Quiz #6, Fall 2016 - Form 1

Let
$$a_n = \frac{2^n + 1}{3^n}.$$

1. (6 points) For the sequence $\{a_n\}$, determine the least upper bound (l.u.b.) and greatest lower bound (g.l.b.).

$$\{a_n\} = \frac{2+1}{3}, \frac{2^2+1}{3^2}, \frac{2^3+1}{3^3}, \frac{2^4+1}{3^4} \cdots$$

$$= 1, \frac{5}{9}, \frac{9}{27}, \frac{17}{81}, \cdots$$

$$\boxed{\text{l.u.b.} = 1, \quad \text{g.l.b.} = 0}$$

2. (6 points) Does the sequence $\{a_n\}$ converge or diverge? Explain your answer mathematically and find the limit if it converges.

yes: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2^n + 1}{3^n} \quad \left[\frac{\infty}{\infty}\right]$

$\overset{L'H}{=} \lim_{n \to \infty} \frac{2^n \ln 2}{3^n \ln 3} = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n \cdot \frac{\ln 2}{\ln 3} = 0 \cdot \frac{\ln 2}{\ln 3} = 0$

The sequence $\boxed{\text{converges to } 0}$.

3. (8 points) Evaluate the sum of the infinite series $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} \frac{2^n + 1}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \frac{2/3}{1 - 2/3} + \frac{1/3}{1 - 1/3}$$

$$= \frac{2/3}{1/3} + \frac{1/3}{2/3} = 2 + \frac{1}{2}$$

$$= \boxed{\frac{5}{2}}$$

Quiz #6, Fall 2016 - Form 2

Let
$$a_n = \frac{3^n + 1}{4^n}.$$

1. (6 points) For the sequence $\{a_n\}$, determine the least upper bound (l.u.b.) and greatest lower bound (g.l.b.).

$$\left\{\frac{3^n + 1}{4^n}\right\} = \frac{3^1 + 1}{4^1}, \frac{3^2 + 1}{4^2}, \frac{3^3 + 1}{4^3}, \frac{3^4 + 1}{4^4}, \text{----}$$

$$= 1, \frac{10}{16}, \frac{28}{64}, \frac{82}{256}, \text{----}$$

$$\boxed{\text{l.u.b.} = 1, \quad \text{g.l.b.} = 0}$$

2. (6 points) Does the sequence $\{a_n\}$ converge or diverge? Explain your answer mathematically and find the limit if it converges.

$$\lim_{n \to \infty} \frac{3^n + 1}{4^n} \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{n \to \infty} \frac{3^n \ln 3}{4^n \ln 4} = \frac{\ln 3}{\ln 4} \lim_{n \to \infty} \left(\frac{3}{4}\right)^n$$

$$= \frac{\ln 3}{\ln 4} \cdot 0 = 0$$

So the sequence $\boxed{\text{converges to } 0}$.

3. (8 points) Evaluate the sum of the infinite series $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} \frac{3^n + 1}{4^n} = \sum_{n=1}^{\infty} \frac{3^n}{4^n} + \sum_{n=1}^{\infty} \frac{1}{4^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= \frac{3/4}{1 - 3/4} + \frac{1/4}{1 - 1/4}$$

$$= \frac{3/4}{1/4} + \frac{1/4}{3/4} = 3 + \frac{1}{3}$$

$$= \boxed{\frac{10}{3}}$$