SOCIAL NETWORK ANALYTICS

Game Theory

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Introduction to Game Theory and Game Theory Types

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Game Theory

Central idea:

Game Theoretic Models are very natural for modeling social networks

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Social network nodes are rational, intelligent

Social networks form in a decentralized way

Strategic interactions among social network nodes

It would be interesting to explore Game Theoretic Models for analyzing social networks

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Example 1: Discovering Communities

Example 2: Finding Influential Nodes

Example 3: Monitizing Social Networks

Ramasuri Narayanam. Game Theoretic Models for Social Network Analysis, Ph.D. Dissertation, CSA, IISc, November 2010

Why are social networks Important?

- Diffusion of Information and Innovations
- To understand spread of diseases (Epidemiology)
- E-Commerce and E-Business (selling patterns, marketing)
- Job Finding (through referrals)
- Determine Influential Players (scientists, innovators, employees, customers, companies, genes, etc.)
- Build effective social and political campaigns
- Predict future events
- Crack terrorist/criminal networks
- Track alumni, etc…

Game Theory: History

- The discipline of game theory was pioneered in the early 20th century by mathematicians Ernst Zermelo (1913) and John von Neumann (1928).
- The breakthrough came with John von Neumann and Oscar Morgenstern's book, **Theory of games and economic behavior**, published in 1944.
- This was followed by important work by John Nash (1950-51) and Lloyd Shapley (1953).


John von Neumann


John Nash

What is Game theory?

- Game theory is the study of the ways in which *interacting choices* of *economic agents* produce *outcomes* with respect to the *preferences* (or *utilities*) of those agents, where the outcomes in question might have been intended by none of the agents.
- Game theory is the study of mathematical models of strategic interaction among rational decision-makers.
- Game theory is a mathematical framework for rigorous study of conflict and cooperation among rational, intelligent agents.
- Game theory is the process of modeling the strategic interaction between two or more players in a situation containing set rules and outcomes.

Note:
1. **Rational -** ವಿಚಾರಶಕ್ತಿಯುಳ್ಳ, ತರ್ಕಬದ್ಧ, ತರ್ಕಾಧಾರಿತ, ವಿವೇಕಯುಕ್ತ, ವಿವೇಚನೆಯಿಂದ ಕೂಡಿದ, ತರ್ಕಸಮ್ಮತವಾದ, ಯುಕ್ತಾಯುಕ್ತಪರಿಜ್ಞಾನವುಳ್ಳ, ತೀರ್ಮಾನಿಸಬಲ್ಲ
   - based on clear thought and reason.
   - based on or in accordance with reason or logic.
2. **Strategy -** ತಂತ್ರ, ಯುದ್ಧ, ವ್ಯೂಹ, ಯೋಜನೆಯನ್ನು ಸಾಧಿಸಲು ಅನುಸರಿಸುವ ನೀತಿ
   - an elaborate and systematic plan of action.
   - a plan of action designed to achieve a long-term or overall aim.
3. **Strategic** – ಕಾರ್ಯತಂತ್ರದ - relating to the identification of long-term or overall aims and interests and the means of achieving them.

Game theory: Applications

- Game theory has a wide range of applications, including psychology, evolutionary biology, war, politics, economics, business, etc..
- Despite its many advances, game theory is still a young and developing science.

Applications of Game Theory

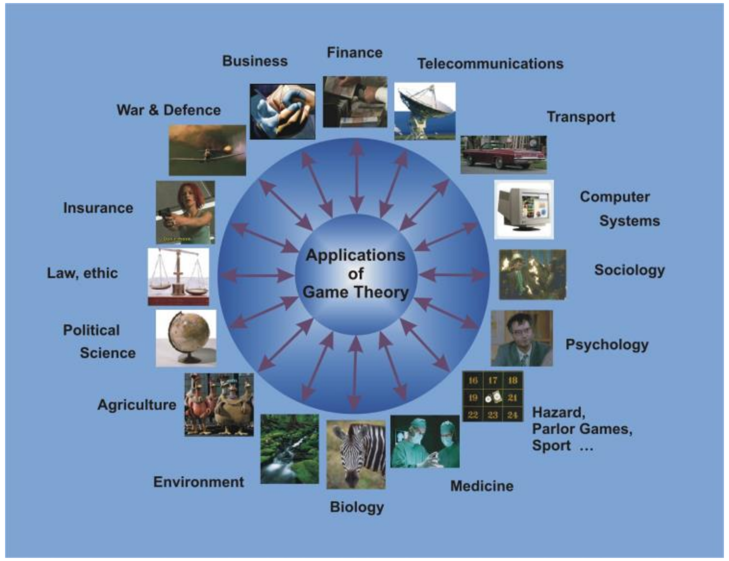
Microeconomics, Sociology, Evolutionary Biology

Auctions and Market Design: Spectrum Auctions, Procurement Markets, Double Auctions

Industrial Engineering, Supply Chain Management, E-Commerce, Resource Allocation

CS: Algorithmic Game Theory, Internet and Network Economics, Protocol Design, etc.

There has been a surge of interest in applying Game Theory to SNA and KDD Problems



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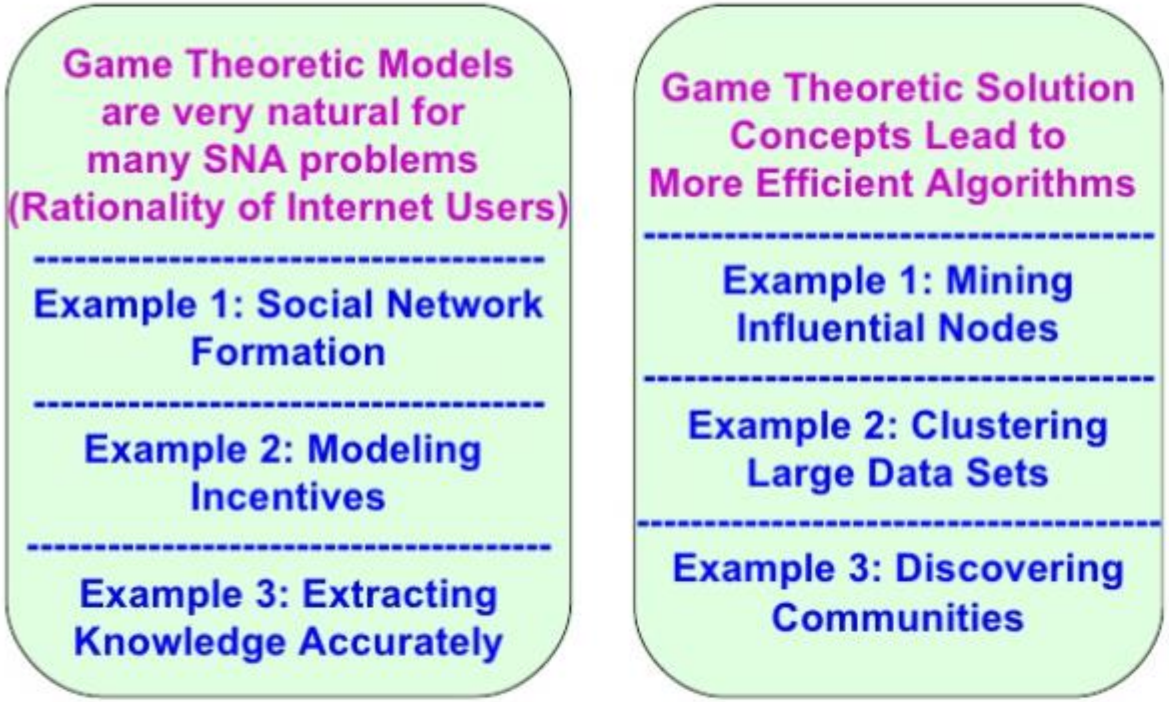
Game Theory applications in Social Networks

1. Application for Information diffusion
2. Application for Community detection
3. Application for Behaviour analysis
4. Application for Social network security
   - Access control
   - Privacy strategy

Ref: "A Survey of Game Theory as Applied to Social Networks" - Xinfang Song, Wei Jiang, Xiaohui Liu, Hui Lu, Zhihong Tian, and Xiaojiang Du

Game Theory applications in Social Networks

Game Theory in SNA: Two viewpoints



Ref: https://www.slideshare.net/municsaa/lecture5social-network-analysisby-dr-y-narahari-9501678

Game Theory in Business

- Game theory is applied for determining different strategies in the business world.
- Game theory offers valuable tools for solving strategy problems. **Many business strategies are short or long-term plans to achieve sustainable profitability**.
- **A business can often successfully position in the market with right strategy** and **a business will suffer in the long run with wrong strategy**.

Table-10: Payoff Matrix for Simultaneous Move Games			
	Organization Y		
Organization X		Outsource	Not to Outsource
	Outsource	20,10	30,0
	Not to Outsource	10,15	25,5

Game Theory in Political affairs

- Game theory is widely used in political affairs, which is focused on the areas of
  - international politics,
  - war strategy, war bargaining,
  - social choice theory,
  - strategic voting,
  - political economy etc.
- Game theory is an effective tool in the hands of diplomats and politicians to analysis any situation of conflict between individuals, companies, states, political parties.

Game Theory: Types of Players

- **The nature of the players varies depending on the context** in which the game theoretic language is invoked:
  1. In evolutionary biology players are non-thinking living organisms;
  2. In computer science players are artificial agents;
  3. In behavioral game theory players are "ordinary" human beings, etc.

  Traditionally, however, **game theory has focused on interaction among intelligent, sophisticated and rational individuals**.

Game Theory - Types

Game Theory can be divided into 5 main types of games:

1. Cooperative vs Non-Cooperative Games:

- **In cooperative games, participants can establish alliances in order to maximise their chances to win the game (eg. negotiations).**
- Cooperative games are the one in which players are convinced to adopt a particular strategy through negotiations and agreements between players.
- The Cooperative game theory assumes that the players can communicate, form coalitions and sign binding agreements.

Game Theory - Types

Game Theory can be divided into 5 main types of games:

1. Cooperative vs Non-Cooperative Games:

- **Non-cooperative game theory** models **situations where the players are either unable to communicate** or are **able to communicate but cannot sign binding contracts.**
- **An example of the cooperative game** is the interaction among firms in an industry in an environment where antitrust laws make it illegal for firms to reach agreements concerning prices or production quotas or other forms of collusive behavior.
- In non-cooperative games, participants can't instead form alliances (eg. wars).

Game Theory - Types

2. Symmetric vs Asymmetric Games:

- **In symmetric games, strategies adopted by all players are same**. Symmetry can exist in short-term games only because in long-term games the number of options with a player increases.

- The decisions in a symmetric game depend on the strategies used, not on the players of the game. Even in case of interchanging players, the decisions remain the same in symmetric games.

- **Example** of symmetric games is prisoner's dilemma.

Game Theory - Types

2. Symmetric vs Asymmetric Games:

- **Asymmetric games are the one in which strategies adopted by players are different.**
- In asymmetric games,
  - the strategy that provides benefit to one player may not be equally beneficial for the other player.
  - Decision making depends on the different types of strategies and decision of players.
- **Example:** An entry of new organization in a market because different organizations adopt different strategies to enter in the same market.

Game Theory - Types

3. Perfect vs Imperfect Information Games:

- In Perfect Information games all the players can see the other players moves (eg. chess).
- Instead, in Imperfect Information games, the other players' moves are hidden (eg. card games).

Game Theory - Types

4. Simultaneous vs Sequential Games:

- **Simultaneous games** are the one in which the move of two players (the strategy adopted by two players) is simultaneous. In simultaneous move, players do not have knowledge about the move of other players.
- **Sequential games** are the one in which players are aware about the moves of players who have already adopted a strategy. Games such as chess, infinite chess, backgammon, tic-tac-toe and Go are examples of sequential games.
- Simultaneous games are represented in normal form(payoff matrices) while sequential games are represented in extensive form(Decision trees).

Table-10: Payoff Matrix for Simultaneous Move Games

		Organization Y	
Organization X		**Outsource**	**Not to Outsource**
	Outsource	20,10	30,0
	Not to Outsource	10,15	25,5

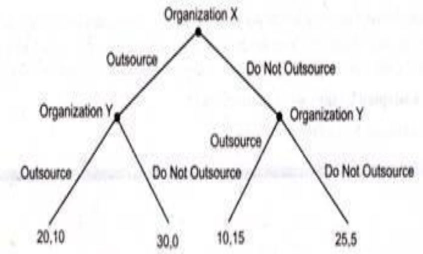


Figure-3: Sequential Move Game

Game Theory - Types

4. Simultaneous vs Sequential Games:

- A **sequential game** is a game where one player chooses their action before the others choose theirs. Importantly, the later players must have some information of the first's choice, otherwise the difference in time would have no strategic effect. **Sequential games hence are governed by the time axis, and represented in the form of decision trees.**
- Sequential games with perfect information can be analysed mathematically using combinatorial game theory.
- Decision trees show the sequence in which players act and the number of times that they can each make a decision. Decision trees also provide information on what each player knows or does not know at the point in time they decide on an action to take. Payoffs of each player are also given at the decision nodes of the tree.

Chess is an example of a sequential game.

Game Theory - Types

4a. Normal Form and Extensive Form Games:

- **Normal form games** refer to the description of game in the form of matrix. In other words, when the payoff and strategies of a game are represented in a tabular form, it is termed as normal form games.
- **Normal form games help in identifying the dominated strategies and Nash equilibrium.**
- In normal form games, the matrix demonstrates the strategies adopted by the different players of the game and their possible outcomes.

		Steven: Split		Steven: Steal	
Sarah	Split	Sarah gets $50,000	Steven gets $50,000	Sarah gets nothing	Steven gets $100,000
Sarah	Steal	Sarah gets $100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Table 1.1

Game Theory - Types

4a. Normal Form and Extensive Form Games:

- On the other hand, extensive form games are the one in which the description of game is done in the form of a decision tree.

- Extensive form games help in the representation of events that can occur by chance. These games consist of a tree-like structure in which the names of players are represented on different nodes.

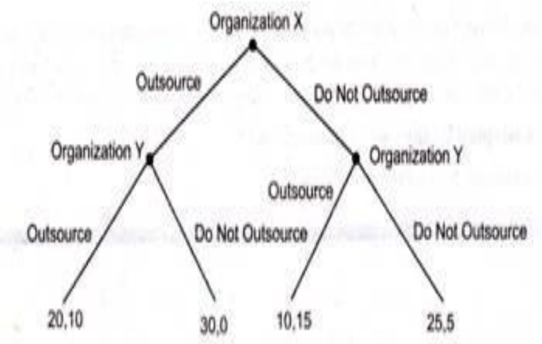


Figure-3: Sequential Move Game

Game Theory - Types

4a. Normal Form and Extensive Form Games:

- **A repeated game is an extensive form game** that **consists of a number of repetitions of some base game** (called a **stage game**).
  The stage game is usually one of the well-studied 2-person games.

- Repeated games capture the idea that a player will have to take into account the impact of his or her current action on the future actions of other players; this impact is sometimes called his or her reputation.

- Single stage game or single shot game are names for non-repeated games.

Game Theory - Types

5. Constant Sum, Zero Sum, and Non-Zero Sum Games:

- **Constant sum game** is the one in which the sum of outcome of all the players remains constant even if the outcomes are different.
- **Zero sum game** is **a type of constant sum game in which the sum of outcomes of all players is zero**.
- **In zero sum game,**
  - **the strategies of different players cannot affect the available resources.**
  - **the gain of one player is always equal to the loss of the other player**.
- **Examples of zero sum games are chess and gambling**. In these games, the gain of one player results in the loss of the other player.

Game Theory - Types

5. Constant Sum, Zero Sum, and Non-Zero Sum Games:

- **Non-zero sum games** are the games in which sum of the outcomes of all the players is not zero.
- A non-zero sum game can be transformed to zero sum game by adding one dummy player. The losses of dummy player are overridden by the net earnings of players.
- **Example:** Cooperative games are the example of non-zero sum games. This is because in cooperative games, either every player wins or loses.

Game Theory: Example

- **Game theory deals with interactive situations where two or more individuals, called players, make decisions that jointly determine the final outcome.**
- To see an example point your browser to the following video
  - Video1
  - Video2
  - Video3
  - Video4

Game Theory: Example

- The video shows **a British daytime TV game show**. In it each of two players, Sarah and Steve, has to pick one of two balls:
  - Inside one ball appears the word '**split**' and inside the other the word '**steal**' (each player is first asked to secretly check which of the two balls in front of him/her is the split ball and which is the steal ball).
  - They make their decisions simultaneously.
  - The possible outcomes are shown in Table 1.1

		Steven: Split		Steven: Steal	
Sarah	Split	Sarah gets $50,000	Steven gets $50,000	Sarah gets nothing	Steven gets $100,000
Sarah	Steal	Sarah gets $100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Table 1.1

Game Theory: Example

- The possible outcomes are shown in Table 1.1

Sarah \ Steven	Split	Steal
Split	Sarah gets \$50,000 / Steven gets \$50,000	Sarah gets nothing / Steven gets \$100,000
Steal	Sarah gets \$100,000 / Steven gets nothing	Sarah gets nothing / Steven gets nothing

Table 1.1

Game Theory: Example

		Steven: Split	Steven: Steal
Sarah	Split	Sarah gets \$50,000 / Steven gets \$50,000	Sarah gets nothing / Steven gets \$100,000
Sarah	Steal	Sarah gets \$100,000 / Steven gets nothing	Sarah gets nothing / Steven gets nothing

Table 1.1

- Let us denote the outcomes as follows:

o_1 :	Sarah gets \$50,000 and Steven gets \$50,000.
o_2 :	Sarah gets nothing and Steven gets \$100,000.
o_3 :	Sarah gets \$100,000 and Steven gets nothing.
o_4 :	Sarah gets nothing and Steven gets nothing.

Table 1.2

Game Theory: Example

o_1 : Sarah gets \$50, 000 and Steven gets \$50, 000.
o_2 : Sarah gets nothing and Steven gets \$100, 000.
o_3 : Sarah gets \$100, 000 and Steven gets nothing.
o_4 : Sarah gets nothing and Steven gets nothing.

Table 1.2

- If, **Sarah is selfish and greedy** – in the sense that, in evaluating the outcomes, she focuses exclusively on what she herself gets and prefers more money to less – then **her ranking of the outcomes** is as follows: $\mathbf{o_3 \succ o_1 \succ o_2 \sim o_4}$
- But there are other possibilities. For example, **Sarah might be fair-minded** and view the outcome where both get \$50,000 as better than all the other outcomes. For instance, her ranking could be $\mathbf{o_1 \succ o_3 \succ o_2 \succ o_4}$;

References

1. Game Theory - An open access textbook with 165 solved exercises - Giacomo Bonanno
2. Wikipedia – Current Literature
3. https://www.economicsdiscussion.net/game-theory/5-types-of-games-in-game-theory-with-diagram/3827

SOCIAL NETWORK ANALYTICS

Game Frame

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Game frame:

➢ **The definition of game-frame is as follows.**

Definition 1.1. A *game-frame in strategic form* is a list of four items (a quadruple) $\langle I, (S_1, S_2, ..., S_n), O, f \rangle$ where:

- $I = \{1,2,...,n\}$ is a set of *players* ($n \geq 2$).
- $(S_1, S_2, ..., S_n)$ is a list of sets, one for each player. For every Player $i \in I$, S_i is the set of *strategies* (or possible choices) of Player *i*. We denote by S the Cartesian product of these sets: $S = S_1 \times S_2 \times ... \times S_n$; thus an element of S is a list $s = (s_1, s_2, ..., s_n)$ consisting of one strategy for each player. We call S the set of *strategy profiles*.
- O is a set of *outcomes*.
- $f : S \to O$ is a function that associates with every strategy profile s an outcome $f(s) \in O$.

Game frame: $\langle I, (S_1, S_2, \dots, S_n), O, f \rangle$

➢ The definition of game-frame is as follows. Cont..

Using the notation of Definition 1.1, the situation illustrated in Table 1.1 is the following game-frame in strategic form:

- $I = \{1,2\}$ (letting Sarah be Player 1 and Steven Player 2),
- $(S_1, S_2) = (\{Split, Steal\}, \{Split, Steal\})$; thus $S_1 = S_2 = \{Split, Steal\}$, so that the set of strategy profiles is $S = \{(Split, Split), (Split, Steal), (Steal, Split), (Steal, Steal)\}$,
- O is the set of outcomes listed in Table 1.2,
- f is the following function:

$$\begin{array}{ccccc} s: & (Split, Split) & (Split, Steal) & (Steal, Split) & (Steal, Steal) \\ f(s): & o_1 & o_2 & o_3 & o_4 \end{array}$$

(that is, $f((Split, Split)) = o_1$, $f((Split, Steal)) = o_2$, etc.).

		Steven: Split		Steven: Steal	
Sarah	Split	Sarah gets \$50,000	Steven gets \$50,000	Sarah gets nothing	Steven gets \$100,000
Sarah	Steal	Sarah gets \$100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Table 1.1

o_1 : Sarah gets \$50, 000 and Steven gets \$50, 000.
o_2 : Sarah gets nothing and Steven gets \$100, 000.
o_3 : Sarah gets \$100, 000 and Steven gets nothing.
o_4 : Sarah gets nothing and Steven gets nothing.

Table 1.2

Game frame:

From a game-frame one obtains a game by adding, for each player, her preferences over (or ranking of) the possible outcomes. We use the following notation.

Notation	Meaning
$o \succ_i o'$	Player i considers outcome o to be *better than* outcome o'
$o \sim_i o'$	Player i considers o to be *just as good as* o' (that is, Player i is indifferent between o and o')
$o \succsim_i o'$	Player i considers o to be *at least as good as* o' (that is, either better than or just as good as)

Table 1.3

Game frame:

Notation	Meaning
$o \succ_i o'$	Player i considers outcome o to be *better than* outcome o'
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$o \succsim_i o'$	Player i considers o to be *at least as good as* o' (that is, either better than or just as good as)

Table 1.3

For example, if M denotes 'Mexican food' and J 'Japanese food', $M \succ_{Alice} J$ means that Alice prefers Mexican food to Japanese food and $M \sim_{Bob} J$ means that Bob is indifferent between the two.

Game frame:

Remark 1.1. The "at least as good" relation $\succsim$ is sufficient to capture also strict preference $\succ$ and indifference $\sim$. In fact, starting from $\succsim$, one can define strict preference as follows: $o \succ_i o'$ if and only if $o \succsim_i o'$ and $o' \not\succsim_i o$ and one can define indifference as follows: $o \sim_i o'$ if and only if $o \succsim_i o'$ and $o' \succsim_i o$.

the "at least as good" relation $\succsim_i$ of Player i – which embodies her preferences over (or ranking of) the outcomes – is *complete* (for every two outcomes o_1 and o_2, either $o_1 \succsim_i o_2$ or $o_2 \succsim_i o_1$) and *transitive* (if $o_1 \succsim_i o_2$ and $o_2 \succsim_i o_3$ then $o_1 \succsim_i o_3$).

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Game frame:

There are (at least) three ways of representing, or expressing, a complete and transitive ranking of a set of outcomes *O*.

For example, suppose that $O = \{o_1, o_2, o_3, o_4, o_5\}$ and that we want to represent the following ranking:
o_3 is the best outcome, it is better than o_5, which is just as good as o_1, o_1 is better than o_4, which, in turn, is better than o_2 (so that o_2 is the worst outcome).

We can represent this ranking in one of the following ways.

1. By using the notation of Table 1.3: $o_3 \succ o_5 \sim o_1 \succ o_4 \succ o_2$.

Notation	Meaning
$o \succ_i o'$	Player *i* considers outcome *o* to be *better than* outcome *o'* (equivalently, Player *i* *prefers* *o* to *o'*)
$o \sim_i o'$	Player *i* considers *o* to be *just as good as* *o'* (equivalently, Player *i* is indifferent between *o* and *o'*)
$o \succsim_i o'$	Player *i* considers *o* to be *at least as good as* *o'* (that is, either better than or just as good as)

Table 1.3

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Game frame:

There are (at least) three ways of representing, or expressing, a complete and transitive ranking of a set of outcomes *O*.

We can represent this ranking in one of the following ways.

Notation	Meaning
$o \succ_i o'$	Player i considers outcome o to be *better than* outcome o' (equivalently, Player i *prefers* o to o')
$o \sim_i o'$	Player i considers o to be *just as good as* o' (equivalently, Player i is indifferent between o and o')
$o \succsim_i o'$	Player i considers o to be *at least as good as* o' (that is, either better than or just as good as)

Table 1.3

2. **By listing the outcomes in a column, starting with the best at the top and proceeding down to the worst**, thus using the convention that if outcome o is listed above outcome o' then o is preferred to o', while if o and o' are written next to each other, then they are considered to be just as good:

best	o_3
	o_1, o_5
	o_4
worst	o_2

There are (at least) three ways of representing, or expressing, a complete and transitive ranking of a set of outcomes *O*.

We can represent this ranking in one of the following ways.

3. By assigning a number to each outcome, with the convention that *if the number assigned to o is greater than the number assigned to o' then o is preferred to o'* and if two outcomes are assigned the same number then they are considered to be just as good.

For example, we could choose the following numbers:

o_1	o_2	o_3	o_4	o_5
6	1	8	2	6

Such an assignment of numbers is called a *utility function*.
A useful way of thinking of utility is as an "index of satisfaction":
the higher the index the better the outcome;
however, this suggestion is just to aid memory and should be taken with a grain of salt, because a utility function does not measure anything and, furthermore, as explained below, the actual numbers used as utility indices are completely arbitrary.

o_1 : Sarah gets \$50,000 and Steven gets \$50,000.
o_2 : Sarah gets nothing and Steven gets \$100,000.
o_3 : Sarah gets \$100,000 and Steven gets nothing.
o_4 : Sarah gets nothing and Steven gets nothing.

Table 1.2

Game frame:

Definition 1.2. Given a complete and transitive ranking $\succsim$ of a finite set of outcomes O, a function $U : O \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of real numbers) is said to be an *ordinal utility function that represents the ranking* $\succsim$ if, for every two outcomes o and o', $U(o) > U(o')$ if and only if $o \succ o'$ and $U(o) = U(o')$ if and only if $o \sim o'$. The number $U(o)$ is called the *utility of outcome* o.

Example:

Player-1 ranking of Outcomes: $o_3 \succ o_1 \succ o_2 \sim o_4$

outcome → / utility function ↓	o_1	o_2	o_3	o_4
f	5	2	10	2
g	0.8	0.7	1	0.7
h	27	1	100	1

		Steven	
Sarah		split	steal
	split	5 , _	2 , _
	steal	10 , _	2 , _

		Steven			
		Split		Steal	
Sarah	Split	Sarah gets \$50,000	Steven gets \$50,000	Sarah gets nothing	Steven gets \$100,000
	Steal	Sarah gets \$100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Table 1.1

o_1 : Sarah gets \$50,000 and Steven gets \$50,000.

o_2 : Sarah gets nothing and Steven gets \$100,000.

o_3 : Sarah gets \$100,000 and Steven gets nothing.

o_4 : Sarah gets nothing and Steven gets nothing.

Table 1.2

Game frame:

Remark:

Note that the statement "**for Alice the utility of Mexican food is 10**" is in itself a meaningless statement; on the other hand, what would be a meaningful statement is "**for Alice the utility of Mexican food is 10 and the utility of Japanese food is 5**", because such a statement conveys the information that she prefers Mexican food to Japanese food.

The two numbers 10 and 5 have no other meaning besides the fact that 10 is greater than 5: for example we *cannot*, and should not, infer from these numbers that she considers Mexican food twice as good as Japanese food.

Game frame:

- There is an infinite number of utility functions that represent the same ranking. For instance, the following are equivalent ways of representing the ranking $o_3 \succ o_1 \succ o_2 \sim o_4$ (f, g and h are three out of the many possible utility functions):

outcome → *utility function* ↓	o_1	o_2	o_3	o_4
f	5	2	10	2
g	0.8	0.7	1	0.7
h	27	1	100	1

- **Utility functions are a particularly convenient way of representing preferences**. In fact, by using utility functions one can give a more condensed representation of games.

Game frame:

Definition 1.3. An *ordinal game in strategic form* is a quintuple $\langle I,(S_1,\ldots,S_n),O,f,(\succsim_1,\ldots,\succsim_n)\rangle$ where:

- $\langle I,(S_1,\ldots,S_n),O,f\rangle$ is a game-frame in strategic form (Definition 1.1) and
- for every Player $i \in I$, $\succsim_i$ is a complete and transitive ranking of the set of outcomes O.

If we replace each ranking $\succsim_i$ with a utility function U_i that represents it, and we assign, to each strategy profile s, Player i's utility of $f(s)$ (recall that $f(s)$ is the outcome associated with s) then we obtain a function $\pi_i : S \to \mathbb{R}$ called Player i's *payoff function*. Thus $\pi_i(s) = U_i(f(s))$.[4] Having done so, we obtain a triple $\langle I,(S_1,\ldots,S_n),(\pi_1,\ldots,\pi_n)\rangle$ called a *reduced-form ordinal game in strategic form* ('reduced-form' because some information is lost, namely the specification of the possible outcomes).

$$\boxed{\begin{array}{l} f : S \to O \\ U : O \to \mathbb{R} \\ \pi_i(s) = U_i(f(s)) \end{array}}$$

S – Set of strategy profiles

Strategy profile (s) – A list consisting of one strategy for each player

Game frame:

- For example,

		Steven: Split		Steven: Steal	
Sarah	Split	Sarah gets \$50,000	Steven gets \$50,000	Sarah gets nothing	Steven gets \$100,000
	Steal	Sarah gets \$100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Table 1.1

o_1 : Sarah gets \$50,000 and Steven gets \$50,000.
o_2 : Sarah gets nothing and Steven gets \$100,000.
o_3 : Sarah gets \$100,000 and Steven gets nothing.
o_4 : Sarah gets nothing and Steven gets nothing.

Table 1.2

$$f : S \to O$$
$$U : O \to \mathbb{R}$$
$$\pi_i(s) = U_i(f(s))$$

S – Set of strategy profiles
Strategy profile – A list consisting of one strategy for each player

- Let us add the information that **both players are selfish and greedy** (that is,

  Player 1's ranking is $o_3 \succ_1 o_1 \succ_1 o_2 \sim_1 o_4$ and

  Player 2's ranking is $o_2 \succ_2 o_1 \succ_2 o_3 \sim_2 o_4$

  and **let us represent their rankings with the following utility functions**

  (note, again, that **the choice of numbers 2, 3 and 4 for utilities is arbitrary**: any other three numbers would do):

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Game frame: Let us represent their rankings with the following utility functions

Player 1's ranking: $o_3 \succ_1 o_1 \succ_1 o_2 \sim_1 o_4$

Player 2's ranking: $o_2 \succ_2 o_1 \succ_2 o_3 \sim_2 o_4$

outcome $\rightarrow$ *utility function* $\downarrow$	o_1	o_2	o_3	o_4
U_1 (Player 1)	3	2	4	2
U_2 (Player 2)	3	4	2	2

		Steven Split		Steven Steal	
Sarah	Split	Sarah gets \$50,000	Steven gets \$50,000	Sarah gets nothing	Steven gets \$100,000
	Steal	Sarah gets \$100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Table 1.1

o_1 : Sarah gets \$50,000 and Steven gets \$50,000.

o_2 : Sarah gets nothing and Steven gets \$100,000.

o_3 : Sarah gets \$100,000 and Steven gets nothing.

o_4 : Sarah gets nothing and Steven gets nothing.

Table 1.2

Then we obtain the following **reduced-form game**, where **in each cell the first number is the payoff of Player 1** and **the second number is the payoff of Player 2.**

		Player 2 (Steven) Split		Player 2 (Steven) Steal	
Player 1 (Sarah)	Split	3	3	2	4
	Steal	4	2	2	2

Table 1.4

Game Theory

➢ On the other hand, if **Player 1 is fair-minded and benevolent** (that is, her ranking is $o_1 \succ_1 o_3 \succ_1 o_2 \succ_1 o_4$), while **Player 2 is selfish and greedy** and represent these rankings with the following utility functions,

outcome → / utility function ↓	o_1	o_2	o_3	o_4
U_1 (Player 1)	4	2	3	1
U_2 (Player 2)	3	4	2	2

then we obtain the following reduced-form game:

		Player 2 (Steven)	
		Split	Steal
Player 1 (Sarah)	Split	4 3	2 4
	Steal	3 2	1 2

Table 1.5

		Steven: Split		Steven: Steal	
Sarah	Split	Sarah gets \$50,000	Steven gets \$50,000	Sarah gets nothing	Steven gets \$100,000
	Steal	Sarah gets \$100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Table 1.1

o_1 : Sarah gets \$50,000 and Steven gets \$50,000.
o_2 : Sarah gets nothing and Steven gets \$100,000.
o_3 : Sarah gets \$100,000 and Steven gets nothing.
o_4 : Sarah gets nothing and Steven gets nothing.

Table 1.2

Game Frame: Summary

A game is any situation with the following three aspects

1. There is a set of participants, whom we call the **players**.
2. Each player has a set of options for how to behave; we will refer to these as the player's possible **strategies**.
3. For each choice of strategies, each player receives a **payoff** that can depend on the strategies selected by everyone. The payoffs will generally be numbers, with each player preferring larger payoffs to smaller payoffs.

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Figure 6.1: Exam or Presentation?

References

1. Game Theory - An open access textbook with 165 solved exercises - Giacomo Bonanno
2. Wikipedia – Current Literature

SOCIAL NETWORK ANALYTICS

Strict, Weak and Equivalent Dominance

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How to Solve Strategic Games?

There are three main concepts to solve strategic games:

1. Dominant Strategies & Dominant Strategy Equilibrium
2. Dominated Strategies & Iterative Elimination of Dominated Strategies
3. Nash Equilibrium

Strict and weak dominance

- In this section **we define two relations on the set of strategies of a player.**

1. The first relation is called **“strict dominance”.**

- Let us focus our attention on one player, say Player 1, and select two of her strategies, say A and B.
- We say that **A** strictly dominates **B** (for Player 1) **if, for every possible strategy profile of the other players**, **strategy A of Player 1, in conjunction with the strategies selected by the other players, yields a payoff for Player 1 which is greater than the payoff associated with strategy B** (in conjunction with the strategies selected by the other players).

Player 2

Player 1		E		F		G	
	A	3	...	2	...	1	...
	B	2	...	1	...	0	...
	C	3	...	2	...	1	...
	D	2	...	0	...	0	...

Table 1.6

Strict and weak dominance

- For example, consider the following two-player game, where only the payoffs of Player 1 are shown:

Player 2

Player 1		E	F	G
	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Table 1.6

- **In this game for Player 1 strategy A strictly dominates strategy B:**
  - **if Player 2 selects E** then A in conjunction with E gives Player 1 a payoff of 3, while B in conjunction with E gives her only a payoff of 2,

Strict and weak dominance

		Player 2		
		E	*F*	*G*
Player 1	*A*	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Table 1.6

- **if Player 2 selects F** then A in conjunction with F gives Player 1 a payoff of 2, while B in conjunction with F gives her only a payoff of 1,
- **if Player 2 selects G** then A in conjunction with G gives Player 1 a payoff of 1, while B in conjunction with G gives her only a payoff of 0.

- In the game of Table 1.6 we also have that **A strictly dominates D** and **C strictly dominates D**; however, it is not the case that B strictly dominates D because, in conjunction with strategy E of Player 2, B and D yield the same payoff for Player 1.

Strict and weak dominance

2. The second relation is called **"weak dominance".**

- The definition is similar to that of strict dominance, but **we replace 'greater than' with 'greater than or equal to' while insisting on at least one strict inequality**:

- **A** weakly dominates **B (for Player 1)** if, for every possible strategy profile of the other players, **strategy A of Player 1, in conjunction with the strategies selected by the other players, yields a payoff for Player 1 which is greater than or equal to the payoff associated with strategy B** (in conjunction with the strategies selected by the other players) and, furthermore, **there is at least one strategy profile of the other players against which strategy A gives a larger payoff to Player 1 than strategy B**.

Player 2

Player 1	E	F	G
A	3 ...	2 ...	1 ...
B	2 ...	1 ...	0 ...
C	3 ...	2 ...	1 ...
D	2 ...	0 ...	0 ...

Table 1.6

Example: B weakly dominates D

Strict and weak dominance

- In the example of Table 1.6, we have that, while it is not true that B strictly dominates D, it is true that **B weakly dominates D**:

Player 2

Player 1		*E*	*F*	*G*
	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Table 1.6

Strict and weak dominance

- For example, suppose that there are three players and the strategy sets are as follows:

S_1 = {a, b, c}, S_2 = {d, e} and S_3 = {f, g}.

Then one possible strategy profile is **s = (b, d, g)** (thus **s_1 = b , s_2 = d** and **s_3 = g**).

If we focus on, say, Player 2 then we will denote by **s_{-2} the sub-profile consisting of the strategies of the players other than player 2**: in this case **s_{-2} =(b, g)**.

This gives us an alternative way of denoting strategy profile **s**, namely as **(s_2 , s_{-2}).**

Continuing our example where s = (b, d, g) , letting s_{-2}= (b, g), we can denote s also by (d, s_{-2}) and we can write the result of replacing Player 2's strategy d with her strategy e in s by (e, s_{-2}) ; thus (d, s_{-2})=(b, d, g) while (e, s_{-2}) = (b, e, g).

Strict and weak dominance

In general, given a Player i, we denote by ***S_{-i}* the set of strategy profiles of the players other than *i*** (that is, S_{-i} is the Cartesian product of the strategy sets of the other players)

$$S_1 = \{a, b, c\}, \ S_2 = \{d, e\} \text{ and } S_3 = \{f, g\}.$$

In the above example we have that

$$\mathbf{S_{-2}} = \mathbf{S_1} \times \mathbf{S_3} = \{\mathbf{a, b, c}\} \times \{\mathbf{f, g}\} = \{(a, f), (a, g), (b, f), (b, g), (c, f), (c, g)\}.$$

We denote an element of S_{-i} by s_{-i} .

Game Theory

Definition 1.4. Given an ordinal game in strategic form, let i be a Player and a and b two of her strategies ($a, b \in S_i$). We say that, for Player i,

- *a strictly dominates b* (or *b is strictly dominated by a*) if, in every situation (that is, no matter what the other players do), a gives Player i a payoff which is *greater* than the payoff that b gives. Formally: for every $s_{-i} \in S_{-i}$, $\pi_i(a, s_{-i}) > \pi_i(b, s_{-i})$.
- *a weakly dominates b* (or *b is weakly dominated by a*) if, in every situation, a gives Player i a payoff which is *greater than or equal to* the payoff that b gives and, furthermore, there is at least one situation where a gives a greater payoff than b. Formally: for every $s_{-i} \in S_{-i}$, $\pi_i(a, s_{-i}) \geq \pi_i(b, s_{-i})$ and there exists an $\bar{s}_{-i} \in S_{-i}$ such that $\pi_i(a, \bar{s}_{-i}) \geq \pi_i(b, \bar{s}_{-i})$.
- *a is equivalent to b* if, in every situation, a and b give Player i the same payoff. Formally: for every $s_{-i} \in S_{-i}$, $\pi_i(a, s_{-i}) = \pi_i(b, s_{-i})$.

Player 2

Player 1	E	F	G
A	3 ...	2 ...	1 ...
B	2 ...	1 ...	0 ...
C	3 ...	2 ...	1 ...
D	2 ...	0 ...	0 ...

Table 1.6

$S_1 = \{A, B, C, D\}$, $S_2 = \{E, F, G\}$

Note: s_{-i} the sub-profile consisting of the strategies of the players other than player i.
S_{-i} the set of strategy profiles of the players other than player i.

Strict and weak dominance

		Player 2 E	Player 2 F	Player 2 G
Player 1	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

For example, in the game of Table 1.6 reproduced above, we have that

- A strictly dominates B.
- A and C are equivalent.
- A strictly dominates D.
- B is strictly dominated by C.
- **B weakly (but not strictly) dominates D.**
- **C strictly dominates D.**

Strict and weak dominance

- **Exercise:** In the following games, find strategies that are strictly dominating or weakly dominating other strategies and equivalent strategies.

Player-2 (columns), Player-1 (rows)

	D	E	F
A	2 , 0	3 , 1	0 , 0
B	1 , 1	2 , 2	1 , 2
C	3 , 2	2 , 2	2 , 1

Game-1

	M	N	O
I	2 , 2	1 , 2	2 , 1
J	1 , 3	2 , 3	1 , 4
K	1 , 1	2 , 2	2 , 1

Game-2

	U	V
X	2 , 1	3 , 0
Y	2 , 1	1 , 2
Z	2 , 2	1 , 2

Game-3

Strict and weak dominance

Remark 1.3.

- Note that if strategy **a strictly dominates strategy b** then it also satisfies the conditions for weak dominance, that is, **‘a strictly dominates b’ implies ‘a weakly dominates b**’.

- The expression ‘**a weakly dominates b’** will be interpreted as **‘a dominates b weakly but not strictly’.**

Strict and weak dominance

Remark 1.3. cont…

- The expression '**a dominates b**' can be understood as '**a is better than b** '. The next term we define is '**dominant**' which can be understood as '**best**'.

  Thus one cannot meaningfully say "**a dominates**" because one needs to name another strategy that is dominated by a; for example, one would have to say "**a dominates b** ".

  On the other hand, one can meaningfully say "**a is dominant**" because it is like saying "**a is best**", which means "**a is better than every other strategy**".

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2. Wikipedia – Current Literature

SOCIAL NETWORK ANALYTICS

Dominant strategies and Dominant Strategy Equilibriums

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Strictly and Weakly dominant Strategies:

Definition 1.5. Given an ordinal game in strategic form, let i be a Player and a one of her strategies ($a \in S_i$). We say that, for Player i,

- *a is a strictly dominant* strategy if a strictly dominates every other strategy of Player i.

A strategy is strictly dominant if choosing it always gives a *better* outcome than choosing an alternative strategy, regardless of which moves other players make.

		Newsweek	
Times		Covid-19	Budget
	Covid-19	35 35	70 30
	Budget	30 70	15 15

Strictly and Weakly dominant Strategies:

Definition 1.5. Given an ordinal game in strategic form, let i be a Player and a one of her strategies ($a \in S_i$). We say that, for Player i,

- *a is a weakly dominant* strategy if, for every other strategy x of Player i, one of the following is true: (1) either a weakly dominates x or (2) a is equivalent to x.

A strategy is weakly dominant if choosing it always gives an outcome that is *as good as or better* than choosing an alternative strategy.

The dominant strategy for a player is one that produces the best payoff for that player regardless of the strategies employed by other players.

Player 2

Player 1		E	F	G
	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Table 1.6

A is weakly dominant strategy, because
A weakly dominates both B and D, and A is equivalent to C.

Similarly, C is weakly dominant strategy, because
C weakly dominates both B and D, and C is equivalent to A.

Strictly and Weakly dominant Strategies:

- **Exercise 1:** In the following game, find ***strictly dominant*** *and* ***weakly dominant*** *strategies.*

Player 2

Player 1		E	F	G
	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Table 1.6

Strictly and Weakly dominant Strategies:

- **Exercise 1:** In the following game, find ***strictly dominant*** *and* ***weakly dominant*** *strategies.*

Player 2

Player 1		E	F	G
	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Table 1.6

in Table 1.6, *A* and *C* are both weakly dominant strategies for Player 1. Note that if a player has two or more strategies that are weakly dominant, then any two of those strategies must be equivalent. On the other hand, there can be at most one strictly dominant strategy.

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Strictly and Weakly dominant Strategies:

- **Exercise 2:** In the following games, find ***strictly dominant*** *and* ***weakly dominant*** *strategies.*

Player-2 (columns), Player-1 (rows)

	D	E	F
A	2 , 0	3 , 1	0 , 0
B	1 , 1	2 , 2	1 , 2
C	3 , 2	2 , 2	2 , 1

Game-1

	M	N	O
I	2 , 2	1 , 2	2 , 1
J	1 , 3	2 , 3	1 , 4
K	1 , 1	2 , 2	2 , 1

Game-2

	U	V
X	2 , 1	3 , 0
Y	2 , 1	1 , 2
Z	2 , 2	1 , 2

Game-3

Dominant Strategy Equilibriums:

Definition 1.6. Given an ordinal game in strategic form, let $s = (s_1, \ldots, s_n)$ be a strategy profile. We say that

- s *is a strict dominant-strategy equilibrium* if, for every Player i, s_i is a strictly dominant strategy.
- s *is a weak dominant-strategy equilibrium* if, for every Player i, s_i is a weakly dominant strategy and, furthermore, for at least one Player j, s_j is not a strictly dominant strategy.

If all players have a strict dominant strategy, then it is natural for them to choose the dominant strategies and we reach a **strict dominant strategy equilibrium.**

Dominant Strategy Equilibriums:

Example: In the following games, find ***dominant strategy equilibriums***.

		Player 2 (Steven) Split	Player 2 (Steven) Steal
Player 1 (Sarah)	Split	3, 3	2, 4
	Steal	4, 2	2, 2

Table 1.4

		Player 2 (Steven) Split	Player 2 (Steven) Steal
Player 1 (Sarah)	Split	4, 3	2, 4
	Steal	3, 2	1, 2

Table 1.5

Dominant Strategy Equilibriums:

Example: In the following games, find ***dominant strategy equilibriums***.

Player 1 (Sarah) \ Player 2 (Steven)	Split	Steal
Split	3, 3	2, 4
Steal	4, 2	2, 2

Table 1.4

Player 1 (Sarah) \ Player 2 (Steven)	Split	Steal
Split	4, 3	2, 4
Steal	3, 2	1, 2

Table 1.5

In the game of Table 1.4, ***Steal*** **is a weakly dominant strategy for each player** and thus **(*Steal, Steal*) is a weak dominant-strategy equilibrium.**

In the game of Table 1.5, ***Split*** **is a strictly dominant strategy for Player 1**, while ***Steal*** **is a weakly (but not strictly) dominant strategy for Player 2** and thus **(*Split, Steal*) is a weak dominant-strategy equilibrium.**

Dominant Strategy Equilibriums:

- **Exercise 1:** In the following games, find ***dominant strategy equilibriums***.

Player-2 (columns), Player-1 (rows)

	D	E	F
A	2 , 0	3 , 1	0 , 0
B	1 , 1	2 , 2	1 , 2
C	3 , 2	2 , 2	2 , 1

Game-1

	M	N	O
I	2 , 2	1 , 2	2 , 1
J	1 , 3	2 , 3	1 , 4
K	1 , 1	2 , 2	2 , 1

Game-2

	U	V
X	2 , 1	3 , 0
Y	2 , 1	1 , 2
Z	2 , 2	1 , 2

Game-3

Dominant Strategy Equilibriums:

- **Exercise 2:** In the following game, find ***dominant strategy equilibriums***.

		Newsweek	
Times		Covid-19	Budget
	Covid-19	35 , 35	70 , 30
	Budget	30 , 70	15 , 15

Dominant Strategy Equilibriums:

- **Exercise 2:** In the following game, find ***dominant strategy equilibriums***.

		Newsweek	
Times		Covid-19	Budget
	Covid-19	35 , 35	70 , 30
	Budget	30 , 70	15 , 15

The Covid-19 story is a dominant strategy for both Times and Newsweek. Therefore (Covid-19, Covid-19) is a dominant strategy equilibrium yielding both magazines a market share of 35 percent.

Dominant Strategy Equilibriums:

- **Exercise 3:** In the following game, find ***dominant strategy equilibriums***.

		Player 2	
		X	Y
Player 1	A	5,2	4,2
	B	3,1	3,2
	C	2,1	4,1
	D	4,3	5,4

Dominant Strategy Equilibriums:

- **Exercise 3:** In the following game, find ***dominant strategy equilibriums***.

Player 2

		X	Y
Player 1	A	5,2	4,2
	B	3,1	3,2
	C	2,1	4,1
	D	4,3	5,4

Here Player 1 does not have a single strategy that "beats" every other strategy. Therefore she does not have a dominant strategy. On the other hand Y is a dominant strategy for Player 2. Hence, ***no dominant strategy equilibrium.***

Dominant Strategy Equilibriums:

- **Exercise 4:** In the following game, find ***dominant strategy equilibriums***.

Payoff is in USD in millions		Firm B	
		Hire a Lawyer	No Lawyer
Firm A	Hire a Lawyer	45,45	70,25
	No Lawyer	25,70	50,50

Dominant Strategy Equilibriums:

- **Exercise 4:** In the following game, find ***dominant strategy equilibriums***.

Payoff is in USD in millions		**Firm B**	
		Hire a Lawyer	No Lawyer
Firm A	Hire a Lawyer	45,45	70,25
	No Lawyer	25,70	50,50

Hire a Lawyer is a dominant strategy for both Firm A and Firm B.
Therefore (Hire a Lawyer , Hire a Lawyer) is a dominant strategy equilibrium yielding both Firm A and Firm B a payoff of 45.

Exercise 1.4. There are two players. Each player is given an unmarked envelope and asked to put in it either nothing or \$300 of his own money or \$600 of his own money. A referee collects the envelopes, opens them, gathers all the money, then adds 50% of that amount (using his own money) and divides the total into two equal parts which he then distributes to the players.

(a) Represent this game frame with two alternative tables: the first table showing in each cell the amount of money distributed to Player 1 and the amount of money distributed to Player 2, the second table showing the change in wealth of each player (money received minus contribution).

(b) Suppose that Player 1 has some animosity towards the referee and ranks the outcomes in terms of how much money the referee loses (the more, the better), while Player 2 is selfish and greedy and ranks the outcomes in terms of her own net gain. Represent the corresponding game using a table.

(c) Is there a strict dominant-strategy equilibrium?

Exercise 1.6. There are three players. Each player is given an unmarked envelope and asked to put in it either nothing or \$3 of his own money or \$6 of his own money. A referee collects the envelopes, opens them, gathers all the money and then doubles the amount (using his own money) and divides the total into three equal parts which he then distributes to the players. For example, if Players 1 and 2 put nothing and Player 3 puts \$6, then the referee adds another \$6 so that the total becomes \$12, divides this sum into three equal parts and gives \$4 to each player. Each player is selfish and greedy, in the sense that he ranks the outcomes exclusively in terms of his net change in wealth (what he gets from the referee minus what he contributed).

(a) Represent this game by means of a set of tables. (Do not treat the referee as a player.)

(b) For each player and each pair of strategies determine if one of the two dominates the other and specify if it is weak or strict dominance.

(c) Is there a strict dominant-strategy equilibrium?

Exercise 1.6 : Solution

Exercise 1.6. (a) The game is as follows:

Player 3: 0

Player 1 \ Player 2	0	3	6
0	0 0 0	2 –1 2	4 –2 4
3	–1 2 2	1 1 4	3 0 6
6	–2 4 4	0 3 6	2 2 8

Player 3: 3

Player 1 \ Player 2	0	3	6
0	2 2 –1	4 1 1	6 0 3
3	1 4 1	3 3 3	5 2 5
6	0 6 3	2 5 5	4 4 7

Player 3: 6

Player 1 \ Player 2	0	3	6
0	4 4 –2	6 3 0	8 2 2
3	3 6 0	5 5 2	7 4 4
6	2 8 2	4 7 4	6 6 6

(b) For Player 1, 3 strictly dominates 6, 0 strictly dominates 6, 0 strictly dominates 3 (the same is true for every player).

(c) The strict dominant-strategy equilibrium is (0,0,0) (everybody contributes 0).

References

1. Game Theory - An open access textbook with 165 solved exercises - Giacomo Bonanno
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SOCIAL NETWORK ANALYTICS

Prisoner's Dilemma and Pareto Superior

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Prisoner's Dilemma

- The Prisoner's Dilemma is an example of a game with a **strict dominant-strategy equilibrium**.

- The prisoner's dilemma, one of the most famous game theories, was conceptualized by Merrill Flood and Melvin Dresher at the Rand Corporation in 1950.

- **Prisoner's Dilemma was later formalized and named by Princeton mathematician, Albert William Tucker.**

Prisoner’s Dilemma

- The prisoner’s dilemma basically provides **a framework for understanding how to strike a balance between cooperation and competition** and is **a useful tool for strategic decision-making**.

- Prisoner’s Dilemma finds application in diverse areas ranging from business, finance, economics, and political science to philosophy, psychology, biology, and sociology.

Prisoner's Dilemma

- A prisoner's dilemma describes a situation where, two players acting strategically will ultimately result in a suboptimal choice for both.

- **In business, understanding the structure of certain decisions as prisoner's dilemmas can result in more favorable outcomes.** **This set-up allows one to balance both competition and cooperation for mutual benefit.**

Prisoner's Dilemma Basics

The prisoner's dilemma scenario works as follows:

- Two suspects have been apprehended for a crime and are now in separate rooms in a police station, with no means of communicating with each other.
- The prosecutor has separately told them the following:
  - **If you confess** and agree to testify(give evidence) against the other suspect, who does not confess, the charges against you will be dropped and you will go scot-free(unpunished).
  - **If you do not confess** but the other suspect does, you will be convicted and the prosecution will seek the maximum sentence of three years.
  - **If both of you confess**, you will both be sentenced to two years in prison.
  - **If neither of you confesses**, you will both be charged with misdemeanors and will be sentenced to one year in prison.

- What should the suspects do? This is the essence of the prisoner's dilemma.

Prisoner’s Dilemma: Evaluating Best Course of Action

- Let’s begin by constructing a payoff matrix as shown in the table below.
- The “payoff” here is shown in terms of the length of a prison sentence (as symbolized by the negative sign; the higher the number the better).
- The terms “cooperate” and “defect” refer to the suspects
  - cooperating with each other (as for example, if neither of them confesses) or
  - defecting (i.e., not cooperating with the other player, which is the case where suspect confesses).
- The first numeral in cells (a) through (d) shows the payoff for Suspect A, while the second numeral shows it for Suspect B.

		Suspect B	
		Cooperate	**Defect**
Suspect A	**Cooperate**	(a) -1, -1	(c) -3, 0
	Defect	(b) 0, -3	(d) -2, -2

Cooperate –> Stay silent
Defect –> Confess

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Prisoner's Dilemma

- **The dominant strategy here is for each player to defect (i.e., confess) since confessing would minimize the average length of time spent in prison.** Here are the possible outcomes:
  - If A and B cooperate and stay mum, both get one year in prison—as shown in the cell (a).
  - If A confesses but B does not, A goes free and B gets three years—represented in the cell (b).
  - If A does not confess but B confesses, A gets three years and B goes free—see cell (c).
  - If A and B both confess, both get two years in prison—as the cell (d) shows.

  So if A confesses, they either go free or get two years in prison. But if they do not confess, they either get one year or three years in prison. B faces exactly the same dilemma.
- Clearly, the best strategy is to confess, regardless of what the other suspect does.

		Suspect B	
		Cooperate	**Defect**
Suspect A	**Cooperate**	(a) -1, -1	(c) -3, 0
	Defect	(b) 0, -3	(d) -2, -2

Cooperate –> Stay silent
Defect –> Confess

Prisoner's Dilemma

- **An instance of the Prisoner's Dilemma is the following.**
  - Doug and Ed work for the same company and the annual party is approaching.
  - They know that they are the only two candidates for the **best-worker-of-the-year prize** and at the moment they are tied;
    however, only one person can be awarded the prize and thus, unless one of them manages to outperform the other, nobody will receive the prize.
  - Each chooses between exerting **Normal effort** or **Extra effort** (that is, work overtime) before the party.
  - The game-frame is as shown in Table 1.7 below.

Prisoner's Dilemma

The game-frame is as shown in Table 1.7 below.

		Player 2 (Ed)	
		Normal effort	Extra effort
Player 1 (Doug)	Normal effort	o_1	o_2
	Extra effort	o_3	o_4

o_1 : nobody gets the prize and nobody sacrifices family time

o_2 : Ed gets the prize and sacrifices family time, Doug does not

o_3 : Doug gets the prize and sacrifices family time, Ed does not

o_4 : nobody gets the prize and both sacrifice family time

Table 1.7

Prisoner's Dilemma

- Doug and Ed rankings are as follows:

$$o_3 \succ_{Doug} o_1 \succ_{Doug} o_4 \succ_{Doug} o_2 \quad \text{and} \quad o_2 \succ_{Ed} o_1 \succ_{Ed} o_4 \succ_{Ed} o_3 .$$

- Using utility function with values from the set {0,1,2,3} we can represent the game in reduced form as follows:

		Player 2 (Ed) Normal effort	Player 2 (Ed) Extra effort
Player 1 (Doug)	Normal effort	2 2	0 3
Player 1 (Doug)	Extra effort	3 0	1 1

Table 1.8
The Prisoner's Dilemma game

o_1 : nobody gets the prize and nobody sacrifices family time
o_2 : Ed gets the prize and sacrifices family time, Doug does not
o_3 : Doug gets the prize and sacrifices family time, Ed does not
o_4 : nobody gets the prize and both sacrifice family time

- In this game exerting _____ effort is a strictly dominant strategy for every player; thus **(____ effort, ____ effort) is a strict dominant-strategy equilibrium.**

Prisoner's Dilemma

- Doug and Ed rankings are as follows:

$$o_3 \succ_{Doug} o_1 \succ_{Doug} o_4 \succ_{Doug} o_2 \quad \text{and} \quad o_2 \succ_{Ed} o_1 \succ_{Ed} o_4 \succ_{Ed} o_3 .$$

- Using utility function with values from the set {0,1,2,3} we can represent the game in reduced form as follows:

		Player 2 (Ed) Normal effort		Player 2 (Ed) Extra effort	
Player 1 (Doug)	Normal effort	2	2	0	3
	Extra effort	3	0	1	1

Table 1.8
The Prisoner's Dilemma game

o_1 : nobody gets the prize and nobody sacrifices family time
o_2 : Ed gets the prize and sacrifices family time, Doug does not
o_3 : Doug gets the prize and sacrifices family time, Ed does not
o_4 : nobody gets the prize and both sacrifice family time

- In this game exerting **Extra effort** is a strictly dominant strategy for every player; thus **(Extra effort, Extra effort) is a strict dominant-strategy equilibrium.**

Prisoner’s Dilemma: Application

The prisoners’ dilemma has applications to economics and business.

- Consider two firms, say Coca-Cola and Pepsi, selling similar products. Each must decide on a pricing strategy.
- They best exploit their joint market power when both charge a high price; each makes a profit of ten million dollars per month.
- If one sets a competitive low price, it wins a lot of customers away from the rival. Suppose its profit rises to twelve million dollars, and that of the rival falls to seven million.
- If both set low prices, the profit of each is nine million dollars.
- Here, the low-price strategy is akin to the prisoner’s confession, and the high-price akin to keeping silent. Then low-price is each firm’s strict-dominant strategy, thus **(low-price, low-price) is a strict dominant-strategy equilibrium.**

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Pareto superior:

Definition 1.7. Given an ordinal game in strategic form, let o and o' be two outcomes. We say that o is *strictly Pareto superior* to o' if every player prefers o to o' (that is, if $o \succ_i o'$ for every Player i). We say that o is *weakly Pareto superior* to o' if every player considers o to be at least as good as o' and at least one player prefers o to o' (that is, if $o \succsim_i o'$ for every Player i and there is a Player j such that $o \succ_j o'$).

In reduced-form games, this definition can be extended to strategy profiles as follows. If s and s' are two strategy profiles, then s is *strictly Pareto superior* to s' if $\pi_i(s) > \pi_i(s')$ for every Player i and s is *weakly Pareto superior* to s' if $\pi_i(s) \geq \pi_i(s')$ for every Player i and, furthermore, there is a Player j such that $\pi_j(s) > \pi_j(s')$.

o_1 : nobody gets the prize and nobody sacrifices family time
o_2 : Ed gets the prize and sacrifices family time, Doug does not
o_3 : Doug gets the prize and sacrifices family time, Ed does not
o_4 : nobody gets the prize and both sacrifice family time

		Player 2 (Ed) Normal effort		Player 2 (Ed) Extra effort	
Player 1 (Doug)	Normal effort	2	2	0	3
	Extra effort	3	0	1	1

Table 1.8
The Prisoner's Dilemma game

Pareto superior:

o_1 : nobody gets the prize and nobody sacrifices family time

o_2 : Ed gets the prize and sacrifices family time, Doug does not

o_3 : Doug gets the prize and sacrifices family time, Ed does not

o_4 : nobody gets the prize and both sacrifice family time

Table 1.7

		Player 2 (Ed): Normal effort	Player 2 (Ed): Extra effort
Player 1 (Doug)	Normal effort	2 2	0 3
	Extra effort	3 0	1 1

Table 1.8

The Prisoner's Dilemma game

For example, in the Prisoner's Dilemma game of Table 1.8, outcome ___ is strictly Pareto superior to ___ or, in terms of strategy profiles, (__________ , __________) is strictly Pareto superior to (__________ , __________).

Pareto superior:

o_1 : nobody gets the prize and nobody sacrifices family time
o_2 : Ed gets the prize and sacrifices family time, Doug does not
o_3 : Doug gets the prize and sacrifices family time, Ed does not
o_4 : nobody gets the prize and both sacrifice family time

Table 1.7

		Player 2 (Ed) Normal effort		Player 2 (Ed) Extra effort	
Player 1 (Doug)	Normal effort	2	2	0	3
	Extra effort	3	0	1	1

Table 1.8
The Prisoner's Dilemma game

For example, in the Prisoner's Dilemma game of Table 1.8, outcome o_1 is strictly Pareto superior to o_4 or, in terms of strategy profiles, (*Normal effort*, *Normal effort*) is strictly Pareto superior to (*Extra effort*, *Extra effort*).

References

1. Game Theory - An open access textbook with 165 solved exercises - Giacomo Bonanno
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SOCIAL NETWORK ANALYTICS

Iterated deletion procedures:

Iterated Deletion of Strictly Dominated Strategies (IDSDS)

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How to Solve Strategic Games?

There are three main concepts to solve strategic games:

1. Dominant Strategies & Dominant Strategy Equilibrium
2. Dominated Strategies & Iterative Elimination of Dominated Strategies
3. Nash Equilibrium

Iterated deletion procedures

- If in a game a player has a (weakly or strictly) dominant strategy then the player ought to choose that strategy: in the case of strict dominance choosing any other strategy guarantees that the player will do worse and in the case of weak dominance no other strategy can give a better outcome, no matter what the other players do.
- Unfortunately, games that have a dominant-strategy equilibrium are not very common.
- **What should a player do when she does not have a dominant strategy?**

We shall consider two iterative deletion procedures that can help solve some games.

Iterated deletion procedures: IDSDS

- **IDSDS.** The **Iterated Deletion of Strictly Dominated Strategies** (IDSDS) is the following procedure or algorithm.
  - Given a finite ordinal strategic-form game **G**,
  - let $\mathbf{G}^1$ be the game obtained by removing from G, for every Player i, those strategies of Player i (if any) that are strictly dominated in G by some other strategy;
  - let $\mathbf{G}^2$ be the game obtained by removing from G^1, for every Player i, those strategies of Player i (if any) that are strictly dominated in G^1 by some other strategy, and so on.
  - Let $\mathbf{G}^\infty$ be the output of this procedure. Since the initial game G is finite, G^∞ will be obtained in a finite number of steps.

Iterated deletion procedures: IDSDS

- If G^∞ **contains a single strategy profile** (this is not the case in the example of Figure 1.10) then **we call that strategy profile the iterated strict dominant-strategy equilibrium**.

Note: **A dominant strategy equilibrium is a Nash equilibrium.**

- If G^∞ **contains two or more strategy profiles** then **we refer to those strategy profiles merely as the output of the IDSDS procedure**.
- For example, in the game of Figure 1.10 the output of the IDSDS procedure is the set of strategy profiles **{ (A, e), (A, f), (B, e), (B, f) }**.

Iterated deletion procedures: IDSDS

- Apply IDSDS for the following game

		Player 2 e	f	g	h
Player 1	A	6 , 3	4 , 4	4 , 1	3 , 0
	B	5 , 4	6 , 3	0 , 2	5 , 1
	C	5 , 0	3 , 2	6 , 1	4 , 0
	D	2 , 0	2 , 3	3 , 3	6 , 1

Iterated deletion procedures: IDSDS

$G = G^0$

Player 1 \ Player 2	e	f	g	h
A	6 , 3	4 , 4	4 , 1	3 , 0
B	5 , 4	6 , 3	0 , 2	5 , 1
C	5 , 0	3 , 2	6 , 1	4 , 0
D	2 , 0	2 , 3	3 , 3	6 , 1

delete h (dominated by g)

G^1

Player 1 \ Player 2	e	f	g
A	6 , 3	4 , 4	4 , 1
B	5 , 4	6 , 3	0 , 2
C	5 , 0	3 , 2	6 , 1
D	2 , 0	2 , 3	3 , 3

delete D (dominated by C)

G^2

Player 1 \ Player 2	e	f	g
A	6 , 3	4 , 4	4 , 1
B	5 , 4	6 , 3	0 , 2
C	5 , 0	3 , 2	6 , 1

delete g (dominated by f)

G^3

Player 1 \ Player 2	e	f
A	6 , 3	4 , 4
B	5 , 4	6 , 3
C	5 , 0	3 , 2

delete C (dominated by A)

$G^4 = G^\infty$

Player 1 \ Player 2	e	f
A	6 , 3	4 , 4
B	5 , 4	6 , 3

Figure 1.10

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Iterated deletion procedures: IDSDS

- **What is the significance of the output of the IDSDS procedure?**
  - Consider game G of Figure 1.10. Since, for Player 2, h is strictly dominated by g, if Player 2 is rational she will not play h.
  - Thus, if Player 1 believes that Player 2 is rational then he believes that Player 2 will not play h, that is, he restricts attention to game G^1; since, in G^1, D is strictly dominated by C for Player 1, if Player 1 is rational he will not play D.
  - It follows that if Player 2 believes that Player 1 is rational and that Player 1 believes that Player 2 is rational, then Player 2 restricts attention to game G^2 where rationality requires that Player 2 not play g, etc.

		Player 2			
		e	*f*	*g*	*h*
Player 1	*A*	6 , 3	4 , 4	4 , 1	3 , 0
	B	5 , 4	6 , 3	0 , 2	5 , 1
	C	5 , 0	3 , 2	6 , 1	4 , 0
	D	2 , 0	2 , 3	3 , 3	6 , 1

		Player 2		
		e	*f*	*g*
Player 1	*A*	6 , 3	4 , 4	4 , 1
	B	5 , 4	6 , 3	0 , 2
	C	5 , 0	3 , 2	6 , 1
	D	2 , 0	2 , 3	3 , 3

		Player 2		
		e	*f*	*g*
Player 1	*A*	6 , 3	4 , 4	4 , 1
	B	5 , 4	6 , 3	0 , 2
	C	5 , 0	3 , 2	6 , 1

Iterated deletion procedures: IDSDS

➢ **What is the significance of the output of the IDSDS procedure?**

- **if the players are rational and there is common belief of rationality then only strategy profiles that survive the IDSDS procedure can be played;** the converse is also true: any strategy profile that survives the IDSDS procedure is compatible with common belief of rationality.

		Player 2	
		e	*f*
Player 1	*A*	6 , 3	4 , 4
	B	5 , 4	6 , 3

$$G^4 = G^\infty$$

- **Remark**. In finite games, **the order in which strictly dominated strategies are deleted is irrelevant**, in the sense that **any sequence of deletions of strictly dominated strategies leads to the same output.**

Iterated deletion procedures:

- **Exercise 1.10.**

  (a) **Apply the IDSDS procedure** (Iterated Deletion of Strictly Dominated Strategies) to the following game.

Player 2 (Bob)

Player 1 (Antonia)	1	3	5
2	3, 2	3, 2	2, 3
4	3, 2	2, 3	1, 1
6	2, 3	1, 1	1, 1

Iterated deletion procedures:

- **Exercise 1.11. Apply the IDSDS procedure** to the following game. Is there a strict iterated dominant-strategy equilibrium?

		Player 2		
		d	e	f
Player 1	a	8 , 6	0 , 9	3 , 8
	b	3 , 2	2 , 1	4 , 3
	c	2 , 8	1 , 5	3 , 1

Note: When you eliminate strictly dominated strategies, order of deleting does not matter.

Iterated deletion procedures:

- **Exercise 1.11. Apply the IDSDS procedure** to the following game. Is there a strict iterated dominant-strategy equilibrium?

Player 2 (columns), Player 1 (rows)

	d	*e*	*f*
a	8 , 6	0 , 9	3 , 8
b	3 , 2	2 , 1	4 , 3
c	2 , 8	1 , 5	3 , 1

Solution:

In this game *c is strictly dominated by b; after deleting c, d becomes strictly* dominated by *f; after deleting d, a becomes strictly dominated by b; after* deleting *a, e becomes strictly dominated by f; deletion of e leads to only one* strategy profile, namely (*b,f*). ***Thus (b,f) is the iterated strict dominant-strategy*** **equilibrium**.

Iterated deletion procedures:

Exercise 1.13. Consider the following game:

Player 2

Player 1		D		E		F	
	a	2	3	2	2	3	1
	b	2	0	3	1	1	0
	c	1	4	2	0	0	4

(a) Apply the IDSDS procedure to it. Is there an iterated strict dominant-strategy equilibrium?

Iterated deletion procedures:

Exercise 1.13. Consider the following game:

		Player 2: *D*		*E*		*F*	
Player 1	*a*	2	3	2	2	3	1
	b	2	0	3	1	1	0
	c	1	4	2	0	0	4

(a) Apply the IDSDS procedure to it. Is there an iterated strict dominant-strategy equilibrium?

Solution: IDSDS: In this game *c is strictly dominated by b; after deleting c, F becomes strictly* dominated by E*; after deleting F,* ***the output of IDSDS is (a,D), (a,E), (b,D), (b,E)***

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SOCIAL NETWORK ANALYTICS

Iterated deletion procedures:

Iterated Deletion of Weakly Dominated Strategies (IDWDS)

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Iterated deletion procedures: IDWDS

- **IDWDS. The Iterated Deletion of Weakly Dominated Strategies (IDWDS)** is a weakening of IDSDS in that it allows the deletion also of weakly dominated strategies.
  However, this procedure has to be defined carefully, since **in this case the order of deletion can matter**.
  To see this, consider the game shown in Figure 1.11.

		Player 2: L	Player 2: R
Player 1	A	4 , 0	0 , 0
	T	3 , 2	2 , 2
	M	1 , 1	0 , 0
	B	0 , 0	1 , 1

Figure 1.11

Note: If the iterated elimination of weakly dominated strategies leaves exactly one strategy for each player, the resulting strategy profile is a Nash equilibrium.

Iterated deletion procedures: IDWDS

Player 2

Player 1		L	R
	A	4 , 0	0 , 0
	T	3 , 2	2 , 2
	M	1 , 1	0 , 0
	B	0 , 0	1 , 1

Figure 1.11

Since M is strictly dominated by T for Player 1, we can delete it and obtain

Player 2

Player 1		L	R
	A	4 , 0	0 , 0
	T	3 , 2	2 , 2
	B	0 , 0	1 , 1

Iterated deletion procedures: IDWDS

Player 2

Player 1	L	R
A	4 , 0	0 , 0
T	3 , 2	2 , 2
B	0 , 0	1 , 1

Now L is weakly dominated by R for Player 2. Deleting L we are left with

Player 2

Player 1	R
A	0 , 0
T	2 , 2
B	1 , 1

Now A and B are strictly dominated by T. Deleting them we are left with

$\boxed{(T,R)}$ with corresponding payoffs (2,2).

Iterated deletion procedures: IDWDS

Alternatively, going back to the game of Figure 1.11, we could note that B is strictly dominated by T; deleting B we are left with

Player 2

Player 1	L	R
A	4 , 0	0 , 0
T	3 , 2	2 , 2
M	1 , 1	0 , 0
B	0 , 0	1 , 1

Figure 1.11

Player 2

Player 1	L	R
A	4 , 0	0 , 0
T	3 , 2	2 , 2
M	1 , 1	0 , 0

Now R is weakly dominated by L for Player 2. Deleting R we are left with

Player 2

Player 1	L
A	4 , 0
T	3 , 2
M	1 , 1

Iterated deletion procedures: IDWDS

Player 2

		L
Player 1	A	4 , 0
	T	3 , 2
	M	1 , 1

Now T and M are strictly dominated by A and deleting them leads to $\boxed{\text{(A,L)}}$ with corresponding payoffs (4,0). Since one order of deletion leads to (T,R) with payoffs (2,2) and the other to (A,L) with payoffs (4,0), the procedure is not well defined: the output of a well-defined procedure should be unique.

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Iterated deletion procedures: IDWDS

Definition 1.8 (IDWDS).
In order to avoid the problem illustrated above, the IDWDS procedure is defined as follows:
at every step identify, for every player, all the strategies that are weakly (or strictly) dominated and then delete all such strategies in that step.

If the output of the IDWDS procedure is a single strategy profile then we call that strategy profile the ***iterated weak dominant-strategy equilibrium*** (otherwise we just use the expression 'output of the IDWDS procedure').

For example, the IDWDS procedure when applied to the game of Figure 1.11 leads to the following output:

Player 1 \ Player 2	L	R
A	4 , 0	0 , 0
T	3 , 2	2 , 2

Hence the game of Figure 1.11 does not have an iterated weak dominant-strategy equilibrium.

Player 1 \ Player 2	L	R
A	4 , 0	0 , 0
T	3 , 2	2 , 2
M	1 , 1	0 , 0
B	0 , 0	1 , 1

Figure 1.11

Iterated deletion procedures: IDWDS

- **Exercise 1.10.**

  (b) **Apply the IDWDS procedure** (Iterated Deletion of Weakly Dominated Strategies) to the following game.

Player 2 (Bob)

Player 1 (Antonia)	1	3	5
2	3, 2	3, 2	2, 3
4	3, 2	2, 3	1, 1
6	2, 3	1, 1	1, 1

Iterated deletion procedures: IDWDS

Exercise 1.13. Consider the following game:

Player 2

Player 1		D		E		F	
	a	2	3	2	2	3	1
	b	2	0	3	1	1	0
	c	1	4	2	0	0	4

(b) Apply the IDWDS procedure to it. Is there an iterated weak dominant-strategy equilibrium?

Iterated deletion procedures: IDWDS

Exercise 1.13. Consider the following game:

		Player 2: D		E		F	
Player 1	a	2	3	2	2	3	1
	b	2	0	3	1	1	0
	c	1	4	2	0	0	4

(b) Apply the IDWDS procedure to it. Is there an iterated weak dominant-strategy equilibrium?

Solution:

IDWDS: In this game *c is strictly dominated by b; after deleting c, F becomes strictly* dominated by E*; after deleting F, a becomes weakly dominated by b; after* deleting *a, D becomes strictly dominated by E; deletion of D leads to only one* strategy profile, namely (*b,E*)*.* ***Thus (b,E) is the iterated weak dominant-strategy*** **equilibrium**.

Iterated deletion procedures:

Exercise Consider the following game:

	N	C	J
N	73, 25	57, 42	66, 32
C	80, 26	35, 12	32, 54
J	28, 27	63, 31	54, 29

(a) Apply the IDSDS procedure to it. Is there an iterated strict dominant-strategy equilibrium?

(b) Apply the IDWDS procedure to it. Is there an iterated weak dominant-strategy equilibrium?

Iterated deletion procedures:

- **Exercise:** Apply the iterated elimination of dominated strategies to the following normal form games.
- Note that in some cases there may remain more that one strategy for each player. Say exactly in what order you eliminated rows and columns.

(a)

	a	b	c
A	2,12	1,10	1,11
B	0,12	0,10	0,11
C	0,12	0,10	0,13

(b)

	a	b	c
A	1,1	−2,0	4,−1
B	0,3	3,1	5,4
C	1,5	4,2	6,2

Iterated deletion procedures:

- **Exercise:** Apply the iterated elimination of dominated strategies to the following normal form games.
- Note that in some cases there may remain more that one strategy for each player. Say exactly in what order you eliminated rows and columns.

(c)	N_2	C_2	J_2
N_1	73,25	57,42	66,32
C_1	80,26	35,12	32,54
J_1	28,27	63,31	54,29

(d)	a	b	c	d	e
A	63, −1	28, −1	−2, 0	−2, 45	−3, 19
B	32, 1	2, 2	2, 5	33, 0	2, 3
C	54, 1	95, −1	0, 2	4, −1	0, 4
D	1, −33	−3, 43	−1, 39	1, −12	−1, 17
E	−22, 0	1, −13	−1, 88	−2, −57	−3, 72

Iterated deletion procedures:

- **Exercise:** Apply the iterated elimination of dominated strategies to the following normal form games.
- Note that in some cases there may remain more that one strategy for each player. Say exactly in what order you eliminated rows and columns.

(e)

	a	b	c	d	e
A	0, 1	0, 1	0, 1	0, 1	0, 1
B	0.81, 0, 19	0.20, 0.80	0.20, 0.80	0.20, 0.80	0.20, 0.80
C	0.81, 0.19	0.49, 0.51	0.40, 0.60	0.40, 0.60	0.40, 0.60
D	0.81, 0.19	0.49, 0.51	0.25, 0.75	0.60, 0.40	0.60, 0.40
E	0.81, 0.19	0.49, 0.51	0.25, 0.75	0.09, 0.91	0.80, 0.20
F	0.81, 0.19	0.49, 0.51	0.25, 0.75	0.09, 0.91	0.01, 0.99

(f)

	a	b	c	d	e	f	g	h
A	−1, 1	−1, 1	−1, 1	−1, 1	1, −1	1, −1	1, −1	1, −1
B	1, −1	0, 0	1, −1	0, 0	1, −1	0, 0	1, −1	0, 0
C	−1, 1	−1, 1	0, 0	0, 0	−1, 1	−1, 1	0, 0	0, 0

Iterated deletion procedures:

- **Exercise:** Apply the iterated elimination of dominated strategies to the following normal form games.
- Note that in some cases there may remain more that one strategy for each player. Say exactly in what order you eliminated rows and columns.

(g)

	0	1	2	3	4	5
0	4, 5	4, 14	4, 13	4, 12	4, 11	4, 10
1	13, 5	3, 4	3, 13	3, 12	3, 11	3, 10
2	12, 5	12, 4	2, 3	2, 12	2, 11	2, 10
3	11, 5	11, 4	11, 3	1, 2	1, 11	1, 10
4	10, 5	10, 4	10, 3	10, 2	0, 1	0, 10

Iterated deletion procedures:

- **Exercise:** Consider the following game in normal form

	A	B	C	D
a	$0,-1$	$4,4$	$0,0$	$2,0$
b	$0,3$	$0,0$	$4,4$	$1,0$
c	$5,2$	$2,0$	$1,3$	$1,3$
d	$4,4$	$1,0$	$0,1$	$0,5$

(a) Iteratively eliminate all strictly dominated strategies; state the assumptions necessary for each elimination.

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Nash Equilibrium

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Nash Equilibrium

- Games where either the IDSDS procedure or the IDWDS procedure leads to a unique strategy profile are not very common.

- How can one then "solve" games that are not solved by either procedure?

  The notion of Nash equilibrium offers a more general alternative.

➢Defining Nash Equilibrium.

- In 1950, John Nash proposed a simple but powerful principle for reasoning about behavior in general games.
- **Nash Equilibrium underlying principle is the following:** **even when there are no dominant strategies, we should expect players to use strategies that are best responses to each other.**
- More precisely, suppose that **Player 1 chooses a strategy S** and **Player 2 chooses a strategy T**.
  We say that this pair of strategies (S, T) is a Nash equilibrium if S is a best response to T, and T is a best response to S.

➢Defining Nash Equilibrium. Cont…

- This is not a concept that can be derived purely from rationality on the part of the players; instead, **it is an equilibrium concept**.
- **The idea is that if the players choose strategies that are best responses to each other, then no player has an incentive to deviate to an alternative strategy - so the system is in a kind of equilibrium state, with no force pushing it toward a different outcome.**
- Nash shared the 1994 Nobel Prize in Economics for his development and analysis of this idea.

Nash Equilibrium

- **In a Nash equilibrium, every person in a group makes the best decision for herself**, based on what she thinks the others will do.

  And **no-one can do better by changing strategy**: every member of the group is doing as well as they possibly can.

Nash Equilibrium

- **It is a solution concept of a non-cooperative game involving two or more players** in which **each player is assumed to know the equilibrium strategies of the other players,** and **no player has anything to gain by changing only his own strategy**.

- If each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitutes a **Nash equilibrium**.

- **The Nash equilibrium is one of the foundational concepts in game theory**.

Nash Equilibrium

Definition 1.9. Given an ordinal game in strategic form with two players, a strategy profile $s^* = (s_1^*, s_2^*) \in S_1 \times S_2$ is a Nash equilibrium if the following two conditions are satisfied:

(1) for every $s_1 \in S_1$, $\pi_1(s_1^*, s_2^*) \geq \pi_1(s_1, s_2^*)$ (or stated in terms of outcomes and preferences, $f(s_1^*, s_2^*) \succsim_1 f(s_1, s_2^*)$) and

(2) for every $s_2 \in S_2$, $\pi_2(s_1^*, s_2^*) \geq \pi_1(s_1^*, s_2)$ (or, $f(s_1^*, s_2^*) \succsim_2 f(s_1^*, s_2)$).

Player 2

Player 1	L	C	R
T	3 , 2	0 , 0	1 , 1
M	3 , 0	1 , 5	4 , 4
B	1 , 0	2 , 3	3 , 0

Figure 1.12

S_1= (T, M, B) **S_2= (L, C, R)**

s_1= **T** or **M** or **B**

s_2= **L** or **C** or **R**

$s^* = (s_1^*, s_2^*)$ **= (T, L) or (B, C)**

Nash Equilibrium

For example, in the game of Figure 1.12 there are two Nash equilibria: (T, L) and (B, C).

		Player 2 L	C	R
Player 1	T	3 , 2	0 , 0	1 , 1
	M	3 , 0	1 , 5	4 , 4
	B	1 , 0	2 , 3	3 , 0

Figure 1.12

Nash Equilibrium

There are several possible interpretations of this definition.

'No regret' interpretation: s^* is a Nash equilibrium if there is no player who, after observing the opponent's choice, regrets his own choice (in the sense that he could have done better with a different strategy of his, given the observed strategy of the opponent).

'Self-enforcing agreement' interpretation: imagine that the players are able to communicate before playing the game and reach a non-binding agreement expressed as a strategy profile s^*; then no player will have an incentive to deviate from the agreement, if she believes that the other player will follow the agreement, if and only if s^* is a Nash equilibrium.

Nash Equilibrium

'Viable recommendation' interpretation: imagine that a third party makes a public recommendation to each player on what strategy to play; then no player will have an incentive to deviate from the recommendation, if she believes that the other players will follow the recommendation, if and only if the recommended strategy profile is a Nash equilibrium.

'Transparency of reason' interpretation: if players are all "equally rational" and Player 2 reaches the conclusion that she should play y, then Player 1 must be able to duplicate Player 2's reasoning process and come to the same conclusion; it follows that Player 1's choice of strategy is not rational unless it is a strategy x that is optimal against y. A similar argument applies to Player 2's choice of strategy (y must be optimal against x) and thus (x,y) is a Nash equilibrium.

It is clear that all of the above interpretations are mere rewording of the formal definition of Nash equilibrium in terms of the inequalities of Definition 1.9.

Nash Equilibrium

The generalization of Definition 1.9 to games with more than two players is straightforward.

Definition 1.10. Given an ordinal game in strategic form with n players, a strategy profile $s^* \in S$ is a Nash equilibrium if the following inequalities are satisfied:

$$\text{for every Player } i,\ \pi_i(s^*) \geq \pi_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \text{ for all } s_i \in S_i.$$

The reader should convince himself/herself that a (weak or strict) dominant strategy equilibrium is a Nash equilibrium and the same is true of a (weak or strict) iterated dominant-strategy equilibrium.

Nash Equilibrium

Definition 1.11. Consider an ordinal game in strategic form, a Player i and a strategy profile $\bar{s}_{-i} \in S_{-i}$ of the players other than i. A strategy $s_i \in S_i$ of Player i is a *best reply (or best response) to* $\bar{s}_{-i}$ if $\pi_i(s_i, \bar{s}_{-i}) \geq \pi_i(s_i', \bar{s}_{-i})$ for every $s_i' \in S_i$.

For example, in the game of Figure 1.12, reproduced below, for Player 1 there are two best replies to L, namely M and T, while the unique best reply to C is B and the unique best reply to R is M; for Player 2 the best reply to T is L, the best reply to M is C and the best reply to B is C.

Player 2

Player 1		L	C	R
	T	3 , 2	0 , 0	1 , 1
	M	3 , 0	1 , 5	4 , 4
	B	1 , 0	2 , 3	3 , 0

Nash Equilibrium

For example, in the game of Figure 1.12, reproduced below, for Player 1 there are two best replies to L, namely M and T, while the unique best reply to C is B and the unique best reply to R is M; for Player 2 the best reply to T is L, the best reply to M is C and the best reply to B is C.

		Player 2		
		L	C	R
Player 1	T	3 , 2	0 , 0	1 , 1
	M	3 , 0	1 , 5	4 , 4
	B	1 , 0	2 , 3	3 , 0

Remark 1.6. Using the notion of best reply, an alternative definition of Nash equilibrium is as follows: $\bar{s} \in S$ is a Nash equilibrium if and only if, for every Player i, $\bar{s}_i \in S_i$ is a best reply to $\bar{s}_{-i} \in S_{-i}$.

Nash Equilibrium

- **A quick way to find the Nash equilibria of a two-player game is as follows:**
  - **In each column of the table underline the largest payoff of Player 1 in that column** (if there are several instances, underline them all) and **in each row underline the largest payoff of Player 2 in that row; if a cell has both payoffs underlined then the corresponding strategy profile is a Nash equilibrium.**
  - **Underlining of the maximum payoff of Player 1 in a given column identifies the best reply of Player 1 to the strategy of Player 2 that labels that column and similarly for Player 2.**
  - This procedure is illustrated in Figure 1.13, where there is a unique Nash equilibrium, namely *(B,E)*.

	E	*F*	*G*	*H*
A	4 , 0	3 , 2	2 , 3	4 , 1
B	4 , 2	2 , 1	1 , 2	0 , 2
C	3 , 6	5 , 5	3 , 1	5 , 0
D	2 , 3	3 , 2	1 , 2	3 , 3

Nash Equilibrium

- It must be noted that any dominant strategy equilibrium is always a ***Nash equilibrium***. However, not all Nash equilibria are dominant strategy equilibria.

- In a Nash equilibrium, each player's strategy is a best response to the other player's strategies.

Nash Equilibrium

Example 1:

Player 2

Player 1		E	F	G	H
	A	4 , 0	3 , 2	2 , 3	4 , 1
	B	4 , 2	2 , 1	1 , 2	0 , 2
	C	3 , 6	5 , 5	3 , 1	5 , 0
	D	2 , 3	3 , 2	1 , 2	3 , 3

Figure 1.13

Nash Equilibrium

Example 2: In a three-player game the procedure for finding the Nash equilibria is the same, with the necessary adaptation for Player 3: in each cell underline the payoff of Player 3 if and only if her payoff is the largest of all her payoffs in the same cell across different tables. This is illustrated in Figure 1.14, where there is a unique Nash equilibrium, namely (*B*,*R*,*W*).

Player 3 chooses *W*

Player 1 \ Player 2	*L*	*R*
T	0 , 0 , $\underline{0}$	2 , $\underline{8}$, $\underline{6}$
B	$\underline{5}$, 3 , $\underline{2}$	$\underline{3}$, $\underline{4}$, $\underline{2}$

Player 3 chooses *E*

Player 1 \ Player 2	*L*	*R*
T	0 , 0 , $\underline{0}$	$\underline{1}$, $\underline{2}$, 5
B	$\underline{1}$, $\underline{6}$, 1	0 , 0 , 1

Figure 1.14

Nash Equilibrium

Exercise 1.14. Find the Nash equilibria of the game of Exercise 1.2.

Player 2 (Bob)

Player 1 (Antonia)	0	1	2
2	0, 2	0, 3	0, 2
4	0, 2	5, 5	4, 2
6	4, 2	3, 7	2, 0

Nash Equilibrium

Exercise 1.14. Find the Nash equilibria of the game of Exercise 1.2.

Player 2 (Bob)

Player 1 (Antonia)	0	1	2
2	0 2	0 3	0 2
4	0 2	5 5	4 2
6	4 2	3 7	2 0

Solution: There is only one Nash equilibrium, namely (4,1) with payoffs (5,5).

Nash Equilibrium

Exercise 1.15. Find the Nash equilibria of the games of Exercise 1.3 (b).

The game of part (b) of Exercise 1.3 is as follows:

P = Press

Charlie: P

		Bob: P			Bob: not P		
Alice	P	1	0	0	0	1	0
Alice	not P	0	1	0	1	0	0

Charlie: not P

		Bob: P			Bob: not P		
Alice	P	0	1	0	1	0	0
Alice	not P	1	0	0	0	0	1

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Nash Equilibrium

Exercise 1.15. Find the Nash equilibria of the games of Exercise 1.3 (b).

The game of part (b) of Exercise 1.3 is as follows:

P = Press

Charlie: P

Alice \ Bob	P			not P		
P	1	0	0	0	1	0
not P	0	1	0	1	0	0

Charlie: not P

Alice \ Bob	P			not P		
P	0	1	0	1	0	0
not P	1	0	0	0	0	1

Solution: This game has only one Nash equilibrium, namely **(not P, P, not P)**.

Nash Equilibrium

Exercise 1.15. Find the Nash equilibria of the games of Exercise 1.3 (c).

The game of part (c) of Exercise 1.3 is as follows:

P = Press

		Bob: P			Bob: not P		
Alice	P	1	0	0	0	2	1
Alice	not P	0	2	1	1	0	0

Charlie: P

		Bob: P			Bob: not P		
Alice	P	0	2	1	1	0	0
Alice	not P	1	0	0	0	1	2

Charlie: not P

Nash Equilibrium

Exercise 1.15. Find the Nash equilibria of the games of Exercise 1.3 (c).

The game of part (c) of Exercise 1.3 is as follows:

P = Press

		Bob: P			Bob: not P		
Alice	P	1	0	0	0	2	1
Alice	not P	0	2	1	1	0	0

Charlie: P

		Bob: P			Bob: not P		
Alice	P	0	2	1	1	0	0
Alice	not P	1	0	0	0	1	2

Charlie: not P

Solution: This game does not have any Nash equilibria.

Nash Equilibrium

Exercise 1.14. **1.17**. Find the Nash equilibria of the game of Exercise 1.6.

Player 1 \ Player 2	0	3	6
0	0 0 0	2 –1 2	4 –2 4
3	–1 2 2	1 1 4	3 0 6
6	–2 4 4	0 3 6	2 2 8

Player 3: 0

Player 1 \ Player 2	0	3	6
0	2 2 –1	4 1 1	6 0 3
3	1 4 1	3 3 3	5 2 5
6	0 6 3	2 5 5	4 4 7

Player 3: 3

Player 1 \ Player 2	0	3	6
0	4 4 –2	6 3 0	8 2 2
3	3 6 0	5 5 2	7 4 4
6	2 8 2	4 7 4	6 6 6

Player 3: 6

Nash Equilibrium

Exercise 1.14. 1.17. Find the Nash equilibria of the game of Exercise 1.6.

Player 3: 0

Player 1 \ Player 2	0	3	6
0	0 0 0	2 −1 2	4 −2 4
3	−1 2 2	1 1 4	3 0 6
6	−2 4 4	0 3 6	2 2 8

Player 3: 3

Player 1 \ Player 2	0	3	6
0	2 2 −1	4 1 1	6 0 3
3	1 4 1	3 3 3	5 2 5
6	0 6 3	2 5 5	4 4 7

Player 3: 6

Player 1 \ Player 2	0	3	6
0	4 4 −2	6 3 0	8 2 2
3	3 6 0	5 5 2	7 4 4
6	2 8 2	4 7 4	6 6 6

Solution: This game has only one Nash equilibrium, namely (0,0,0).

Nash Equilibrium

Exercise 1.18. Find the Nash equilibria of the game of Exercise 1.7.

Player 1 (value $30M)	$10M	$20M	$30M	$40M	$50M
$10M	0 , 40	0 , 40	0 , 40	0 , 40	0 , 40
$20M	20 , 0	0 , 30	0 , 30	0 , 30	0 , 30
$30M	20 , 0	10 , 0	0 , 20	0 , 20	0 , 20
$40M	20 , 0	10 , 0	0 , 0	0 , 10	0 , 10
$50M	20 , 0	10 , 0	0 , 0	–10 , 0	0 , 0

Nash Equilibrium

Exercise 1.18. Find the Nash equilibria of the game of Exercise 1.7.

Player 1 (value $30M)

	$10M	$20M	$30M	$40M	$50M
$10M	0 , 40	0 , 40	0 , 40	0 , 40	0 , 40
$20M	20 , 0	0 , 30	0 , 30	0 , 30	0 , 30
$30M	20 , 0	10 , 0	0 , 20	0 , 20	0 , 20
$40M	20 , 0	10 , 0	0 , 0	0 , 10	0 , 10
$50M	20 , 0	10 , 0	0 , 0	–10 , 0	0 , 0

Solution: This game has 15 Nash equilibria: (10,30), (10,40), (10,50), (20,30), (20,40), (20,50), (30,30), (30,40), (30,50), (40,40), (40,50), (50,10), (50,20), (50,30), (50,50).

Nash Equilibrium

Exercise 1.20. Find the Nash equilibria of the game of part (b) of Exercise 1.12.

Carla votes for A

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	2,0,1
B	2,0,1	0,1,2	0,1,2
C	2,0,1	1,2,0	1,2,0

Carla votes for B

ANN \ BOB	A	B	C
A	2,0,1	0,1,2	2,0,1
B	0,1,2	0,1,2	0,1,2
C	1,2,0	0,1,2	1,2,0

Carla votes for C

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	1,2,0
B	0,1,2	0,1,2	1,2,0
C	1,2,0	1,2,0	1,2,0

Nash Equilibrium

Exercise 1.20. Find the Nash equilibria of the game of part (b) of Exercise 1.12.

BOB (Carla votes for A)

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	2,0,1
B	2,0,1	0,1,2	0,1,2
C	2,0,1	1,2,0	1,2,0

Carla votes for A

BOB (Carla votes for B)

ANN \ BOB	A	B	C
A	2,0,1	0,1,2	2,0,1
B	0,1,2	0,1,2	0,1,2
C	1,2,0	0,1,2	1,2,0

Carla votes for B

BOB (Carla votes for C)

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	1,2,0
B	0,1,2	0,1,2	1,2,0
C	1,2,0	1,2,0	1,2,0

Carla votes for C

Solution: There are 5 Nash equilibria: (A,A,A), (B,B,B), (C,C,C), (A,C,A) and (A,B,B).

Nash Equilibrium

Exercise 1.21. Find the Nash equilibria of the game of Exercise 1.13.

Player 2

Player 1		D		E		F	
	a	2	3	2	2	3	1
	b	2	0	3	1	1	0
	c	1	4	2	0	0	4

Nash Equilibrium

Exercise 1.21. Find the Nash equilibria of the game of Exercise 1.13.

Player 2

		D		E		F	
Player 1	a	2	3	2	2	3	1
	b	2	0	3	1	1	0
	c	1	4	2	0	0	4

Solution: There are 2 Nash equilibria: (a,D) and (b,E).

Nash Equilibrium

Best responses/ Best replies

The concept of the *best response* is important for determining and understanding equilibriums. A player's best response refers to the respective strategies of the opponent, i.e. for each strategy of player 2 there exists one (or more) best response(s) for player 1.

Let us take another look at the prisoner's dilemma:

Prisoner 1 \ Prisoner 2	A: Confess	B: Not Confess
A: Confess	-5 (Prisoner 1), -5 (Prisoner 2)	0 (Prisoner 1), -10 (Prisoner 2)
B: Not confess	-10 (Prisoner 1), 0 (Prisoner 2)	-1 (Prisoner 1), -1 (Prisoner 2)

If prisoner 2 confesses, the best response for prisoner 1 is to confess as well. If prisoner 2 does not confess, again, the best response to this for prisoner 1 is to confess.

Nash Equilibrium

Finding Nash equilibria using best response functions

a. The *Prisoner's Dilemma* and *BoS* are shown in Figure 6.1; *Matching Pennies* a
the two-player *Stag Hunt* are shown in Figure 6.2.

	Quiet	*Fink*
Quiet	$2, 2$	$0, 3^*$
Fink	$3^*, 0$	$1^*, 1^*$

Prisoner's Dilemma

	Bach	*Stravinsky*
Bach	$2^*, 1^*$	$0, 0$
Stravinsky	$0, 0$	$1^*, 2^*$

BoS

Figure 6.1 The best response functions in the *Prisoner's Dilemma* (left) and in *BoS* (right).

	Head	*Tail*
Head	$1^*, -1$	$-1, 1^*$
Tail	$-1, 1^*$	$1^*, -1$

Matching Pennies

	Stag	*Hare*
Stag	$2^*, 2^*$	$0, 1$
Hare	$1, 0$	$1^*, 1^*$

Stag Hunt

Figure 6.2 The best response functions in *Matching Pennies* (left) and the *Stag Hunt* (right).

Nash Equilibrium

Best responses/ Best replies

b. The best response functions are indicated in Figure 6.3. The Nash equilibria are (T, C), (M, L), and (B, R).

	L	C	R
T	$2\ ,2$	$1^*,3^*$	$0^*,1$
M	$3^*,1^*$	$0\ ,0$	$0^*,0$
B	$1\ ,0^*$	$0\ ,0^*$	$0^*,0^*$

Figure 6.3 The game in Exercise 37.1.

Nash Equilibrium

47.2 Nash equilibrium and weakly dominated actions

The only Nash equilibrium of the game in Figure 8.2 is (T, L). The action T is weakly dominated by M and the action L is weakly dominated by C. (There are of course many other games that satisfy the conditions.)

	L	C	R
T	1, 1	0, 1	0, 0
M	1, 0	2, 1	1, 2
B	0, 0	1, 1	2, 0

Figure 8.2 A game with a unique Nash equilibrium, in which both players' equilibrium actions are weakly dominated. (The unique Nash equilibrium is (T, L).)

Nash Equilibrium

Assignment: Reading

- **Nashpy**
  - A Python library for the computation of Nash Equilibria
  - https://nashpy.readthedocs.io/en/stable/
- **Nashpy Documentation**

Assignment: Activity

- Building and finding the equilibrium for a simple game
  - Installing Nashpy
  - Creating a game
  - Calculating the utility of a pair of strategies
  - Computing Nash equilibria

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SOCIAL NETWORK ANALYTICS

Pareto Optimality and Social Optimality

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Pareto Optimality

- When the strategies from game theory are discussed, they are often mentioned from a player's perspective.
  When the strategies are formed from an observer's angle whose main motive is to wish for the best outcome for every player;
  that is, **when strategies are formed from a socially balanced viewpoint, then the outcome is known as a Pareto Optimal outcome.**

- We've defined some games, and thought about how to play them. **Now let's examine the games from the outside.**

- **From the point of view of an outside observer, can some outcomes of a game be said to be better than others?**
- **Are there situations where we can still prefer one outcome to another?**

Pareto Optimality

- **Pareto efficiency:** Economic state where resources are allocated in the most efficient manner.

- **Pareto efficiency has nothing to do with equality**. In fact, Pareto efficiency can occur even when situations are unfair.

- Pareto efficiency is when an economy has its resources and goods allocated to the maximum level of efficiency, and no change can be made without making someone worse off.

Pareto Optimality

- **Pareto efficiency** or **Pareto optimality** is a situation where no individual or preference criterion can be better off without making at least one individual or preference criterion worse off.
- A situation is called **Pareto optimal** or **Pareto efficient** if no change could lead to improved satisfaction for some agent without some other agent losing.
- **Pareto optimal** outcome cannot be improved upon without hurting at least one player.
- Pareto optimality cannot be used to make predictions regarding how agents will behave. Rather, **Pareto optimality provides a minimal criterion for whether or not an outcome is good from the perspective of social welfare**.

Pareto Optimality

- **Definition:**

  "An outcome is said to be Pareto optimal if it cannot be Pareto dominated by any other outcome.
  To be specific on choosing a Pareto outcome, it is evident that no other outcome can prove to be better than this outcome for all the players.
  In addition to this, **one player strictly chooses the Pareto optimal outcome over any other outcome."**

Definition 2.2 (Pareto dominated). *An action profile $a \in A$ is Pareto dominated by action profile $a' \in A$ if $u_i(a') \geq u_i(a)$ for all agents $i \in N$ and $u_i(a') > u_i(a)$ for some agent $i \in N$.*

Definition 2.3 (Pareto optimality). *An action profile $a \in A$ is Pareto optimal if there is no action profile $a' \in A$ that Pareto dominates a.*

Pareto Optimality

- **Example 1:**

Prisoner's Dilemma

Agent 1 \ Agent 2	C	D
C	3, 3	0, 5
D	5, 0	1, 1

Pareto Optimality

➢ **Example 1:**

Prisoner's Dilemma

Agent 1 \ Agent 2	C	D
C	3, 3	0, 5
D	5, 0	1, 1

- **Strategy profile (*C,C*) is Pareto optimal (No profile dominates Strategy profile (C,C))**
  - No profile gives both players a higher payoff
- **Strategy profile (*D,C*) is Pareto optimal (No profile dominates Strategy profile (D,C))**
  - No profile gives player-1 a higher payoff
- **Strategy profile (C,*D*) is Pareto optimal (No profile dominates Strategy profile (C,D))**
  - No profile gives player-2 a higher payoff
- Strategy profile (*D,D*) is **not pareto optimal** because it is pareto dominated by Strategy profile(*C,C*). But ironically, Strategy profile (*D,D*) is the dominant strategy equilibrium.

Pareto Optimality

- **Example 2:**

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Pareto Optimality

➢ Example 2:

- The outcome in which you and your partner **both study for the exam** is **not Pareto-optimal**, since the outcome in which you **both prepare for the presentation is strictly better for both of you**.
- **The two outcomes in which exactly one of you prepares for the presentation are also Pareto-optimal.** In this case, although one of you is doing badly, there is no alternate choice of strategies in which *everyone* is doing at least as well.

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
You	*Exam*	92, 86	88, 88

(Presentation, Presentation) is Pareto optimal - No profile gives both players a higher payoff
(Exam, Presentation) is Pareto optimal - No profile gives player-1 a higher payoff
(Presentation, Exam) is Pareto optimal - No profile gives player-2 a higher payoff

Pareto Optimality

➤ Example 3: **The Coordination game:**

The scenario in this game is comparable to two people walking on pavement from opposite directions.

If both choose to stick to their respective right or left, it will prove to be advantageous to both.

However, if either of them deviates from this choice, they are prone to collision. The payoff matrix for this game is as follows:

		Player-2	
		Left	Right
Player-1	Left	1, 1	0, 0
	Right	0, 0	1, 1

Pareto Optimality

➢ Example 3: **The Coordination game:**

The scenario in this game is comparable to two people walking on pavement from opposite directions.

If both choose to stick to their respective right or left, it will prove to be advantageous to both.

However, if either of them deviates from this choice, they are prone to collision. The payoff matrix for this game is as follows:

		Player-2	
		Left	Right
Player-1	Left	1, 1	0, 0
	Right	0, 0	1, 1

From the matrix above, it is clear that the outcomes (1, 1) are Pareto optimal.

Pareto Optimality

- **Example 4: The Battle of the Sexes:** This can be thought off as a situation between a husband and a wife. **The husband puts forward an idea of going to a boxing match, and the wife for obvious reasons prefers shopping over boxing.** They have different interests, but most importantly both want to spend the day together. This means that the wife would go to the boxing match with her husband, where she would end up getting a payoff of one along with the husband enjoying his payoff of two, rather than going shopping alone, where both would just receive a zero payoff. The payoff matrix for this game looks something like this:

		Husband	
		Boxing	Shopping
Wife	Boxing	1, 2	0, 0
	Shopping	0, 0	2, 1

Pareto Optimality

- **Example 4: The Battle of the Sexes:** This can be thought off as a situation between a husband and a wife. **The husband puts forward an idea of going to a boxing match, and the wife for obvious reasons prefers shopping over boxing.** They have different interests, but most importantly both want to spend the day together. This means that the wife would go to the boxing match with her husband, where she would end up getting a payoff of one along with the husband enjoying his payoff of two, rather than going shopping alone, where both would just receive a zero payoff. The payoff matrix for this game looks something like this:

		Husband	
Wife		Boxing	Shopping
	Boxing	1, 2	0, 0
	Shopping	0, 0	2, 1

From the matrix above, it is clear that the outcomes (2, 1) & (1, 2) are Pareto optimal.

Pareto Optimality

- **Example 4: The Battle of the Sexes:** This can be thought off as a situation between a husband and a wife. **The husband puts forward an idea of going to a boxing match, and the wife for obvious reasons prefers shopping over boxing.** They have different interests, but most importantly both want to spend the day together. This means that the wife would go to the boxing match with her husband, where she would end up getting a payoff of one along with the husband enjoying his payoff of two, rather than going shopping alone, where both would just receive a zero payoff. The payoff matrix for this game looks something like this:

		Husband	
		Boxing	Shopping
Wife	Boxing	1, 2	0, 0
	Shopping	0, 0	2, 1

(Shopping, Shopping) is Pareto optimal - No profile gives player-1 a higher payoff
(Boxing, Boxing) is Pareto optimal - No profile gives player-2 a higher payoff

Pareto Optimality

➢ Example 5: **The Matching Rupees game:**

The objective of the game is different for the two players. Both players are given coins with two faces a head and a tail. Player one must try and match his rupee with that of player two and player two must make sure his/her rupee doesn't match player one.
So, in cases where the rupees match, player one gets some positive payoff along with player two getting an equal negative payoff.
Conversely, on a mismatch player two is rewarded a positive payoff and player one a negative. The given payoff matrix better demonstrates this:

		Player-2	
Player-1		Heads	Tails
	Heads		
	Tails		

Pareto Optimality

- **Example 5: The Matching Rupees game:**

In this game, if player one chooses to play heads, player two would obviously respond with tails. Again if player two chose tails, player one would be interested to play tails in order to win and these choices would repeat themselves in a cyclic manner.

Therefore, **it is evident that each player's decision is directly influenced by the other's and there is no dominant strategy any player would choose to win.**

		Player-2	
Player-1		Heads	Tails
	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

Pareto Optimality

➢ Example 5: **The Matching Rupees game:**

As a result, **all the outcomes in the payoff matrix are essentially Pareto optimal which is quite common in case of zero-sum games.**

(Heads, Heads) and (Tails, Tails) are Pareto optimal - No profile gives player-1 a higher payoff
(Heads, Tails) and (Tails, Heads) are Pareto optimal - No profile gives player-2 a higher payoff

		Player-2	
		Heads	Tails
Player-1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

Pareto Optimality

➢ **Exercise: Q & A**

1. Can a game have more than one Pareto-optimal outcome?
2. Does every game have at least one Pareto-optimal outcome?

Social Optimality

- A stronger condition that is even simpler to state is social optimality. **A choice of strategies — one by each player — is a social welfare maximizer (or socially optimal) if it maximizes the sum of the players' payoffs.**
- **Example:** In the Exam-or-Presentation Game, the social optimum is achieved by the outcome in which both you and your partner prepare for the presentation, which produces a combined payoff of $90 + 90 = 180$.

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

Social Optimality

- **Outcomes that are socially optimal must also be Pareto-optimal**: if such an outcome weren't Pareto-optimal, there would be a different outcome in which all payoffs were at least as large, and one was larger — and this would be an outcome with a larger sum of payoffs.
- **On the other hand, a Pareto-optimal outcome need not be socially optimal**. For example, the Exam-or-Presentation Game has three outcomes that are Pareto-optimal, but only one of these is the social optimum.

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

References

1. Game Theory - An open access textbook with 165 solved exercises - Giacomo Bonanno

2. “Networks, Crowds, and Markets: Reasoning About a Highly Connected World”, D Easley and J Kleinberg, Cambridge University Press, 2010.

3. https://www.cs.umd.edu/~nau/cmsc421/game-theory.pdf

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SOCIAL NETWORK ANALYTICS

Multiple equilibria:

Coordination Games and Hawk-Dove Game

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A coordination game

- **A coordination game is one where both players are better off cooperating.**

- In game theory, **coordination games are a class of games with multiple pure strategy Nash equilibria in which players choose the same or corresponding strategies.**

A coordination game

- A perennial question in economics concerns **the conditions under which individuals cooperate to achieve an efficient outcome**.

This question is, especially, relevant to situations in which there are multiple equilibria.

- Consider the following examples:

		Player-2	
		Left	Right
Player-1	Top	80, 80	0, 0
	Bottom	0, 0	100, 100

		Player-2	
		Left	Right
Player-1	Top	80, 80	80, 0
	Bottom	0, 80	100, 100

A coordination game

Example:

	Left	Right
Up	A, a	C, c
Down	B, b	D, d

Fig. 1: *2-player coordination game*

- If this game is a coordination game, then the following inequalities hold in the payoff matrix
  - for player 1 (rows): **A > B, D > C**, and
  - for player 2 (columns): **a > c, d > b.** See Fig. 1.
- In this game the strategy profiles {Left, Up} and {Right, Down} are pure Nash equilibria.

A coordination game

- **Choosing sides:**

- The simplest type of coordination game is one where players are rewarded entirely for how much they cooperate.
- A good example of this is two oncoming drivers deciding what side of the road to drive on so that they don't crash. Fill out the payoff matrix for this situation.
- What are the Nash equilibrium(s)? Are there any dominant strategies?

		Driver-2	
Driver-1		Left	Right
	Left		
	Right		

A coordination game

➢ **Choosing sides:**

- This game has two pure Nash equilibria: either both swerve to the left, or both swerve to the right.
- **Both solutions are Pareto efficient**. This is not true for all coordination games.

		Driver-2	
		Left	Right
Driver-1	Left	10, 10	0, 0
	Right	0, 0	10, 10

A coordination game

➢ **Pure (or common interest) Coordination Game:**

- Pure Coordination is **the game where the players both prefer the same Nash equilibrium outcome.**
- In this game(Fig. 3) **both prefer partying** over **both staying at home** to watch TV.
- The **{Party, Party}** outcome **Pareto dominates** the **{Home, Home}** outcome, just as both Pareto dominate the other two outcomes, {Party, Home} and {Home, Party}.

	Party	Home
Party	10, 10	0, 0
Home	0, 0	5, 5

***Fig. 3**: Pure coordination game*

A coordination game

- **Battle of the Sexes:**

- A husband and wife have agreed to go out for the night, but as they are leaving work, they realize that they forgot to decide whether to go to the opera or the football game.
  Unable to contact each other, they must choose where to go. If they each get a value 1 for going to their preferred activity (football for the husband, opera for the wife) and a value 2 for being in the same place as their spouse, fill out the payoff matrix.
  Hint: Each spouse should have each value between 0 and 3 show up once as a payoff.
- **What are the Nash equilibrium(s)? Are there any dominant strategies?**
- **How do you think this will turn out compared to other the other coordination games? (Which activity is each spouse more likely to go to?)**

A coordination game

➢ **Battle of the Sexes:**

- What are the Nash equilibrium(s)? Are there any dominant strategies?
- How do you think this will turn out compared to other the other coordination games? (Which activity is each spouse more likely to go to?)

		Wife	
Husband		Opera	Football
	Opera	2, 3	0, 0
	Football	1, 1	3, 2

A coordination game

- The **stag hunt game** in Fig. 5 shows a situation in which **both players (hunters) can benefit if they cooperate (hunting a stag)**.
  However, cooperation might fail, because each hunter has an alternative which is safer because it does not require cooperation to succeed (hunting a rabbit).
- This example of the potential conflict between safety and social cooperation is originally due to Jean-Jacques Rousseau.

Player-2

		Stag	Hare
Player-1	 Stag	10, 10	0, 8
	 Hare	8, 0	7, 7

Fig. 5: *Stag hunt*

A coordination game

- **Consider two people who wish to go out together, but who, agree on the more desirable concert—say they both prefer Bach.**
- A strategic game that models this situation is shown in Figure 29.1; it is an example of a **coordination game**.
- **The game has two Nash equilibria: (Bach, Bach) and (Stravinsky, Stravinsky).**
- In particular, the action pair (Stravinsky, Stravinsky) in which both people choose their less-preferred concert is a Nash equilibrium.

	Bach	*Stravinsky*
Bach	2, 2	0, 0
Stravinsky	0, 0	1, 1

Figure 29.1 *A coordination game.*

A coordination game

Exercise 1:

- **Two siblings want to meet up after school at either the beach or at home. They both would hate to not have a chance to hang out, and both prefer the beach.**
  - Fill out the payoff matrix for this situation.
  - What are the Nash equilibrium(s)? Are there any dominant strategies?
  - Do you think this will have a better outcome than the simple coordination case? Why or why not?
- Why do you think this sort of coordination game is called a "pure coordination" game?

		Sibling 2	
		Beach	Home
Sibling 1	Beach	o_1	o_2
	Home	o_3	o_4

$$o_1 \succ o_4 \succ o_2 \sim o_3$$

A coordination game

Exercise 1:

- **Two siblings want to meet up after school at either the beach or at home. They both would hate to not have a chance to hang out, and both prefer the beach.**
- Fill out the payoff matrix for this situation.
- What are the Nash equilibrium(s)? Are there any dominant strategies?
- Do you think this will have a better outcome than the simple coordination case? Why or why not?
- Why do you think this sort of coordination game is called a "pure coordination" game?

		Sibling 2	
		Beach	Home
Sibling 1	Beach	2, 2	0, 0
	Home	0, 0	1, 1

$$o_1 \succ o_4 \succ o_2 \sim o_3$$

Hawk–Dove Game

- **Two animals are fighting over some prey**.
  - Each can be passive or aggressive.
  - Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome when its opponent is passive to that in which its opponent is aggressive.

- **Formulate this situation as a strategic game and find its Nash equilibria.**

Hawk–Dove Game

- The contestants can be either Hawk or Dove. These are two subtypes or morphs of one species with different strategies.

  The Hawk first displays aggression, then escalates into a fight until it either wins or is injured (loses).

  The Dove first displays aggression, but if faced with major escalation runs for safety. If not faced with such escalation, the Dove attempts to share the resource.

Hawk–Dove Game

- Given that
  - **the resource is given the value V,**
  - **the damage from losing a fight is given cost C:**
- If a Hawk meets a Dove he gets the full resource V to himself.
- If a Hawk meets a Hawk – half the time he wins, half the time he loses...so his average outcome is then V/2 minus C/2.
- If a Dove meets a Hawk he will back off and get nothing – 0.
- If a Dove meets a Dove both share the resource and get V/2.

Payoff Matrix for Hawk Dove Game

	meets Hawk	**meets Dove**
if Hawk	(V/2 – C/2) (V/2 – C/2)	V 0
if Dove	0 V	V/2 V/2

Hawk–Dove Game

- If C=4 and V=6, write Payoff Matrix for Hawk Dove Game.

Hawk–Dove Game

- If C=4 and V=6, write Payoff Matrix for Hawk Dove Game.

		Bird-2	
		Hawk	Dove
Bird-1	Hawk	1, 1	6, 0
	Dove	0, 6	3, 3

References

1. Game Theory - An open access textbook with 165 solved exercises - Giacomo Bonanno
2. “Networks, Crowds, and Markets: Reasoning About a Highly Connected World”, D Easley and J Kleinberg, Cambridge University Press, 2010.

SOCIAL NETWORK ANALYTICS

Mixed Strategies & Mixed strategy equilibria

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Mixed strategies

What is a Pure Strategy?

- A pure strategy provides a complete definition of how a player will play a game.
  A pure strategy determines the move a player will make for any situation they could face.
- In pure strategies, the player assigns 100% probability to one plan of action/strategy.
- A pure strategy can be seen as a mixed strategy where one of the probabilities is 1 and the others are all 0.

Mixed strategies

What is a mixed strategy?

- **A mixed strategy exists in a strategic game, when the player does not choose one definite action/strategy, but rather, chooses according to a probability distribution over his actions/strategies.**
- **Example:**
  Imagine you are in Nandos, and you are considering of choosing Lemon & Herb or Wild Herb sauce for your chicken.
  The spiciness of the sauce is equal, but you are indifferent between the two flavours and thus you decide to toss a coin.
  The probability of you choosing Lemon & Herb is 50%, or 1/2.
  The probability of choosing Wild Herb is also 50%, or 1/2.

- *Note: In pure strategies, the player assigns 100% probability to one plan of action.*

Mixed strategies

What is a mixed strategy?

- A mixed strategy is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy.
- A **mixed strategy** is one in which each strategy is played with fixed probability.
- A **mixed strategy** is a probability distribution over the set of strategies.

- One can regard **a pure strategy** as a degenerate case of a mixed strategy, **in which that particular pure strategy is selected with probability** ***1*** and **every other strategy with probability** ***0*.**

	1 \ 2	probability ½	½
probability		h	t
½	H	+1, -1	-1, +1
½	T	-1, +1	+1, -1

Mixed strategies

- A **totally mixed strategy** is a mixed strategy in which the player assigns a strictly positive probability to every pure strategy.

	1 \ 2	h	t
	probability	½	½
probability			
½	H	+1, -1	-1, +1
½	T	-1, +1	+1, -1

Mixed strategies

➢Consider the payoff matrix given below (known as a coordination game).

- **If player-1 opts to play A with probability 1 (i.e. play A for sure), then he is said to be playing a pure strategy.**

- **If player-2 opts to flip a coin and play A if the coin lands heads and B if the coin lands tails, then he is said to be playing a mixed strategy, and not a pure strategy.**

		Player - 2	
		A	B
Player - 1	A	1, 1	0, 0
	B	0, 0	1, 1

Mixed strategies

- In his famous paper, John Nash proved that there is an equilibrium for every finite game.

- **One can divide Nash equilibria into two types.**

1. **Pure strategy Nash equilibria** are Nash equilibria where all players are playing pure strategies.
   A Nash equilibrium in which no player randomizes is called a **pure strategy Nash equilibrium**.

2. **Mixed strategy Nash equilibria** are equilibria where at least one player is playing a mixed/randomized strategy and no player being able to increase his or her expected payoff by playing an alternate strategy.

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Mixed strategies

- **John Nash proved that every finite game has a Nash equilibrium, not all have pure strategy Nash equilibria.**
- For an example of a game that does not have a Nash equilibrium in pure strategies, see Matching pennies.
- **Many games do have pure strategy Nash equilibria** (e.g. the Coordination game, the Prisoner's dilemma, the Stag hunt).
- **Games can also have both pure strategy and mixed strategy equilibria.** An easy example is the pure coordination game, where in addition to the pure strategy nash equilibriums (A,A) and (B,B), a **mixed equilibrium** exists in which both players play either strategy with probability 1/2.
- If no equilibrium exists in pure strategies, one must exist in mixed strategies.

		Player 2	
		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

		Player - 2	
		A	B
Player - 1	A	1, 1	0, 0
	B	0, 0	1, 1

Mixed strategies

- **Why do we bother about mixed strategies?**

- **There are cases where a pure strategy equilibrium does not exist.**
- If we play these games, we should be "unpredictable. Therefore, we have to find the probability for which the player would be willing to randomize between his actions.
  This probability will depend on the expected payoffs of the player for each of his actions.

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Matching Pennies

Goalie

		$Left$	$Right$
Kicker	$Left$	$-1, 1$	$1, -1$
	$Right$	$1, -1$	$-1, 1$

Penalty kick

Randomize: To make something random (so that it happens or is chosen by chance)

Mixed strategies

Assignment: Q & A

1. What games require or admit randomization as part of their solution?

SOCIAL NETWORK ANALYTICS

Computing mixed strategy equilibria

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Computing Mixed Strategy Equilibria in 2×2 Games

- **Solution criterion:** Each pure strategy in a mixed strategy equilibrium pays the same at equilibrium.

  For example: $EU_1(H) = EU_1(T)$ $EU_2(h) = EU_2(t)$

- In equilibrium, each player's probability distribution(i.e., mixed strategies) makes all others indifferent between their pure strategies.
  - This is only true for 2×2 games.

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2 (probability)	h (y)	t (1- y)	EU_1:
H	+1, -1	-1, +1	2y - 1
T	-1, +1	+1, -1	1 - 2y

$EU_1(H) = y \times 1 + (1- y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1- y) \times 1 = 1 - 2y$

Need to calculate player 2's expected utility from player 1's mixed strategy

probability	1 \ 2	h	t
x	H	+1, -1	-1, +1
1- x	T	-1, +1	+1, -1
	EU_2:	1 - 2x	2x - 1

$EU_2(h) = x \times -1 + (1- x) \times 1 = 1 - 2x$

$EU_2(t) = x \times 1 + (1- x) \times -1 = 2x - 1$

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

- To compute a mixed strategy, let the player-2 go with strategy **h** with probability y, and the player-1 go with strategy H with probability x.

Matching Pennies:
What about mixed strategies?

		probability	y	1- y
	1	2	h	t
probability				
x	H		+1, -1	-1, +1
1- x	T		-1, +1	+1, -1

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

- Suppose that Player-1 believes Player-2 plays Heads(h) with probability y.
  Then **if Player-1 plays Heads(H)**, Player-1 gets 1 with probability y and –1 with probability (1 – y), for an expected value of **2y – 1**.
- Similarly, **if Player-1 plays Tails(T)**, Player-1 gets –1 with probability y (when Player-2 plays Heads(h)), and 1 with probability (1 – y), for an expected value of **1 – 2y**.
- This is summarized in Table "Mixed strategy in matching pennies".

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2	probability: y	1- y	
	h	t	EU_1:
H	+1, -1	-1, +1	2y - 1
T	-1, +1	+1, -1	1 - 2y

$EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y$

Table: "Mixed strategy in matching pennies"

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Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

- If **(2y – 1) > (1 – 2y)**, then **Player-1 is better off, on average, playing Heads than Tails.**
  Similarly, if **(2y – 1) < (1 – 2y)**, then Player-1 is better off playing Tails than Heads.

- If, on the other hand, **2y – 1 = 1 – 2y**, then Player-1 gets the same payoff no matter what Player-1 does.
  In this case, Player-1 could play Heads, could play Tails, or could flip a coin and randomize Player-1's play.

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2 (probability)	h (y)	t (1- y)	EU_1:
H	+1, -1	-1, +1	2y - 1
T	-1, +1	+1, -1	1 - 2y

$EU_1(H) = y \times 1 + (1- y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1- y) \times 1 = 1 - 2y$

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

Need to calculate player 2's expected utility from player 1's mixed strategy

probability	1 \ 2	h	t
x	H	+1, -1	-1, +1
1- x	T	-1, +1	+1, -1
	EU_2:	1 - 2x	2x - 1

$EU_2(h) = x \times -1 + (1- x) \times 1 = 1 - 2x$

$EU_2(t) = x \times 1 + (1- x) \times -1 = 2x - 1$

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

In equilibrium, Player 1 is willing to randomize only when he is indifferent between H and T

$$EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1$$
$$EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y$$

In equilibrium: $EU_1(H) = EU_1(T)$

$\therefore \quad 2y - 1 = 1 - 2y$

$\Rightarrow \quad 4y = 2$

$\Rightarrow \quad y = ½$

$\Rightarrow \quad 1 - y = 1 - ½ = ½$

$\therefore \quad y = 1 - y = ½$

Need to calculate player 1's expected utility from player 2's mixed strategy

probability	y	1- y	
1 \ 2	h	t	EU_1:
H	+1, -1	-1, +1	2y - 1
T	-1, +1	+1, -1	1 - 2y

$EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y$

Randomize: To make something random (so that it happens or is chosen by chance)

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

- Note that randomization requires equality of expected payoffs.

  For example: $EU_1(H) = EU_1(T)$

- If a player is supposed to randomize over strategy *H* or strategy *T*, then both of these strategies must produce the same expected payoff.
  Otherwise, the player would prefer one of them and wouldn't play the other.

In equilibrium, Player 1 is willing to randomize only when he is indifferent between H and T

$$EU_1(H) = y \times 1 + (1-y) \times -1 = 2y - 1$$
$$EU_1(T) = y \times -1 + (1-y) \times 1 = 1 - 2y$$

In equilibrium: $EU_1(H) = EU_1(T)$

$\therefore \quad 2y - 1 = 1 - 2y$

$\Rightarrow \quad 4y = 2$

$\Rightarrow \quad y = ½$

$\Rightarrow \quad 1 - y = 1 - ½ = ½$

$\therefore \quad y = 1 - y = ½$

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

Similarly, Player 2 is willing to randomize only when she is indifferent between h and t

Player 1's Conditions:
$EU_1(H) = EU_1(T)$

Player 2's Conditions:
$EU_2(h) = x \times -1 + (1- x) \times 1 = 1 - 2x$
$EU_2(t) = x \times 1 + (1- x) \times -1 = 2x - 1$

In equilibrium: $EU_2(h) = EU_2(t)$

$\therefore \quad 1 - 2x = 2x - 1$

$\Rightarrow \quad x = ½ \quad \text{and} \quad 1 - x = 1 - ½ = ½$

$\therefore \quad x = 1 - x = ½$

Need to calculate player 2's expected utility from player 1's mixed strategy

probability	1 \ 2	h	t
x	H	+1, -1	-1, +1
1- x	T	-1, +1	+1, -1
	EU_2:	1 - 2x	2x - 1

$EU_2(h) = x \times -1 + (1- x) \times 1 = 1 - 2x$
$EU_2(t) = x \times 1 + (1- x) \times -1 = 2x - 1$

Computing Mixed Strategy Equilibria

- **Computing a mixed strategy has one element that often appears confusing.**

- Suppose, if **Player-1 is willing to randomize**.
  Then Player-1's payoffs must be equal for all strategies that Player-1 plays with positive probability.
  For example: $\mathbf{EU_1(H) = EU_1(T)}$ (i.e., $\mathbf{2y\text{-}1 = 1\text{-}2y}$)

- But that **equality in Player-1's payoffs doesn't determine the probabilities with which Player-1 plays the various rows**.
  Instead, that **equality in Player-1's payoffs will determine the probabilities with which Player-2 plays the various columns**.
  The reason is that **it is Player-2's probabilities that determine the expected payoffs for Player-1**; **if Player-1 is going to randomize**, then Player-2's probabilities must be such that Player-1 is willing to randomize.

Need to calculate player 1's expected utility from player 2's mixed strategy

probability (1 \ 2)	y	1- y	
	h	t	EU_1:
H	+1, -1	-1, +1	2y - 1
T	-1, +1	+1, -1	1 - 2y

$EU_1(H) = y \times 1 + (1\text{-}y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1\text{-}y) \times 1 = 1 - 2y$

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

- ➢ **Computing a mixed strategy has one element that often appears confusing.**

- ○ **For example,** we computed the payoff to Player-1 of playing Heads(H), which was 2y – 1, where *y* was the probability that Player-2 played Heads(h).
  Similarly, the payoff to Player-1 of playing Tails(T) was 1 – 2y. **Player-1 is willing to randomize if these are equal**, which solves for $y = ½$.

Need to calculate player 1's expected utility from player 2's mixed strategy

probability	y	1- y	
1 \ 2	h	t	EU_1:
H	+1, -1	-1, +1	2y - 1
T	-1, +1	+1, -1	1 - 2y

$EU_1(H) = y \times 1 + (1- y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1- y) \times 1 = 1 - 2y$

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

- Equilibrium with Mixed Strategies is a pair of strategies (now probabilities) so that each is a best response to the other.

- The pair of strategies x = 1/2 and y = 1/2 is the only possibility for a Mixed-strategy Nash equilibrium. This pair of strategies (x, y) = (½, ½) in fact do form best responses to each other.

- As a result, (x, y) = (½, ½) is the unique Nash equilibrium for the mixed-strategy version of Matching Pennies.

(x, y) = (½, ½)

$$2y-1 = 0 \qquad 1-2x = 0$$
$$1-2y = 0 \qquad 2x-1 = 0$$

Matching Pennies:
Equilibrium in mixed strategies

probability	1 \ 2	probability ½	½	
		h	t	EU_1:
½	H	+1, -1	-1, +1	0
				=
½	T	-1, +1	+1, -1	0
	EU_2:	0	= 0	

Each is playing a best response to the other!

Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

$(x, y) = (½, ½)$

$2y-1 = 0$	$1-2x = 0$
$1-2y = 0$	$2x-1 = 0$

Matching Pennies:
Equilibrium in mixed strategies

probability (1)	1 \ 2	h (probability ½)	t (probability ½)	EU_1:
½	H	+1, -1	-1, +1	0
				||
½	T	-1, +1	+1, -1	0
	EU_2:	0	= 0	

Each is playing a best response to the other!

Mixed strategies are not intuitive:
You randomize to make me indifferent.

Row randomizes to make Column indifferent.

Column randomizes to make Row indifferent.

Then each is playing a best response to the other.

Computing Mixed Strategy Equilibria

Interpreting the Mixed-Strategy Equilibrium for Matching Pennies:

- **Having derived the Nash equilibrium for this game, it's useful to think about what it means, and how we can apply this reasoning to games in general.**
- First, let's picture a concrete setting in which two people actually sit down to play Matching Pennies, and **each of them commits to behaving randomly according to probabilities x and y respectively.**
- **The choice of y = 1/2 by Player-2 makes Player 1 indifferent between playing H or T**: the strategy y = 1/2 is effectively "non-exploitable" by Player-1.
- This was in fact our original intuition for introducing randomization: **each player wants their behavior to be unpredictable to the other, so that their behavior can't be taken advantage of.**

Matching Pennies:
Equilibrium in mixed strategies

	probability	½	½	
probability	1 \ 2	h	t	EU_1:
½	H	+1, -1	-1, +1	0
				‖
½	T	-1, +1	+1, -1	0
EU_2:		0	= 0	

Each is playing a best response to the other!

Computing Mixed Strategy Equilibria

Interpreting the Mixed-Strategy Equilibrium for Matching Pennies:

- This notion of indifference is a general principle behind the computation of mixed-strategy equilibria in two-player, **two-strategy games when there are no equilibria involving pure strategies: each player should randomize so as to make the other player indifferent between their two alternatives.**
- **This way, neither player's behavior can be exploited by a pure strategy, and the two choices of probabilities are best responses to each other.**

Matching Pennies:
Equilibrium in mixed strategies

		probability ½	½	
probability	1 \ 2	h	t	EU_1:
½	H	+1, -1	-1, +1	0
				‖
½	T	-1, +1	+1, -1	0
EU_2:		0	= 0	

Each is playing a best response to the other!

Computing Mixed Strategy Equilibria

- **Example 2:** Find mixed strategy equilibrium for the following game.

		Woman	
		Baseball	**Ballet**
Man	**Baseball**	3, 2	1, 1
	Ballet	0, 0	2, 3

This game has two pure strategy Nash equilibria: (Baseball, Baseball) and (Ballet, Ballet). Is there a mixed strategy?

Computing Mixed Strategy Equilibria

- **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman	
		Baseball (p)	**Ballet (1-p)**
Man	**Baseball (q)**	3, 2	1, 1
	Ballet (1-q)	0, 0	2, 3

To compute a mixed strategy, let the Woman go to the Baseball game with probability p, and the Man go to the Baseball game with probability q.
Figure above contains the computation of the mixed strategy payoffs for each player.

Computing Mixed Strategy Equilibria

- **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman	
		Baseball (p)	Ballet (1-p)
Man	Baseball (q)	3, 2	1, 1
	Ballet (1-q)	0, 0	2, 3

$1+2p$

For example, if the Man (row player) goes to the Baseball game, he gets 3 when the Woman goes to the Baseball game (with probability *p*), and otherwise gets 1, for an expected payoff of $3p + 1(1-p) = 1 + 2p$.

Computing Mixed Strategy Equilibria

➢ **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman		
		Baseball (p)	**Ballet (1-p)**	
Man	**Baseball (q)**	3, 2	1, 1	**1+2p**
	Ballet (1-q)	0, 0	2, 3	**2-2p**
		2q	**3-2q**	

A mixed strategy in the battle of the sexes game requires both parties to randomize.

The Man's indifference between going to the Baseball game and to the Ballet requires $1 + 2p = 2 - 2p$, which yields $p = 1/4$ and **1-p=3/4.**

The Woman's indifference between going to the Baseball game and to the Ballet requires **2q = 3 – 2q**, which yields $q = 3/4$ and **1-q=1/4.**

Computing Mixed Strategy Equilibria

- **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman		
		Baseball (p)	Ballet (1-p)	
Man	Baseball (q)	3, 2	1, 1	1+2p
	Ballet (1-q)	0, 0	2, 3	2-2p
		2q	3-2q	

The pair of strategies p = 1/4 and q = 3/4 is the Mixed-strategy Nash equilibrium.

The strategy p=1/4 makes the Man indifferent between the two events because he prefers to be with the Woman, but he also likes to be at the Baseball game.
To make up for the advantage that the game holds for him, the Woman has to be at the Ballet more often.

Computing Mixed Strategy Equilibria

➢ **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman		
		Baseball (p)	Ballet (1-p)	
Man	Baseball (q)	3, 2	1, 1	$1+2p = 1\frac{1}{2}$
	Ballet (1-q)	0, 0	2, 3	$2-2p = 1\frac{1}{2}$
		$2q = 1\frac{1}{2}$	$3-2q = 1\frac{1}{2}$	

The probability that the Man goes to the Baseball game is ¾, and he goes to the Ballet ¼ of the time. These are independent probabilities, so to get **the probability that both go to the Baseball game**, we multiply the probabilities, which yields **3/16**.

Computing Mixed Strategy Equilibria

➢ **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman	
		Baseball (p=¼)	**Ballet (1-p=¾)**
Man	**Baseball (q=¾)**	3/16	9/16
	Ballet (1-q=¼)	1/16	3/16

Table above "**Mixed strategy probabilities**" fills in the probabilities for all four possible outcomes.
Note that more than half of the time (Baseball, Ballet) is the outcome of the mixed strategy and the two people are not together. This lack of coordination is generally a feature of mixed strategy equilibria.

Computing Mixed Strategy Equilibria

➢ **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman	
Man		Baseball ($p=\frac{1}{4}$)	Ballet ($1-p=\frac{3}{4}$)
	Baseball ($q=\frac{3}{4}$)	3/16	9/16
	Ballet ($1-q=\frac{1}{4}$)	1/16	3/16

The expected payoffs for both players are readily computed as well. The Man's payoff is $1 + 2p = 2 - 2p$, and since $p = \frac{1}{4}$, the Man obtains $1\frac{1}{2}$.
A similar calculation shows that the Woman's payoff is the same. Thus, both do worse than coordinating on their less preferred outcome.

Mixed Strategy and Computing Mixed Strategy Equilibria

KEY TAKEAWAYS

- A mixed strategy Nash equilibrium involves at least one player playing a randomized strategy and no player being able to increase his or her expected payoff by playing an alternate strategy.
- A Nash equilibrium without randomization is called a pure strategy Nash equilibrium.
- If a player is supposed to randomize over two strategies, then both must produce the same expected payoff.
- The matching pennies game has a mixed strategy and no pure strategy.
- The battle of the sexes game has a mixed strategy and two pure strategies.

SOCIAL NETWORK ANALYTICS

Computing Mixed Strategy equilibria: Examples

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Computing Mixed Strategy Equilibria

Example 1: Matching Pennies game

- Equilibrium with Mixed Strategies is a pair of strategies (now probabilities) so that each is a best response to the other.
- The pair of strategies x = 1/2 and y = 1/2 is the only possibility for a Mixed-strategy Nash equilibrium.
  This pair of strategies (x, y) = (½, ½) in fact do form best responses to each other.
- As a result, (x, y) = (½, ½) is the unique Nash equilibrium for the mixed-strategy version of Matching Pennies.

(x, y) = (½, ½)

2y-1 = 0   1-2x = 0
1-2y = 0   2x-1 = 0

Matching Pennies:
Equilibrium in mixed strategies

probability	1 \ 2	h (probability ½)	t (probability ½)	EU_1:
½	H	+1, -1	-1, +1	0
				‖
½	T	-1, +1	+1, -1	0
EU_2:		0	= 0	

Each is playing a best response to the other!

Computing Mixed Strategy Equilibria

- **Example 2:** Find mixed strategy equilibrium for the following game.

Solution:

		Woman		
		Baseball (p)	Ballet (1-p)	
Man	Baseball (q)	3, 2	1, 1	1+2p
	Ballet (1-q)	0, 0	2, 3	2-2p
		2q	3-2q	

The pair of strategies p = 1/4 and q = 3/4 is the Mixed-strategy Nash equilibrium.

The strategy p=1/4 makes the Man indifferent between the two events because he prefers to be with the Woman, but he also likes to be at the Baseball game.
To make up for the advantage that the game holds for him, the Woman has to be at the Ballet more often.

Computing Mixed Strategy Equilibria

➢ **Example 3:** Find mixed strategy equilibrium for the following game.

Market Niche: The payoff matrix

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	100, 0
Stay Out	0, 100	0, 0

Computing Mixed Strategy Equilibria

- **Example 3:**

Market Niche: Two pure strategy equilibria

Mutual best responses form an equilibrium.

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	100, 0 ★
Stay Out	0, 100 ★	0, 0

Computing Mixed Strategy Equilibria

➢ **Example 3:**

Market Niche:
What about mixed strategies?

probability	1 \ 2	y	1- y
		e	s
x	E	-50, -50	100, 0
1- x	S	0, 100	0, 0

x, y between 0 and 1

That is, $0 \le x \le 1$ and $0 \le y \le 1$

Computing Mixed Strategy Equilibria

- **Example 3:**

Need to calculate firm 1's expected utility from firm 2's mixed strategy

1 \ 2 (probability)	e (y)	s ($1-y$)	EU_1:
E	-50, -50	100, 0	100 - 150y
S	0, 100	0, 0	0

$EU_1(E) = y \times -50 + (1-y) \times 100 = 100 - 150y$

$EU_1(S) = y \times 0 + (1-y) \times 0 = 0$

Computing Mixed Strategy Equilibria

- **Example 3:**

Need to calculate firm 2's expected utility from firm 1's mixed strategy

probability	1 \ 2	e	s
x	E	-50, -50	100, 0
1- x	S	0, 100	0, 0
	EU_2:	100-150x	0

$EU_2(e) = x \times -50 + (1-x) \times 100 = 100 - 150x$

$EU_2(s) = x \times 0 + (1-x) \times 0 = 0$

Computing Mixed Strategy Equilibria

- **Example 3:**

In equilibrium, Firm 1 is willing to randomize only when it is indifferent between E and S

$$EU_1(E) = y \times -50 + (1-y) \times 100 = 100 - 150y$$
$$EU_1(S) = y \times 0 + (1-y) \times 0 = 0$$

In equilibrium: $EU_1(E) = EU_1(S)$

$$\therefore \quad 100 - 150y = 0$$
$$\Rightarrow \quad 150y = 100$$
$$\Rightarrow \quad y = 2/3$$
$$\Rightarrow \quad 1 - y = 1 - 2/3 = 1/3$$

$$\therefore \quad y = 2/3 \text{ and } 1 - y = 1/3$$

Computing Mixed Strategy Equilibria

➢ Example 3:

Similarly, Firm 2 is willing to randomize only when it is indifferent between h and t

Firm 1's Conditions:
$EU_1(E) = EU_1(S)$

Firm 2's Conditions:
$EU_2(e) = x \times -50 + (1- x) \times 100 = 100 - 150x$
$EU_2(s) = x \times 0 + (1- x) \times 0 = 0$

In equilibrium: $EU_2(e) = EU_2(s)$

$\therefore \quad 100 - 150x = 0$

$\Rightarrow \quad 150x = 100$

$\therefore \quad x = 2/3$ and $1 - x = 1/3$

Computing Mixed Strategy Equilibria

➢ Example 3:

- ➢ The pair of strategies x = 2/3 and y = 2/3 is the Mixed-strategy Nash equilibrium.
- ➢ The choice of y = 2/3 by Firm-2 makes Firm-1 indifferent between playing E or S

Market Niche:
Equilibrium in mixed strategies

1 \ 2	probability	2/3	1/3	
probability		e	s	EU_1:
2/3	E	-50, -50	100, 0	0
				||
1/3	S	0, 100	0, 0	0
EU_2:		0	= 0	

Each firm is playing a best response to the other!

Computing Mixed Strategy Equilibria

- **Exercise 1:** Find mixed strategy equilibrium for the following game.

		Player 2	
		Heads	Tails
Player 1	Right	4 3	2 1
	Left	1 2	3 2

Computing Mixed Strategy Equilibria

➢ **Exercise 1:** Find mixed strategy equilibrium for the following game.

Solution:

		Player 2	
		Heads	Tails
Player 1	Right (x)	3, 4	1, 2
	Left (1-x)	2, 1	2, 3

$EU_1(\text{Right}) = 3y + 1(1 - y) = 1 + 2y$

$EU_1(\text{Left}) = 2y + 2(1 - y) = 2$

$EU_2(\text{Right}) = 4x + 1(1 - x) = 1 + 3x$ $\quad EU_2(\text{Tails}) = 2x + 3(1 - x) = 3 - x$

So $1 + 3x = 3 - x$. Thus $x = 1/2$.

Hence **Player 1 will reach the mixed strategy equilibrium when they play Right and Left with equal probability.** A similar computation shows that Player 2 should also be equally inclined to play Heads and Tails at the equilibrium.

Computing Mixed Strategy Equilibria

- **Exercise 1:** Find mixed strategy equilibrium for the following game.

Solution:

		Player 2	
		Heads	Tails
Player 1	Right (x)	3, 4	1, 2
	Left (1-x)	2, 1	2, 3

$EU_1(Right) = 3y + 1(1 - y) = 1 + 2y$

$EU_1(Left) = 2y + 2(1 - y) = 2$

$EU_2(Right) = 4x + 1(1 - x) = 1 + 3x$ $EU_2(Tails) = 2x + 3(1 - x) = 3 - x$

So $1 + 3x = 3 – x$. Thus $x = 1/2$.

The pair of strategies x = 1/2 and y = 1/2 is the Mixed-strategy Nash equilibrium.

The strategy x=1/2 makes the Player-2 indifferent between the two strategies Heads and Tails.

Computing Mixed Strategy Equilibria

- **Exercise 2:** Find mixed strategy equilibrium for the following game.

Asymmetrical Market Niche:
The payoff matrix

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	150, 0
Stay Out	0, 100	0, 0

Market Niche: The payoff matrix

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	100, 0
Stay Out	0, 100	0, 0

Computing Mixed Strategy Equilibria

- **Exercise 2:** Find mixed strategy equilibrium for the following game.

Solution:

Asymmetrical Market Niche:
Two pure strategy equilibria

Firm 1 \ Firm 2	Enter	Stay Out
Enter	-50, -50	150, 0 ★
Stay Out	0, 100 ★	0, 0

Computing Mixed Strategy Equilibria

- **Exercise 2:** Find mixed strategy equilibrium for the following game.

Solution:

Asymmetrical Market Niche:
What about mixed strategies?

probability (1 \\ 2)		y	1 - y
probability	1 \\ 2	e	s
x	E	-50, -50	150, 0
1 - x	S	0, 100	0, 0

x, y between 0 and 1

That is, $0 \leq x \leq 1$ and $0 \leq y \leq 1$

Need to calculate each firm's expected utility from the firm's mixed strategy

probability (1 \\ 2)		y	1 - y	
probability	1 \\ 2	e	s	EU_1:
x	E	-50, -50	150, 0	150 - 200y
1 - x	S	0, 100	0, 0	0
	EU_2:	100 - 150x	0	

Computing Mixed Strategy Equilibria

- **Exercise 2:** Find mixed strategy equilibrium for the following game.

Solution:

In equilibrium, Firm 1 is willing to randomize only when it is indifferent between E and S

$EU_1(E) = y \times -50 + (1-y) \times 150 = 150 - 200y$
$EU_1(S) = y \times 0 + (1-y) \times 0 = 0$

In equilibrium: $EU_1(E) = EU_1(S)$

$\therefore \quad 150 - 200y = 0$

$\Rightarrow \quad 200y = 150$

$\Rightarrow \quad y = 3/4$

$\Rightarrow \quad 1 - y = 1 - 3/4 = 1/4$

$\therefore \quad y = 3/4 \text{ and } 1 - y = 1/4$

Similarly, Firm 2 is willing to randomize only when it is indifferent between e and s

Firm 1's Conditions:
$EU_1(E) = EU_1(S)$

Firm 2's Conditions:
$EU_2(e) = x \times -50 + (1-x) \times 100 = 100 - 150x$
$EU_2(s) = x \times 0 + (1-x) \times 0 = 0$

In equilibrium: $EU_2(e) = EU_2(s)$

$\therefore \quad 100 - 150x = 0$

$\Rightarrow \quad 150x = 100$

$\therefore \quad x = 2/3 \text{ and } 1 - x = 1/3$

Computing Mixed Strategy Equilibria

- **Exercise 2:** Find mixed strategy equilibrium for the following game.

Solution:

- **The pair of strategies x = 2/3 and y = 3/4 is the Mixed-strategy Nash equilibrium.**
- **The choice of y = 3/4 by Player-2 makes Player 1 indifferent between playing E or S.**
- **Each firm(player) wants their behavior to be unpredictable to the other, so that their behavior can't be taken advantage of.**

Asymmetrical Market Niche:
Equilibrium in mixed strategies

probability 1 \ 2 probability		3/4 e	1/4 s	EU_1:
2/3	E	-50, -50	150, 0	0
				||
1/3	S	0, 100	0, 0	0
EU_2:		0	= 0	

Each firm is playing a best response to the other!

Asymmetrical Market Niche:
Equilibrium in mixed strategies

Although the two pure strategy equilibria (E,s) and (S,e) did not change in Asymmetrical Market Niche, the mixed strategies equilibrium did change.

Computing Mixed Strategy Equilibria

Exercise 3:

(a) Find all pure-strategy Nash equilibria for this game.

(b) This game also has a mixed-strategy Nash equilibrium; find the probabilities the players use in this equilibrium, together with an explanation for your answer.

The payoff matrix

player 1 \ player 2	drive straight ahead	swerve
drive straight ahead	-10, -10	1, -1
swerve	-1, 1	0, 0

Computing Mixed Strategy Equilibria

- **Exercise 4:** Find mixed strategy equilibrium for the following game.

Solution:

Everyday Low Pricing:
The payoff matrix

Retailer 1 \ Retailer 2	Normal price np	Sale price sp
NP	7500, 7500	7500, 8500
SP	8500, 7500	5500, 5500

Computing Mixed Strategy Equilibria

- **Assignment:**

1. Let q be the probability that Row plays Heads. Show that Column is willing to randomize, if and only if $q = ½$. (Hint: First compute Column's expected payoff when Column plays Heads, and then compute Column's expected payoff when Column plays Tails. These must be equal for Column to randomize.)
2. Show that in the rock, paper, scissors game there are no pure strategy equilibria. Show that playing all three actions with equal likelihood is a mixed strategy equilibrium.
3. If you multiply a player's payoff by a positive constant, the equilibria of the game do not change. Is this true or false, and why?

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Pareto Optimality

- The definition Pareto-optimality, named after the Italian economist Vilfredo Pareto who worked in the late 1800's and early 1900's.

- **Definition 2:**
  "A choice of strategies — one by each player — is Pareto-optimal if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff."

Pareto Optimality

➢ Definition 3:

- **Strategy profile *s*** **Pareto dominates** **a strategy profile *s'*** if
  - no agent gets a worse payoff with *s* than with *s'*, i.e., $U_i(s) \geq U_i(s')$ for all i, and
  - at least one agent gets a better payoff with s than with s', i.e., $U_i(s) > U_i(s')$ for at least one i.

- **Strategy profile *s*** **is** **Pareto optimal**, or **strictly Pareto efficient**, if there's no strategy *s'* that Pareto dominates *s*.
- **Every game has at least one Pareto optimal profile**. Always at least one Pareto optimal profile in which the strategies are pure.

Example 1:

s=(5, 5, 6)

s'=(5, 5, 5)

A coordination game

- In game theory, **battle of the sexes** (**BoS**) is a two-player coordination game.
  - Imagine a couple that agreed to meet this evening, but cannot recall if they will be attending the opera or a football match (and the fact that they forgot is common knowledge).
  - The husband would prefer to go to the football game. The wife would rather go to the opera. Both would prefer to go to the same place rather than different ones. If they cannot communicate, where should they go?

husband

		Opera	Football
wife	Opera	3,2	0,0
	Football	0,0	2,3

Battle of the Sexes 1

Game Theory

Exercise 1.4. There are two players. Each player is given an unmarked envelope and asked to put in it either nothing or $300 of his own money or $600 of his own money. A referee collects the envelopes, opens them, gathers all the money, then adds 50% of that amount (using his own money) and divides the total into two equal parts which l

(a) Represent this … e showing in each cell the … amount of money distribu… in wealth of each player (m…

		Player-2		
		0	300	600
Player-1	0	0 0	225 225	450 450
	300	225 225	450 450	675 675
	600	450 450	675 675	900 900

(b) Suppose that Player 1 has some animosity towards the referee and ranks the outcomes in terms of how much money the referee loses (the more, the better), while Player 2 is selfish and greedy and ranks the outcomes in terms of her own net gain. Represent the corresponding game using a table.

(c) Is there a strict dominant-strategy equilibrium?

Mixed strategies

- For example, we computed the payoff(expected utility) to Row of playing Heads, which was 2y – 1, where *y* was the probability that Column played Heads.
  Similarly, the payoff to Row of playing Tails was 1 – 2y.
  Row is willing to randomize if these are equal, which solves for *y* = ½.

Need to calculate player 1's expected utility from player 2's mixed strategy

1 \ 2 (probability)	h (y)	t (1- y)	EU_1:
H	+1, -1	-1, +1	2y - 1
T	-1, +1	+1, -1	1 - 2y

$EU_1(H) = y \times 1 + (1- y) \times -1 = 2y - 1$

$EU_1(T) = y \times -1 + (1- y) \times 1 = 1 - 2y$