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Author(s)	Zubrinic, Darko
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Lapidus zeta functions of arbitrary fractals and compact sets in Euclidean spaces

Darko Žubrinić, University of Zagreb, Croatia

This is a short survey of some of the results obtained in an extensive joint work with Michel L. Lapidus, University of California, Riverside, USA, and Goran Radunović, Univeristy of Zagreb, Croatia; see [LaRaZu].

A new class of zeta functions, for which I propose the name indicated in the title, has been discovered by Professor Michel L. Lapidus in 2009, during my lecture delivered at a conference at the University of Catania, Italy. The discovery has been disclosed to me immediately after the lecture, and it was a starting point of our joint work. The new zeta functions represent a bridge between the geometry of fractal sets and complex analysis.

Assume that A is a nonempty, bounded set in $\mathbb{R}^N$, and let $d(x, A)$ denote the Euclidean distance from $x \in \mathbb{R}^N$ to A. Fixing any $\delta > 0$, let A_δ be an open δ-neighbourhood of A. Then the *Lapidus zeta function*[1] (in [LaRaZu] we call it the *distance zeta function*) is defined as follows:

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} dx. \tag{1}$$

Here s is a complex number, and the integral is understood in the sense of Lebesgue. This zeta function has several remarkable properties.

The abscissa of convergence of ζ_A (i.e. the infimum of $\sigma \in \mathbb{R}$ such that ζ_A is analytic on the half-plane $\{\mathrm{Re}\, s > \sigma\}$), *denoted by* $D(\zeta_A)$, *is equal to the upper box dimension of* A (also called the upper Minkowski dimension of A), *which we denote by* $\overline{\dim}_B A$. *This bound is optimal.* In particular, this enables us to compute $\overline{\dim}_B A$ using ζ_A, which we illustrate by several examples: a-strings, generalized Cantor sets, and geometric chirps, etc. A basic reference dealing with the notion of box (and Hausdorff) dimensions of fractal sets in Euclidean spaces is Falconner [Fal1]. For the reader's convenience, we recall that the *upper and lower* r*-dimensional Minkowski contents of a bounded set* A *in* $\mathbb{R}^N$ *are defined respectively by*

$$\mathcal{M}^{*r}(A) = \limsup_{t\to\infty} \frac{|A_t|}{t^{N-r}},$$

$$\mathcal{M}_*^r(A) = \limsup_{t\to\infty} \frac{|A_t|}{t^{N-r}}.$$

Here $|A_t|$ denotes the N-dimensional Lebesuge measure of A. The *upper and lower box dimensions of* A *are then defined respectively by*

$$\overline{\dim}_B A = \inf\{r > 0 : \mathcal{M}^{*r}(A) = 0\},$$

$$\underline{\dim}_B A = \inf\{r > 0 : \mathcal{M}_*^r(A) = 0\}.$$

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[1]In this overview, definitions and main results are printed in italic font.

If $\overline{\dim}_B A = \underline{\dim}_B A$, the common value is denoted by $D := \dim_B A$, and we call it the *box dimension of A*.

It known that there exist fractal sets A in $\mathbb{R}^N$ such that $\underline{\dim}_B A < \overline{\dim}_B A$; see e.g. [Fal1]. Moreover, it is possible to construct fractal sets A such that the gap between lower and upper box dimensions is maximal possible, that is,

$$\underline{\dim}_B A = 0 \quad \text{and} \quad \overline{\dim}_B A = N.$$

The construction of such sets is described in [Zu], and they represent a subclass of the so called *zigzagging fractals.*

It is interesting that the residue of the distance zeta function of a fractal set A is closely related to the Minkowski content of A. More precisely, *if A is Minkowski measurable, and ζ_A possesses a meromorphic extension to a neighbourhood of $D := \dim_B A$, then the residue of ζ_A at $s = D$ is equal to*

$$\operatorname{res}(\zeta_A, D) = (N - D)\mathcal{M}^D(A).$$

Recall that A is said to be *Minkowski measurable* if there exists the limit

$$\mathcal{M}^D(A) := \lim_{t\to 0} \frac{|A_t|}{t^{N-D}},$$

and it is contained in $(0, \infty)$. This limit is called the D-dimensional *Minkowski content* of A. Then necessarily $D = \dim_B A$, that is, the upper and lower box dimensions of A coincide.

The Lapidus zeta function of A is closely related to the *tube zeta function* $\tilde{\zeta}_A$, that we define as follows:

$$\tilde{\zeta}_A(s) = \int_0^\delta t^{s-N-1}|A_t|\, dt.$$

Here $\delta > 0$ is also fixed. More precisely, for any s such that $\operatorname{Re} s > \overline{\dim}_B A$, the following identity holds:

$$\zeta_A(s) = \delta^{s-N}|A_\delta| + (N - s)\tilde{\zeta}_A(s).$$

Let us first consider a class of Minkowski measurable sets. *If $A \subset \mathbb{R}^N$ is such that its tube function $t \mapsto |A_t|$ satisfies*

$$|A_t| = t^{N-D}(\mathcal{M} + O(t^\alpha)) \quad \text{as } t \to 0,$$

where $D \in [0, N]$, $\mathcal{M} > 0$, $\alpha > 0$, then $\tilde{\zeta}_A$ (and hence ζ_A as well) possesses a unique meromorphic extension to the half-plane $\{\operatorname{Re} s > D - \alpha\}$. Furthermore, $s = D$ is the only pole of $\tilde{\zeta}_A$ in this half-plane, and

$$\operatorname{res}(\tilde{\zeta}_A, D) = \mathcal{M}^D(A) = \mathcal{M}.$$

Now we pass to the case of a class of Minkowski non-measurable sets. *If $A \subset \mathbb{R}^N$ is such that*

$$|A_t| = t^{N-D}(G(\log t^{-1}) + O(t^\alpha)) \quad \text{as } t \to 0,$$

where $D \in [0, N]$, G *is a periodic function with the minimal period equal to* $T > 0$,[2] $\alpha > 0$, *then* ζ_A *possesses a meromorphic extension to the half-plane* $\{\mathrm{Re}\, s > D - \alpha\}$.[3] *Furthermore, the set of poles of* $\tilde{\zeta}_A$ *in this half-plane is equal to*

$$\{s_k := D + \frac{2\pi}{T} ki : \hat{G}_0(\frac{k}{T}) \neq 0,\ \ k \in \mathbb{Z}\}$$

where $\hat{G}_0$ *is the Fourier transform of the function* G_0, *equal to* G *truncated to* $[0, T]$,[4] *and*

$$\mathrm{res}(\tilde{\zeta}_A, s_k) = \frac{1}{T} \hat{G}_0(\frac{k}{T}).$$

In particular, the residue of the tube zeta function of A *at* D *is equal to the average of* G *on* $[0, T]$, *that is,*

$$\mathrm{res}(\tilde{\zeta}_A, D) = \frac{1}{T} \int_0^T G(\tau)\, d\tau.$$

and

$$\mathcal{M}_*^D(A) < \mathrm{res}(\tilde{\zeta}_A, D) < \mathcal{M}^{*D}(A).$$

These and other results from [LaRaZu] represent nontrivial extensions of many results from the fundamental monograph by Lapidus and van Frankenhuijsen [La-vFr3], that were obtained in the context of *fractal strings*, and which are essentially one-dimensional objects.

Furthermore, it is possible to extend our theory to *generalized fractal drums* (A, Ω) in $\mathbb{R}^N$, where A is a nonempty subset, and Ω is of finite N-dimensional volume, such that $\Omega \subset A_\delta$ for some $\delta > 0$. The zeta function of the generalized fractal drum is defined by

$$\zeta_A(s; \Omega) = \int_\Omega d(x, A)^{s-N} dx.$$

Its abscissa of convergence is equal to the *relative upper box dimension* $\overline{\dim}_B(A, \Omega)$, defined in [Zu]. We illustrate it on a class of *unbounded geometric chirps* (A, Ω). Furthermore, it turns out that for relative fractal drums (A, Ω), contrary to the case of single sets A, the value of the relative box dimension $\dim_B(A, \Omega)$, if it exists, can be even negative. This is a consequence of the definition of the *relative upper r-dimensional Minkowski content*:

$$\mathcal{M}^{*r}(A, \Omega) := \limsup \frac{|A_t \cap \Omega|}{t^{N-r}}.$$

in which we take $r \in \mathbb{R}$, that is, we allow negative values of r as well. The relative lower r-dimensional Minkowski content is defined analogously. Then

[2] In particular, G is nonconstant.

[3] This meromorphic extension is necessarily uniquely determined. It is worth noting that the indicated meromorphic extension range of the distance zeta function of A depends in essential way on the information of the *second term* of the asymptotic exapnsion of the tube zeta function $t \mapsto |A_t|$, that is, on the value of the parameter α.

[4] That is, G_0 is equal to G on $[0, T]$, and to zero outside of it.

the notion of the *relative upper box dimension* $\overline{\dim}_B(A,\Omega)$ (and relative lower box dimension $\underline{\dim}_B(A,\Omega)$) of the relative fractal drum (A,Ω) is defined similarly as in the case of a single set A, as the infimum of all $r \in \mathbb{R}$ such that $\mathcal{M}^{*r}(A,\Omega)$.

Minkowski contents are important in analysis of some classes of singular integrals. For example, it can be shown (see [Zu]) that if $D = \dim_B(A,\Omega)$ exists and $\mathcal{M}_*^D(A,\Omega) > 0$, then for any $\gamma > 0$,

$$\int_\Omega d(x,A)^{-\gamma}dx < \infty \quad \Leftrightarrow \quad \gamma < N - \dim_B(A,\Omega).$$

It is possible to show that the equivalence does not hold in general if $M_*^D(A,\Omega) = 0$; see [Zu].

It is possible to construct some simple relative fractal drums for which $\dim_B(A,\Omega) < -\infty$, and even dim)$B(A,\Omega) = -\infty$. For example, if (A,Ω) is a relative fractal drum in the plane, such that $A = \{0\}$ and

$$\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 : 0 < y < x^\alpha,\ x \in (0,1)\},$$

and $\alpha > 1$, then $\dim_B(A,\Omega_\alpha) = 1 - \alpha < 0$.

If (A,Ω) is any relative fractal drum in $\mathbb{R}^N$, then the upper box dimension of the drum is equal to the abscissa of convergence of the corresponding Lapidus zeta function, that is,

$$\overline{\dim}_B(A,\Omega) = D(\zeta_A(\,\cdot\,,\Omega)).$$

This enables us to compute the upper box dimension of a given relative fractal drum (A,Ω) by computing its Lapidus zeta function, that is, by using complex analysis.

There is a simple and natural sufficient condition for the upper box dimension of a relative fractal drum to be nonnegative. We say that a *relative fractal drum (A,Ω) in $\mathbb{R}^N$ has the cone property at a point $a \in \overline{A} \cap \overline{\Omega}$*, if there exists $r > 0$ such that Ω contains a cone $K_r(a,G)$ with the vertex at a. We can show that *if there exists a point point $a \in \overline{A} \cap \overline{\Omega}$, in which the relative fractal drum (A,Ω) satisfies the cone property, then $D(\zeta_A(\,\cdot\,\Omega)) \geq 0$, or equivalently,* $\overline{\dim}_B(A,\Omega) \geq 0$. We mention in passing that it is possible to substantially generalize this result to relative fractal drums (A,Ω) satisfying the so called *lacunary cone property in a point $a \in a \in \overline{A} \cap \overline{\Omega}$.*

We hope that some of the obtained results will be useful in the study of box dimension of radial chirp-like surfaces in $\mathbb{R}^{N+1}$, appearing as graphs of solutions of p-Laplace equations defined in a punctured ball in $\mathbb{R}^N$, $N \geq 2$. A related result dealing with p-Laplace equations on an annular domain in $\mathbb{R}^N$ can be seen in our joint work [NaTaPaZu].

We close this short review by indicating some interesting consequences from transcendental number theory the theory of fractal sets. To describe the simplest nontrivial situation, assume that A is a bounded subset of the real line, such that its tube function has the following form:

$$|A_t| = t^{1-D}(F(t) + o(1)) \quad \text{as } t \to 0^+,$$

where

$$F(t) = G_1(\log 1/t) + G_2(\log 1/t),$$

$D \in (0,1)$, and the function $G(\tau) := G_1(\tau) + G_2(\tau)$ is the sum of two periodic functions having periods T_1 and T_2, such that their quotient T_1/T_2 is transcendental. We say for short that the function G is *transcdentendally quasi-periodic.* Furthermore, we also say in this case that the set A *is transcendentally quasi-periodic.* In LaRaZu we show that *such sets can be effectively constructed*, using generalized Cantor sets, defined by two auxilliary parameters, introduced and studied in [Zu]. We say for short that A possesses two transcendentally incommensurable periods T_1 and T_2. The construction uses a classic 1931 result by Gel'fond and Schneider from the transcendental number theory; see e.g. Baker [Ba].

Using Baker's theory of transcdenental numbers, this construction can be further extended in a nontrivial way. As a consequence, *it is possible to construct bounded sets in* $\mathbb{R}$ *possessing arbitrarily many transcendetally incommensurable periods* $T_1, \dots, T_n$. Moreover, due to Baker's result, *it is possible to achieve that the periods are algebraically linearly independent.*

We also obtain some new results for *fractal sprays* in $\mathbb{R}^N$. This notion has been introduced by M. L. Lapidus; see [La-vFr3] and the references therein. These new results are formulated in the context of relative fractal drums (which in turn represent a natural extension of the notion of fractal strings),[5] *average Minkowski contents of fractal sets* (also introduced M. L. Lapidus; see the same reference for more information), *weighted Lapidus zeta functions of fractal sets, geometric chirps, fractal nests*, etc. At the end of [LaRaZu] we offer a *classification of bounded sets A in Euclidean spaces*, based on asymptotic properties of their corresponding tube functions $t \mapsto |A_t|$, in particular, on the properties of their Minkowski contents. The basic classification in this spirit has been introduced in [Zu]:

(1) a bounded set A in $\mathbb{R}^N$ is said to be *Minkowski nondegenerate* if there exists $D := \dim_B A$, and the D-dimensional Minkowski contents are nondegenerate, that is,
$$0 < \mathcal{M}_*^D(A) \le \mathcal{M}^{*D}(A) < \infty;$$

(2) a bounded set A is said to be *Minkowski degenerate*, if it is not Minkowski nondegenerate, that is, either $\underline{\dim}_B A < \overline{\dim}_B A$, or there exists $D = \dim_B A$ such that either $\mathcal{M}_*^D(A) = 0$ or $\mathcal{M}^{*D}(A) = \infty$.

A finer classification of Minkowski nondegenerate sets is provided in [LaRaZu]. A special case of Minkowski nondegenerate sets are for example Minkowski measurable sets, periodic sets (or lattice sets), nonperiodic sets (or nonlatice sets), quasi-periodic sets, etc.

[5] The notion of *generalized fractal strings*, also due to M. L. Lapidus, already exists in the literature; see [La-vFr3] and the references therein.

We also sketch a history of the study of Minkowski contents of fractal sets, indicating some directions for further research, and we propose a set of open problems.

A PDF of my 2012 lecture at the RIMS in Kyoto, containing additional information, is available at the Internet address indicated in [PDF1]. Another PDF, containing a sketch of my lecture delivered at the Okayama University of Science, dedicated to fractal analysis of trajectories of dynamical systems, can be found at the Internet address indicated at [PDF2]. Some of personal impressions of Professor Mervan Pašice and myself, during our stay at the Okayama University of Science and at the prestigious Research Institute of Mathematical Sciences (RIMS) of the Kyoto University, can be seen in [Web].

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