Supporting Information for

Analytic relationship of relative synchronizability to network structure and motifs

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This PDF file includes:

Supporting text
Figs. S1 to S2
SI References

Supporting Information Text

1. Linearization of non-linear dynamics

In this appendix, we provide a more detailed exploration of non-linear systems which may be linearized into a form akin to the linear systems in Main text Eq. (1-2) that we analyse. This linearization is as per the Master Stability Function approach (1, 2).

We present dynamics in the noise-free case only in this appendix for simplicity, given the equivalence under which we show the deviation from synchronization for stochastic and deterministic dynamics can be considered here (see SI Appendix, section 6). Alternatively, one could add a noise term in the update equations here (as per Main text Eq. (3)), and that noise term would carry through to the final linearizations.

A. Coupling via difference of functions $h(y_j) - h(y_i)$. The Master Stability Function approach (1, 2) considers, following the presentation by Arenas et al. (3, sec. 4) (with adaptations and use of scalars $y_i(t)$ instead of vectors $\vec{y}_i(t)$ for simplicity), continuous-time systems $\vec{y}(t)$ of the general form:

$$\frac{dy_i(t)}{dt} = f(y_i(t)) + \sum_{j=1}^{N} C_{ji}(h(y_j(t)) - h(y_i(t))), \tag{1}$$

$$= f(y_i(t)) - h(y_i(t)) + \sum_{j=1}^{N} C_{ji} h(y_j(t)), \tag{2}$$

where f and h may be linear or non-linear mappings. Note that the simplification to Eq. (2) is valid when ψ_0 is an eigenvalue of C with $\lambda_0 = 1$ giving:

$$\psi_0 C = \psi_0 \Leftrightarrow \sum_{j=1}^{N} C_{ji} = 1, \forall i. \tag{3}$$

In this case there exists a completely synchronized state $y_i(t) = s(t), \forall i$ (3), where (beginning with condition Eq. (3)):

$$\frac{ds(t)}{dt} = f(s(t)) - h(s(t)) + \sum_{j=1}^{N} C_{ji} h(s(t)), \tag{4}$$

$$= f(s(t)). \tag{5}$$

Considering fluctuations around this synchronized solution, we write $y_i(t) = s(t) + x_i(t)$ for perturbations $x_i(t)$. We then take Taylor expansions to first order in $x_i(t)$, being $f(y_i(t)) = f(s(t)) + f'(s(t))x_i(t)$ and $h(y_i(t)) = h(s(t)) + h'(s(t))x_i(t)$, giving:

$$\frac{dy_i(t)}{dt} = \frac{ds(t)}{dt} + \frac{dx_i(t)}{dt},$$

$$\frac{dy_i(t)}{dt} = f(s(t)) - h(s(t)) + (f'(s(t)) - h'(s(t)))x_i(t) + \sum_{j=1}^{N} C_{ji}(h(s(t) + h'(s(t))x_j(t)).$$

Then using Eq. (4) we have:

$$\frac{dx_i(t)}{dt} = (f'(s(t)) - h'(s(t)))x_i(t) + \sum_{j=1}^{N} C_{ji} h'(s(t))x_j(t). \tag{6}$$

Where the synchronized solution $s(t)$ is either a fixed point or the derivatives f' and h' do not vary along $s(t)$, then the f' and h' are absorbed into constants or the coupling matrix and we have linear dynamics for $\vec{x}(t)$ consistent with those we analyse of Main text Eq. (1).

A special case exists where $f(s(t))$ is a constant (i.e. $f'(s(t)) = 0$), then if the $h'(s(t))$ is constant it affects a constant factor akin to changing θ in Main text Eq. (1) and our main result for $\langle \sigma^2 \rangle$ in Main text Eq. (15) (but no change to the coupling matrix C).

Alternatively, where f' and h' change with time these may introduce time-varying coupling into the linearization Eq. (6). Lu et al. (4–6) have described how time-varying coefficients may be handled, or one might consider maximum Lyapunov exponents as per (1).

Linear stability of the synchronized state in Eq. (6) is a necessary condition for synchronization of the non-linear system Eq. (2) (3), and as such synchronization of this linearized system is the focus for much study of synchronization of the non-linear system. This approach has been used to study synchronization in, for example, Rössler dynamics (3). The formalism also applies for discrete-time non-linear coupled map lattices, as presented in (7) with application to the logistic map.

B. Coupling via function of differences $h(y_j - y_i)$. Moreover, the above formalism has also been applied to linearize the Kuramoto model (8, 9):

$$\frac{d\phi_i(t)}{dt} = \omega_i + \sum_{j=1}^{N} C_{ji} \sin\left(\phi_j(t) - \phi_i(t)\right), \tag{7}$$

in (10, 11). Strictly speaking, on comparison to Eq. (1) though Eq. (7) considers coupling via a function $h()$ of differences rather than being a difference of functions $h()$. The validity of directly using the formalism in Eq. (1) rests on small differences $\phi_j(t) - \phi_i(t)$ (with the sin() function being linearized before the terms are separated).

Thus, to generalize for more direct applicability to Eq. (7), one considers variants of Eq. (1) in a similar fashion, e.g.:

$$\frac{dy_i(t)}{dt} = f(y_i(t)) + \sum_{j=1}^{N} C_{ji} h(y_j(t) - y_i(t)). \tag{8}$$

B.1. Homogeneous synchronized solution. In the first case, we model fluctuations $y_i(t) = s(t) + x_i(t)$ around the homogeneous synchronized solution $y_i(t) = s(t), \forall i$, when it exists and take Taylor expansions, this time around:

$$h(y_j(t) - y_i(t)) = h(0) + h'(0)(x_j(t) - x_i(t)),$$

to first order, giving:

$$\frac{dy_i(t)}{dt} = f(s(t)) + f'(s(t))x_i(t) + \sum_{j=1}^{N} C_{ji} h(0) +$$
$$\sum_{j=1}^{N} C_{ji} h'(0)(x_j(t) - x_i(t)),$$

(although usually $h(0) = 0$) and since

$$\frac{ds(t)}{dt} = f(s(t)) + \sum_{j=1}^{N} C_{ji} h(0)$$

on the homogeneuous synchronized solution here then we are left with:

$$\begin{aligned}\frac{dx_i(t)}{dt} &= f'(s(t))x_i(t) + \sum_{j=1}^{N} C_{ji} h'(0)(x_j(t) - x_i(t)),\\ &= (f'(s(t)) - h'(0))x_i(t) + \sum_{j=1}^{N} C_{ji} h'(0) x_j(t),\end{aligned} \tag{9}$$

with the final simplification valid for condition Eq. (3). This results in a realization consistent with those we analyse of Main text Eq. (1) in the same way as per SI Appendix, section A.

As a special case, wherever $f(s(t))$ is a constant we have $f'(s(t)) = 0$, leaving:

$$\frac{dx_i(t)}{dt} = -h'(0)x_i(t) + \sum_{j=1}^{N} C_{ji} h'(0) x_j(t), \tag{10}$$

and again there is no change to the coupling matrix for the linearization, with the $h'(0)$ term affecting a constant factor akin to changing θ in Main text Eq. (1). This is the case for linearization of the Kuramoto model for the homogeneous synchronized solution $s(t)$, and moreover we note that with $h(x) = \sin(x)$ for it then $h'(0) = 1$ and so $\theta = 1$.

B.2. Generalized synchronization with a heterogeneous solution. Finally, we consider a second case of generalized non-linear synchronization to an attractor $\vec{s}(t)$ of Eq. (8) with homogeneous rates of change $\frac{ds_i(t)}{dt} = \Theta$ leading to constant differences $s_j(t) - s_i(t) = S_{ji}$, even though we may have heterogeneous values for $s_i(t)$. Here, we model fluctuations $y_i(t) = s_i(t) + x_i(t)$ around the heterogeneous solution $\vec{s}(t)$, and take Taylor expansions to first order, this time around

$$h(y_j(t) - y_i(t)) = h(S_{ji}) + h'(S_{ji})(x_j(t) - x_i(t)),$$

to first order, giving:

$$\frac{dy_i(t)}{dt} = f(s_i(t)) + f'(s_i(t))x_i(t) + \sum_{j=1}^{N} C_{ji} h(S_{ji}) +$$
$$\sum_{j=1}^{N} C_{ji} h'(S_{ji})(x_j(t) - x_i(t)),$$

and since

$$\frac{ds_i(t)}{dt} = f(s_i(t)) + \sum_{j=1}^{N} C_{ji} h(S_{ji})$$

on the synchronized solution then we are left with:

$$\frac{dx_i(t)}{dt} = f'(s_i(t)) x_i(t) + \sum_{j=1}^{N} C_{ji} h'(S_{ji})(x_j(t) - x_i(t)). \tag{11}$$

Again, this results in a realization consistent with those we analyse of Main text Eq. (1) in the same way as per the previous sub-sections. Importantly, this means that our analysis of the steady-state deviation from a homogeneous synchronized state in the linearization provides insight into the deviation from a potentially heterogeneous generalized synchronization solution of the non-linear system here.

Again as a special case, wherever $f(s_i(t))$ is a constant we have $f'(s_i(t)) = 0$, leaving:

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{N} C_{ji} h'(S_{ji})(x_j(t) - x_i(t)), \tag{12}$$

with constants $h'(S_{ji})$ being absorbed into the coupling matrix C.

Now, this case can be applied in full generality to the Kuramoto model in Eq. (7) to analyze generalized synchronization arising to heterogeneuous attractor solution $\vec{s}(t)$ (with synchronized $\frac{d\theta_i}{dt} = \Theta$). In other words, this applies to generalized synchronization to an attractor with common frequency $\frac{d\theta_i}{dt}$ albeit with heterogeneous phases θ_i, which can result from heterogeneous natural frequencies ω_i. Moreover, we note that with $h(x) = \sin(x)$ here then $h'(x) = \cos(x)$, and so we can approximate $h'(S_{ji}) = 1$ for attractors with a relatively wide range of angular differences S_{ji} on the synchronized solution. This leads to:

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{N} C_{ji}(x_j(t) - x_i(t)), \tag{13}$$

$$= -x_i(t) + \sum_{j=1}^{N} C_{ji} x_j(t), \tag{14}$$

for condition Eq. (3). This again results in no change to the coupling matrix for the linearization, and moreover leaves $\theta = 1$ in Main text Eq. (1).

2. Properties of the centering projection

First we define the *centering matrix* (12), the projection $U = I - G$ where G is the *averaging operator* $G_{ij} = 1/N$ (for $N \times N$ matrices). The operator U removes the average of the given vector it acts on:

$$\vec{x}U = \vec{x} - \bar{x}\vec{\psi_0}, \tag{15}$$

such that for any vector $\vec{x}$, the average of $\vec{x}U$ is 0. This serves to project $\vec{x}$ onto the space *orthogonal* to the synchronized state vector $\vec{\psi_0} = [1, 1, \ldots, 1]$. Clearly we have $U^T = U$, $U^m = U$ ($m \in \mathbb{N}$) and $GU = UG = 0$. In the next lemmata we list a number of useful properties of U:

Lemma 1.

$$\vec{\psi_0}U = 0. \tag{16}$$

Proof. $\vec{\psi_0}U = \vec{\psi_0} - \bar{\vec{\psi_0}}\vec{\psi_0} = 0$. □

Lemma 2. *For $\vec{v} \perp \vec{\psi_0}$, we have:*

$$\vec{v}U = \vec{v}. \tag{17}$$

Proof. Since $\vec{v} \perp \vec{\psi_0}$, we have $\vec{v} \cdot \vec{\psi_0} = 0$ which implies:

$$\frac{1}{N}\sum_j v_j = 0.$$

This means nothing but:

$$\vec{v}G = 0,$$

and therefore:

$$\vec{v}U = \vec{v}(I - G) = \vec{v}.$$

□

Lemma 3. *For a matrix C with eigenvector $\vec{\psi_0}$ we have the following properties:*

$$UCU = CU, \tag{18}$$
$$UC^T U = UC^T, \tag{19}$$
$$C^m U = (CU)^m,\ m \in \mathbb{N}, \tag{20}$$
$$U(C^m)^T = ((CU)^m)^T,\ m \in \mathbb{N}. \tag{21}$$

Proof. Since G is composed of rows $1/N\vec{\psi_0}$, and $\vec{\psi_0}$ is an eigenvector of C with eigenvalue say λ_0 it is easy to see that:

$$GC = \lambda_0 G.$$

Now we have:

$$UCU = (C - GC)U = (C - \lambda_0 G)U = CU,$$

where we used that $GU = 0$. This proves Eq. (18), while Eq. (19) follows in a similar way by using $UG = 0$ or directly by transposing Eq. (18).

Next, we observe that:

$$C^m U = C^{m-1}CU = C^{m-1}UCU, \tag{22}$$

where we used Eq. (18) in the last equality. Iterating this process $m-1$ times in total directly gives Eq. (20). Again Eq. (21) follows in a similar way, or by directly transposing Eq. (20). □

Lemma 4. *If $\vec{v}U = 0$, then $\exists d, \vec{v} = d\vec{\psi_0}$.*

Proof. If $\vec{v}U = 0$:

$$\vec{v}U = \vec{v}(I - G) = 0, \tag{23}$$
$$\therefore \vec{v} = \vec{v}G, \tag{24}$$
$$\therefore \forall i:\ v_i = \frac{1}{N}\sum_i v_i, \tag{25}$$
$$\text{i.e. } \exists d,\ \vec{v} = d\vec{\psi_0}. \tag{26}$$

□

Lemma 5. *The trace of a given matrix A after the centering projections are applied to produce* $\mathrm{trace}(UAU)$ *is given by:*

$$\mathrm{trace}(UAU) = \sum_i A_{ii} - \frac{1}{N}\sum_{i,j} A_{ij} \tag{27}$$

Proof. We have:

$$\begin{aligned}
\mathrm{trace}(UAU) &= \mathrm{trace}((I-G)A(I-G)), \\
&= \mathrm{trace}(A - AG - GA + GAG), \\
&= \sum_i A_{ii} - \frac{2}{N}\sum_{i,j} A_{ij} \\
&\quad + \sum_i \left(\frac{1}{N^2} \sum_{k,l} A_{kl} \right), \\
&= \sum_i A_{ii} - \frac{1}{N}\sum_{i,j} A_{ij}
\end{aligned}$$

□

3. Spectral radius after centering projection

Theorem 1. *For a matrix A with eigenvalues $|\lambda_A| < 1$, except for $\lambda_0 = 1$ corresponding to the zero-mode eigenvector $\vec{\psi_0}$, we have $|\lambda_{AU}| < 1$ for all eigenvalues of AU (i.e. $\rho(AU) < 1$).*

Proof. By Lemma 7 AU only has eigenvalues $\lambda_{AU} = 0$ (including for eigenvector $\vec{\psi_0}$ by Lemma 6) or $\lambda_{AU} \neq 0$ for some eigenvectors $\vec{w}$ with $\vec{w} \cdot \vec{\psi_0} = 0$.

Then by Lemma 8 each eigenvector $\vec{w}$ of AU with $\vec{w} \cdot \vec{\psi_0} = 0$ has an eigenvalue λ_{AU} which is also an eigenvalue of A for some eigenvector $\vec{v} \neq \vec{\psi_0}$, and so by the assumptions of the theorem $|\lambda_{AU}| < 1$.

Therefore, $\rho(AU) < 1$. □

Lemma 6. $\vec{\psi_0} A = \lambda_0 \vec{\psi_0} \rightarrow \vec{\psi_0} AU = 0$.

Proof.

$$\vec{\psi_0} AU = \lambda_0 \vec{\psi_0} U, \quad [28]$$
$$= 0, \quad [29]$$

via Lemma 1. □

Lemma 7. $\vec{\psi_0} A = \lambda_0 \vec{\psi_0}$, $\vec{v} AU = \lambda_v \vec{v} \rightarrow \lambda_v = 0$ *or else* $\lambda_v \neq 0$ *with* $\vec{v} \cdot \vec{\psi_0} = 0$.

Proof. Consider a general eigenvector $\vec{v} = a\vec{w} + b\vec{\psi_0}$ of AU with eigenvalue λ_v for some $\vec{w} \perp \vec{\psi_0}$, and scalars a and b. By definition $\vec{v} AU = \lambda_v \vec{v}$, so via Eq. (18):

$$\vec{v} UAU = \vec{v} AU = \lambda_v \vec{v}. \quad [30]$$

Now, via Lemma 1:

$$\vec{v} UAU = (a\vec{w} + b\vec{\psi_0}) UAU = a\vec{w} AU, \quad [31]$$

but also:

$$\lambda_v \vec{v} = a\lambda_v \vec{w} + b\lambda_v \vec{\psi_0}, \quad [32]$$

so substituting Eq. (31) and Eq. (32) into Eq. (30) we have:

$$a\vec{w} AU = a\lambda_v \vec{w} + b\lambda_v \vec{\psi_0}, \quad [33]$$
$$(\times U:) \quad a\vec{w} AUU = a\lambda_v \vec{w} U + b\lambda_v \vec{\psi_0} U, \quad [34]$$
$$a\vec{w} AU = a\lambda_v \vec{w}, \quad [35]$$

since U is idempotent and via Lemma 1 and Lemma 2.

From Eq. (30) and Eq. (31) we have $a\vec{w} AU = \lambda_v \vec{v}$ and then substituting into Eq. (35) we have:

$$\lambda_v \vec{v} = a\lambda_v \vec{w}. \quad [36]$$

So either $\lambda_v = 0$, or $\lambda_v \neq 0$ with $\vec{v} \perp \vec{\psi_0}$ (since $\vec{w} \perp \vec{\psi_0}$). (Note if $a = 0$, then $\vec{v} = 0$ and we still have $\vec{v} \cdot \vec{\psi_0} = 0$.) □

Lemma 8. *For a matrix A with eigenvalues $|\lambda_A| < 1$, except for $\lambda_0 = 1$ corresponding to the zero-mode eigenvector $\vec{\psi_0}$, if $\vec{w}$ with $\vec{w} \cdot \vec{\psi_0} = 0$ is an eigenvector of AU with eigenvalue λ_w, then λ_w is also an eigenvalue of A for some eigenvector $\vec{v} \neq \vec{\psi_0}$.*

Proof. We have:

$$\vec{w} AU = \lambda_w \vec{w} = \lambda_w \vec{w} U, \quad [37]$$

via Lemma 2, so:

$$(\vec{w} A - \lambda_w \vec{w}) U = 0. \quad [38]$$

Now, this implies via Lemma 4:

$$(\vec{w} A - \lambda_w \vec{w}) = d\vec{\psi_0}, \quad [39]$$

for some scalar d.

If $d = 0$, $\vec{w} A = \lambda_w \vec{w}$ and λ_w is an eigenvalue of A with eigenvector $\vec{w} \neq \vec{\psi_0}$ (since $\vec{w} \perp \vec{\psi_0}$), as required.

If $d \neq 0$, then $\vec{v} = \frac{\lambda_w - \lambda_0}{d}\vec{w} + \vec{\psi_0}$ is an eigenvector of A with eigenvalue λ_w (proof by substitution). Here, if $\lambda_w \neq \lambda_0$ then $\vec{v} \neq \vec{\psi_0}$, as required. We can then show that if $\lambda_w = \lambda_0$ (in which case both equal 1, and we would have $\vec{v} = \vec{\psi_0}$) cannot occur since it leads to a contradiction, as follows. Eq. (39) with $\lambda_w = 1$ gives:

$$\vec{w}A - \vec{w} = d\psi_0, \tag{40}$$

$$(\times A:) \quad \vec{w}A^2 - \vec{w}A = d\psi_0, \tag{41}$$

and summing these two equations gives:

$$\vec{w}A^2 - \vec{w} = 2d\psi_0. \tag{42}$$

With further iterations we have:

$$\vec{w}A^n - \vec{w} = nd\psi_0, \tag{43}$$

for integers $n \geq 1$. Now, taking the limit as $n \to \infty$, the RHS clearly diverges for any $d \neq 0$, whereas the LHS does not (via Gelfand's thereom (13), the norm of A^n scales as $\rho(A)^n = 1$ as $n \to \infty$). This contradiction implies that we cannot have $d \neq 0$ with $\lambda_w = \lambda_0 = 1$ here.

As such, we have shown that λ_w is also an eigenvalue of A for some eigenvector $\vec{v} \neq \vec{\psi_0}$ here. □

4. Centering projected covariance matrix in presence of zero-mode eigenvalue

Here we demonstrate how to write a convergent form for $\Omega_U = U^T \Omega U$ when the zero-mode $\vec{\psi_0}$ is an eigenvector of C with eigenvalue $\lambda_0 = 1$.

A. Continuous-time case. Modifying the derivation by Barnett et al. (14) for Ω for the continuous-time process, we first approximate Main text Eq. (1) by the discrete-time process:

$$\vec{x}(t + dt) = \vec{x}(t)[I - \theta(I - C)dt] + \zeta\vec{r}(t)\sqrt{dt}, \tag{44}$$

where $\vec{r}(t)$ is uncorrelated mean-zero unit-variance Gaussian noise. We then right multiply by U, along with the substitution $K = I - \theta(I - C)dt$:

$$\vec{x}(t + dt)U = \vec{x}(t)KU + \zeta\vec{r}(t)U\sqrt{dt}, \tag{45}$$

and then, since K also has $\vec{\psi_0}$ as an eigenvector with eigenvalue 1, we use Eq. (18) to restate:

$$\vec{x}(t + dt)U = \vec{x}(t)UKU + \zeta\vec{r}(t)U\sqrt{dt}, \tag{46}$$

We left multiply Eq. (46) by its transpose, and average over the ensemble at a given time t to obtain:

$$\begin{aligned} U^T\overline{\vec{x}^T(t+dt)\vec{x}(t+dt)}U =& U^T K^T U^T \overline{\vec{x}^T(t)\vec{x}(t)}UKU \\ &+ \zeta^2 U^T U dt, \end{aligned} \tag{47}$$

since $\overline{\vec{r}^T(t)\vec{r}(t)} = I$ for all t, and all cross-terms vanish because $\vec{r}(t)$ is uncorrelated with $\vec{x}(t)$.

Extending (14), we require stationarity of $\vec{x}(t)U$ such that $\Omega_U = U^T\overline{\vec{x}^T(t)\vec{x}(t)}U = U^T\overline{\vec{x}^T(t+1)\vec{x}(t+1)}U$. Stationarity of $\vec{x}(t)U$ in Eq. (45) for fixed dt requires $\rho(KU) < 1$. This is satisfied where we have $|\lambda_K| < 1$ for all eigenvalues λ_K of K except that corresponding to $\vec{\psi_0}$, via Theorem 1 in SI Appendix, section 3. Now, the eigenvalues of K are $\lambda_K = 1 - \theta(1 - \lambda_C)dt$ with λ_C the eigenvalues of C. As such, we need $|1 - \theta(1 - \lambda_C)dt| < 1$ which in the continuous limit gives the stationarity condition $\mathrm{Re}(\lambda_C) < 1$, for all eigenvalues λ_C of C except that corresponding to $\vec{\psi_0}$.

Then we have the following expression (from Eq. (47)):

$$\Omega_U = UK^T\Omega_U KU + \zeta^2 U dt. \tag{48}$$

We then substitute $K = I - \theta(I - C)dt$ back in:

$$\begin{aligned} \Omega_U &= U(I - \theta(I - C^T)dt)\Omega_U(I - \theta(I - C)dt)U + \zeta^2 U dt, \\ &= U\Omega_U U - U(I - C^T)\Omega_U U\theta dt - U\Omega_U(I - C)U\theta dt \\ &\quad + \zeta^2 U dt + O(dt^2), \\ &= \Omega_U - U(I - C^T)\Omega_U\theta dt - \Omega_U(I - C)U\theta dt \\ &\quad + \zeta^2 U dt + O(dt^2), \\ 0 &= -\Omega_U dt + UC^T\Omega_U dt - \Omega_U dt + \Omega_U CU dt \\ &\quad + \frac{\zeta^2}{\theta} U dt + O(dt^2), \end{aligned}$$

and consider terms at highest order $O(dt)$ to find that in the continuous limit $dt \to 0$:

$$2\Omega_U = \frac{\zeta^2}{\theta} U + UC^T \Omega_U + \Omega_U CU, \quad [49]$$

$$= \frac{\zeta^2}{\theta} U + (CU)^T \Omega_U + \Omega_U CU. \quad [50]$$

In the special case when C is symmetric (and with CU having no eigenvalue equal to 1), we can solve for Ω_U to obtain:

$$\Omega_U = \frac{\zeta^2}{2\theta} \frac{U}{I - CU}. \quad [51]$$

Otherwise, we obtain the following power series solution:

$$\begin{aligned} 2\Omega_U = & \frac{\zeta^2}{\theta} U + \frac{\zeta^2}{2\theta}((CU)^T U + U(CU)) + \\ & \frac{\zeta^2}{4\theta}\left[((CU)^T)^2 U + 2(CU)^T U(CU) + U(CU)^2\right] + \ldots, \\ = & \frac{\zeta^2}{\theta} \sum_{m=0}^{\infty} 2^{-m} \sum_{u=0}^{m} \binom{m}{u} ((CU)^u)^T U (CU)^{m-u}, \end{aligned} \quad [52]$$

insofar as it converges. Then, we simplify via Eq. (20-21), and then Eq. (18):

$$2\Omega_U = \frac{\zeta^2}{\theta} \sum_{m=0}^{\infty} 2^{-m} \sum_{u=0}^{m} \binom{m}{u} U(C^u)^T U C^{m-u} U, \quad [53]$$

$$= \frac{\zeta^2}{\theta} \sum_{m=0}^{\infty} 2^{-m} \sum_{u=0}^{m} \binom{m}{u} U(C^u)^T C^{m-u} U. \quad [54]$$

Finally, as per (14), the above stationarity condition does not guarantee convergence of Eq. (52-54). As such, we briefly demonstrate that a sufficient condition for convergence is $|\lambda_C| < 1$, for all eigenvalues λ_C of C except that corresponding to $\vec{\psi}_0$ (which implies the stationarity condition). For any matrix norm $||\cdot||$ (13) applied to Eq. (54), we have:

$$2||\Omega_U|| \leq \frac{\zeta^2}{\theta} \sum_{m=0}^{\infty} 2^{-m} \sum_{u=0}^{m} \binom{m}{u} ||((CU)^u)^T||\ ||(CU)^{m-u}||, \quad [55]$$

using Eq. (20-21), that the product of norms is greater than the norm of the products, and that the sum of norms is greater than the norm of the sum. Similarly, we have:

$$2||\Omega_U|| \leq \frac{\zeta^2}{\theta} \sum_{m=0}^{\infty} 2^{-m} \sum_{u=0}^{m} \binom{m}{u} ||(CU)^T||^u\ ||CU||^{m-u}. \quad [56]$$

It is well known that for all $\epsilon > 0$ there exists a matrix norm $||\cdot||$ such that $||A|| \leq \rho(A) + \epsilon$ (13, Lemma 5.6.10). Noting $\rho((CU)^T) = \rho(CU)$, we observe that for any $\epsilon > 0$ there exists a matrix norm such that:

$$2||\Omega_U|| \leq \frac{\zeta^2}{\theta} \sum_{m=0}^{\infty} (\rho(CU) + \epsilon)^m. \quad [57]$$

Then, if we have $\rho(CU) < 1$ and choose ϵ such that $(\rho(CU) + \epsilon)$ remains < 1, then Eq. (57) converges, leaving $||\Omega_U||$ finite. As such, convergence of this sum of norms then implies convergence of the matrix sum for Ω_U (13), under the condition $\rho(CU) < 1$. This is satisfied where $|\lambda_C| < 1$ for all eigenvalues λ_C of C except that corresponding to $\vec{\psi}_0$, via Theorem 1 in SI Appendix, section 3.

B. Discrete-time case. Using a parallel analysis for the discrete-time process, we right multiply Main text Eq. (2) by U:

$$\vec{x}(t+1)U = \vec{x}(t)CU + \zeta\vec{r}(t)U, \quad [58]$$

and use Eq. (18) to restate:

$$\vec{x}(t+1)U = \vec{x}(t)UCU + \zeta\vec{r}(t)U. \quad [59]$$

We then left multiply Eq. (59) by its transpose, and average over the ensemble at a given time t to obtain:

$$U^T \overline{\vec{x}^T(t+1)\vec{x}(t+1)} U = U^T C^T U^T \overline{\vec{x}^T(t)\vec{x}(t)} UCU + \zeta^2 U^T U, \quad [60]$$

since $\overline{\vec{r}^T(t)\vec{r}(t)} = I$ for all t, and all cross-terms vanish because $\vec{r}(t)$ is uncorrelated with $\vec{x}(t)$. As above, we require stationarity of $\vec{x}(t)U$ such that $\Omega_U = U^T\overline{\vec{x}^T(t)\vec{x}(t)}U = U^T\overline{\vec{x}^T(t+1)\vec{x}(t+1)}U$. For stationarity of $\vec{x}(t)U$ in Eq. (59) we need $\rho(CU) < 1$, which is met when we have $|\lambda_C| < 1$ for all eigenvalues of C except that corresponding to $\vec{\psi}_0$ (via Theorem 1 in SI Appendix, section 3).

Then we have the following expression for $\Omega_U = U^T\Omega U$:

$$\begin{aligned} \Omega_U &= UC^T\Omega_U CU + \zeta^2 U, && [61] \\ &= (CU)^T\Omega_U CU + \zeta^2 U. && [62] \end{aligned}$$

In the special case when C is symmetric (and since we have $\rho(CU) < 1$ from the stationarity condition above), we can solve for Ω_U to obtain:

$$\Omega_U = \frac{\zeta^2 U}{I - (CU)^2}. \quad [63]$$

Otherwise, we then obtain the following power series solution:

$$\begin{aligned} \Omega_U &= \zeta^2\left[U + (CU)^T U(CU) + ((CU)^2)^T U(CU)^2 + \ldots\right], \\ &= \zeta^2\sum_{u=0}^{\infty}((CU)^u)^T U(CU)^u, && [64] \end{aligned}$$

insofar as it converges. We again simplify via Eq. (20-21), and Eq. (18) to obtain:

$$\Omega_U = \zeta^2\sum_{u=0}^{\infty} U(C^u)^T C^u U. \quad [65]$$

Unlike the continuous case, the stationarity condition here does indeed guarantee convergence of Eq. (64-65). This is demonstrated via a similar argument with matrix norms as was used for the continuous case.

5. $\langle\sigma^2\rangle$ for discrete time

Expanding Main text Eq. (16) in a similar way to the continuous-time process in Section 4.A via Main text Eq. (20) we obtain:

$$\langle\sigma^2\rangle = \frac{\zeta^2}{N}\sum_{u=0}^{\infty}\left(\sum_i((C^u)^T C^u)_{ii} - \frac{1}{N}\sum_{i,j}((C^u)^T C^u)_{ij}\right),$$

so:

$$\begin{aligned} \langle\sigma^2\rangle &= \frac{\zeta^2}{N}\sum_{u=0}^{\infty}\left(\sum_{i,k} C^u_{ki}C^u_{ki} - \frac{1}{N}\sum_{i,j,k} C^u_{ki}C^u_{kj}\right), \\ &= \frac{\zeta^2}{N}\sum_{u=0}^{\infty}\left(\sum_{i,k}\mathbf{w}^{k\to i,u}_{k\to i,u} - \frac{1}{N}\sum_{i,j,k}\mathbf{w}^{k\to j,u}_{k\to i,u}\right), && [66] \\ &= \zeta^2(1-\frac{1}{N}) + \frac{\zeta^2}{N}\sum_{u=1}^{\infty}\sum_{i,k}\left(\mathbf{w}^{k\to i,u}_{k\to i,u} - \frac{1}{N}\sum_j \mathbf{w}^{k\to j,u}_{k\to i,u}\right). && [67] \end{aligned}$$

using Main text Eq. (19).

6. Total integrated/summed deviation from synchronized state for deterministic dynamics

A. Single perturbation response in continuous-time. We can define deterministic continuous-time dynamics in response to a single perturbation or initial condition $\zeta\vec{r}(0)$ (sampled from uncorrelated unit-variance Gaussian noise $\vec{r}$ as before for discrete-time) at time $t = 0$ as follows:

$$\begin{aligned} \vec{x}(0) &= \zeta\vec{r}(0); && [68] \\ d\vec{x}(t) &= -\vec{x}(t)(I - C)\theta dt, \forall t \geq 0 && [69] \end{aligned}$$

(c.f. Main text Eq. (1)). We can solve for $\vec{x}(t)$ as:

$$\vec{x}(t) = \zeta\vec{r}(0)e^{-(I-C)\theta t}. \quad [70]$$

Then we define the expected integral of mean deviations away from synchronization (summed over all time) experienced by the system due to this single perturbation as:

$$D = \left\langle \int_0^\infty \sigma^2(t) dt \right\rangle, \qquad [71]$$

with $\sigma^2(t)$ defined already in Main text Eq. (4).

Theorem 2. *For continuous-time dynamics: The expected integral of mean deviations D (Eq. (71)) away from synchronization (integrated over all time) experienced by the system due to a single perturbation (with subsequent deterministic dynamics in Eq. (69-70)), is equal to the expected steady state deviation $\langle \sigma^2 \rangle$ (Main text Eq. (5)) from synchronisation due to stochastic perturbations at every time point (stochastic dynamics in Main text Eq. (1)). This is under the condition that $|\lambda_C| < 1$ for all eigenvalues λ_C of C except that corresponding to $\vec{\psi}_0$ which may have $\lambda_0 = 1$.*

Proof. The expectation in Eq. (71) is taken over the ensemble of samples of $\vec{r}(0)$, and we can exchange the expectation and integral (via Fubini's theorem since $|\sigma^2(t)| = \sigma^2(t)$, when the result is finite):

$$D = \int_0^\infty \left\langle \sigma^2(t) \right\rangle dt. \qquad [72]$$

Using the same substitution made in Main text Eq. (8) for $\sigma^2(t)$, we then have:

$$\begin{aligned} D &= \frac{1}{N} \int_0^\infty \left\langle \mathrm{trace}(U\vec{x}(t)^T \vec{x}(t) U) \right\rangle dt, && [73] \\ &= \frac{\zeta^2}{N} \int_0^\infty \mathrm{trace}(\left\langle U (e^{-(I-C)\theta t})^T \vec{r}(0)^T \vec{r}(0) e^{-(I-C)\theta t} U \right\rangle) dt, && [74] \\ &= \frac{\zeta^2}{N} \int_0^\infty \mathrm{trace}(U (e^{-(I-C)\theta t})^T \left\langle \vec{r}(0)^T \vec{r}(0) \right\rangle e^{-(I-C)\theta t} U) dt, && [75] \\ &= \frac{\zeta^2}{N} \int_0^\infty \mathrm{trace}(U (e^{-(I-C)\theta t})^T e^{-(I-C)\theta t} U) dt, && [76] \end{aligned}$$

since $\left\langle \vec{r}(0)^T \vec{r}(0) \right\rangle = I$ (as per any value of t in the stochastic case).

Then, as for the solution for Ω_U in SI Appendix, section 4, we make a discrete-time approximation, here specifically taking a Riemann integral with time intervals dt (indexed by v) and exchanging the trace and summation operations:

$$\begin{aligned} D &= \frac{1}{N} \mathrm{trace} \left(\zeta^2 \sum_{v=0}^{\infty} U (e^{-(I-C)v\theta dt})^T e^{-(I-C)v\theta dt} U dt \right), && [77] \\ &= \frac{1}{N} \mathrm{trace}\,(S), && [78] \end{aligned}$$

with S defined by identification with Eq. (77). We can then write:

$$\begin{aligned} S &= \zeta^2 U dt + \zeta^2 \sum_{v=1}^{\infty} U (e^{-(I-C)v\theta dt})^T e^{-(I-C)v\theta dt} U dt, && [79] \\ &= \zeta^2 U dt + \zeta^2 \left(\sum_{v=1}^{\infty} U (e^{-(I-C)\theta dt})^T U (e^{-(I-C)(v-1)\theta dt})^T e^{-(I-C)(v-1)\theta dt} U e^{-(I-C)\theta dt} U dt \right), && [80] \end{aligned}$$

using Eq. (22) and its transpose. (Use of that identity is valid since the matrix exponential $e^{-(I-C)v\theta dt}$ will have ψ_0 as an eigenvector when C does; this can be demonstrated via the power series expansion of the matrix exponential, being a sum of I or $I - C$ terms). We then have:

$$\begin{aligned} S &= \zeta^2 U dt + \zeta^2 U (e^{-(I-C)\theta dt})^T \left(\sum_{v=1}^{\infty} U (e^{-(I-C)(v-1)\theta dt})^T e^{-(I-C)(v-1)\theta dt} U dt \right) e^{-(I-C)\theta dt} U, \\ &= \zeta^2 U dt + \zeta^2 U (e^{-(I-C)\theta dt})^T \left(\sum_{v=0}^{\infty} U (e^{-(I-C)v\theta dt})^T e^{-(I-C)v\theta dt} U dt \right) e^{-(I-C)\theta dt} U, \\ &= \zeta^2 U dt + U (e^{-(I-C)\theta dt})^T S e^{-(I-C)\theta dt} U. && [81] \end{aligned}$$

We can then make a power series expansion of the matrix exponentials, being:

$$
\begin{aligned}
e^{-(I-C)\theta dt} &= I + \sum_{q=1}^{\infty} \frac{(-1)^q (I-C)^q (\theta dt)^q}{q!}, \\
&= I - \theta(I-C)dt + O(dt^2), \\
&= K + O(dt^2),
\end{aligned} \tag{82}
$$

with $K = I - \theta(I-C)dt$ as previously defined in SI Appendix, section 4. So substituting Eq. (82) back into Eq. (81) we have:

$$
S = \zeta^2 U dt + U(K + O(dt^2))^T S (K + O(dt^2)) U.
$$

and by comparison to Eq. (48) for Ω_U in SI Appendix, section 4, we can see that the process to solve for S when we consider terms at the highest order $O(dt)$ in the continuous limit $dt \to 0$ will result in the same solution as for Ω_U (noting that $USU = S$ also), being Eq. (54) / Main text Eq. (12):

$$
S = \frac{\zeta^2}{2\theta} \sum_{m=0}^{\infty} 2^{-m} \sum_{u=0}^{m} \binom{m}{u} U (C^u)^T C^{m-u} U \tag{83}
$$

with the same convergence conditions for the solution.

As such, this solution for S can then be substituted back into Eq. (78) to obtain, after exchanging the sum and trace operations:

$$
D = \frac{\zeta^2}{2\theta} \sum_{m=0}^{\infty} \frac{2^{-m}}{N} \sum_{u=0}^{m} \binom{m}{u} \mathrm{trace}\left(U (C^u)^T C^{m-u} U \right), \tag{84}
$$

We see that Eq. (84) matches the form for $\langle \sigma^2 \rangle$ in Main text Eq. (15), and clearly has the same convergence conditions. □

B. Single perturbation response in discrete-time. We can define the deterministic dynamics in response to a single perturbation $\zeta\vec{r}(0)$ (sampled from uncorrelated unit-variance Gaussian noise $\vec{r}$ as before) at time $t = 0$ as follows:

$$
\vec{x}(0) = \zeta \vec{r}(0); \tag{85}
$$

$$
\vec{x}(t+1) = \vec{x}(t) C, \forall (t+1) \geq 1 \tag{86}
$$

(c.f. Main text Eq. (2)). We can recusively expand Eq. (86) down to $\vec{x}(1)$ to write:

$$
\vec{x}(t) = \zeta \vec{r}(0) C^t. \tag{87}
$$

Then we define the expected sum of mean deviations away from synchronization (summed over all time) experienced by the system due to this single perturbation as:

$$
D = \left\langle \sum_{t=0}^{\infty} \sigma^2(t) \right\rangle, \tag{88}
$$

with $\sigma^2(t)$ defined already in Main text Eq. (4).

Theorem 3. *For discrete-time dynamics: The expected sum of mean deviations D (Eq. (88)) away from synchronization (summed over all time) experienced by the system due to a single perturbation (deterministic dynamics in Eq. (86-87)), is equal to the expected steady state deviation $\langle \sigma^2 \rangle$ (Main text Eq. (5)) from synchronisation due to stochastic perturbations at every time point (stochastic dynamics in Main text Eq. (2)). This is under the condition that $|\lambda_C| < 1$ for all eigenvalues λ_C of C except that corresponding to $\vec{\psi}_0$ which may have $\lambda_0 = 1$.*

Proof. The expectation in Eq. (88) is taken over the ensemble of samples of $\vec{r}(0)$, and we can exchange the expectation and sum:

$$
D = \sum_{t=0}^{\infty} \left\langle \sigma^2(t) \right\rangle, \tag{89}
$$

and using the same substitution made in Main text Eq. (8) for $\sigma^2(t)$, we have:

$$D = \frac{1}{N}\sum_{t=0}^{\infty}\left\langle \mathrm{trace}(U\vec{x}(t)^T\vec{x}(t)U)\right\rangle, \quad [90]$$

$$= \frac{\zeta^2}{N}\sum_{t=0}^{\infty}\mathrm{trace}(\left\langle U(C^t)^T\vec{r}(0)^T\vec{r}(0)C^tU\right\rangle), \quad [91]$$

$$= \frac{\zeta^2}{N}\sum_{t=0}^{\infty}\mathrm{trace}(U(C^t)^T\left\langle \vec{r}(0)^T\vec{r}(0)\right\rangle C^tU), \quad [92]$$

$$= \frac{\zeta^2}{N}\sum_{t=0}^{\infty}\mathrm{trace}(U(C^t)^TC^tU), \quad [93]$$

since $\left\langle \vec{r}(0)^T\vec{r}(0)\right\rangle = I$ (as per any value of t in the stochastic case). We see that Eq. (93) matches the form for $\left\langle \sigma^2\right\rangle$ in Main text Eq. (16), and clearly has the same convergence conditions. $\square$

7. Proof of convergent forms for symmetric C

Starting from Main text Eq. (15) for continuous time, with $\theta = \zeta = 1$ and symmetric $C = C^T$ we have:

$$\begin{aligned}\left\langle \sigma^2\right\rangle &= \sum_{m=0}^{\infty}\frac{2^{-m-1}}{N}\sum_{u=0}^{m}\binom{m}{u}\mathrm{trace}\left(UC^mU\right), \\ &= \sum_{m=0}^{\infty}\frac{1}{2N}\mathrm{trace}\left(UC^mU\right), \\ &= \frac{1}{2N}\sum_{m=0}^{\infty}\mathrm{trace}\left((CU)^m\right),\end{aligned}$$

via Eq. (18) and Eq. (20). Then, since:

- the trace of a matrix is equal to the sum of its eigenvalues;
- the eigenspectrum λ_{CU} of CU is the same as that of C, λ_C, except λ_0 for ψ_0 if ψ_0 is an eigenvector, which will have $\lambda_0 \to 0$ for CU (see Lemma 6 and Lemma 8 in SI Appendix, section 3); and
- the eigenvalues of C^m are those of C raised to the power m,

we have:

$$\begin{aligned}\left\langle \sigma^2\right\rangle &= \frac{1}{2N}\sum_{m=0}^{\infty}\sum_{\lambda\neq\lambda_0}\lambda_C^m, \\ &= \frac{1}{2N}\sum_{\lambda\neq\lambda_0}\sum_{m=0}^{\infty}\lambda_C^m, \\ &= \frac{1}{2N}\sum_{\lambda\neq\lambda_0}\frac{1}{1-\lambda_C}, \quad [94]\end{aligned}$$

with the last step valid since $|\lambda_C| < 1$ for all λ_C corresponding to eigenvectors other than ψ_0 (being the domain of validity of our solution in Main text Eq. (12)).

Indeed, one can see a simpler route to this same result via Eq. (51), however we have chosen to start from Main text Eq. (15) to demonstrate explicitly how the power series converges.

Similarly, starting from Main text Eq. (16) for discrete time, with $\zeta = 1$ and symmetric $C = C^T$ we have:

$$\begin{aligned}\left\langle \sigma^2\right\rangle &= \frac{1}{N}\sum_{u=0}^{\infty}\mathrm{trace}\left(UC^{2u}U\right). \\ &= \frac{1}{N}\sum_{u=0}^{\infty}\mathrm{trace}\left((CU)^{2u}\right),\end{aligned}$$

via Eq. (18) and Eq. (20). Then, following similar reasoning as for the continuous-time case, we have:

$$\begin{aligned}\left\langle \sigma^2 \right\rangle &= \frac{1}{N} \sum_{u=0}^{\infty} \sum_{\lambda \neq \lambda_0} \lambda_C^{2u}, \\ &= \frac{1}{N} \sum_{\lambda \neq \lambda_0} \sum_{u=0}^{\infty} (\lambda_C^2)^u, \\ &= \frac{1}{N} \sum_{\lambda \neq \lambda_0} \frac{1}{1-\lambda_C^2},\end{aligned} \quad [95]$$

with the last step valid since $|\lambda_C| < 1$ for all λ_C corresponding to eigenvectors other than ψ_0 (being the domain of validity of our solution in Main text Eq. (14)).

8. Supplementary figures

Fig. S1 repeats the experiment in Section C to validate our analytic solution with numerical experiments, however for the discrete-time dynamics of Main text Eq. (2). Otherwise, all parameters are as per the experiment for continuous-time dynamics in Section C. We can see that the conclusions for this experiment mirror those for continuous-time, being:

- The empirical results converge exponentially to the expected analytic values as the number of samples (which $\left\langle \sigma^2 \right\rangle_E$ is averaged over) increases, validating the analytic solution;
- The convergence is observed across all values of network randomisation parameter p; and
- There is a smaller relative error for random networks for the same number of samples L.

Fig. S2 reproduces Fig. 4(a), but with the y-axis expanded to show the full extent of the error bars. The error bars are large in this regime because with $d = 2$ a small amount of randomization p has a relatively higher probability (as compared to larger d and larger p) of making the network almost disconnected, and therefore driving $\left\langle \sigma^2 \right\rangle$ significantly higher for some network samples.

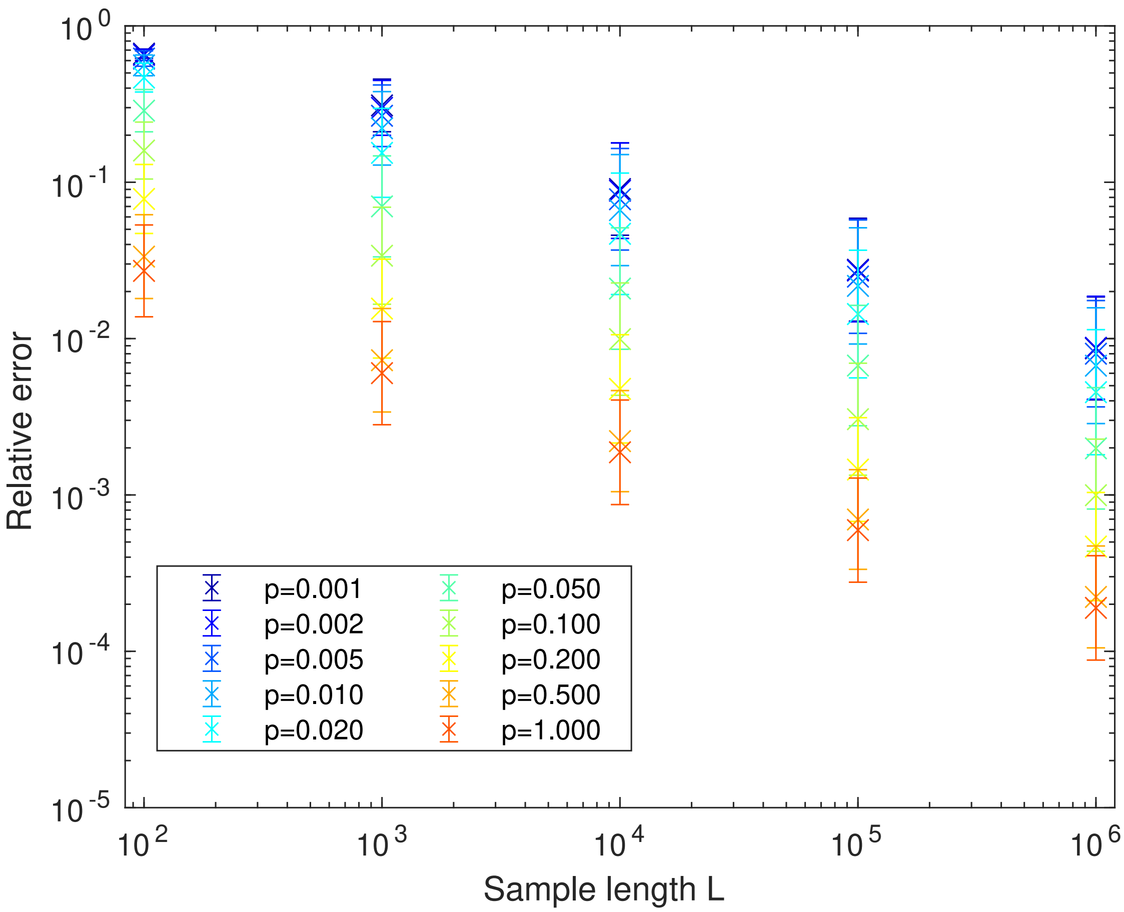

Fig. S1. *Numerical results validating* our analytic expression for average steady-state distance from synchronisation $\left\langle \sigma^2 \right\rangle$ for discrete-time dynamics against empirical measurements, throughout a small-world transition on an $N = 100$ ring network with network randomization parameter p. Parameters for the network connectivity matrix are described in Section 3.C, with $d = 4$ and $c = 0.5$, being the same as per the experiment in Section C except for the use of discrete-time dynamics. The plot shows the relative absolute error of the empirical measurements $\left\langle \sigma^2 \right\rangle_E$, averaged over 2000 network realisations, versus the number of time series samples L for the empirical measurements for each network. Error bars represent the standard deviation across realisations in log space.

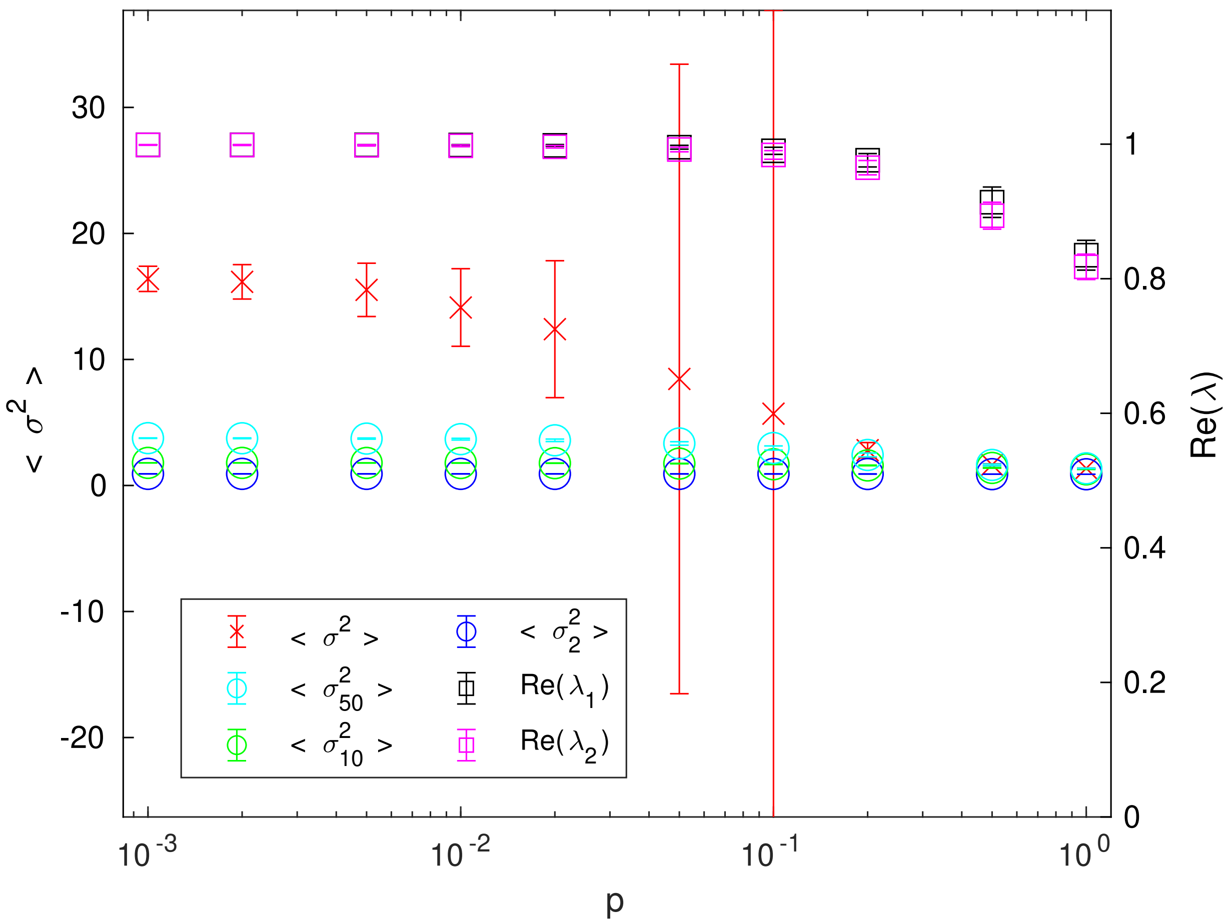

Fig. S2. *Numerical results for* $d = 2$*,* $c = 0.5$ following Fig. 4(a), with the y-axis expanded to show the full extent of the error bars.

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