_gS$ as the elements of S left multiplied by g in our compact topological group, and similarly for right multiplication and S_g then we find that each of these three sets have the same measures under the Haar measure. There is an intrinsic motivation for this, which is that groups often represent symmetries of different systems, and therefore heuristically we would likely want measures to be symmetric under the action of the group as well, but this justification for importance may still be too abstracted and at first glance, invariance may seem mildly arcane or unimportant; however consider what this actually means on specific spaces.

If we return to the understanding of the Lebesgue measure on $\mathbb{R}$ as *length*, then $\mathbb{R}$ under addition is a topological group. What invariance tells us here on an interval is that $m([0,1]_r) = m([0,1])$, which is to say $m([r, r+1]) = m([0,1])$, or essentially if we slide the interval up or down the number line without stretching it, it keeps the same length. In this context, it is clear why our mathematical definition of a length function or a size function should naturally want to be translation invariant.

These measures, or size functions, do not exist solely for their own sake, either. There are several external appeals for the construction of the Haar measure on any compact topological group. The construction of measures for these spaces allows us automatically to integrate any continuous function on any compact group. This follows almost immediately from the construction of such a measure. Additionally, Haar measures are the unique invariant Borel probability measure on the topology. Therefore if we wish to understand the group as a probability space with an invariant measure (e.g. studying random events invariant under rotations), something one might naturally desire when studying symmetry, the Haar measure is the unique such measure. Additionally, the Haar measure is a relative nice measure that, in conjunction with a Radon-Nikodym derivative, can express a integration over a large number of probability spaces over the group. The existence of these measures can inform us about topological groups more generally as well. For example we will prove that no countable topological group is compact (see Section 7.1). We can also use these measures to generate measures on large classes of spaces using group actions, and we can generate group invariant metrics for compact topological metric groups.

2 Measures

The goal of this portion of the paper is to acquire the necessary machinery to construct Haar measures. This means that sometimes the more computational aspects of certain topics will be covered, such as the constructions of certain functions, sets or other objects but some of their properties will be left unproven with a citation to a text with a proof. In this section we will introduce some definitions and theorems that aid in the construction of a Haar measure, but will assert several properties which are important but nevertheless relate more directly to studies of measures generally than the study of Haar measures. This is because many of the measure theoretic concepts in the beginning of this chapter should generally be familiar to a reader of this text, or any student who had taken an introductory real analysis course; however, this section is included here for clarity of notation and definitions.

2.1 Borel σ-Algebra

If the goal of this section is to establish the set-functions known as measures, we must first define what measures act upon. We naturally want the idea of size to be additive and to have some idea of subtraction as well, which would indicate that we would want unions, intersections, and complements of measurable sets to be measurable. This intuition matches up nicely with the domain of measures, namely σ-algebra.

Defintion 2.1. [5] Let X be a set. Then $\mathcal{A}$ is a *σ-algebra* if for any countable index set I of some sequence $A_i \in \mathcal{A}$, we have that

[1] This concept can be extended more broadly to *locally compact* topological groups, but the construction of such measures is outside the scope of this paper, and generally even more involved than the compact case.

1. $\bigcap_{i\in I} A_i \in \mathcal{A}$
2. $\bigcup_{i\in I} A_i \in \mathcal{A}$
3. $A_i^c \in \mathcal{A}$

In general, if $\mathcal{F}$ is a family of sets in X, then $\sigma(\mathcal{F})$ denotes the σ-algebra *generated* by $\mathcal{F}$, which is the smallest σ-Algebra that contains all elements of $\mathcal{F}$. The existence of such a σ-algebra is not trivial, and is outside the scope of this paper, but is often proved using the $\pi - \lambda$ theorem.

With our domain now defined, let us proceed to provide an exact definition of these set functions which roughly correspond to size, called measures

Defintion 2.2. [5] A *Measure* is a function from a σ-algebra $\mathcal{A}$ with $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and for some countable index set I and a disjoint sequence of sets $A_i \in \mathcal{A}$, we have

$$\mu\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} \mu(A_i)$$

We identify a topological space with its measure as a *measure space* and write $(X, \mathcal{F}, \mu)$ where X is the space, μ is the measure, and $\mathcal{F}$ is the σ-algebra.

Measures have a particularly helpful use, which is in the integration of functions. In order to integrate continuous functions to $\mathbb{R}$, it helps to have a measure over the closed sets of a space, since then $f^{-1}(\{f(x)\})$ is closed and can be measured (A reader is expected to know the basics of measure theoretic integration. If not, refer to [5]). This of course, given the condition on complements, means that the open sets of the space will also be measurable. Correspondingly, we will define

Defintion 2.3. The *Borel σ-Algebra* of a topological space X with topology $\mathcal{T}$ is the σ-algebra $\mathcal{B}(X) := \sigma(\mathcal{T})$. Correspondingly, a *Borel measure* on a space X is a measure on the Borel σ-algebra of X, $\mu : \mathcal{B}(X) \to \mathbb{R} \cup \{\infty\}$ such that if $S \subset X$ is compact, then $\mu(S) < \infty$.

Note that every open and closed set is thus contained in the Borel σ-algebra, because if U is open then $U \in \mathcal{T} \subset \sigma(\mathcal{T})$ and if U is closed then $U^c \in \mathcal{T}$, and therefore by closure under complements we have $U = (U^c)^c \in \mathcal{T}$. Building a Borel σ-Algebra on a generic space can be somewhat complicated, but it does exist and is the σ-algebra generated by the topology, i.e. the collection of open sets, on a given space. This lines up with our desire for a measure for the purpose of integration.

2.2 Regular Measure

Defintion 2.4. [3, p. 455] Let $\mathcal{F}$ be a σ-algebra on a topological space X. Then a measure μ is deemed *regular* if the following two conditions are met:

- (Outer Regularity) If $U \in \mathcal{F}$, then

$$\mu(U) = \inf\{\mu(O) \mid U \in \mathcal{F}, U \subset O \text{ is open}\}$$

- (Inner Regularity) If $U \subset X$ is open and $U \in \mathcal{F}$ then

$$\mu(U) = \sup\{\mu(K) \mid K \in \mathcal{F}, K \subset U \text{ is compact}\}$$

A regular Borel measure is called a *Radon measure*.

This concept is useful for estimating sets in a space, and it tells us that our notion of size acts as we expect it to: For outer regularity, we know that a sequence of smaller and smaller sets that approaches a specific set will, in its limit, have the size of that set. Similarly, for inner regularity, we are essentially saying sequences of larger and larger sets that approach a specific set will, in the limit, have the same size as the specific set. These are natural qualities that we should want any well behaved measure to have.

2.3 Extension of Premeasures

Finding measures can be kind of difficult. First there is the issue of finding a σ-algebra and understanding the qualities of the sets in the σ-algebra to a sufficient extent to define a function on them. Then, there is the difficulty of guaranteeing that a set function behaves according to the definition of a measure. Therefore, we might want to set ourselves a lower bar, so to speak, by defining a different set function from which we can induce a measure. This set function is called a *premeasure* and this subsection will focus on premeasures and their extension to full measures.

Defintion 2.5. [3, p. 352] If A is an collection of sets then $\mu_0 : A \to [0, \infty]$ is a *premeasure* on A if $\mu_0(\varnothing) = 0$ and for any at most countable disjoint sequence $U_i \in A$ such that $\bigcup_{i=1}^{\infty} U_i \in A$ we have

$$\mu_0\left(\bigcup_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} \mu_0(U_i)$$

and if $E_i \in A$ are a countable sequence of sets (not necessarily disjoint) then we have

$$\mu_0\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} \mu_0(E_i) \tag{1}$$

The first condition is called finite additivity, and the second condition is called countably monotone

We can now induce this premeasure to provide us with a set function defined on the entire power set of the space X, from which we will cut away sets to define our σ-algebra and our measure.

Defintion 2.6. [3, p. 349] If μ_0 is a premeasure on $(X, \mathcal{S})$ for some $\mathcal{S} \subset P(X)$, then define the function μ^* as $\mu^*(\varnothing) = 0$ and for any $E \subseteq X$

$$\mu^*(E) := \inf_{\{E_k\}} \sum_{k=1}^{\infty} \mu_0(E_k) \tag{2}$$

where each $\{E_k\}$ is a sequence of sets in $\mathcal{S}$ which cover E as the *Outer Measure induced by* μ_0. A set E is called measurable with respect to an outer measure μ^* if for every $A \subset E$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

With this outer measure defined, we take only the measurable sets and they constitute a σ-algebra. We can then restrict the set function to this σ-algebra by the following theorem, which allows us to get measures from premeasures.

Theorem 2.1 (The Caratheodory-Hahn Theorem)**.** [3, p. 356]
Let μ_0 be a premeasure on a collection $\mathcal{A}$ of sets which are subsets of X Then the *Caratheodory measure* $\bar{\mu}$ induced by μ_0 is defined as the restriction of the outer measure to its measurable sets[2] Furthermore, for any $A \in \mathcal{A}$ we have $\bar{\mu}(A) = \mu_0(A)$, and if μ_0 is σ-finite then $\bar{\mu}$ is σ-finite and $\bar{\mu}$ is the unique measure on the σ-algebra of μ^* measurable sets.

We will not prove this statement, nor the following statement about Radon measures, because their proofs are largely measure theoretic and have little to do with topology or Haar measures in particular. View the cited pages above for a full proof.

Defintion 2.7. Let $(X, \mathcal{T})$ be a topological space. a premeasure $\mu : \mathcal{T} \to [0, \infty]$ is called a *Radon premeasure* if

- If U is open and $\overline{U}$ is compact, then $\mu(U) < \infty$
- For each open set O,
$$\mu(O) = \sup\left\{\mu(U) \mid U \text{ open and } \overline{U} \subset O \text{ is compact}\right\}$$

These conditions are sufficient to state that the measure induced by the premeasure μ_0 is a Radon measure. This is a fact contained in [3].

[2]The fact that the collection of measurable sets of an outer measure form a σ-algebra is implicit in this statement. A proof can be found in [3, p. 347]

3 Riesz Markov Theorem

In this section of the paper, we will first explore the notion of Linear Functionals and dual spaces ,i.e. the spaces of linear functionals. Using these concepts, we will be able to construct measures that conform with the value of linear functionals on the space of continuous functions with compact support.

3.1 Linear Functionals

The general idea of a linear function is a familiar one. It is a function T such $T(ax + by) = aT(x) + bT(y)$, so we can see that it preserves the operations of addition and scalar multiplication. A typical example of linear functions are the matrices in $\mathbb{R}^n$. We will define this concept more generally, though, so that we may use it later. The motivation here is that integration is actually a linear functional on the space of continuous functions, since $\int af + bg = a \int f + b \int g$. Therefore if we learn some technology regarding these linear functionals, we might be able to apply it to integration (and therefore measures) in the future. Therefore lets define the spaces on which we can have linear functionals.

Defintion 3.1. [3, p. 253] A linear space X is an Abelian group with addition and real scalar multiplication that scalar multiplication is distributive and associative. A linear space is equipped with a norm is a *normed linear space* and has the metric defined by $d(u, v) = \|u - v\|$.

We may additionally want limits to exist under our linear functionals, so that they are well behaved. Therefore we define the concept of Banach Space:

Defintion 3.2. A *Banach Space* is a normed linear space such that it is complete under the norm metric.

Example 3.1. *Continuous functions on compact support*
Let $C(X)$ denote the set of continuous functions from X to $\mathbb{R}$. Then $C_c(X)$ denotes the set of continuous functions of *compact support*, defined as

$$C_c(X) = \{f \in C(X) \mid \overline{\{x \in X \mid f(x) \neq 0\}} \text{ is compact}\}$$

along with the standard supremum norm of a function, $\|f\|_\infty = \sup_{x \in X} |f(x)|$

Defintion 3.3. [3, p. 256] A *linear functional* is a function $T : X \to \mathbb{R}$ such that for all $x, y \in X$ and $a, b \in \mathbb{R}$ where X is a linear space

$$T(a \cdot x + b \cdot y) = aT(x) + bT(y)$$

A linear functional is considered *positive* if whenever $v > 0$, then $L(v) > 0$.

3.2 Dual Space

We can equip the set of these linear functionals with a topology themselves. This can allow us to have continuous functions that output linear functionals, or perform other topological operations on the space. Hence, we will define the space of linear functions over another space below:

Defintion 3.4. [3, p. 157] If X is some normed linear space equipped with the norm $\|\cdot\|$, then define the *Dual Space* of X, denoted X^* as the linear space of Linear Functions on X equipped with the topology induced by the norm $\|\cdot\|_*$ defined as, for any $T \in X^*$

$$\|T\|_* := \sup\{|T(x)| \mid x \in X, \|x\| \leqslant 1\}$$

We will leave the fact that the norm above is, in fact, a norm as an exercise.

3.3 Riesz-Markov Theorem

We are now arriving at the motivation for these linear functionals in general. Using linear functionals on the space of continuous functions, we can actually find measures such that they accord in some sense with the value of the functional. This sense will be made rigorous in Theorem 3.1. In order to get there, we must first prove an introductory lemma about measures which will be useful for a later uniqueness claim.

Lemma 1. [3, p. 457] Let X be a compact Hausdorff space, and μ and ν be Radon measures on $\mathcal{F}$, the Borel σ-algebra in X such that for any $f \in C_c(X)$,

$$\int_X f \ d\mu = \int_X f \ d\nu$$

Then $\mu = \nu$

Proof. Because by the definition of the Radon Measure, we know that any set U can be approximated by above by open sets, which can in turn be approximately from below by compact sets, if we show that the measures agree on compact sets, then we can approximate any sets to be arbitrarily close and therefore we are done. Hence, we will show that for any compact set S, we have $\mu(S) = \nu(S)$.

By the outer regularity properties of μ and ν from Definition 2.4, we know that there exists some open sets A, B such that $\mu(A \backslash S) < \epsilon$ and $\nu(B \backslash S) < \epsilon$ (Otherwise, the measure of S would exceed its own measure by ϵ by outer regularity). Then, define $O = A \cap B$ and note that

$$\mu(A \cap B \backslash S) < \epsilon \text{ and } \nu(A \cap B \backslash S) < \epsilon \tag{3}$$

then, by Urysohn's lemma [3, p. 240], since the space is compact Hausdorff, we know the space is normal and therefore there exists some function f such that on S, $f = 1$, on $X \backslash (A \cap B)$, $f = 0$, and on $(A \cap B) \backslash S$, $f \in [0, 1]$[3] Because X is a continuous space, then $\overline{\{x \in X \mid f(x) \neq 0\}}$ is compact because it is a closed subset of a compact set. Hence, $f \in C_c$. By our assumption that μ and ν integrate each function $f \in C_c(X)$ equally, we know that

$$\int_X f\mu = \int_X f\nu \tag{4}$$

Then, because $f = 0$ on $X \backslash (A \cap B)$ and $f = 1$ on S we know that

$$\begin{aligned} \int_X fd\mu = \int_{A \cap B} fd\mu = \int_S fd\mu + \int_{(A \cap B) \backslash S} fd\mu = \mu(S) + \int_{(A \cap B) \backslash S} fd\mu \\ \int_X fd\nu = \int_{A \cap B} fd\nu = \int_S fd\nu + \int_{(A \cap B) \backslash S} fd\nu = \nu(S) + \int_{(A \cap B) \backslash S} fd\nu \end{aligned} \tag{5}$$

and therefore by equations (3) and (4) we know that

$$-\epsilon < \mu(S) - \nu(S) = \int_{(A \cap B) \backslash S} fd\nu - \int_{(A \cap B) \backslash S} fd\mu < \epsilon$$

therefore we know by (3)

$$|\nu(S) - \mu(S)| < \epsilon$$

for any compact set S and arbitrarily small ϵ. Hence, $\mu(S) = \nu(S)$ on compact sets. Therefore, as we stated earlier, we can use the regularity of Radon measures to approximate any other set and therefore $\mu = \nu$ ☐

With this lemma, we are now ready to prove one of the most substantial theorems in this paper. I have tried to aid presentation with figures, so please make note of them when they are cited, as they are particularly helpful for understanding how this proof functions.

Theorem 3.1 (Riesz-Markov Theorem)**.** [3, p. 458]
Let X be a compact Hausdorff space, and I is a positive linear functional on $C_c(X)$. Then there is a *unique* Radon measure μ on $\mathcal{B}(X)$ such that for any $f \in C_c(X)$, the space of compact supported continuous functions, we have

$$I(f) = \int_X f \ d\mu$$

[3]Urysohn's Lemma is a result somewhat outside the scope of this paper, which states that in a normal topological space, for any disjoint close sets A, B and $[a.b] \subset \mathbb{R}$, there exists some continuous function such that $f = b$ on B and $f = a$ on A. Its proof is long and not particularly enlightening with respect to measures; it generally relies on a good amount of point-set topology, and focuses the construction of ascending open sets, then making a function which assigns points values based on their placement in the sequence of ascending sets.

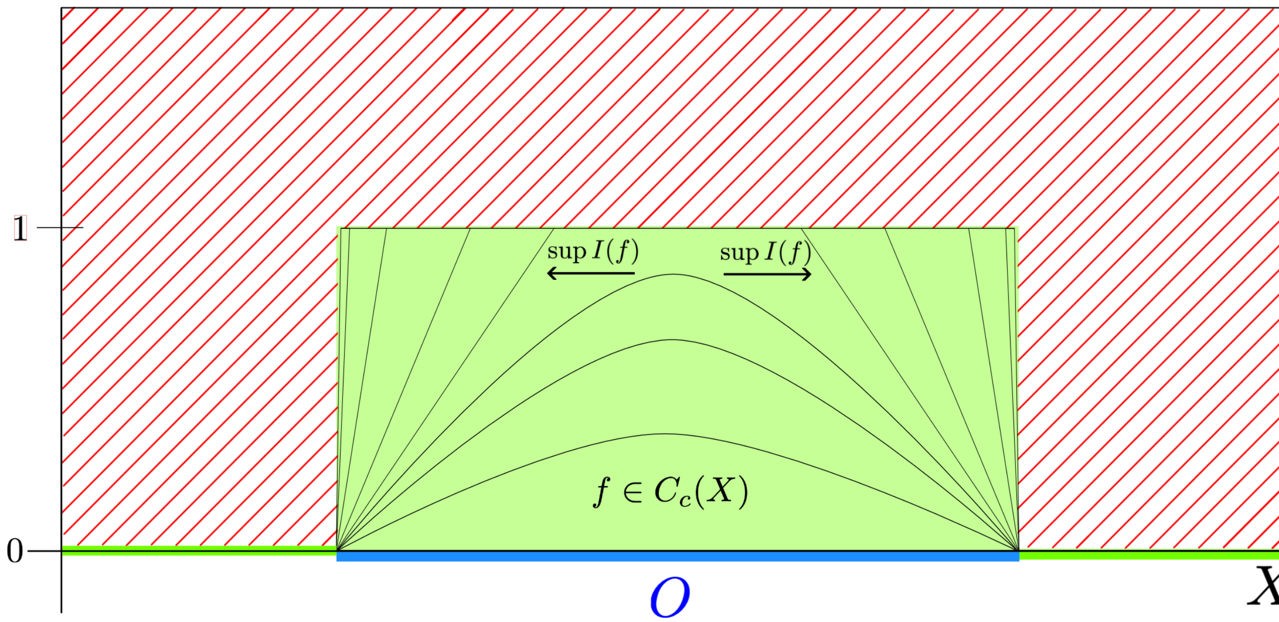


Figure 1: This figure shows how we are defining our premeasure, μ_0, in equation (6). The green bounding boxes represent the values f can take for different inputs in the space X, the blue box on the X axis represents the set being measured, O, and we can think of μ_0 as measuring the limit of I as these functions approach the outline of the box. μ_0 would therefore be representing the output of the indicator function under I as a limit of continuous functions. This corresponds with our notion of integration and measures under the Lebesgue measure on $\mathbb{R}$.

Proof. First, let us construct the premeasure μ_0. Let $\mu_0(\varnothing) = 0$ and then for any open set $O \subset X$, define

$$\mu_0(O) = \sup \{I(f) \mid f \in C_c(X), 0 \leqslant f \leqslant 1, \text{support } f \subseteq O\} \tag{6}$$

This should intuitively make sense. In the terms of integration with respect to the Lebesgue measure over $\mathbb{R}$, this supremum would approach the integration of the indicator function over the set, approximating the indicator function arbitrarily well. For more help with the intuition, look at figure 1. If we denote μ as the restriction of the extension of this premeasure to Borel sets, then our goal is to prove that μ is a Radon measure. In order to do this, we will prove some of the properties necessary under the premeasures section of this paper, subsection 2.3. Uniqueness will follow in certain respects from the Lemma 1.

We must first prove that μ_0 is a premeasure. There are several important properties here. First, we must prove countable monotonicity, one of the properties in Definition 2.5. Hence, assume a sequence of open sets, O_i which cover an open set O, let K be the support of $f \in C_c(X)$ where f is a function whose support lies in O. Students of analysis will be familiar with the idea of a partition of unity, i.e. a finite collection ψ_i of k functions on K such that $\sum_{i=1}^{k} \psi_i(x) = 1$ for every $x \in K$, and that take values in $[0, 1]$ and are 0 outside their respective O_i.

Now, because f has support K, we know that $f = \sum_{i=1}^{k} \psi_i \cdot f = 1 \cdot f$ on k and 0 everywhere else. Hence,

$$I(f) = I\left(\sum_{i=1}^{n} \psi_i \cdot f\right) = \sum_{i=1}^{n} I(\psi_i \cdot f) \leqslant \sum_{i=1}^{n} \mu_0(O_i) \leqslant \sum_{i=1}^{\infty} \mu_0(O_i) \tag{7}$$

Because this holds for each f, we may take the supremum over f and find that $\mu_0(O) \leqslant \sum_{i=1}^{\infty} O_i$, which gives us countable monotonicity.

Now, we will prove finite additivity for two disjoint open sets O_1 and O_2, which allows for repeated use and the application to any finite number of open sets. Suppose $O = O_1 \cup O_2$. We already know from the previous part of the proof that

$$\mu(O) \leqslant \mu(O_1) + \mu(O_2) \tag{8}$$

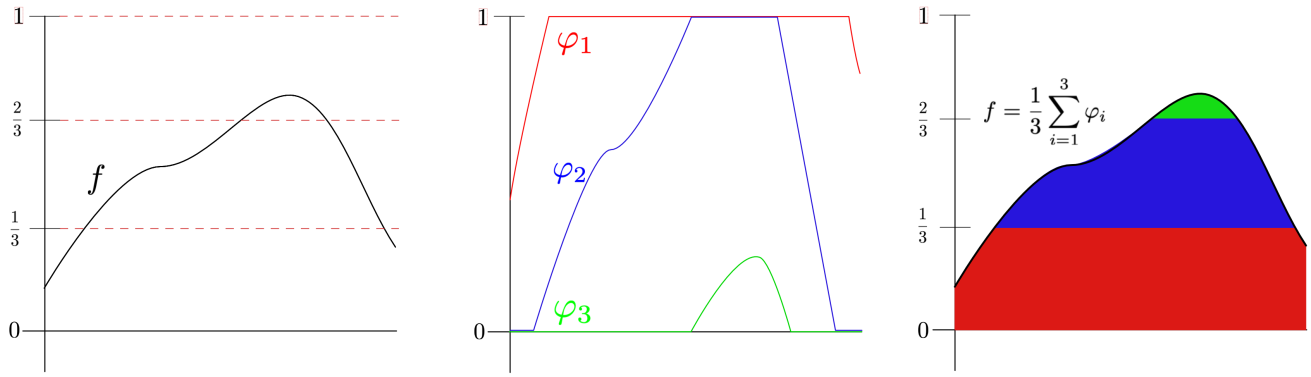


Figure 2: This graphic demonstrates the method of constructing the φ. Here, n is picked to be 3. On the left most graph, we see the partitioning of the $[0, 1]$ interval along with f, in the middle graph we see the φ_i, each of which can be pictured as the function constrained to their respective interval, then stretched out ot the entire $[0, 1]$ interval, and on the right most graph,, the black line above the color region represents the sum of the φ with the colors demonstrating the contribution from each φ in the sum.

so we simply need to prove the other inequality Then we know that f on O is equal to $\dot{f} \mid_{O_1} + \dot{f} \mid_{O_2}$ where $\dot{f}$ denotes an extension of f to 0 outside its domain. Fo each i, we know $f \mid_{O_i} \in C_c(X)$ and therefore we find that

$$\mu_0(O) \geqslant I(f) = I(f \mid_{O_1}) + I(f \mid_{O_2}) \tag{9}$$

and hence taking the supremum over all f by the definition of μ_0 in equation (6) gives us the desired inequality

$$\mu_0(O) \geqslant \mu(O_1) + \mu(O_2)$$

so along with equation (8) we have finite additivity.

Next, we need to certify that μ_0 has regularity, i.e. that it is a *Radon* premeasure. Inner regularity is established from the definition of the measure, since we may take K as the support of the measure and therefore there must be some open set U such that $U \subseteq K \subseteq O$ by local compactness, and outer regularity is established by taking progressively smaller open sets by Hausdorffness around the closure of O and then picking a function which is 1 on the closure of O and 0 outside the progressively smaller open sets.

We now know from our section on the exploration of premeasures that μ, the restriction of μ_0's extension to the Borel sets, matches μ_0 on open sets and is a Radon measure. We now only need to prove that for any $f \in C_c(X)$ with $f(x) \in [0, 1]$ for all x we have

$$I(f) = \int_X f d\mu$$

In order to prove this, let f be such a function and pick $n \in \mathbb{N}$. Now define the functions $\{\varphi_i\}_{i=1}^n$ where $\varphi_i : X \to [0, 1]$ with

$$\varphi_i(x) = \begin{cases} 1 & \text{if } f(x) \in \left[\frac{i}{n}, 1\right] \\ nf(x) - (k-1) & \text{if } f(x) \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \\ 0 & \text{if } f(x) \in \left[0, \frac{k}{n}\right] \end{cases}$$

These functions essentially truncate f to an interval ranging between integer multiples of $1/n$, so that when f is outside those integer values, the new function stops at the last value f obtained in the interval, and then stretches that interval to be 0 to 1, and moves the function along with it. It is clear from figure 2 that $f = \sum_{i=1}^n \frac{\varphi_i}{n}$, is and that the φ_i are continuous.

X is compact and Hausdorff, so it is locally compact. Therefore, find an open set O such that the support of f is contained in O and the closure of O is compact. Define a sequence of sets O_i by $O_0 = O$, $O_{n+1} = \varnothing$ and for other O_i define

$$O_i = \left\{ x \in O \;\middle|\; f(x) > \frac{i-1}{1} \right\} \tag{10}$$

We can see that the support of φ_i is a subset of $\bar{O}_i \subset O_{i-1}$ and that $\varphi = 1$ on O_{i+1} which allows by the monotonicity of i that

$$\mu(O_{i+1}) \leqslant I(\varphi_i) \leqslant \mu(O_{i-1}) = \mu(O_i) + [\mu(O_{i-1}) - \mu(O_i)]$$
$$\text{and} \tag{11}$$
$$\mu(O_{i+1}) \leqslant \int \varphi_i d\mu \leqslant \mu(O_{i-1}) = \mu(O_i) + [\mu(O_{i-1}) - \mu(O_i)]\,.$$

These two inequalities arrive at the result that

$$-2\mu(O) \leqslant \sum_{i=1}^{n} \left[I(\varphi_i) - \int_X \varphi_i d\mu \right] \leqslant 2\mu(O) \tag{12}$$

which tells us that

$$\left| I(f) - \int_X f d\mu \right| = \left| I\left(\frac{1}{n}\sum_{i=1}^{n} \varphi_i\right) - \int \frac{1}{n}\sum_{i=1}^{n} \varphi_i d\mu \right| < \frac{2\mu(O)}{n}$$

and therefore pick n arbitrarily large and because $\bar{O}$ was compact, $\mu(O) < \infty$. Hence the difference between I and the integral of f is arbitrarily small, so they are equal. This proof is omitting some detail, but for full proof refer to [3, p. 459] . ☐

4 Topological Group

As we move into this section of the paper, we are beginning to approach topics specific to our end goal, the construction of a Haar measure on compact groups. Since in the previous sections, we have found ways create measures, it makes sense that here we should try to understand on what spaces we are assuring the existence of a measure , namely compact topological groups:

Defintion 4.1. Let G be a group with a Hausdorff topology, with the group operation $g \cdot h$, inverse function g^{-1} and identity e. G is a *topological group* if the functions

$$m : G \times G \to G; \quad m(g, h) = gh \quad \text{and} \quad i : G \to G; \quad i(g) = g^{-1}$$

are each continuous functions. We will often write $m(x, y)$ as $x \cdot y$ and $i(g)$ as g^{-1} for ease of notation.

The imposition of continuity on the group operation and inversion imply that the topology of the group in certain manners respects the group's actions on itself. This is codified in several important properties of topological groups we will touch upon, later in subsection 4.1 however for now, let us understand a particularly important example of a topological group, which we will use a good deal for the construction of the Haar measure on other topological groups.

Example 4.1. Suppose that $\mathcal{L}(E)$ is the space of linear functions over a Banach space E. Then the *General Linear Group*, denoted $\mathrm{GL}(E)$, is defined as

$$\mathrm{GL}(E) = \{S \in \mathcal{L}(E) \mid S \text{ is invertible}\}$$

and $\mathrm{GL}(E)$ is a group, with the operation composition of functions, denoted by $\circ$ and the inversion function ${}^{-1} : S \to S^{-1}$ as given by the definition of $\mathrm{GL}(E)$.

Whenever E is a field, for example, $\mathrm{GL}(E)$ takes the form of multiplication by a constant in E, equivalent to the 1 dimensional matrix. Similarly, if E is field then $\mathrm{GL}(E^n)$ is group of matrix multiplications by matrices with non-zero determinants.

As for proof that $\mathrm{GL}(E)$ is a group, we have the existence of inverses by definition, associativity, or the fact that $A \circ (B \circ C) = (A \circ B) \circ C$ in generality for any functions, so it holds in the specific instance for the elements of the general linear groups, and the identity clearly exists, and it is the identity map. Hence $\mathrm{GL}(E)$ is a group. Now, let us demonstrate that it does have a topology as well which respects its group operations, allowing it to be a topological group.

Theorem 4.1. [3, p. 478] Let E be a Banach space. Then $\mathrm{GL}(E)$ is a topological group with respect to the composition and inverses under the subspace topology induced by the operator norm on $\mathcal{L}(E)$.

Proof. First, note that immediately from the definition of the norm, we have

$$\|S \circ T\| \leqslant \|S\| \circ \|T\| \tag{13}$$

For any $T, T', S, S' \in \mathrm{GL}(E)$, and $\epsilon > 0$ by the triangle inequality and equation (13) we have

$$d(T \circ S, T' \circ S') = \|T \circ S - T' \circ S'\| \leqslant \|T\| \cdot \|S - S'\| + \|T - T'\| \cdot \|S'\| \leqslant \|T\| \cdot d(S, S') + \|S\| \cdot d(T, T')$$

and therefore pick $\delta = \min\left\{\frac{\epsilon}{2\|T\|}, \frac{\epsilon}{2\|S\|}\right\}$, which grants that $d(T \circ S, T' \circ S') < \|T\|\frac{\epsilon}{2\|T\|} + \|S\|\frac{\epsilon}{2\|S\|} = \epsilon$, and therefore composition is continuous.

The continuity of the inverse is slightly more involved. First, we let $S \in \mathrm{GL}(E)$ and $\|S - \mathrm{Id}\| < 1$. Then we claim that $\|(\mathrm{Id} - S)^{-1}\| \leqslant (1 - \|S\|)^{-1}$. Because $\|S\| < 1$, the power series $\sum_{i=1}^{n} \|S\|^k$ converges, which is a Cauchy sequence so by completeness we know that $\sum_{k=1}^{\infty} S^k$ converges in $\mathcal{L}(E)$. Then we know that

$$(\mathrm{Id} - S) \circ \left(\sum_{k=1}^{n} S^k\right) = \mathrm{Id} - S^{n+1} \tag{14}$$

and therefore the sequence $\sum_{k=1}^{\infty} S^k$ converges to $(\mathrm{Id} - S)^{-1}$, since S^{n+1} eventually goes to zero, and therefore the product above equals Id in the limit. By the geometric series limiting formula, we get that $\|\sum_{k=1}^{\infty} S^k\| \leqslant \frac{1}{1-\|S\|}$, and therefore the desired result.

Now, we know that

$$S^{-1} - \mathrm{Id} = (\mathrm{Id} - S)S^{-1} = (\mathrm{Id} - S)(\mathrm{Id} - (\mathrm{Id} - S))^{-1}$$

which along with equations (14) and (13) gives us

$$d(S^{-1}, \mathrm{Id}^{-1}) = \|S^{-1} - \mathrm{id}\| = \|(\mathrm{Id} - S)(\mathrm{Id} - (\mathrm{Id} - S))^{-1}\| \leqslant \|\mathrm{Id} - S\| \frac{1}{1 - \|\mathbf{1} - S\|)} = \frac{d(\mathrm{Id}, S)}{1 - d(\mathrm{Id}, S)} \tag{15}$$

therefore pick $\delta > 0$ such that $\frac{\delta}{1-\delta} < \epsilon$ for any ϵ and therefore we know that the inverse function is continuous at the identity.

Now for some other points $S, T \in \mathrm{GL}(E)$, ensure that δ is such that $d(S, T) < \|T^{-1}\|^{-1} < \delta$ and insert $T^{-1}S$ into equation (15) to find that

$$\|S^{-1}T - \mathrm{Id}\| \leqslant \frac{\|T^{-1}S - \mathrm{Id}\|}{1 - \|T^{-1}S - \mathrm{Id}\|}$$

which along side the trivial algebraic identities $S^{-1} - T^{-1} = (S^{-1}T - \mathrm{Id})T^{-1}$ and $T^{-1}S - \mathrm{Id} = T^{-1}(S - T)$, we find that

$$\|S^{-1} - T^{-1}\| \leqslant \frac{\|T^{-1}\|^2 \cdot \|T - S\|}{1 - \|T^{-1}\| \cdot \|T - S\|}$$

which implies continuity at T, and therefore we find that inversion is continuous everywhere. This tells us that $\mathrm{GL}(E)$ is a topological group under the norm metric, as we sought to prove. ☐

4.1 Translations in Topological Groups

Because the elements of any topological group are group elements, they act on the other elements by multiplication. This leads to natural understandings of methods of shifting functions, set, and measures around a group called *translations*. Here we will briefly define these translations since they are one of the more interesting aspects of topological groups and one of the ways in which topological groups are unique. Generally, each translation takes an object, whether that is a set, a function, or a measure, and multiplies whatever is contained within that object by elements $g \in G$. For sets, we will multiply the elements by g, for functions, we will multiply the input of the function by g, and for measures, we will translate the set inside the measure by g. As we define these, we will think about them in terms of groups to understand why these translations appear as a relatively natural concept that one might want to study.

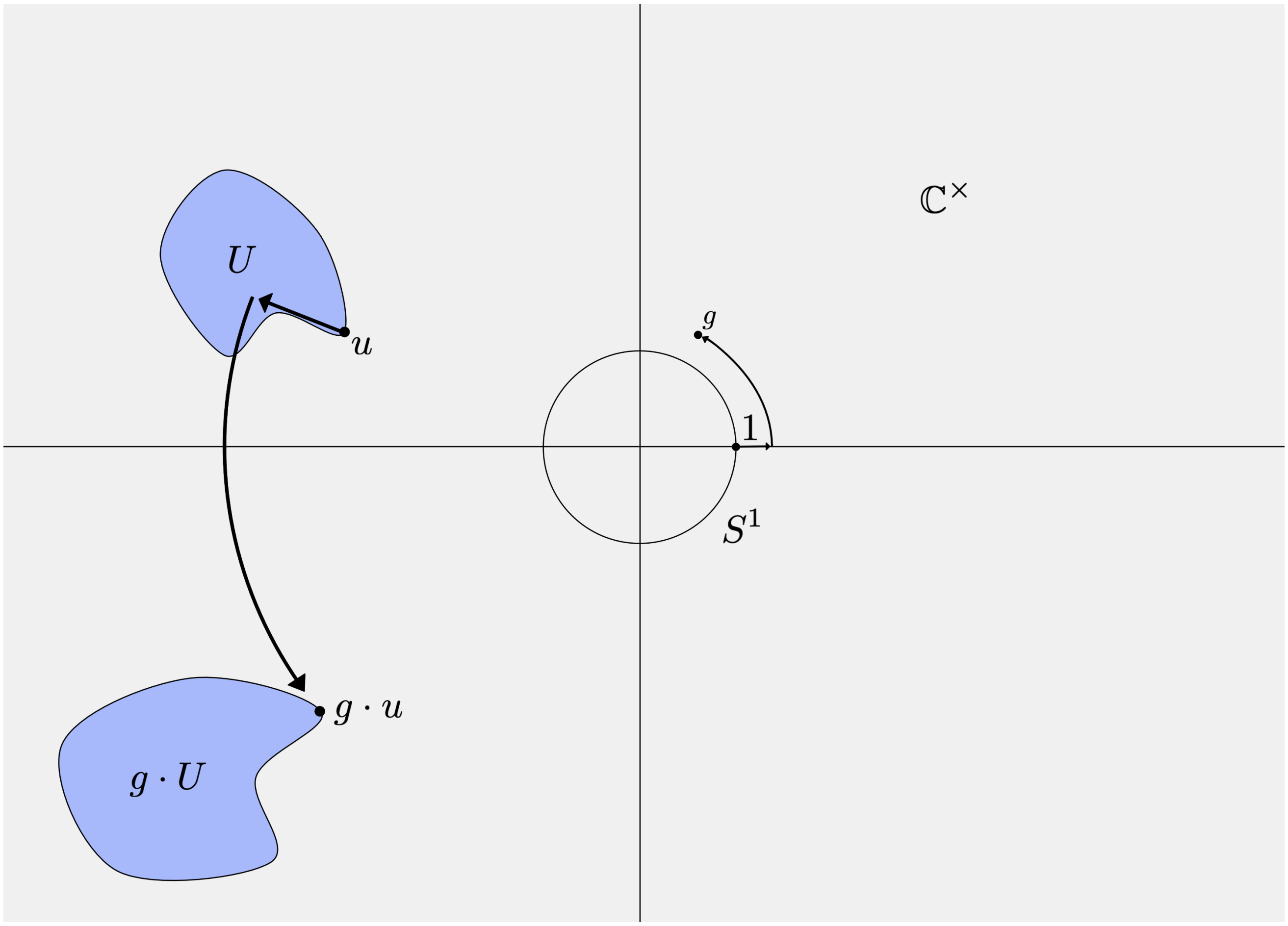


Figure 3: This figure depicts a set translation by the element in $\mathbb{C}^\times$ shown as g. The arrow pointing outwards from the origin represents the scaling of U under the set translation to $g \cdot U$, determined in proportion to the scaling of 1 towards g, and the rotation is the same angle as from 1 to g when measured from the center point. This gives an intuitive understanding of the movement of a set under translation by an element in $\mathbb{C}^\times$. The multiplication by g of a single point u at the bottom right of U is shown as well for context.

Defintion 4.2. For any $g \in G$ and $U \subset G$, define the *set translation* of U as

$$g \cdot U := \{g \cdot u \mid u \in U\} \tag{16}$$

This looks and acts very much like *co-sets* from typical group theory. For an intuitive understanding of these set translations we picture g as an element in the multiplicative group $C^\times$, then $g \cdot U$ would be a rotated and scaled, with U rotated by the unit vector in the direction of g and scaled by the magnitude of g. An example of this is provided in Figure 3. One important fact about set-translation is that it preserves open and closed sets. We will first prove this fact, then motivate it somewhat afterwards.

Theorem 4.2. If U is an open subset of the topological space G and $g \in G$ then $g \cdot U$ is open in G.

Proof. By the definition of the topological group, the multiplication operator, denoted by $m : G^2 \to G$ is continuous, and for any $h \in G$, the constant function $h' : G \to G$ defined as $h'(g) = h^{-1}$ is continuous as it is constant. Therefore the composition of these two functions and the identity, $m_h(x) = m(h'(x), x) = m(h^{-1}, x))$ is continuous. Hence, $m_h^{-1}(U) = h \cdot U$ is open as the inverse image of a continuous function. $\square$

This helps give us a sense of the fact that on a topological group, the topology is in some heuristic sense also symmetric in the same ways that the group is, in that an open set around one point can be turned into an open set around any other point.

Defintion 4.3. Let G be a topological group, for any $f \in C(G)$ define the left *translation* by g and right translation by g of f, denoted ${}_gf$ and f_g respectively, as

$$
{}_gf(x) = f(g \cdot x) \text{ and } f_g(x) = f(x \cdot g) \tag{17}
$$

Note that by the result for set translation in Theorem 4.2, we find that if f is continuous, then so are f_g and ${}_gf$. This is because $f^{-1}(g \cdot U) =_g f^{-1}(U)$ and $f^{-1}(U \cdot g) = f_g^{-1}(U)$, which so if U is open then $g \cdot U$ are open and $U \cdot g$ are open and hence so are the preimages of U by the two translates, which gives continuity of the translations of f. Finally, we will define a similar construction for measures:

Defintion 4.4. Let G be a compact topological group. A Borel measure $\mu : \mathcal{B}(G) \to [0, \infty)$ is *left invariant* if ${}_g\mu(A) = \mu(A)$ for all $g \in G$ and $A \in \mathcal{B}(G)$, where ${}_g\mu(A) = \mu(g\cdot)$ is the *left translation* of the measure μ.

It is somehow natural in many instances to want our measures to be invariant under these translations. Because groups are often understood as methods of studying symmetry, it would make sense that we would want our understanding of size to be invariant under the group action, since the new set translation from which we derive the measure of the original set would be symmetrical to the original set. Additionally, because of the rigid structure of a group, $g \cdot A$ is always in bijection with A, so they would always have the same cardinality, and we already stated that the set translations have the same topology as the original sets, which intuitively says that they have the same mechanics of distance. These two facts lead us heuristically to want to assign $g \cdot A$ and A with the same size, which in practice looks like measure invariant under translation. In order to achieve this goal, we will develop a method for G acting on other spaces, including the continuous functions over G through translations.

4.2 Representations of Topological Groups

In the previous section, we tried to understand the action of elements of G on different aspects of G itself. In this section of G, we will use the concept of group homomorphism to understand actions of g on other objects, namely Banach Spaces, as defined in Definition 3.2.

In a representation, each element of a group G is mapped to a linear functional on the space E, sort of like a matrix in $\mathbb{R}^n$, and can stretch, flip and turn the space. For the representation to respect groups well, we need it to preserve the operation of the group G, that is, we need it to be a homomorphism. This implies that we are not simply mapping into the linear functional space of E, but more specifically the general linear group as defined in Definition 4.1.

Defintion 4.5. Let G be a topological group and E be a Banach space. A group homomorphism $\pi : G \to \mathrm{GL}(E)$ is a *representation* of G on E.

Given a representation π on E, we can also create a representation on E^*, the dual space (see Definition 3.4) of E using the representation on E. We can achieve this in the following manner: for any linear functional ψ, our the new representation on E^* will skew the input of E by $\pi(g^{-1})$. We can write this idea down in the following manner:

Defintion 4.6. Let G be a topological group and E a Banach space and $\pi : G \to \mathrm{GL}(E)$ a representation of G on E. Then the *adjoint representation* $\pi^* : G \to \mathrm{GL}(E^*)$ is a representation of G on E^* with

$$
\pi^*(g)\psi = \psi \circ \pi(g^{-1}) \tag{18}
$$

for any $\psi \in E^*$.

The fact that this is a homomorphism follows from the homomorphism properties of the original representation, π. In this sense, the linear function defined by the image of L under the representation of g, which we will denote ψ' will map $\psi'(\pi(gx)) = \psi(\pi(g^{-1}g)\pi(x)) = \psi(\pi(x))$. Therefore, this new representation is essentially transforming the input space so that $\pi(e)$ lies over $\pi(g)$ and everything else moves rigidly alongside that transformation.

Given that we came to the conclusion when considering translations of measures on topological groups G that we might want some sort of invariance property so that the measure of sets is invariant under multiplication by g. Given that in section 3 we established that we can form measures that align with the values of elements of the dual space of $C_c(X)$ for some space X, if we can prove the invariance of some linear functional under a group in some manner, we will be able to generate a measure with an invariance property as well. Therefore, let us define a sort of invariance for representations.

Defintion 4.7. Let G be a topological group, E a Banach space, and $G \to \mathrm{GL}(E)$ a representation of G on E. A subset $K \subset E$ is said to be *invariant* under π if $\pi(g)(K) \subset K$ for all $g \in G$. A point $x \in E$ is *fixed* under π if $\pi(g)x = x$ for all $g \in G$.

An easy example of such invariance could be rotation of the complex plane by the unit circle, with $\pi(g) = g \cdot x$. Then we find that a circle of any radius around center 0 is invariant, since if we let R_ϵ be the ring of radius ϵ around 0, then $a \in R_\epsilon$ implies $\|a\|$, and then if $s \in S^1$, $\pi(s)a = sa$, and $\|sa\| = \|s\|\|a\| = \|a\|$ and hence $sa \in R_\epsilon$. In particular, this implies that $\{0\} = R_\epsilon$ is invariant, which proves that 0 is fixed under the representation. As we continue to build up this idea of invariance, before moving on specifically to the set up which will give us the Haar measure, we must prove that given some invariant subset, there is a fixed point when the input group is compact.

Lemma 2. Let G be a compact topological group, E a Banach space and $\pi : G \to$
a representation of G on E. Assume that there is a nonempty, convex weak-$*$ compact subset K^* of E^* that is invariant under π^*. Then there is a functional $\psi \in K^*$ fixed under π^*

The proof of this lemma is complex and requires a good number of concepts which will not be covered in this paper, including convexity, Zorn's Lemma. A sketch of the proof is provided below, but not in great detail. For a more detailed proof, refer to [3, p. 481].

Proof. If we let $\mathcal{F}$ be the collection of closed (and some other properties) subsets of K^* which are invariant under K^*, we may define a partial ordering by the subset relation. Using Zorn's Lemma and some additional properties assigned to $\mathcal{F}$, we know that there must exist some $K_0^* \in \mathcal{F}$ such that there is no $K' \in \mathcal{F}$ such that $K \subsetneq K_0^*$. This K_0^* must be weak-$*$ compact as the intersection of closed sets in a compact space.

We then claim that K_0^* must consist of precisely one functional. If there are two functionals, one can then construct more functionals which must be in the set K_0^* and use these to construct a closed subset of $K' \subset K_0^*$ such that K' has all the properties necessary and sufficient to be an element of $\mathcal{F}$. Therefore K_0^* is not the minimal such set in $\mathcal{F}$. Because K_0^* is a singleton set and invariant as the intersection of invariant sets, it is itself invariant under π^*, which means that the only element in K_0^* must always be mapped back to itself, and hence is a fixed point under π^*. ☐

4.3 Kakutani's Fixed Point Theorem

Equipped with the results of the previous lemma, about fixed points of general representations of compact topological groups, we will now apply those results to the continuous functions over compact groups to find the linear functional which will give rise to our invariant measure. First, we will specific the representation of which we will find the fixed point of later to establish invariance.

Defintion 4.8. Let G be a compact topological group, with $C(G)$ as the continuous functions to the reals from G, equipped with the norm $\|f\| = \sup_{g \in G} |f(g)|$. The *regular representation* of G on $C(G)$ is $\pi : G \to \mathrm{GL}(C(G))$ where

$$\pi(g)f =_{g^{-1}} f \tag{19}$$

where $f \in C(G)$ and $g, x \in G$, and ${}_{g^{-1}}f$ denotes the left translation in Definition 4.3.

The regular representation is a homomorphism since $\pi(g) \circ \pi(h) f =_{h^{-1}g^{-1}} f =_{(gh)^{-1}} f = \pi(gh)f$. Additionally, the output of this function is a member of the General Linear Group, since $\pi(g)(\alpha f(x) + \beta h(x)) = \alpha f(g \cdot x) + \beta h(g \cdot x) = \alpha \pi(g) f(x) + \beta \pi(g) f(x)$, implying that $\pi(g)$ is a linear functional on $C(G)$, and it is invertible by $\pi(g^{-1})$ since π is a homomorphism.

Lemma 3. If G is a compact group and $\pi : G \to \mathrm{GL}(C(G))$ is the regular representation of G, then for each $f \in C(G)$ the map $\Pi_f : G \to C(G)$ is continuous, given by

$$\Pi_f(g) = [\pi(g)f] =_{g^{-1}} f. \tag{20}$$

Proof. Let $f \in C(G)$. Then for any $x \in G$ take $U_x = f^{-1}((f(x) - \frac{\epsilon}{2}, f(x) + \frac{\epsilon}{2}))$ which is open by the continuity of f and contains x. By the triangle inequality, we know for any $a, b \in U$ we have

$$|f(a) - f(b)| < \epsilon. \tag{21}$$

Now, if $m : G \times G \to G$ is the group multiplication operation, find open sets $E_x \times X_x \subset m^{-1}(U_x)$ such that $(e, x) \in X_x \times E_x$, which exist since these are the basis elements in $G \times G$.

Every $x \in X_x$ by construction, so $\{X_x\}_{x \in G}$ is an open cover of G. Therefore by the compactness of G reduce $\{X_x\}_{x \in G}$ to a finite cover $\{X_{x_i}\}_{i=1}^n$. Now define $E = \bigcap_{i=1}^n E_{x_i}$. E is the finite intersection of open sets containing the identity, and therefore is a neighborhood around the identity, e.

For every $x \in G$, we know that $x \in X_{x_i}$ for some x_i, and therefore because $m(E_{x_i}, X_{x_i}) \subset U$ by our definition of X_x and E_x we know that since $E \subset E_{x_i}$ we have for every $x \in X$, and every $g \in E$ we have $(g, x) \in U$ which, by equation (21) gives us

$$|f(g \cdot x) - f(x)| \leqslant \epsilon \tag{22}$$

But furthermore $E \cap E^{-1} \subset E$, and the inverse is a continuous function so $E' = E \cap E^{-1}$ is an open neighborhood of e, which is also contained in E and therefore again we know that $m(x, E') \subset O$. Hence we can find that

$$|\pi(g)f(x) - \pi(e)f(x)|\left|f(g^{-1} \cdot x) - f(x)\right| \leqslant \epsilon \tag{23}$$

which implies that Π_f is continuous at the identity. Therefore, for any other point $h \in G$, take the neighborhood $h \cdot E'$. This is open by theorem 4.2 and contains h because $e \in E'$, and for any $y \in G$ take $x = h^{-1}y$ and $g \in h \cdot E'$, which implies $h^{-1}g \in E'$. For any ϵ apply equation (23) we find

$$|f(g^{-1}x) - f^{-1}(h^{-1}x)| = |f((h^{-1}g)^{-1}h^{-1}x) - f(h^{-1}x)| < \epsilon$$

which implies that Π_f is continuous.

□

Theorem 4.3 (Kakutani's Fixed Point Theorem)**.** [3, p. 484]
Let G be a compact group and $\pi : G \to \mathrm{GL}(C(G))$ be the regular representation of G on $C(G)$. Then there exists a probability functional $\psi \in [C(G)]^*$, i.e. a linear functional such that $\psi(1(x)) = 1$ and ψ is positive fixed that is under the adjoint action π^*, such that for any $f \in C(G)$ and $g \in G$

$$\psi(f) = \psi(\pi(g)f) \tag{24}$$

Proof. The unit ball in $[C(G)]^*$ is compact in the weak-$*$ topology.[4] Then, let K^* be the subset of probability functionals on $C(G)$. Now for any $f \in C(G)$ with $\|f\|_\infty \leqslant 1$, we have by the positivity and linearity of ψ the inequality

$$-1 = \psi(-1) \leqslant \psi(f) \leqslant \psi_1(1) = 1 \tag{25}$$

since $\psi(1) = 1$ owing to ψ being a probability functional. Therefore we find that $|\psi(f)| \leqslant 1$ and hence K^* is a subset of the closed unit ball in the the weak-$*$ topology on $[C(G)]^*$. It is a closed set because by the definition of the weak-$*$ topology, for every nonnegative $f \in C(G)$, the sets $U_f = \{\psi \in [C(G)]^* \mid \psi(f) \geqslant 0\}$ is closed, and so is the set of functionals $V = \{\psi \in [C(G)]^* \mid \psi(1) = 1\}$. The intersections of these sets give K^*, and hence K^* is a closed subset of a compact space, so K^* is compact and nonempty. We find that K^* is invariant under $\Pi = \pi^*$ and therefore we may apply our two Lemmas 2 and 3 to find that there is a fixed functional $\psi \in K^*$ such that equation (24) holds. This is what we sought to find, and hence we have proven Kakutani's Fixed Point Theorem. □

[4]This is a result known as Alaoglu's Theorem which applies to a certain type of topology called a weak topology. This is somewhat outside the scope of this paper. A full statement of this topology and Alaoglu's theorem and the proof can be found in [3]

5 Existence of Unique Haar Measures for Compact Groups

Perhaps surprisingly, this section of the paper is relatively straightforward We have done much of the dirty work already up to this point. Section 2 provided us with some basic equipment to build measures, Section 3 provided us with the idea of Linear Functionals and how to form measures from linear functionals on continuous function spaces, and finally Section 4 demonstrates how to find an appropriate linear functional. This section closely follows [3, Section 22.3], with some extra explanation and proof.

Lemma 4. Let G be a compact group and μ a Borel measure on $\mathcal{B}(G)$. Remember Definition 4.4, for any $g \in G$ we have the set function, which is the left translation of the measure, ${}_g\mu : \mathcal{B}(G) \to [0, \infty)$ by

$$ {}_g\mu(A) = \mu(g \cdot A) $$

for all $A \in \mathcal{B}(G)$. Then ${}_g\mu$ is a Borel measure, and if μ is Radon, then ${}_g\mu$ is Radon, and for all $f \in C(G)$,

$$ \int_G {}_g f d\mu = \int_G f d_g\mu \tag{26} $$

A right invariant measure is defined similarly, and the right translate is denoted μ_g.

Proof. We already proved $g^{-1}A$ and gA preserve open sets, in Theorem 4.2. In particular, this implies $g \cdot x$ is a homeomorphism and that Borel sets are preserved under $g \cdot x$. Hence, ${}_h\mu$ is defined on the Borel sets as $g \cdot A$ is a Borel set if and only if A is a Borel set.

Additionally, if U_i are some countable collection of disjoint Borel sets,

$$ {}_g\mu\left(\bigcup_{i=1}^{\infty} U_i\right) = \mu\left(g \cdot \bigcup_{i=1}^{\infty} U_i\right) = \mu\left(\bigcup_{i=1}^{\infty} g \cdot U_i\right) = \sum_{i=1}^{\infty} \mu(g \cdot U_i) = \sum_{i=1}^{\infty} {}_g\mu(U_i) $$

and therefore ${}_g\mu$ has countable additivity. □

Now that we know the translations of any measure μ are measures, let us show that there is some measure such that the translations of the measure are always equal to the measure itself.

Lemma 5. Let G be a compact group. Then there is a Radon probability measure that is left invariant and a Radon probability measure that is right invariant.

Proof. By Kakutani's Fixed Point Theorem, Theorem 4.3, we know that there exists a probability linear functional $\psi \in [C(G)]^*$ fixed under the adjoint of the regular representation of G on $C(G)$. Hence, $\psi(1) = 1$ and for any $f \in C(G)$ and $g \in G$

$$ \psi(f) = \psi(\pi(g^{-1})f) = \psi({}_{g^{-1}}f) \tag{27} $$

Now that we have a linear functional on the set of continuous functions (which have compact support because G is compact), we may apply the Riesz-Markov Theorem, Theorem 3.1, which gives a unique Radon measure μ on $\mathcal{B}(G)$ such that for any $f \in C(G)$

$$ \psi(f) = \int_G f d\mu \tag{28} $$

and therefore using equation (27) we may substitute in the left translate of f to find

$$ \int_G f d\mu = \psi(f) = \psi({}_{g^{-1}}f) = \int_G ({}_{g^{-1}}f) d\mu \tag{29} $$

then using Lemma 4 we find that

$$ \psi(f) = \int_G f d_{g^{-1}}\mu \tag{30} $$

and that ${}_{g^{-1}}$ is a Radon measure. Remember, however, that by the Riesz Markov Theorem 3.1, μ was the unique Radon measure that integrated functions to equal their value under the linear functional. Therefore we find that $\mu =_{g^{-1}} \mu$, and hence μ is invariant under the left-translation.

The same argument with all elements of the proof flipped to the opposite side applies for the right-translation. □

Now that we have these left and right invariant measures, we will show that they are one and the same, and additionally because we will have shown that any left invariant measure is equal to all right invariant measures, and vice versa, we will know that all these invariant measures are inf fact the same measure, and therefore we will have uniquenss:

Theorem 5.1. Let G be a compact topological group. Then there is a unique left-invariant probability measure μ on the Borel σ-Algebra, $\mathcal{B}(G)$ and μ is also right invariant. This implies that μ is the unique left and right invariant probability measure. We call μ the *Haar Measure* of G.

Proof. By Lemma 5 let μ be a left-invariant Radon measure and μ' be a right-invariant, and let $f \in C(G)$. Now, define $h : G^2 \to G$ as $h(x, y) = f(x \cdot y)$. h is continuous as a composition of continuous functions. Readers should remember from an analysis class that the product measure $\mu' \times \mu$ is a defined measure on G^2. Because h is continuous over the compact set G^2, we know it is integrable since it is bounded and measurable. We therefore know that

$$\int_{G^2} h d\mu' \times \mu = \int_G \left(\int_G f(x \cdot y) d\mu(y) \right) d\mu'(x) = \int_G \left(\int_G f(y) d\mu(y) \right) d\mu'(x) = \int_G f d\mu \int_G 1 d\mu' = \int f d\mu \tag{31}$$

with the removal of x from $f(x \cdot y)$ from the left invariance of μ, and the integration of 1 over G with respect to μ' equal to 1 because μ' is a probability measure. Similarly we can establish that

$$\int_{G^2} h d\mu' \times \mu = \int_G \left(\int_G f(x \cdot y) d\mu'(x) \right) d\mu(y) = \int_G \left(\int_G f(y) d\mu'(x) \right) d\mu(y) = \int_G f d\mu' \int_G 1 d\mu = \int f d\mu' \tag{32}$$

which demonstrates the same equality for μ', and thus $\int f d\mu = \int f d\mu'$.

This implies that any right-invariant measure equals any left-invariant measure, which means that any left-invariant measure equals any right-invariant measure, and so there is only one such measure. Hence, the Haar measure exists and is both left and right invariant and unique. $\square$

6 Examples and Properties of the Haar Measure

Now that we have Haar measures, let's demonstrate some of their properties and understand why their existence matters.

6.1 Haar Measures of Singleton Sets in Topological Groups

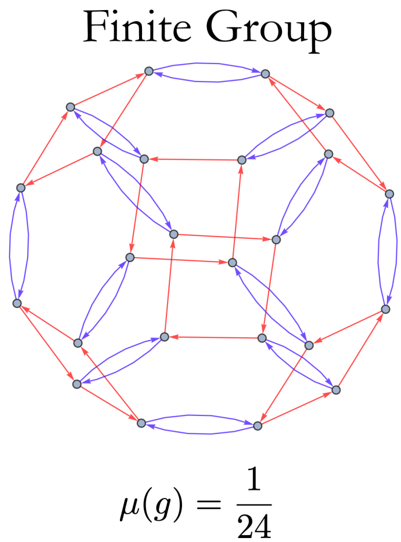
Theorem 6.1. If G is a compact topological group and μ is its Haar measure, then either G is finite and $\mu(\{g\}) = \frac{1}{|G|}$ for any $g \in G$ or G is infinite and $\mu(\{g\}) = 0$ for any $g \in G$.

Proof. Let $\mathcal{G}$ be some finite topological group. Then the group is automatically compact from finiteness, which implies the existence of a probability measure on $\mathcal{G}$. Because G is Hausdorff, it is T_1 and and individual points are closed, and hence for any $g, h \in \mathcal{G}$ we have a

$$m(\{g\}) = m(hg^{-1}\{g\}) = m(\{h\})$$

by the invariance property. This implies that, because $(G) = \bigcup_{g \in \mathcal{G}} \{g\}$, and each $\{g\}$ is disjoint from all other singleton sets, we have $\sum_{g \in \mathcal{G}} m(\{g\}) = 1$ and therefore $m(\{g\}) = \frac{1}{|\mathcal{G}|}$. This also fixes the measure of any $A \subset \mathcal{G}$ as $\frac{|A|}{|\mathcal{G}|}$, as the sum of the measures of the points in A. Therefore the only Haar measure on a finite compact group is the uniform probability measure.

Now, let G be an infinite compact group. Suppose that μ is the Haar measure on G. Then $\mu(\{g\}) = \mu\{h \cdot \{g\}\}$ for any h, and therefore again we find that $\mu(\{g_1\}) = \mu(\{g_2\})$ for any $g_1, g_2 \in G$ since we can simply multiple the singleton set $\{g_1\}$ by $h = g_2 g_1^{-1}$. If $\mu(\{g\}) > 0$ then for some countable sequence of $g_i \in G$ we get $\sum_{i=1}^{\infty} \mu(\{g_i\}) = \mu(G) = 1$ because each singleton set is disjoint. This, however, is impossible, since any constant nonzero sequence sums to ∞. For a visualization of these ideas, refer to figure 4. $\square$



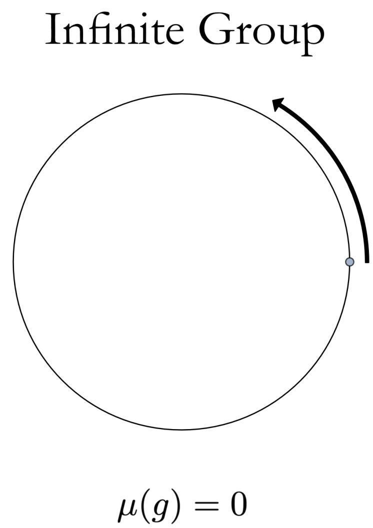


Figure 4: On the left, there is the Cayley graph of S_4 generated by Mathematica with vertices representing the elements of the group, with red arrows representing multiplication by (1234) and blue arrows representing multiplication by (12). We can see that each point is measurable because the space is Hausdorff, so points are Borel as they are closed, and that each point may be translated to any other point via repeated translation by only two elements. Therefore, an invariant measure must hold all points equal. In the case of the infinite group S^1 on the right, there is a similar dynamic where any point may be shifted to any other, but giving any point positive measure will imply that the measure on the circle is not a probability measure, which is a contradiction.

6.2 Haar Measure of Measurable Subgroups

While in generality there is no guarantee on the measurablity of a subgroup of a topological group, one can calculate the measure of a subgroup due to the fact that the co-sets of a subgroup form a partition of G.

Theorem 6.2. If G is a topological group, μ is the Haar measure of G and H is a μ-measurable subgroup, then $\mu(H) = \frac{1}{|G/H|}$ if $|G/H| < \infty$, or $\mu(H) = 0$ if $|G/H| = \infty$.

Proof. Let H be a subgroup of the compact topological group G which is equipped with the Haar measure μ. Immediately, we know that $G = \bigcup_{g\in G} g \cdot H$. Inclusion in one of these co-sets forms an equivalence relation on G, where $a \sim b$ if and only if $b \in a \cdot H$. As proof, obviously $a \in a \cdot H$ since $e \in H$ so $a = ae \in a \cdot H$, further more if $a \sim b$ then $b \in a \cdot H$ and so $b = ah$ for some $h \in H$ and hence $a = bh^{-1}$, which tells us that $a \in b \cdot H$, and therefore by definition that $b \sim a$. Finally, for transitivity suppose $a \sim b$ and $b \sim c$. Then $b = ah$ and $c = bh'$ so $c = ahh'$ and $a \sim c$. Therefore, we know that this forms an equivalence relation. Hence denote the set of (disjoint) equivalence classes as G/H.

Because each element of G/H is a left translation of H, we know that for any $g \cdot H$, $g' \cdot H \in G/H$ we have $\mu(g \cdot H) = \mu(g' \cdot H)$ and therefore because the elements of G/H are disjoint we know that $\sum_{gH\in G/H} \mu(gH) = \mu(G)$ and therefore that either G/H is infinite and $\mu(H) = 0$ or G/H is finite and $\mu(H) = \frac{1}{|G/H|}$. □

6.3 Open Subsets have Positive Measure

Theorem 6.3. If μ is a Haar measure on a compact topological group G, then for any nonempty open set $U \subset G$, the measure of U is positive, i.e. $\mu(U) > 0$.

Proof. First, construct the open cover $\{U_g\}_{g\in G}$ where $U_g := U \cdot g$, the right translate of U. This is obviously an open cover, since for whatever $h \in U$ exists, since U nonempty, so for any $g \in G$ consider the set $U_{h^{-1}g}$. By the definition of the right translate, we have $hh^{-1}g = g \in U_{h^{-1}g}$. Therefore $\{U_g\}$ covers G. Additionally, we know that the translate of an open set is open in a topological group. Therefore we know that the U_g are an open cover of G.

Now, we know that G is compact, therefore reduce U_g to a finite subcover, U_{g_i}. for $1 \leqslant i \leqslant n$. By the properties of a measure, we know that

$$\sum_{i=1}^{n} \mu(U_{g_i}) \geqslant \mu\left(\bigcup_{i=1}^{n} U_{g_i}\right) = \mu(G) = 1$$

Because μ is a Haar measure and U_{g_i} are all translations, we know that $\sum_{i=1}^{n} \mu(U_{g_i}) = n\mu(U)$. Therefore we find by the previous inequality that

$$\mu(U) \geqslant \frac{1}{n}$$

where n is the positive integer denoting the number of sets in the finite subcover of $\{U_g\}$ given by compactness and therefore the Haar measure of any open set is positive. $\square$

Corollary 6.3.1. If H is an open subgroup of a compact group G, then H has finite index.

We know this from subsections 6.2 and 6.3. The latter tells us that $\mu(H) > 0$ and the former then tells us that $\mu(H) = 0$ if $[G : H] = \infty$ and therefore $[G : H]$ cannot be infinite, and must be finite.

6.4 On the Complex Circle

Defintion 6.1. Let S^1 denote the *complex circle*, that is

$$S^1 = \{s \in \mathbb{C} \mid \|c\| = 1\}$$

with the typical norm, $\|c\| = \operatorname{Re}(c)^2 + \operatorname{Im}(c)^2$

Theorem 6.4. S^1 is a compact topological group under the d_2 metric, and therefore there exists a Haar measure μ on S^1.

Proof. Because the nonzero complex numbers are a group under multiplication, we need only to prove that S^1 is a subgroup in order to show that it is a group. Hence, let $a, b \in S^1$. Then $\|a\| = 1 = \|b\|$ and $\|ab\| = \|a\|\|b\| = 1$. Hence S^1 is closed under multiplication, and is a group.

Then we will prove further that S^1 is compact and a topological group in the subspace topology. For compactness, consider the function $f : [0, 1] \to S^1$ where $f(x) = e^{2\pi i \cdot x}$. From basic analysis, we know this function is continuous and the image is all of S^1. Additionally, $[0, 1]$ is compact and the continuous image of compact sets are compact. Hence S^1 is compact. Finally, continuity of complex multiplication is an elementary analytical proof, and will be left to the reader. Inversion which is defined as $a - bi = \frac{1}{a+bi}$ for $a + bi \in S^1$ is also trivially continuous on the unit circle.

Therefore, by Theorem 5.1, we know that there exists a unique measure μ such that $\mu(c\dot{E}) = \mu(E)$ for every $c \in S^1$. $\square$

55 We now know that such a Haar measure exists on the complex sphere, but this proof leaves something to be desired. All we now know is that the Haar measure on the complex unit circle exists. We know nothing as to its properties nor how to do any calculations on it. Therefore, we will construct the measure to demonstrate some of its properties and understand how to perform calculations with it. We will construct it using the concept of premeasures from Subsection 2.3. This will allow us to easily know the value of the Haar measure on any open or closet set, and methods for computing the measure of any Borel set.

Theorem 6.5. Let the *arc* of U denoted $U_\circ \subset S_1$ be defined as $e^{2\pi i \cdot U}$ for any $U \subset [0, 1)$. By the bijectivity of $e^{2\pi i \cdot x}$ on the domain $[0, 1)$ we know that $U_\circ \cup V_\circ = (U \cup V)_\circ$ and $U_\circ \cap V_\circ = (U \cap V)_\circ$ for any $U, V \subset [0, 1)$, and therefore define the *Set of Open Arcs* as

$$\mathcal{A} = \left\{U_\circ \mid U = \bigcup_{i \in I}(a_i, b_i) \cap [0, 1) \text{ such that } a_i, b_i \in \mathbb{R}, \text{ and } I \text{ an index set of any order}\right\} \tag{33}$$

and equip the set of open arcs with the function $\mu_0 : \mathcal{A} \to \mathbb{R}_{\geqslant 0}$ with, for any $A_\circ \in \mathcal{A}$,

$$\mu_0(A_\circ) = \mu_0\left(\bigcup_{i=1}^{n}(a_i, b_i)_\circ\right) = \sum_{i=1}^{n}|b_i - a_i| \tag{34}$$

where the (a_i, b_i) are disjoint representatives of A such that for any $i \neq j$, $(a_i, b_i) \cap (a_j, b_j) = \varnothing$. Then μ_0 is a Radon premeasure, $\mathcal{A}$ is the topology of S^1 and the induced measure is the Haar measure.

We will not prove this theorem, because most of the proof would be analytic or geometric and not particularly illuminating. Essentially, what can be proved is that each open arc can be written as an open set in $\mathbb{C}^\times$ under the typical topology restricted to S^1. Additionally, what this construction is doing can be thought of in a very nature topological manner. The set up here unravels S^1 to the unit interval with equal end points, i.e. $[0,1]/(0 \sim 1)$, and then measures the unraveled arc with the Lebesgue measure on $[0,1]$. The fact that this coincides very closely with the Lebesgue measure ought to tell us that the Haar measure is in many respects a canonical measure of sorts that can be thought of as an analog to the Lebesgue measure in different topological groups.

6.5 On the Quaternions Sphere

In this section of the paper, we will derive the Haar measure for unit Quaternions. The intuition for the multiplicative group of unit quaternions is similar to the intuition for the multiplicative group of unit complex numbers. The quaternion group is actually a subset of the fourth dimensional real numbers, $\mathbb{R}^4$ equipped with a nonstandard form of group operation. If we let $(a,b,c,d) \in \mathbb{R}^4$, then we may rewrite (a,b,c,d) as $a + bi + cj + dk$. The product of two quaternions distributes in the expected manner and real constants commute as well, so $j \cdot r = r \cdot j$, etc. however, the elements i, j, k and their multiples under the reals do not commute. Their products are defined by the relations $i^2 = j^2 = k^2 = -1 = ijk$. This yields the following product table for i, j, k and 1: (remember, all real scalars commute so those products are already defined)

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Using these relations, distributivity, and commutativity of real scalars, the product of two quaternions is defined. We will not go through the rather tedious steps of checking that all group relations are, in fact, group relations and so fourth. Most of it is in one way or another inherited from the quaternion group Q_8 and the properties of the real numbers. We will, however, establish the following norm on the quaternions: $\|a+bi+cj+dk\| = a^2+b^2+c^2+d^2$. This is a norm and does respect quaternion multiplication, and if x and y are quaternions then $\|x\|\|y\| = \|xy\|$. Again, these facts ultimately can be proven by just writing the definitions out. We can additionally find that if we extend the idea of complex conjugation to quaternions, and $x = a + bi + cj + dk$ is a quaternion, then the conjugate of x is $\bar{x} = a - bi - cj - dk$ and $x\bar{x} = \|x\|$, which implies that $x^{-1} = \frac{\bar{x}}{\|x\|}$ is the inverse of x. This is unsurprising, as in many ways it acts just the same as the complex numbers.

We will not prove anything in particular about this example, but we will claim that the unique quaternions are the spherical equivalent of the unit complex number circle, and that the measure on the unit quaternions, which exists since it is a compact group, is essentially an instantiation of the Lebesgue measure.

7 Applications of the Existence of Haar Measures

7.1 There are no compact infinitely countable topological groups

Suppose that G were a countably infinite compact topological group, then let μ be its Haar probability measure and list the elements of G as g_i. This tells us that $\sum_{i=1}^{\infty} \mu(\{g_i\}) = \mu(G) = 1$ since each $\{g_i\}$ is disjoint and thus we may apply the countable additivity of measures. However, because G is an infinite topological group, $\mu(\{g_i\}) = 0$, as calculated in subsection 6.1.Therefore, $\sum_{i=1}^{\infty} 0 = 1$ which is obviously false. Hence no such G can exist.

This is helpful in several respects. First, note that this result implies that there is no topology under which the integers are both a topological group and compact, i.e. any topology under which addition and inverses are continuous on the integers leave $\mathbb{Z}$ as compact. This is also true of $\mathbb{Q}^+$ and $\mathbb{Q}^\times$ without zero.

Further, if we know that a space is compact and countable, then it is not a topological group. Hence, because any subgroup of a topological group is a subgroup, if we have a compact group, there are no closed countable subgroups. If we let $\Xi = \{e^{2\pi i q} \mid q \in \mathbb{Q}\}$ be the roots of unity as a subgroup of S^1, we find that this must not be closed, since otherwise this would be a countable closed subgroup.

Readers may have also noticed similarity in the proofs for measures of single points and measures of measurable subgroups in subsections 6.1 and 6.2 respectively. There is hence a correlated result here for groups which contain subspaces with countable co-set partitions.

Suppose H is a subgroup of a compact topological group G such that H is measurable and G/H is countable. Then by a similar proof to before, $\mu(G) = \sum_{gH \in G/H} \mu(G \cdot H) = 0$, which is a contradiction. Therefore, for any compact group G, any subgroup H such that G/H is countable must not be measurable.

For example, consider the group $\mathbb{QC} := \{q \in \mathbb{C}^\times \mid \|q\| \in \mathbb{Q}\}$ under multiplication. We know this is a subgroup of $\mathbb{C}^\times$ if there is closure under multiplication and inverses, which there is since $\|a\|\|b\| = \|ab\|$ so if $a.b \in \mathbb{QC}$ then $ab \in \mathbb{QC}$, and therefore $\|a^{-1}\| = \frac{1}{\|a\|} \in \mathbb{Q}$.

Now consider S^1 as subgroup of $\mathbb{QC}$. Clearly, $\mathbb{QC}/S^1 = \mathbb{Q}^+$ and is countable, which implies that under any topology in which $\mathbb{QC}$ is compact, S^1 is not measurable. This tells us that under most typologies that make $\mathbb{QC}$ a topological group, $\mathbb{QC}$ is not compact, since intuitively S^1 should be closed under Hausdorff spaces, since we could likely construct a homeomorphism that acts like polar coordinates to $S^1 \times \mathbb{Q}^+$ under some topology, and S^1 would by $S^1 \times \{1\}$ which would be closed if the topology on $\mathbb{Q}^+$ is Hausdorff, which it should be since the topology on $S^1 \times \mathbb{Q}^+$ is Hausdorff. Then S^1 is closed, and closed sets are measurable under the Haar measure.

7.2 Group invariant metrics

In this section, inspired by a series of exercises from [3, p. 488], we will demonstrate that if given a topological group equipped as a metric space, there is a metric which is invariant under G, i.e. $d(x, y) = d(g \cdot x, g \cdot y) = d(x \cdot g, y \cdot g)$. In certain ways, this should not be surprising. If the size of spaces does not chance when translated, we ought to expect the distances in that shape to not change either. To do this, we need to provide some technology for the product measure and the invariance of the integral of translations for continuous functions with respect to the Haar measure. We will take on the latter first.

Theorem 7.1. Let G be a compact topological group and μ be its Haar measure, and f is any measurable function. Then for any $g \in G$,

$$\int_G f(g \cdot x) d\mu(x) = \int_G f d\mu \tag{35}$$

Proof. We know that $\int_G f d\mu = \sup_{s \leqslant f} \int_G s d\mu$ where s are simple functions of the form $s(x) = \sum_{i=1}^n c_i \chi_i(x)$ where χ_i are indicator functions of respective measurable sets E_i. Then, for any s indexed in the supremum, we find that the function $s(g \cdot x) \leqslant f(g \cdot x)$ and that $\int s d\mu = \sum_{i=1}^n c_i \mu(E) = \sum_{i=1}^n c_i \mu(g \cdot E) = \int s(g \cdot x) d\mu$ which implies that the equality in equation (35) holds for simple functions.

Now by the earlier equalities we know

$$\int_G f(g \cdot x) d\mu = \sup_{s \leqslant f(g \cdot x)} \int_G s d\mu = \sup_{s(g \cdot x) \leqslant f(g \cdot x) d} \int_G s(g \cdot x) d\mu = \sup_{s \leqslant f} \int_G s d\mu = \int_G f d\mu$$

therefore we have proven the desired property, as written in equation (35), which concludes the proof. ☐

This is of course not a very surprising result. If we consider $f^{-1}x$ for any x in $\mathbb{R}$, which is measurable because f is a measurable function, we know that $f_g^{-1}(x) = g^{-1} \cdot f^{-1}(x)$, which implies that the preimage of any x has the same measure under any measurable function and its translations, by the invariance of the measure. If we heuristically think of integration as adding up all the measures of these preimage sets, then this theorem is essentially the same as saying sums are commutative, as all g really does is shift around where the $f^{-1}(x)$ sets fall on G.

Now, we will think about the topological group $G \times G$, and the product measure $\mu \times \mu$. We will claim that $\mu \times \mu$ is not only a well defined measure on $G \times G$ but also that it is the product measure.

Theorem 7.2. Let G be a compact topological group with Haar measure μ. Then G^2 is a compact topological group and $\mu \times \mu$ is its Haar measure.

Proof. $G \times G$ is compact as the product of compact spaces, and it is a group as the direct product of two groups. Let $m((g,h),(x,y)) = (gx, hy)$ and $ii(x,y) = (x^{-1}, y^{-1})$. Then we find that each is continuous as each component is independently continuous, which is inherited from the original group G. Hence, $G \times G$ is a compact topological group and therefore there exists some Haar measure ν.

Now we seek to prove that $\nu = \mu \times \mu$. In order to prove this, we must simply prove that $\mu \times \mu$ is invariant, since the product of two Radon measures is itself a Radon measure, and since the product of probability measures is a probability measure. Therefore consider $\mu \times \mu(G \times H)$. By basic measure theory, we know that $\mu \times \mu(G \times H) = \mu(G) \cdot \mu(H)$ [5, p. 277]. Therefore we find that $\mu \times \mu(G \times H) = \mu(G) \cdot \mu(H) = \mu(g \cdot G) \cdot \mu(h \cdot H) = \mu \times \mu((g,h) \cdot G \times H)$. A similar argument follows for right invariance of measurable rectangles. We will not prove this claim, but we claim that this is enough to ensure left and right invariance in general for measurable sets. This essentially follows from the construction of product measures. To read more, see [5, p. 276]. We hence find that by the uniqueness of the Haar measure, $G \times G$ has the Haar measure $\mu \times \mu$. ☐

This is again a not very surprising result either, since in general attributes or functions of direct products are often determined by the component groups. This does have some interesting results though. If we remember that we calculated the Haar measure on S^1 in Subsection 6.4, we can use this product measure rule to compute the Haar measure on the torus $S^1 \times S^1$, which will be generated, essentially, from the product of open arcs on the surface of the torus.

With these two theorems under our belt, we are ready to prove the main result of this subsection. Namely, that a compact metric topological group has a group invariant metric, i.e. a metric such that $d(x,y) = d(g \cdot x, g \cdot y) = d(x \cdot g, y \cdot g)$.

Theorem 7.3. Let (G, d) be a compact topological group equipped with a metric. Then, there exists a group invariant metric $d' : G^2 \to \mathbb{R}_{\geqslant 0}$ defined as

$$d'(x,y) = \int_{G \times G} d(gxh, gyh) d(\mu \times \mu)(g,h) \tag{36}$$

Proof. Because G is a metric space with d, we know that $d : G^2 \to \mathbb{R}$ is continuous, and is therefore integrable. Therefore, define $d'(x,y)$ as in equation (36). Now let us prove this is a metric and that it is invariant. First, we will prove that $d'(x,y) = 0$ if and only if $x = y$. First, if $x = y$ then $\int_{G \times G} d(gxh, gyh) d\mu \times \mu = \int_{G \times G} 0 = 0$ because $x = y$ implies $gxh = gyh$ for any $(g,h) \in G^2$. Furthermore, if $x \neq y$ then $gxh \neq gyh$ for any $(g,h) \in G^2$ which implies that $d(gxh, gyh) > 0$ for all $(g,h) \in G^2$. Because G^2 is compact, the function attains its minimum, $\epsilon > 0$ somewhere, so $\int_{G \times G} d(gxh, gyh) \geqslant \mu \times \mu(G^2)\epsilon > 0$. Therefore, $d'(x,y) = 0$ if and only if $x = y$.

Symmetry is immediate from the definition and the symmetry of d:

$$d'(x,y) = \int_{G^2} d(gxh, gyh) = \int_{G^2} d(gyh, gxh) = \int_{G^2} d(y,x)$$

Similarly, the triangle inequality is immediate.

$$d'(x,y) + d'(y,z) = \int_{G^2} d(gxh, gyh) + d(gyh, gzh) \geqslant \int_{G^2} d(gxh, gzh) = d'(x,z)$$

By these three properties, we know that d' is a metric on G.

Finally we will prove invariance. Invariance in this case follows directly from theorems 7.1 and 7.2. First, to ease notation, let us write $f_{x,y}(g,h) = d(gxh, gyh)$. As the composition of continuous functions, f is always continuous, and therefore integrable. By the previous two theorems, then, we can shift the interior of f by an element of G^2. Therefore, we know that

$$\begin{aligned}\int_{G^2} d(gxh, gyh)d(\mu \times \mu) &= \int_{G^2} f_{x,y}(g,h)d(\mu \times \mu) \\ &= \int_{G^2} f_{x,y}((g,h) \cdot (g', e))d(\mu \times \mu) \\ &= \int_{G^2} f_{x,y}(gg', h)d(\mu \times \mu) \\ &= \int_{G^2} d(gg'xh, gg'yh)d(\mu \times \mu)\end{aligned} \tag{37}$$

with the second equality justified by theorem 7.1, with a similar equation possible for right invariance. Now using equations (36) and (37) we find by definition that $d'(g'x, g'y) = d'(x, y)$ and similarly for right invariance. Therefore d' is an invariant metric. $\square$

This proof gives ua a very satisfying result. Our notion of size that is invariant under translation has given us a notion of distance that is invariant under translation. Intuitively this makes sense, since size in metric spaces is determined by distance and distance in some respect is determined by size of the interval between points. Of course, that is not exactly what went on here, but it is nice that even in an abstract metric space these results hold.

7.3 Measures induced by transitive group action

This section follows closely a section on homogeneous spaces and topological groups in [2, p. 149]. Our motivation here is to extend the result we have achieved for measures on topological groups to a slightly larger class of spaces, namely compact Hausdorff spaces on which G acts transitively and continuously. Obvious these properties hold for G acting on itself by the group operation, so the class of homogeneous groups is larger than the class of topological groups which is our motivating reason for this exploration. We will start the exploration of homogeneous spaces by defining them.

Defintion 7.1. Let G be a compact topological group and K be a compact Hausdorff space. We say K is a *homogeneous* space if there exists a group action $a : G \times K \to K$ such that a is continuous and transitive.

In general, we will write the function $g : K \to K$, $g(k) = a(g, k)$ to represent the continuous homeomorphism from K to itself generated by the group action and g.

In order to prove the existence of the invariant measure on this group, we will first endow the quotient group of G/H with a topology, and we will then proceed to show that there is some H which makes G/H homeomorphic and isomorphic under G to K. First, let us define the topology on G/H, the spaces of co-sets of H in G.

Defintion 7.2. Let G be a compact Topological group and H be a closed subgroup. The quotient set topology is defined as $U \subset G/H$ is open when $U = \{x \cdot H \mid x \in V\}$ for some open set V in G.

We will leave it as a proof to the reader that this topology is compact, Hausdorff and that the map $\pi : G \to G/H$ which takes $\pi(g) = gH$ is continuous. We see trivially that by basic algebra, g acts on G/H with $g(g'H) = (gg')H$.

Theorem 7.4. Let K be a homogeneous space with the topological group G. Then there is a closed subgroup H of G such that there exists a homeomorphism $\varphi : G/H \to K$ such that $\varphi(g(g'H)) = g(\varphi(g'H))$. This is called isomorphic under the action of G.

Proof. For some fixed point $k_0 \in K$, construct the subgroup $H = \{g \in G \mid g(k_0) = k_0\}$. H is closed as the preimage of k_0 under the map $a_{k_0} : G \to K$ defined by $a_{k_0}(g) = a(g, k_0)$ which is continuous as a composition of continuous functions, and H is a subgroup by the definition of a group action.

Now define $\varphi(gH) = g(k_0)$. We will show that this is an operation preserving homeomorphism. This is a well defined function, since if $g_1H = g_2H$ then $g_1 = g_2h$ and $g_1(k_0) = g_2(h(k_0)) = g_2(k_0)$.

Then, for bijectivity, we know that $g_1(k_0) = g_2(k_0)$ if $g_2^{-1}g_1(k_0) = e$ so $g_2^{-1}g_1 \in H$ and $g_1H = g_2H$, which gives injectivity. Now, for surjectivity we know for any $k_1 \in K$ there exists some g' such that $g(k_0) = k_1$ by transitivity. Therefore pick $\varphi(g'H) = g'(k_0) = k_1$ and we have surjectivity.

Now let $U \subset K$ be open, and let $k_1 \in U$. Then there exists some g such that $g(k_0) = k_1$, and so by the continuity of a we know that $a^{-1}(U)$ has some basis set $U_1 \times U_2 \subset G \times K$ such that $k_0 \in U_2$ and therefore for all $u \in U_1$, we have $u(k_0) \in U$. By the definition of the quotient set topology, $\pi(U_1)$ and is a subset of $\varphi^{-1}(U)$ that contains gH. We can repeat this process for every g such that $g(k_0) = k_1$ to find that $\pi^{-1}(U)$ is open since there is an open neighborhood around each point contained in the set.

We therefore know that φ is a continuous bijection from G/H to K, which are compact and Hausdorff spaces and therefore φ is a homeomorphism. We now only have to prove that φ is operation preserving, which is easy since we have $g\varphi(g'H) = g(g'(k_0)) = \varphi(g(g'H))$. Therefore we find that φ is such a function as we sought and the proof is done $\square$

An important fact to note about this proof is that it does not depend on the choice of k_0 and therefore we may construct such a function for *any* $k_0 \in K$. Equipped with this method of passing a homogeneous space to a quotient set spaces of G, we can now proceed to the main result of this section of the paper.

Theorem 7.5. Let K be a homogeneous space acted on by the compact group G. Then there is a unique G-invariant Borel probability measure on K

Proof. Let G/H be the quotient set such that G acting on G/H is isomorphic to G acting on K which exists by Theorem 7.4. Let $\pi : G \to G/H$ map $\pi(g) = gH$ and let μ by the Haar measure on the compact group G, and let φ be the isomorphic homeomorphism between G/H and K. Then define the linear functional $\lambda : C(K) \to \mathbb{R}$ as

$$\lambda(f) = \int_G f(\varphi(\pi(g)))d\mu \tag{38}$$

We can understand this function as taking each element of G, placing it into its co-set gH. Then, mapping the co-set to its bijective respective other in K, and then placing that respective other into f. In some sense, it is like we are taking the function f, which is defined on K, and placing it onto G. So now, lets continue with the proof.

First, we will prove that λ is a positive linear functional. If $f \geqslant 0$ then $\int f \circ g d\mu \geqslant 0$ for any continuous function g, and therefore λ is positive. Next, we will prove that it is linear. Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in C(K)$. Then

$$\lambda(\alpha f + \beta g) = \int_G \alpha(f \circ \varphi \circ \pi) + \beta(g \circ \varphi \circ \pi)d\mu = \alpha \int_G f \circ \varphi \circ \pi d\mu + \beta \int_g g \circ \varphi \circ \pi d\mu = \alpha\lambda(f) + \beta\lambda(g)$$

Therefore $\lambda \in C(K)^*$ and is positive. Therefore we may apply the Riesz-Markov Theorem, Theorem 3.1, to find that there exists a unique measure η such that for any function $f \in C(K)$, we have

$$\lambda(f) = \int_K f d\eta$$

Now let us prove that this is a probability measure, that it is invariant, and that it is unique. For the probability measure, we know that

$$\eta(K) = \int_K 1 d\eta = \lambda(1) = \int_G 1(\varphi(\pi(g)))d\mu = \int_G 1 d\mu = 1$$

and therefore η is a probability measure on K. Now, for invariance, we will use a sequence of equations and provide justifications on the right. For any $s \in G$

$$
\begin{aligned}
\int_K f_s d\eta &= \int_G f(s\varphi(\pi(g)))d\mu \\
&= \int_G f(\varphi(s\pi(g)))d\mu && \text{Since } \varphi \text{ preserves group action} \\
&= \int_G f(\varphi(sgH))d\mu && \text{Definition of } \pi \\
&= \int_G f(\varphi(\pi(sg)))d\mu \\
&= \int_G f(\varphi(\pi(g)))d\mu && \text{Translation invariance of } \mu \\
&= \int_K f d\eta
\end{aligned}
$$

and therefore we find that η is left G-invariant, with a similar argument following for any right-group action. Therefore we know that η is the unique measure that corresponds to λ from Riesz Markov. Hence for the uniqueness of the G-invariant Radon probability measure, let us prove any other G-invariant Radon probability measure must correspond to λ

Now suppose that ν is some other G-invariant Radon probability measure. Then we seek to prove that

$$\int_K f(k)d\nu = \lambda(f) \tag{39}$$

for all $f \in C(K)$. Therefore, first by the translation invariance of ν and the fact that G is a probability measure

$$\int_K f(k)d\nu = \int_K f(g \cdot k)d\nu = \int_G \int_K f(g \cdot k)d\nu d\mu$$

therefore if we let ν' be the corresponding measure to ν on G/H then we have

$$\int_G \int_K f(g \cdot k)d\nu d\mu = \int_G \int_K f(\varphi(g\varphi^{-1}(k))d\nu d\mu = \int_G \int_{G/H} f(\varphi(gk'H))d\nu'(k')d\mu(g)$$

and then by the invariance of the Haar measure we find that this gives us

$$\int_K f(k)d\nu = \int_G \int_{G/H} f(\varphi(gk'H))d\nu'(k')d\mu = \int_G \int_{G/H} f(\varphi(\pi(g\pi^{-1}(k'H))))d\nu'(k')d\mu(g) = \int_G f(\varphi(\pi(g)))d\mu = \lambda(f)$$

This is what we sought to prove in equation (39), since by Theorem 3.1, η is the *unique* measure corresponding to λ and therefore we find that η is unique and the theorem is proven. □

This result instantly gives us some very helpful measures. I won't go into examples here, since they are someone computational, however, if we recall the Haar measures on S^1, T^2 or the unit Quaternions, we can expand each of these to any scaled image of then in their respective larger fiends. S^1 acts on any space of complex numbers which all have the same magnitude, and similarly for the unit Quaternion group. T^2 can act on any scaled version of itself where if we think of a point in T^2 as $(a, b) \in S^1 \times S^1$, then T^2 acts transitively on the Cartesian product of two spaces of complex numbers of the same magnitude. We can think of these homogeneous spaces as the circle of radius r in $\mathbb{C}^\times$, or the sphere of radius r in the quaternions, or the torus defined by two radii, r' and r' where r determines the meridian length and r' determines the longitudinal length of curves.

8 Bibliography

While I did at times read Conway [1] and Rudin [4], I ultimately didn't use much material from them, mostly relying on Royden and Fitzgerald [3] as well as Stein and Shakarchi, [5]. The idea of translation is inspired by Rudin originally, but also appears in some of the other sources, such as Diestel and Spalsbury [2]. This treatment of Haar measures most closely mirrors that of Royden and Fitzgerald, whose work is cited frequently around the paper. The last proof about group action induced measures is very different from the proof in Diestel and Spalsbury, although the proofs before it are not. I did not like their proofs, mainly because it seemed to me that they never actually proved their function was a measure, and it did not use the Riesz Markov Theorem which is actually very useful here.

References

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