LP Solver outline

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Dec 2014

1 LP Formulation

Primal

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & Gx + s = h \\ & s \geq 0 \end{aligned} \tag{1}$$

Dual

$$\begin{aligned} \max \quad & -h^T z - b^T y \\ \text{s.t.} \quad & Ax = b \\ & G^T z + A^T y + c = 0 \\ & z \geq 0 \end{aligned} \tag{2}$$

Where

- A is $n \times k$
- G is $m \times k$
- x is $k \times 1$
- s is $m \times 1$
- z is $m \times 1$
- y is $n \times 1$

2 Central Path for the original formulation

$$\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 & A^T & G^T \\ A & 0 & 0 \\ G & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -c \\ b \\ h \end{bmatrix} \tag{3}$$

$$(z, s) \geq 0, \quad z \bullet s = \mu \tag{4}$$

3 Homogenous Self-dual Embedding

$$\max \quad 0 \tag{5}$$

$$\text{s.t.} \quad \begin{bmatrix} 0 \\ 0 \\ s \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & 0 & h \\ -c^T & -b^T & -h^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \tau \end{bmatrix} \tag{6}$$

$$(z, s, \tau, \kappa) \geq 0 \tag{7}$$

For each feasible solution $z \bullet s = 0$, $\tau\kappa = 0$

4 Central Path for Homogenous Self-dual Embedding

$$\begin{bmatrix} 0 \\ 0 \\ s \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & 0 & h \\ -c^T & -b^T & -h^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \tau \end{bmatrix} \tag{8}$$

$$z \bullet s = \mu, \quad \tau\kappa = \mu \tag{9}$$

$$(z, s, \tau, \kappa) \geq 0 \tag{10}$$

5 Path Following Algorithm

1. Choose an initial point $(x_0, s_0, y_0, z_0, \tau_0, \kappa_0)$, $\quad s_0, z_0, \tau_0, \kappa_0 > 0$

2. Calculated residuals

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s_0 \\ \kappa_0 \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & 0 & h \\ -c^T & -b^T & -h^T & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \tau_0 \end{bmatrix} \tag{11}$$

$$\mu_0 = \frac{s_0^T z_0 + \tau_0\kappa_0}{m+1}$$

3. Solve for affine direction

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s \\ \Delta\kappa \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & 0 & h \\ -c^T & -b^T & -h^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\tau \end{bmatrix} = - \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \tag{12}$$

$$\Delta z \bullet s_0 + z_0 \bullet \Delta s = -z_0 \bullet s_0$$

$$\Delta\tau\kappa_0 + \tau_0\Delta\kappa = -\tau_0\kappa_0$$

4. Compute Step size α, and Centering Parameter σ

$$\alpha = \sup\{\alpha \in [0,1] | (x_0, s_0, y_0, z_0, \tau_0, \kappa_0) + \alpha(\Delta x_a, \Delta s_a, \Delta y_a, \Delta z_a, \Delta\tau_a, \Delta\kappa_a) \geq 0\}$$

$$\mu_a = \frac{(s_0 + \alpha\Delta s_a)^T(z_0 + \alpha\Delta z_a) + (\tau_0 + \alpha\Delta\tau_a)(\kappa_0 + \alpha\Delta\kappa_a)}{m+1}$$

$$\sigma = \frac{\mu_a}{\mu_0} \text{ or } \left(\frac{\mu_a}{\mu_0}\right)^3$$

5. Corrector direction

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s \\ \Delta\kappa \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & 0 & h \\ -c^T & -b^T & -h^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\tau \end{bmatrix} = -(1-\sigma) \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \tag{13}$$

$$\Delta z \bullet s_0 + z_0 \bullet \Delta s = -z_0 \bullet s_0 - \Delta s_a \bullet \Delta z_a + \sigma\mu_0$$

$$\Delta\tau\kappa_0 + \tau_0\Delta\kappa = -\tau_0\kappa_0 - \Delta\tau_a\Delta\kappa_a + \sigma\mu_0$$

6. Update

$$(x_{\text{new}}, s_{\text{new}}, y_{\text{new}}, z_{\text{new}}, \tau_{\text{new}}, \kappa_{\text{new}}) = (x_0, s_0, y_0, z_0, \tau_0, \kappa_0) + \alpha(\Delta x, \Delta s, \Delta y, \Delta z, \Delta\tau, \Delta\kappa)$$

6 Solving Linear Equations

Consider

$$\begin{bmatrix} 0 \\ 0 \\ \Delta s \\ \Delta\kappa \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & 0 & h \\ -c^T & -b^T & -h^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\tau \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{14}$$

$$\Delta z \bullet s_0 + z_0 \bullet \Delta s = q_5$$

$$\Delta\tau\kappa_0 + \tau_0\Delta\kappa = q_6$$

Eliminating $\Delta\kappa$ and Δs:

$$\Delta s = (q_5 - \Delta z \bullet s_0)/z_0$$

$$\Delta\kappa = (q_6 - \Delta\tau\kappa_0)/\tau_0$$

$$\begin{bmatrix} 0 \\ 0 \\ (q_5 - \Delta z \bullet s_0)/z_0 \\ (q_6 - \Delta\tau\kappa_0)/\tau_0 \end{bmatrix} - \begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & 0 & h \\ -c^T & -b^T & -h^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\tau \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{15}$$

Or

$$-\begin{bmatrix} 0 & A^T & G^T & c \\ -A & 0 & 0 & b \\ -G & 0 & D_\tau & h \\ -c^T & -b^T & -h^T & \kappa_0/\tau_0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \tau \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_7 \\ q_8 \end{bmatrix} \tag{16}$$

where $D_\tau = \mathrm{diag}(s_0/z_0)$, $q_7 = q_3 - q_5/z_0$, $q_8 = q_4 - q_6/\tau_0$
eliminating $\Delta\tau$:

$$\Delta\tau = -\frac{q_8 - c^T\Delta x - b^T\Delta y - h^T\Delta z}{\kappa_0/\tau_0}$$

More on Section 7.4 of "The CVXOPT" linear and quadratic cone program solver