SIAM J. OPTIM.
Vol. 11, No. 2, pp. 380–388

ON PRIMAL AND DUAL INFEASIBILITY CERTIFICATES IN A HOMOGENEOUS MODEL FOR CONVEX OPTIMIZATION*

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Abstract. Andersen and Ye [*Math. Programming*, 84 (1999), pp. 375–399] suggested a homogeneous formulation and an interior-point algorithm for solution of the monotone complementarity problem (MCP). The advantage of the homogeneous formulation is that it always has a solution. Moreover, in the case in which the MCP is solvable or is (strongly) infeasible, the solution provides a certificate of optimality or infeasibility. In this note we demonstrate that if the suggested formulation is applied to the Karush–Kuhn–Tucker optimality conditions corresponding to a convex optimization problem, then an infeasibility certificate provides information about whether the primal or dual problem is infeasible given certain assumptions.

Key words. convex programming, homogeneous, self-dual

AMS subject classifications. 90C25, 90C20

PII. S1052623499353145

1. Introduction.

Most interior-point methods for solving convex optimization problems require that the problem has an optimal solution. Clearly, this assumption is not satisfied if the problem is either primal or dual infeasible. In the linear case this problem is addressed by using a homogeneous and self-dual model which was originally suggested by Goldman and Tucker [3] and later generalized to the monotone complementarity problem (MCP) by Andersen and Ye [2]. This larger class of problems contains all convex optimization problems, because the Karush–Kuhn–Tucker conditions corresponding to a convex optimization problem form an MCP.

The main idea of the homogeneous model is to embed the optimization problem in a slightly larger problem which always has a solution. The optimal solution to the embedded problem indicates whether the original problem has an optimal solution. Moreover, in the case in which the original problem has an optimal solution, the optimal solution to the embedded problem can easily be transformed into an optimal solution to the original problem. In the case where the original problem is (strongly) infeasible, then a certificate for the infeasibility is computed. However, in [2] it is not stated whether an infeasibility certificate indicates primal or dual infeasibility when the homogeneous model is applied to the optimality conditions of a convex optimization problem. The main purpose of the present work is to show that an infeasibility certificate in some cases indicates whether the primal or dual problem is infeasible.

The outline of the paper is as follows. In section 2 we present a homogeneous model for convex optimization and state the main lemma. In section 3 we apply the developed theory to convex quadratic and quadratically constrained optimization problems.

*Received by the editors March 3, 1999; accepted for publication (in revised form) May 31, 2000; published electronically October 11, 2000.
http://www.siam.org/journals/siopt/11-2/35314.html

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2. A homogeneous model for convex optimization. The problem of interest is

$$\begin{array}{ll} \text{minimize} & c(x) \\ \text{subject to} & a_i(x) \geq 0, \quad i = 1, \ldots, m, \end{array} \tag{2.1}$$

where $x \in R^n$. The function $c : R^n \to R$ is assumed to be convex, and the component functions $a_i : R^n \to R$, $i = 1, \ldots, m$, are assumed to be concave. All functions are assumed to be once differentiable. Hence, the problem (2.1) minimizes a convex function over a convex set.

Next, define the Lagrange function

$$L(x, y) := c(x) - y^T a(x),$$

and then the Wolfe dual corresponding to (2.1) is

$$\begin{array}{ll} \text{maximize} & L(x, y) \\ \text{subject to} & \nabla_x L(x, y)^T = 0, \\ & y \geq 0. \end{array} \tag{2.2}$$

Combining (2.1) and (2.2) gives the MCP

$$\begin{array}{ll} \text{minimize} & y^T z \\ \text{subject to} & \nabla_x L(x, y)^T = 0, \\ & a(x) = z, \\ & y, z \geq 0, \end{array} \tag{2.3}$$

where $z \in R^m$ is a vector of slack variables. A solution to (2.3) is said to be complementary if the corresponding objective value is zero.

Now when applying the homogeneous model suggested in [1, 2] to this problem we obtain the homogenized MCP

$$\begin{array}{lll} \text{minimize} & z^T y + \tau \kappa & \\ \text{subject to} & \tau \nabla_x L(x/\tau, y/\tau)^T & = 0, \\ & \tau a(x/\tau) & = z, \\ & -x^T \nabla_x L(x/\tau, y/\tau)^T - y^T a(x/\tau) & = \kappa, \\ & z, \tau, y, \kappa \geq 0, & \end{array} \tag{2.4}$$

where τ and κ are two additional variables.

Following [2], we say that (2.4) is asymptotically feasible if and only if a convergent sequence $(x^k, z^k, \tau^k, y^k, \kappa^k)$ exists for $k = 1, 2, \ldots$ such that

$$\lim_{k \to \infty} \begin{pmatrix} \tau^k \nabla_x L(x^k/\tau^k, y^k/\tau^k)^T, \\ \tau^k a(x^k/\tau^k) - z^k, \\ -(x^k)^T \nabla_x L(x^k/\tau^k, y^k/\tau^k)^T - (y^k)^T a(x^k/\tau^k) - \kappa^k \end{pmatrix} = 0 \tag{2.5}$$

and

$$(x^k, z^k, \tau^k, y^k, \kappa^k) \in R^n \times R^m_+ \times R_{++} \times R^m_+ \times R_{++} \quad \forall k, \tag{2.6}$$

where the limit point $(x^*, z^*, \tau^*, y^*, \kappa^*)$ of the sequence is called an asymptotically feasible point. We write R_+ and R_{++} for the nonnegative and positive real line, respectively. If this limit point also satisfies

$$(y^*)^T z^* + \tau^* \kappa^* = 0,$$

it is said to be asymptotically complementary.

THEOREM 2.1. *Equation* (2.4) *is asymptotically feasible, and every asymptotically feasible point is an asymptotically complementary solution.*

Proof. See [1] for the proof. □

Hence, Theorem 2.1 implies that the objective function in (2.4) is redundant, and hence the problem is a feasibility problem.

An asymptotically complementary solution $(x^*, z^*, \tau^*, y^*, \kappa^*)$ is said to be maximally complementary if the number of positive coordinates in $(z^*, \tau^*, y^*, \kappa^*)$ is maximal among asymptotically complementary solutions. Using this definition we can state the following theorem.

THEOREM 2.2. *Let* $(x^*, z^*, \tau^*, y^*, \kappa^*)$ *be any asymptotically feasible and maximally complementary solution to* (2.4). *Equation* (2.3) *has a feasible and complementary solution if and only if* $\tau^* > 0$. *Furthermore, in this case* $(x^*, y^*, z^*)/\tau^*$ *is an optimal solution to* (2.3).

Proof. See [1] for the proof. □

Therefore, in the case $\tau^* > 0$ it can be concluded that (2.1) has an optimal solution. On the other hand if $\kappa^* > 0$, then it can be concluded that a primal-dual optimal solution to (2.1) having zero duality gap does not exist. Moreover, using the following lemma it may be possible to conclude that either the primal or the dual problem is infeasible.

LEMMA 2.3. *Let* $(x^k, z^k, \tau^k, y^k, \kappa^k)$ *be any bounded sequence satisfying* (2.6) *such that*

$$\lim_{k\to\infty}(x^k, z^k, \tau^k, y^k, \kappa^k) = (x^*, z^*, \tau^*, y^*, \kappa^*)$$

is an asymptotically feasible and maximally complementary solution to (2.4). *Given*

$$\lim_{k\to\infty} -(x^k)^T \nabla_x L(x^k/\tau^k, y^k/\tau^k)^T - (y^k)^T a(x^k/\tau^k) = \kappa^* > 0, \tag{2.7}$$

then

$$\lim_{k\to\infty} \sup \;\; (\nabla a(x^k/\tau^k)(x^k/\tau^k) - a(x^k/\tau^k))^T(y^k) > 0 \tag{2.8}$$

or

$$\lim_{k\to\infty} \sup -\nabla c(x^k/\tau^k)x^k > 0 \tag{2.9}$$

holds true.

Moreover, if

$$\lim_{k\to\infty} \tau^k \nabla c(x^k/\tau^k) = 0, \tag{2.10}$$

then the primal problem (2.1) *is infeasible if* (2.8) *holds and the dual problem* (2.2) *is infeasible if* (2.9) *holds.*

Proof. Note that $\kappa^* > 0$ implies that $\tau^* = 0$ by complementarity. Furthermore, if (2.7) is true, then either (2.8) or (2.9) must be true. Now suppose (2.10) holds.

Assume first that (2.8) holds and the primal problem (2.1) has a feasible solution. Let $\bar{x}$ be any feasible solution, and since a is concave, then

$$a(\bar{x}) \leq a(x^k/\tau^k) + \nabla a(x^k/\tau^k)(\bar{x} - x^k/\tau^k),$$

which leads to the contradiction

$$
\begin{aligned}
0 &\le \liminf_{k\to\infty}\ (y^k)^T a(\bar{x}) \\
&\le \liminf_{k\to\infty}\ (y^k)^T (a(x^k/\tau^k) + \nabla a(x^k/\tau^k)(\bar{x} - x^k/\tau^k)) \\
&= \liminf_{k\to\infty}\ -(y^k)^T (\nabla a(x^k/\tau^k)(x^k/\tau^k) - a(x^k/\tau^k)) \\
&< 0.
\end{aligned}
\tag{2.11}
$$

Here the equation follows from the fact that assumption (2.10) and the first equation of (2.5) together imply

$$\lim_{k\to\infty} \nabla a(x^k/\tau^k)^T y^k = 0.$$

Hence, if (2.8) holds, then (2.1) must be infeasible.

Assume next that (2.9) is true and the dual problem (2.2) has a solution denoted $(\bar{x}, \bar{y})$. By convexity we have that

$$
\begin{aligned}
c(0) &\ge c(x^k/\tau^k) + \nabla c(x^k/\tau^k)(0 - x^k/\tau^k) \quad \forall k, \\
c(x^k/\tau^k) &\ge c(\bar{x}) + \nabla c(\bar{x})(x^k/\tau^k - \bar{x}) \quad \forall k, \\
a(x^k/\tau^k) &\le a(\bar{x}) + \nabla a(\bar{x})(x^k/\tau^k - \bar{x}) \quad \forall k.
\end{aligned}
$$

This implies

$$
\begin{aligned}
c(0) - \nabla c(x^k/\tau^k)(0 - x^k/\tau^k) - c(\bar{x}) &\ge c(x^k/\tau^k) - c(\bar{x}) \\
&\ge \nabla c(\bar{x})(x^k/\tau^k - \bar{x}) \\
&= (\nabla a(\bar{x})^T \bar{y})^T (x^k/\tau^k - \bar{x}) \\
&\ge \bar{y}^T (a(x^k/\tau^k) - a(\bar{x})).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tau^k(c(0) - \nabla c(x^k/\tau^k)(0 - x^k/\tau^k) - c(\bar{x})) &\ge \tau^k(c(x^k/\tau^k) - c(\bar{x})) \\
&\ge \tau^k \bar{y}^T (a(x^k/\tau^k) - a(\bar{x})).
\end{aligned}
\tag{2.12}
$$

Given the assumptions, we have that

$$\liminf_{k\to\infty}\ \tau^k(c(0) - \nabla c(x^k/\tau^k)(0 - x^k/\tau^k) - c(\bar{x})) < 0$$

and

$$\limsup_{k\to\infty}\ \tau^k \bar{y}^T (a(x^k/\tau^k) - a(\bar{x})) \ge 0,$$

because $\bar{y} \ge 0$ and $\lim_{k\to\infty} \tau^k a(x^k/\tau^k) \ge 0$. Therefore, taking the limit on both sides of (2.12) leads to a contradiction, implying that the dual problem (2.2) is infeasible. $\square$

3. Applications.

In this section we will show that Lemma 2.3 can be strengthened in the case of quadratic and quadratically constrained optimization problems.

3.1. Quadratically constrained quadratic optimization.

A quadratically constrained optimization problem can be stated as

$$
\begin{aligned}
&\text{minimize} && \tfrac{1}{2} x^T Q^0 x + c^T x \\
&\text{subject to} && \tfrac{1}{2} x^T Q^i x + a_{i:} x \ge b_i, \quad i = 1, \ldots, m,
\end{aligned}
\tag{3.1}
$$

where $a_{i:}$ is the ith row of A. It is assumed Q^0 and $-Q^i \in R^{n\times n}$ are symmetric positive semidefinite matrices $\forall i$'s. Moreover, $A \in R^{m\times n}$ and all other quantities have conforming dimensions. The dual problem corresponding to (3.1) is

$$\begin{array}{lll} & \text{maximize} & b^T y - \left(\frac{1}{2}x^T Q^0 x - \sum_{i=1}^m y_i \frac{1}{2} x^T Q^i x\right) \\ (3.2) & \text{subject to} & Q^0 x + c - A^T y - \sum_{i=1}^m y_i Q^i x = 0, \\ & & y \geq 0, \end{array}$$

and the associated homogeneous model is

$$\begin{array}{ll} & Q^0 x + c\tau - A^T y - \sum_{i=1}^m \frac{y_i}{\tau} Q^i x = 0, \\ (3.3) & \frac{x^T Q^i x}{2\tau} + a_{i:}x - b_i\tau = z_i,\ i = 1,\dots,m, \\ & -\frac{x^T Q^0 x}{\tau} - c^T x + \sum_{i=1}^m \frac{y_i}{2\tau}\frac{x^T Q^i x}{\tau} + b^T y = \kappa, \\ & z, y, \tau, \kappa \geq 0. \end{array}$$

Using the special structure of (3.1) and Lemma 2.3 we can state the following lemma.

LEMMA 3.1. *Let* $(x^k, z^k, \tau^k, y^k, \kappa^k)$ *be any bounded sequence satisfying* (2.6) *such that*

$$\lim_{k\to\infty} (x^k, z^k, \tau^k, y^k, \kappa^k) = (x^*, z^*, \tau^*, y^*, \kappa^*)$$

is an asymptotically feasible and maximally complementary solution to (3.3). *Given*

$$\lim_{k\to\infty}\left(-\frac{(x^k)^T Q^0 x^k}{\tau^k} - c^T x^k + \sum_{i=1}^m \frac{y_i^k}{2\tau^k}\frac{(x^k)^T Q^i x^k}{\tau^k} + b^T y^k\right) = \kappa^* > 0, \tag{3.4}$$

then at least one of

$$b^T y^* + \lim_{k\to\infty} \sup \sum_{i=1}^m \frac{y_i^k}{2\tau^k}\frac{(x^k)^T Q^i x^k}{\tau^k} > 0 \tag{3.5}$$

or

$$c^T x^* < 0 \tag{3.6}$$

holds true. The primal problem (3.1) *is infeasible if* (3.5) *holds. Moreover, the dual problem* (3.2) *is infeasible if* (3.6) *holds.*

Proof. It can be verified that

$$\frac{(x^k)^T Q^0 x^k}{\tau^k} \geq 0 \quad \forall k.$$

Hence it follows that if (3.4) holds, then at least one of the conditions (3.5) or (3.6) is true.

First we prove an intermediate result. We have that

$$
\begin{aligned}
0 &= \lim_{k\to\infty} (x^k)^T \left(Q^0 x^k + c\tau^k - A^T y^k - \sum_{i=1}^m \frac{y_i^k}{\tau^k} Q^i x^k \right) \\
&= \lim_{k\to\infty} \left((x^k)^T \left(Q^0 x^k - \sum_{i=1}^m \frac{y_i^k}{2\tau^k} Q^i x^k + c\tau^k \right) - \sum_{i=1}^m y_i^k \left(\frac{(x^k)^T Q^i x^k}{2\tau^k} + a_{i:} x^k \right) \right) \\
&= \lim_{k\to\infty} (x^k)^T \left(Q^0 x^k - \sum_{i=1}^m \frac{y_i^k}{2\tau^k} Q^i x^k \right)
\end{aligned} \tag{3.7}
$$

because $\lim_{k\to\infty} \tau^k = 0$ and

$$
\lim_{k\to\infty} y_i^k \left(\frac{(x^k)^T Q^i x^k}{2\tau^k} + a_{i:} x^k \right) = \lim_{k\to\infty} y_i^k (z_i^k + b_i \tau^k) = 0.
$$

Given the convexity assumptions,

$$
(x^k)^T Q^0 x^k \geq 0.
$$

This fact in combination with (3.7) leads to the conclusion that

$$
(x^*)^T Q^0 x^* = \lim_{k\to\infty} (x^k)^T Q^0 x^k = 0 \qquad \text{and} \qquad Q^0 x^* = \lim_{k\to\infty} Q^0 x^k = 0.
$$

This, combined with the facts

$$
\lim_{k\to\infty} \left(Q^0 x^k + c\tau^k - A^T y^k - \sum_{i=1}^m \frac{y_i^k}{\tau^k} Q^i x^k \right) = 0
$$

and $\lim_{k\to\infty} \tau^k = 0$, leads to the conclusion

$$
\lim_{k\to\infty} \left(-A^T y^k - \sum_{i=1}^m \frac{y_i^k}{\tau^k} Q^i x^k \right) = 0. \tag{3.8}
$$

First, assume that (3.5) is the case and (3.1) has a feasible solution $\bar{x}$. Therefore,

$$
\begin{aligned}
0 &\leq \sum_{i=1}^m y_i^k \left(\frac{1}{2} \bar{x}^T Q^i \bar{x} + a_{i:} \bar{x} - b_i \right) \\
&\leq \sum_{i=1}^m y_i^k \left(\frac{1}{2} \frac{(x^k)^T Q^i x^k}{(\tau^k)^2} + a_{i:} \bar{x} - b_i + (Q^i x^k / \tau^k)^T (\bar{x} - x^k/\tau^k) \right) \\
&= \sum_{i=1}^m y_i^k \left(-\frac{1}{2} \frac{(x^k)^T Q^i x^k}{(\tau^k)^2} - b_i \right) + \bar{x}^T \left(A^T y^k + \sum_{i=1}^m \frac{y_i^k}{\tau^k} Q^i x^k \right),
\end{aligned}
$$

where the second inequality follows from the concavity assumption. This fact, in combination with (3.5) and (3.8), gives rise to the contradiction

$$
\begin{aligned}
0 &\leq \lim_{k\to\infty} \inf \left(\sum_{i=1}^m y_i^k \left(-\frac{1}{2} \frac{(x^k)^T Q^i x^k}{(\tau^k)^2} - b^T y^k \right) + \bar{x}^T \left(A^T y^k + \sum_{i=1}^m \frac{y_i^k}{\tau^k} Q^i x^k \right) \right) \\
&= -b^T y^* - \lim_{k\to\infty} \sup \sum_{i=1}^m \frac{y_i^k}{2\tau^k} \frac{(x^k)^T Q^i x^k}{\tau^k} \\
&< 0.
\end{aligned}
$$

Hence, given (3.5) then (3.1) must be infeasible.

Second, assume that (3.6) holds and the dual problem has a feasible solution denoted $(\bar{x}, \bar{y})$. Then

$$(x^*)^T \left(Q^0 \bar{x} + c - A^T \bar{y} - \sum_{i=1}^m \bar{y}_i Q^i \bar{x} \right) = 0,$$

from which we obtain

$$c^T x^* = (x^*)^T \left(A^T \bar{y} + \sum_{i=1}^m \bar{y}_i Q^i \bar{x} \right). \tag{3.9}$$

Since

$$\begin{aligned} \lim_{k \to \infty} \tau^k \left(\frac{(x^k)^T Q^i x^k}{2\tau^k} + a_{i:} x^k - \tau^k b_i \right) &= \lim_{k \to \infty} \tau^k z_i^k \\ &= 0 \end{aligned} \tag{3.10}$$

then

$$\lim_{k \to \infty} (x^k)^T Q^i x^k = (x^*)^T Q^i x^* = 0$$

is true. This fact, in combination with (3.9), leads to the contradiction

$$\begin{aligned} 0 &> c^T x^* \\ &= (x^*)^T A^T \bar{y} \\ &\geq 0 \end{aligned}$$

because $\bar{y} \geq 0$ and

$$\begin{aligned} a_{i:} x^* &= \lim_{k \to \infty} a_{i:} x^k \\ &= \lim_{k \to \infty} \left(z_i^k - \frac{(x^k)^T Q^i x^k}{\tau^k} + b_i \tau^k \right) \\ &\geq 0. \end{aligned}$$

Therefore, we can conclude that if (3.6) holds, then (3.2) is infeasible. ◻

3.2. Quadratic optimization. An important special case of quadratically constrained optimization is quadratic optimization, i.e., the case where

$$Q^i = 0, \quad i = 1, \ldots, m.$$

In this case the homogeneous model has the form

$$\begin{aligned} Q^0 x + c\tau - A^T y &= 0, \\ Ax - b\tau &= z, \\ -\frac{x^T Q^0 x}{\tau} - c^T x + b^T y &= \kappa, \\ x, \tau, y, \kappa, &\geq 0. \end{aligned} \tag{3.11}$$

Using Lemma 3.1 we can state the following lemma.

LEMMA 3.2. *Let* $(x^k, z^k, \tau^k, y^k, \kappa^k)$ *be any bounded sequence satisfying* (2.6) *such that*

$$\lim_{k\to\infty}(x^k, z^k, \tau^k, y^k, \kappa^k) = (x^*, z^*, \tau^*, y^*, \kappa^*)$$

is an asymptotically feasible and maximally complementary solution to (3.11). *Given*

$$\lim_{k\to\infty}\left(-\frac{(x^k)^T Q^0 x^k}{\tau^k} - c^T x^k + b^T y^k\right) = \kappa^* > 0, \tag{3.12}$$

then

$$b^T y^* > 0 \tag{3.13}$$

or

$$c^T x^* < 0 \tag{3.14}$$

holds true. The primal problem (3.1) *is infeasible if* (3.13) *holds. Moreover, the dual problem* (3.2) *is infeasible if* (3.14) *holds.*

Proof. This follows immediately from Lemma 3.1. From the proof, note that $Q^0 x^* = 0$.

It can be observed that in the case where the primal problem (3.1) is concluded to be infeasible, y^* satisfies

$$A^T y^* = 0, \quad b^T y^* > 0, \quad y^* \geq 0, \tag{3.15}$$

which by Farkas's lemma implies that

$$\{x : Ax \geq b\} = \emptyset, \tag{3.16}$$

i.e., the problem is infeasible. Observe if the dual problem has a feasible solution; then $(0, y^*)$ is a ray along which dual objective value tends to $+\infty$, i.e., the dual problem (3.2) is unbounded.

Similarly, it can be observed that in the case where the dual problem (3.2) is concluded to be infeasible, an x^* is known such that

$$Ax^* \geq 0, \quad Q^0 x^* = 0, \quad c^T x^* < 0, \tag{3.17}$$

which (once again using Farkas's lemma) implies that

$$\{(x, y) :\ Q^0 x + c - A^T y = 0,\ y \geq 0\} = \emptyset.$$

Note also that if the primal problem has a feasible solution, then x^* is a ray along which the primal objective value tends to $-\infty$, i.e., the primal problem (3.1) is unbounded. ∎

Acknowledgments. The author is thankful for comments made by Tamás Terlaky, the anonymous referees, and the associate editor, Mike Todd, regarding this article.

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