Splines and Finite Element Spaces

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1 Splines

Splines are piecewise polynomial functions that have certain "regularity" properties. These can be defined on all finite intervals, and intervals of the form $(-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$.

We have already encountered linear splines, which are simply continuous, piecewise-linear functions. More general splines are defined similarly to the linear ones. They are labeled by three things: (1) a knot sequence, Δ; (2) the degree k of the polynomial; and, (3) the space C^r, the level of differentiability of the whole spline. The knot sequence is where the polynomial may change. For a linear spline defined on $[0,1]$, the knot sequence $\Delta = \{x_0 = 0 < x_1 < x_2 < \cdots < x_n = 1\}$ is where one linear polynomial meets another. Since the polynomials are linear, $k = 1$. Finally, since the linear splines are continuous, they are in $C^0[0,1]$, so $r = 0$.

Definition 1.1. *We denote the set of splines having knot sequence Δ, degree of polynomial k, and smoothness C^r by $S^\Delta(k, r)$.*

There is a special case in which $k = 0$ and $r = -1$. These are just step functions. Since the polynomials are taken to be constants, $k = 0$. Letting $r = -1$ simply means that the step function is discontinuous at the knots.

With Δ, k, and r fixed, $S^\Delta(k, r)$ is a vector space, which may be finite dimensional or infinitely dimensional. This raises the issue of bases for the spaces.

1.1 Basis Splines – B-Splines

We begin with the following useful notation. The function below is called the *plus function*, for obvious reasons.

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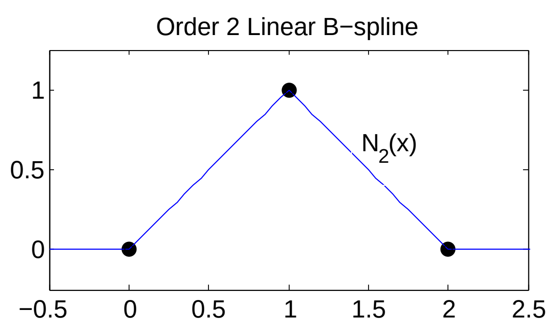
$$(x)_+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0. \end{cases}$$

The plus function is a linear spline, with $\Delta = \mathbb{Z}$, $k = 1$, and $r = 0$. (We remark that the only place the linear function changes is at $x = 0$.) It s defined over $\mathbb{R}$. With it in hand, we can define the order[1] $m = 2$ *cardinal* B-spline:

$$N_2(x) = (x)_+ - 2(x-1)_+ + (x-2)_+ \; . \tag{1.1}$$

The knot sequence for N_2 is the the set of all integers, $\mathbb{Z}$, although changes in the function only occur at $\{0, 1, 2\}$, and N_2 is a linear spline. As the graph below shows, N_2 is a "tent" function.



Proposition 1.2. *Let Δ be an equally spaced knot sequence, with $x_j = \frac{j}{n}$, $j = 0, \ldots, n$. Then $B = \{N_2(nx - j + 1) : j = 0, \ldots, n\}$ is a basis for $S^\Delta(1, 0)$ (the space of linear splines), provided $x \in [0, 1]$.*

Proof. Exercise. $\square$

[1] The *order* of a B-spline is $m = k + 1$.

Example 1.3. Consider $n = 4$. Recall that the values at the corners and endpoints determine the linear spline. So, let y_j be given at $j = 0, 1, 2, 3, 4$. Then, the interpolating spline is

$$s(x) = \sum_{j=0}^{4} y_j N_2(4x - j + 1), \qquad 0 \le x \le 1.$$

The order 1 B-spline is just a "box" of the form $N_1(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & x \notin [0,1) \end{cases}$. It can be used to start an iteration to obtain cardinal B-splines of order $m \ge 2$ and higher. The recurrence formula to be iterated is

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1).$$

From the formula above, one can show that the order m B-splines, N_m, are in $S^{\mathbb{Z}}(m-1, m-2)$, and that the *support* of N_m is precisely $[0, m]$. This feature is important enough that is used to label them.

2 Finite Element Spaces

Let $\Delta := \{x_0 = 0 < x_1 < x_2 < \cdots < x_n = 1\}$ be a knot sequence for $[0, 1]$. It is convenient to define the subintervals $I_j = [x_{j-1}, x_j)$, with $I_n = [x_{n-1}, 1]$. Let $\mathcal{P}_k$ denote the set of polynomials of degree less than or equal to k. By Definition 1.1, the space of splines may be written as follows:

$$S^{\Delta}(k, r) = \{\phi : [0,1] \to \mathbb{R} : \phi|_{I_j} \in \mathcal{P}_k(I_j) \text{ and } \phi \in C^{(r)}([0,1])\} \tag{2.1}$$

When $r = -1$, ϕ is discontinuous.

Consider an equally spaced knot sequence for $[0,1]$, $\Delta = \{j/n \colon j = 0, \ldots, n\}$. The *finite element* spaces[2] $S^{\frac{1}{n}}(k, r)$ are degree k polynomials on each interval and have $r \le k-1$ derivatives that match at the interior knots. We consider the following question: How many parameters are required to describe a function in $S^{\frac{1}{n}}(k, r)$? That is, what is the dimension of this linear space?

There are n intervals and on each interval there are $k+1$ free parameters, since the function is a degree k polynomial there. Therefore, we have $n(k+1)$ free parameters. At each of the $n-1$ knots, the polynomials must smoothly

[2]In the case where Δ is a set of equally spaced knots on $[0,1]$, we will let $S^{\frac{1}{n}}(k, r) := S^{\Delta}(k, r)$.

join, so there are $r+1$ equations that must match (the polynomials across a knot must match and their r derivatives must match). This yields $(n-1)(r+1)$ constraints. Therefore, we have at least $n(k+1)-(n-1)(r+1) = n(k-r)+r+1$ parameters. It follows that the dimension of $S^{\frac{1}{n}}(k,r) = n(k-r)+r+1$ provided that the equations at the knots are independent (which can be shown). We summarize this below[3]

Proposition 2.1. $\dim S^{\frac{1}{n}}(k,r) = n(k-r)+r+1$.

For an example, consider $k=1, r=0$. This is the space $S^{\frac{1}{n}}(1,0)$ which has dimension $n(1-0)+0+1 = n+1$. If we consider $k=m-1$, $r=m-2$, then the dimension $S^{\frac{1}{n}}(m-1,m-2)$ is $n(m-1-m+2)+m-2+1 = n+m-1$.

3 Construction of Cubic Splines

The cubic splines in $S^{\frac{1}{n}}(3,1)$ are differentiable, piecewise cubic polynomials defined on $[0,1]$. Cubic splines can be used to simultaneously interpolate both a function and its derivatives on any set of knots $\{x_j\}_{j=0}^{n}$. That is, if the values $f(x_j)$ and $f'(x_j)$ are known, then there exists a (unique) cubic spline $s \in S^{\frac{1}{n}}(3,1)$ satisfies both $s(x_j) = f(x_j)$ and $s'(x_j) = f'(x_j)$. Returning to $\Delta = \{j/n\}_{j=0}^{n}$, we see that, by Proposition 2.1, the dimension of $S^{\frac{1}{n}}(3,1)$, is $2n+2$, which exactly matches the $2n+2$ pieces of data to be fit.

We construct a basis of functions for $S^{\frac{1}{n}}(3,1)$ by first constructing two interpolating functions. Consider the interval $[0,1]$ and the problem of finding a cubic polynomial $\phi(x)$ such that $\phi(0)=1$, and $\phi(1)=\phi'(1)=\phi'(0)=0$. Then, a polynomial of the form

$$\phi(x) = A(x-1)^3 + B(x-1)^2$$

satisfies $\phi(1)=\phi'(1)=0$. Substituting the values for $\phi(0)=1$ and $\phi'(0)=0$ yields $-A+B=1$ and $3A-2B=0$, which has the solution $A=2$ and $B=3$. Then, after re-arranging, we see that

$$\phi(x) = 2(x-1)^3 + 3(x-1)^2 = (x-1)^2(2x+1).$$

We then extend the function to all of $\mathbb{R}$ as follows:

$$\phi(x) = \begin{cases} (|x|-1)^2(2|x|+1) & |x| \le 1 \\ 0 & |x| > 1, \end{cases} \tag{3.1}$$

[3]The same argument applies to a knot sequence of the form $\Delta = \{x_0 = 0 < x_1 < x_2 < \cdots < x_n = 1\}$. Hence, $\dim S^{\Delta}(k,r) = n(k-r)+r+1$.

By construction, $\phi(0) = 1$ and $\phi'(\pm 1) = \phi'(0) = 0$. Of course, outside of $[-1, 1]$, it is identically 0. It is easy to show that $\phi \in C^{(1)}$, so $\phi \in S^{\mathbb{Z}}(3, 1)$. The function ϕ will be used to interpolate the *values* of a function, while yielding zero derivative data on each of the knots.

We next construct a function ψ that takes zero value at the endpoints, but assumes a derivative value of one at 0. We let ψ be the cubic function

$$\psi(x) = A(x-1)^3 + B(x-1)^2,$$

which already satisfies $\psi(1) = \psi'(1) = 0$. The condition $\psi(0) = 0$ implies $A = B$ and the condition $\psi'(0) = 1$ implies $3A - 2B = 1$. Combining these conditions yields the function

$$\psi(x) = x(x-1)^2.$$

We then extend it to all of $\mathbb{R}$:

$$\psi(x) = \begin{cases} x(|x|-1)^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \tag{3.2}$$

As in the case of ϕ, we have $\psi \in S^{\mathbb{Z}}(3, 1)$, but this time $\psi(0) = 0$ and $\psi'(0) = 1$.

We now construct a set of functions that will form a basis for $S^{\frac{1}{n}}(3, 1)$. We begin by changing scale in ϕ and ψ, which are defined in (3.1) and (3.2), and then translating the resulting functions. For ϕ, we define

$$\phi_j(x) := \phi(nx - j). \tag{3.3}$$

Notice that $\phi_0(x) = \phi(nx)$ and $\phi_j(x) = \phi(n(x - \frac{j}{n})) = \phi_0(x - \frac{j}{n})$. That is, $\phi_j(x)$ is $\phi_0(x)$ translated by $\frac{j}{n}$, that $\phi_j(x)$ is supported on the interval $[\frac{j-1}{n}, \frac{j+1}{n}]$, and that the conditions used to define ϕ – i.e., $\phi(0) = 1$, $\phi'(0) = 0$ and so on – imply that $\phi_j(k/n) = \delta_{j,k}$ and that $\phi_j'(k/n) = 0$.

To construct ψ_j basis functions from ψ, we first consider the derivative of $\psi(nx - j)$. We note that

$$\tfrac{d}{dx}(\psi(nx-j))\big|_{x=\frac{j}{n}} = n\psi'(nx-j)\big|_{x=\frac{j}{n}} = n\psi'(0) = n.$$

From this computation, in order to have $\psi_j'(k/n) = 1$, must scale $\psi(nx - j)$ by n. Consequently, we define

$$\psi_j(x) = \frac{1}{n}\psi(nx - j) \tag{3.4}$$

and we see the the support of ψ_j is also contained in the interval $[\frac{j-1}{n}, \frac{j+1}{n}]$. Applying the conditions imposed on ψ, we see that $\psi_j(k/n) = 0$ and that $\psi_j'(k/n) = \delta_{j,k}$.

4 Interpolation with Cubic Splines

We consider the problem of interpolating a function f and its derivative at a set of $n+1$ equally spaced knots, using the cubic splines constructed in the previous section. We begin by showing that $\{\phi_j, \psi_j\}_{j=0}^n$ is a basis for $S^{\frac{1}{n}}(3,1)$.

We note that there are $n+1$ of each type, which gives a total of $2n+2$ functions in the set. Since this is the dimension of $S^{\frac{1}{n}}(3,1)$, it suffices to show that the set $\{\phi_j, \psi_j\}_{j=0}^n$ is linearly independent.

Consider a linear combination of the cubic splines, $s(x) = \sum_{j=0}^n \alpha_j \phi_j(x) + \beta_j \psi_j(x)$. Using $\phi_j(k/n) = \delta_{j,k}, \phi_j(k/n) = 0$ and $\psi_j(k/n) = 0, \psi_j'(k/n) = \delta_{k,j}$, we see that

$$s(k/n) = \sum_{j=0}^n \alpha_j \underbrace{\phi_j(k/n)}_{\delta_{j,k}} + \beta_j \underbrace{\psi_j(k/n)}_{0} = \alpha_k \tag{4.1}$$

$$s'(k/n) = \sum_{j=0}^n \alpha_j \underbrace{\phi_j'(k/n)}_{0} + \beta_j \underbrace{\psi_j'(k/n)}_{\delta_{j,k}} = \beta_k, \tag{4.2}$$

As usual, showing linear independence amounts to showing that $s(x) \equiv 0$ implies that the α_j's and β_j's are all 0. Note that if $s \equiv 0$, then so is s'. Hence, the previous equation implies that $\alpha_k = s(k/n) = 0$ and $\beta_k = s'(k/n) = 0$. Since the α_j's and β_j's are all 0, the set $\{\phi_j, \psi_j\}_{j=0}^n$ is linearly independent, and hence is a basis for $S^{\frac{1}{n}}(3,1)$.

Solving the interpolation problem stated at the start of this section is now actually very easy to do; just set

$$s(x) = \sum_{j=0}^n f(j/n)\phi_j(x) + f'(j/n)\psi_j(x). \tag{4.3}$$

By (4.1), we have $s(k/n) = f(k/n)$ and $s'(k/n) = f'(k/n)$. Hence, s in (4.3) (uniquely) solves the interpolation problem.

5 Finite Element Methods and Galerkin Methods

Consider the problem of finding the "smoothest" function in $S^{\frac{1}{n}}(3,1)$ such that at the knots x_j, $s(x_j) = f_j$ for $j = 0, \ldots, n$. To define "smoothest", we seek a function s that minimizes

$$\|s\|^2 := \int_0^1 (s''(x))^2 \, dx \tag{5.1}$$

over all $s \in S^{\frac{1}{n}}(3,1)$ for which $s(x_j) = f_j$ for $j = 0, \ldots, n$.

Since s is a piecewise cubic function, s'' exists and is piecewise continuous. Therefore, the equation (5.1) is well defined for all of $s \in S^{\frac{1}{n}}(3,1)$. In fact, it can be shown that (5.1) is an inner product on the set of functions in $S^{\frac{1}{n}}(3,1)$ that are zero at the endpoints.

Any function $s \in S^{\frac{1}{n}}(3,1)$ such that $s(x_j) = f_j$ can be written in the form

$$s(x) = \sum_{j=0}^{n} f_j \phi_j(x) - \sum_{j=0}^{n} \alpha_j \psi_j(x).$$

Let $f = \sum_{j=0}^{n} f_j \phi_j(x)$. We seek to find coefficients α that minimize the norm of s. That is, we want to solve the problem

$$\min_{g \in \operatorname{span}(\psi_j)} \|f - g\|. \tag{5.2}$$

This is a least-squares problem that can be dealt with by solving the associated normal equations. We expand $g = \sum_{j=0}^{n} \alpha_j \psi_j$ and we seek to find coefficients α_j such that

$$\langle f - g, \psi_k \rangle = 0 \tag{5.3}$$

for $k = 0, \ldots, n$. Expanding g in terms of the ψ_k functions, we see this yields a system of equations

$$\sum_{j=0}^{n} \alpha_j \underbrace{\langle \psi_j, \psi_k \rangle}_{G_{j,k}} = \langle f, \psi_k \rangle. \tag{5.4}$$

The matrix G is a Gram matrix for the linearly independent ψ_j's. Consequently, it's invertible. Due to the compact support of ψ_k, we see that

$$\langle \psi_j, \psi_k \rangle = \int_0^1 \psi_j''(x) \psi_k''(x)\, dx = \int_{[\frac{j-1}{n}, \frac{j+1}{n}] \cap [\frac{k-1}{n}, \frac{k+1}{n}]} \psi_j''(x) \psi_k''(x)\, dx. \tag{5.5}$$

This integral is nonzero only for $k = j - 1$, $k = j$ or $k = j + 1$. Therefore, G is a tridiagonal matrix, and the system (5.4) is also "tridiagonal." Such systems are easy to solve numerically.

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