Local characterization of rigid motions in 2D Cartesian grid

K. Pluta[1], Y. Kenmochi[1], P. Romon[2], N. Passat[3]

[1] Université Paris-Est, LIGM
[2] Université Paris-Est, LAMA
[3] Université de Reims, CReSTIC

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Agenda

1. Introduction to digitized rigid transformations
2. Neighbourhood motion maps
3. Remainder partitioning
4. Bijectivity
5. Neighbourhood motion map graph
6. Conclusion and perspectives

Introduction to digitized rigid transformations

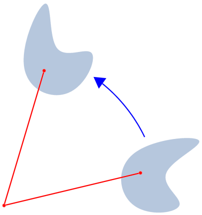
Rigid transformations in $\mathbb{R}^2$

$$\left| \begin{array}{lll} \mathcal{U} : \mathbb{R}^2 & \to & \mathbb{R}^2 \\ \mathbf{x} & \mapsto & \mathbf{R}\mathbf{x} + \mathbf{t} \end{array} \right.$$

$\mathbf{x}$ – point

$\mathbf{R}$ – rotation matrix

$\mathbf{t}$ – translation vector



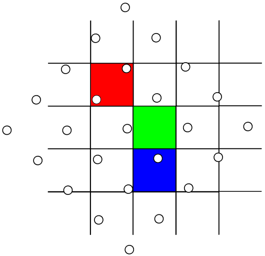
Properties:

- distance and angle preserving maps
- bijective
- inverse: $\mathcal{T} = \mathcal{U}^{-1}$ is also a rigid transformation

Rigid transformations in $\mathbb{Z}^2$

$$U = \mathcal{D} \circ \mathcal{U}_{|\mathbb{Z}^2}$$

where $\mathcal{D}$ is a digitization e.g. rounding function



Properties:

- in general, does not preserve distances and angles
- not always injective
- not always surjective $U(\mathbb{Z}^2) \subsetneq \mathbb{Z}^2$

Related work

Digitized rotations

- Bertrand Nouvel and Eric Rémila. Characterization of Bijective Discretized Rotations.
- Bertrand Nouvel and Eric Rémila. Configurations induced by discrete rotations: Periodicity and quasi-periodicity properties.
- Tristan Roussillon and David Cœurjolly. Characterization of bijective discretized rotations by Gaussian integers

Digitized rigid transformations

- Phuc Ngo, Yukiko Kenmochi, Nicolas Passat, and Hugues Talbot.

  Combinatorial structure of rigid transformations in 2D digital images

Contributions

- Study of local configurations induced by digitized rigid transformations in 2D
- Study of bijective digitized rigid transformations
- Verification algorithm of non-bijective digitized rigid transformations
- Link to the previous work of Ngo *et al.*

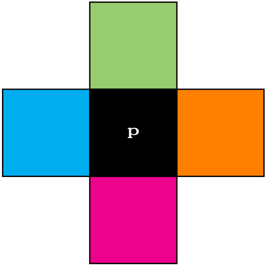
Neighbourhood motion maps

Neighbourhood

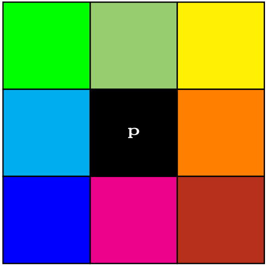
Definition

Let $\mathbf{p} \in \mathbb{Z}^2$ and $r \in \mathbb{R}_+$. The *neighbourhood of* $\mathbf{p} = (p, q)$ *(of radius* r*)*, noted $\mathcal{N}_r(\mathbf{p})$, is the set of all integer points contained in the closed ball of radius r centred at $\mathbf{p}$

$$\mathcal{N}_r(\mathbf{p}) = \{\mathbf{p} + \mathbf{d} \in \mathbb{Z}^2 \mid |\mathbf{d}| \leq r\}.$$



$\mathcal{N}_1(\mathbf{p})$ a.k.a.
4-neighbourhood



$\mathcal{N}_{\sqrt{2}}(\mathbf{p})$ a.k.a.
8-neighbourhood

Neighbourhood motion maps

Definition

Let $\mathbf{p} \in \mathbb{Z}^2$ and $r \in \mathbb{R}_+$. The *neighbourhood motion map* of $\mathbf{p}$ with respect to U and r is the function $\mathcal{G}_r^U(\mathbf{p}) : \mathcal{N}_r(\mathbf{0}) \to \mathcal{N}_{r'}(\mathbf{0})$ (with $r' \geq r$) defined as

$$\mathcal{G}_r^U(\mathbf{p}) : \mathbf{d} \mapsto U(\mathbf{p} + \mathbf{d}) - U(\mathbf{p}).$$

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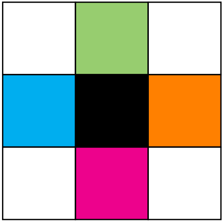

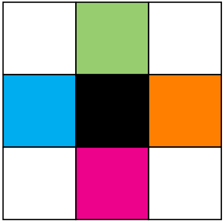

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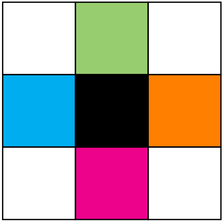

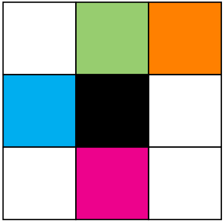

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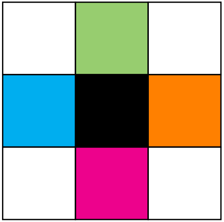
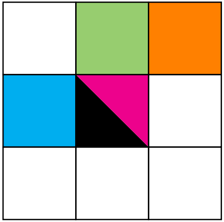
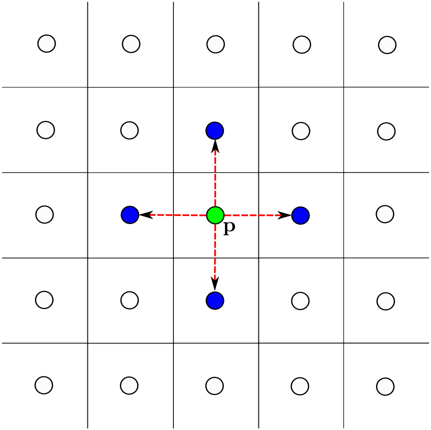
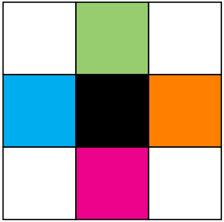




Remainder partitioning

Critical rigid transformations step-by-step





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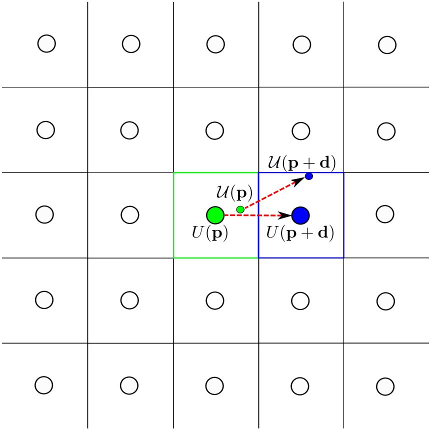

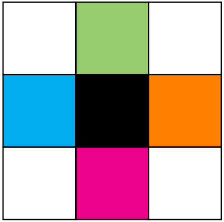

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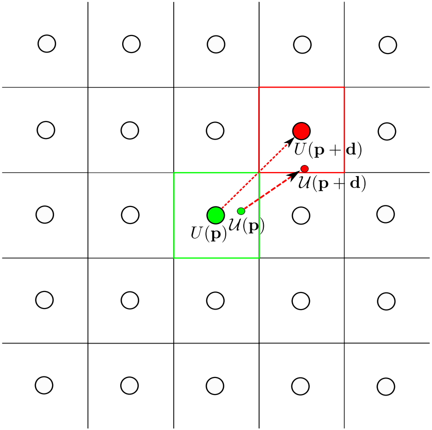

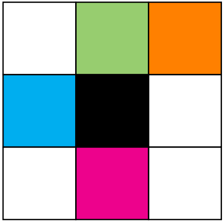

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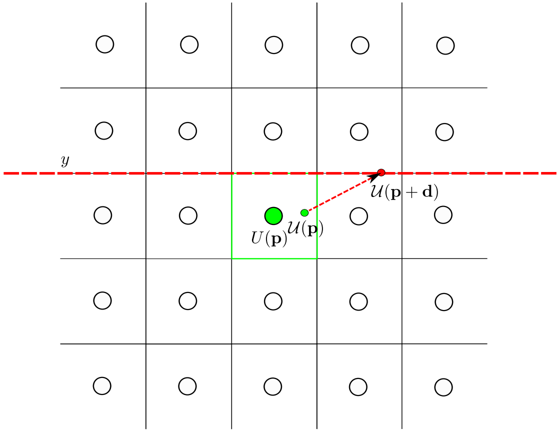
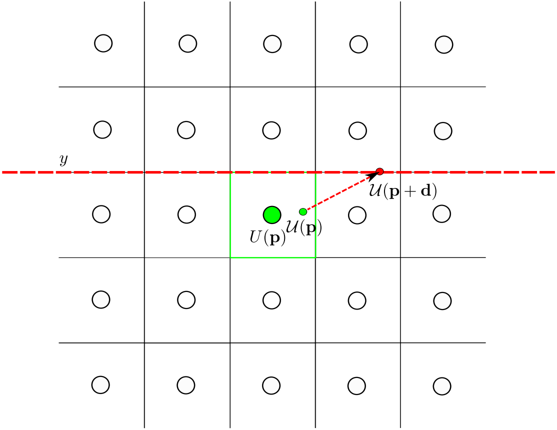
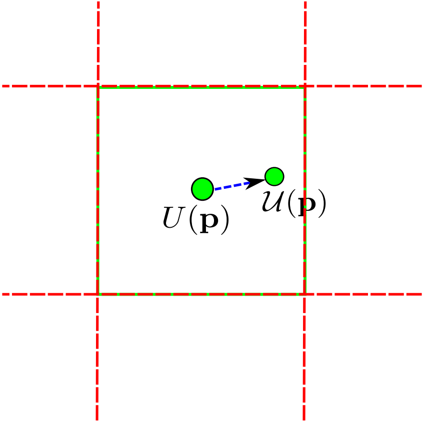


Critical rigid transformations step-by-step

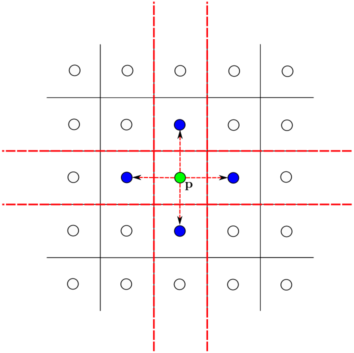


In particular, it happens when $\mathcal{U}(\mathbf{p}) - U(\mathbf{p}) + \sin\theta - \frac{1}{2} = 0$

Remainder map



Remainder map is defined as $\rho(\mathbf{p}) = \mathcal{U}(\mathbf{p}) - U(\mathbf{p}) \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2$



More generally,

$$x + u\cos\theta - v\sin\theta = k_x + \frac{1}{2}$$

$$y + u\sin\theta + v\cos\theta = k_y + \frac{1}{2}$$

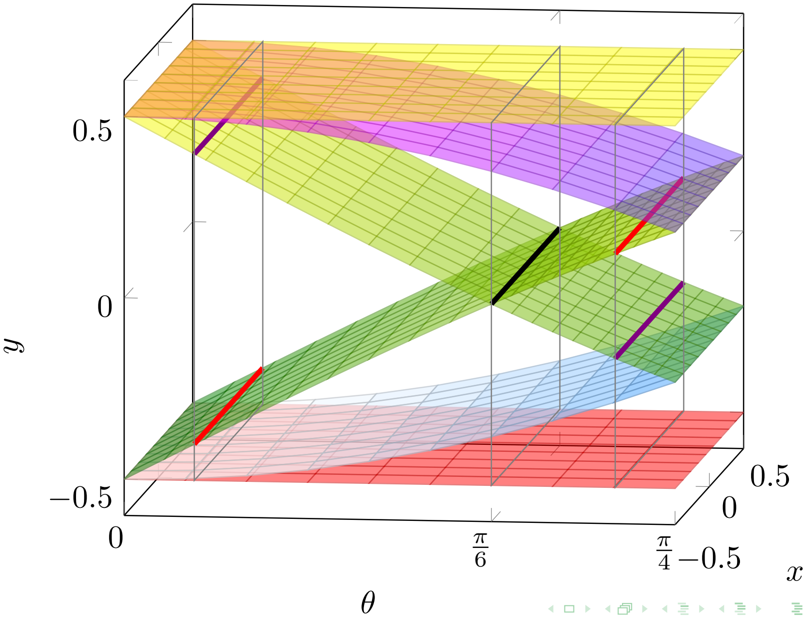
where $\rho(\mathbf{p}) = (x, y)^t$ and $k_x, k_y \in \mathbb{Z}$.

Critical lines in remainder range

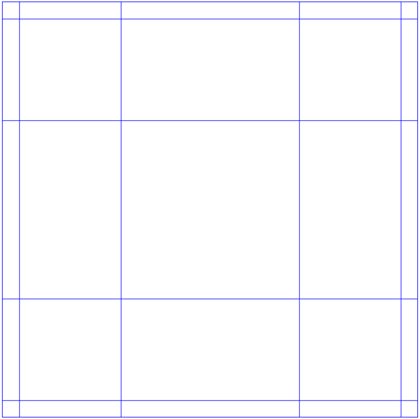
$$x + u\cos\theta - v\sin\theta = k_x + \frac{1}{2}$$

$$y + u\sin\theta + v\cos\theta = k_y + \frac{1}{2}$$

4-neighbourhood case: $\mathcal{G}_1^U$

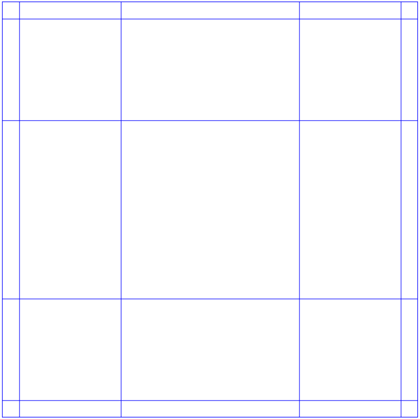


Frames



Each region bounded by critical lines is called a frame.

Frames

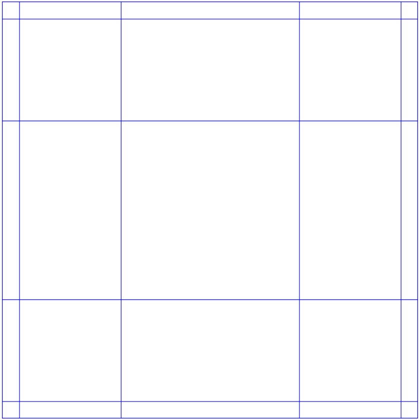


Each region bounded by critical lines is called a frame.

Proposition

For any $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^2$, $\mathcal{G}_r^U(\mathbf{p}) = \mathcal{G}_r^U(\mathbf{q})$ *iff* $\rho(\mathbf{p})$ *and* $\rho(\mathbf{q})$ *are in the same frame.*

Frames

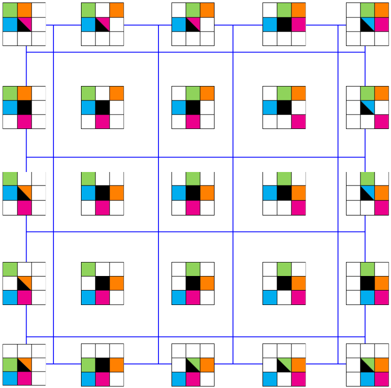


Each region bounded by critical lines is called a frame.

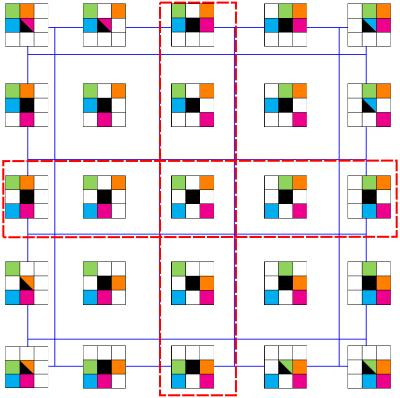
Remark

For a given θ we have at most 25 frames in the 4-neighbourhood case.

Frames and neighbourhood motion maps

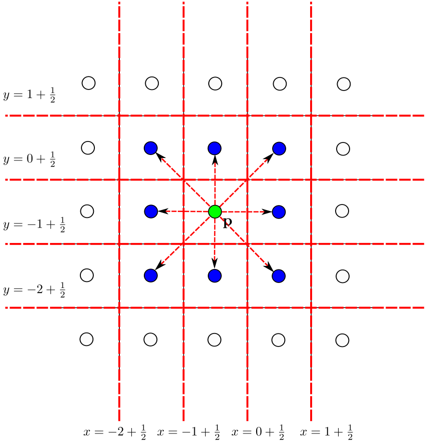


$0 < \theta < \frac{\pi}{6}$



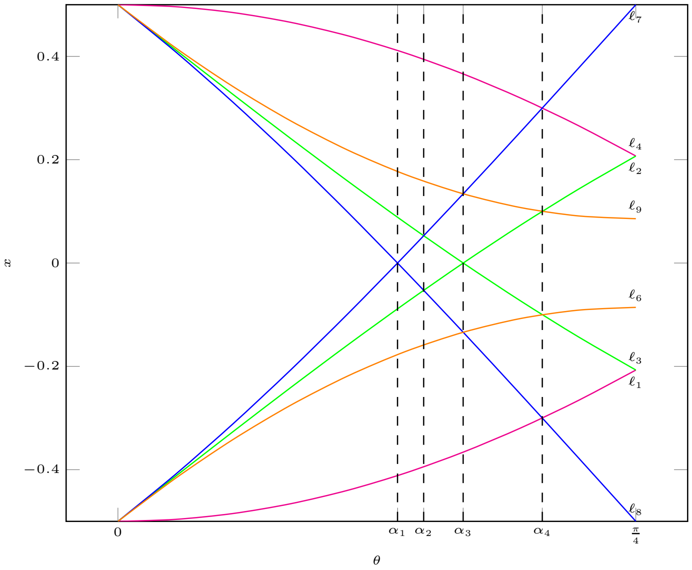
$\frac{\pi}{6} < \theta < \frac{\pi}{4}$

8-neighbourhood case: $\mathcal{G}^U_{\sqrt{2}}$



For a given θ we have at most 81 frames.

8-neighbourhood case: $\mathcal{G}^U_{\sqrt{2}}$



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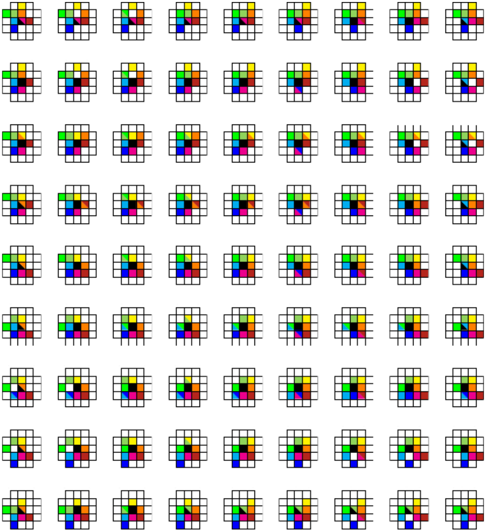

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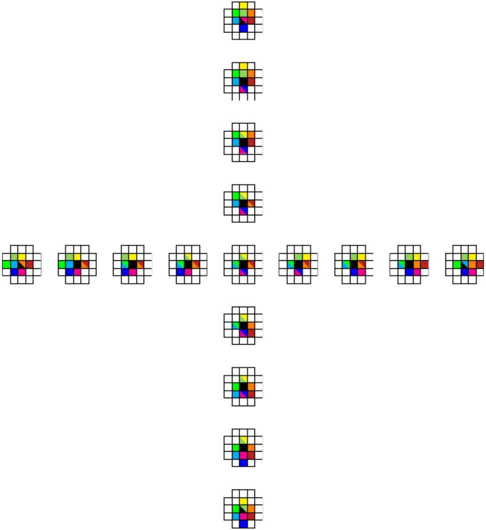

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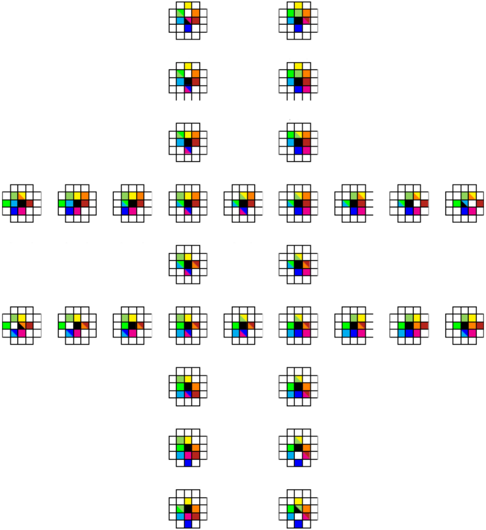

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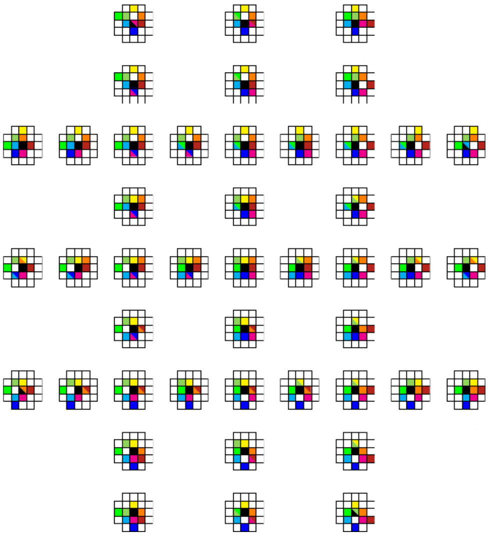

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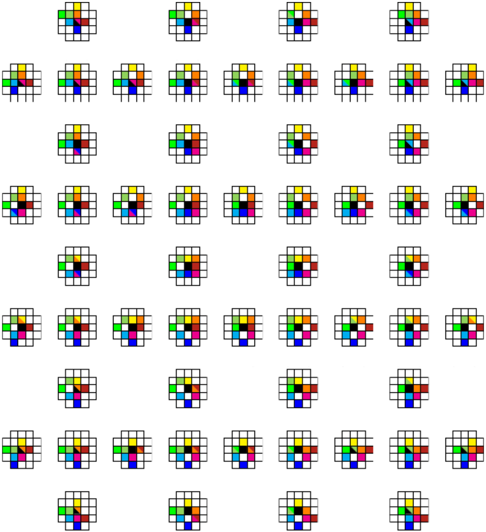


Four and eight neighbourhood motion maps in numbers

4-neighbourhood

In total 34 different neighbourhood motion maps.

8-neighbourhood

In total 231 different neighbourhood motion maps.

Bijectivity

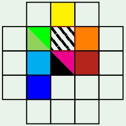
Which digitized rigid transformations are bijective?

Theorem (Nouvel and Rémila '06)

Digitized rotation is one-to-one iff is onto. Thus bijectivity is equivalent to surjectivity or injectivity.

Surjectivity

Example of non-surjectivity.



Neighbourhood motion maps which corresponds to one of the non-surjective frames.

In particular, there exists $\mathbf{q}' = U(\mathbf{p}) + (0,1)^t$ with no preimages.

Surjectivity

Lemma (Nouvel and Rémila '06)

If $\rho(\mathbf{p})$ is in one of the non-surjective frames, then there exists $\mathbf{q}'$ such that it has no preimages.

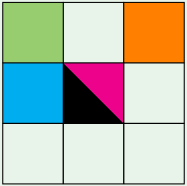
Surjectivity

Proposition (Nouvel and Rémila '06)

A digitized rigid transformation U is surjective if there exists no $\mathbf{p} \in \mathbb{Z}^2$ such that $\rho(\mathbf{p})$ belongs to one of the non-surjective frames.

Injectivity

Example of non-injectivity



Neighbourhood motion maps which corresponds to a non-injective frame.

A point $\mathbf{q}'$ will have two preimages $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^2$, such that $|\mathbf{p} - \mathbf{q}| = 1$, and $\mathbf{q}' = U(\mathbf{p}) = U(\mathbf{q})$.

Injectivity

Lemma (Nouvel and Rémila '06)

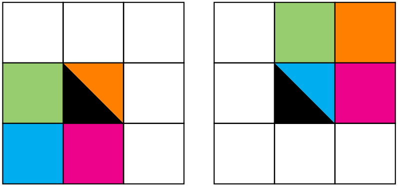
If $\rho(\mathbf{p})$ is in one of the non-injective frames, then $U(\mathbf{p})$ has two preimages.

Injectivity

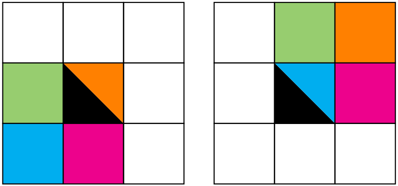
Proposition (Nouvel and Rémila '06)

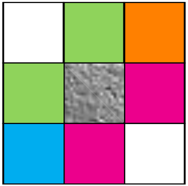
A digitized rigid transformation U is injective if there exists no $\mathbf{p} \in \mathbb{Z}^2$ such that $\rho(\mathbf{p})$ belongs to one of the non-injective frames.

Neighbourhood motion maps and non-injectivity



Neighbourhood motion maps and non-injectivity





$$\mathcal{G}_1^U(\mathbf{p},\mathbf{q}) = \mathcal{G}_1^U(\mathbf{q}) \sqcup \mathcal{G}_1^U(\mathbf{p}) : \mathcal{N}_1(\mathbf{0}) \sqcup \mathcal{N}_1(\mathbf{0}) \to \mathcal{N}_{\sqrt{2}}(\mathbf{0})\,.$$

Neighbourhood motion maps and non-injectivity

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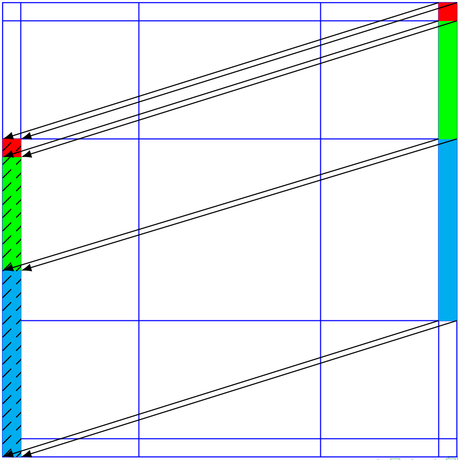
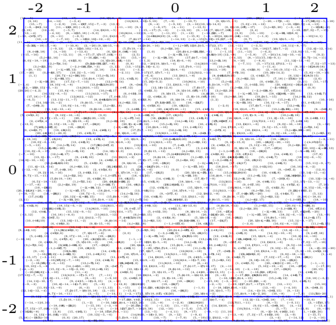


Bijectivity

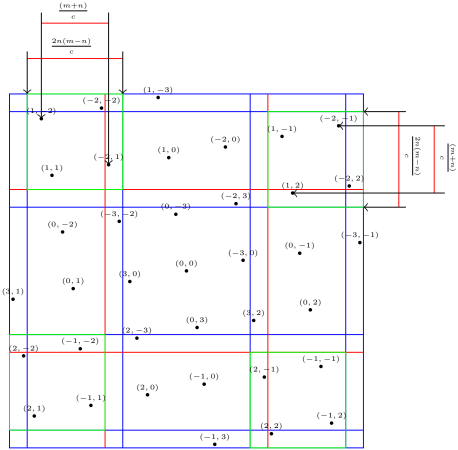
Which digitized rigid transformations are bijective? Case 1: Irrational rotations followed by translations.



Answer: No

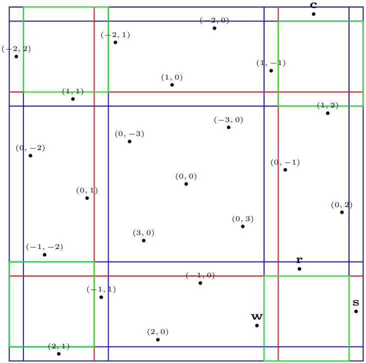
Bijectivity

Which digitized rigid transformations are bijective? Case 2: Rational rotations followed by translations?



Answer: No in general

Which digitized rigid transformations are bijective? Case 3: Rational rotations defined by Pythagorean twins followed by translations?



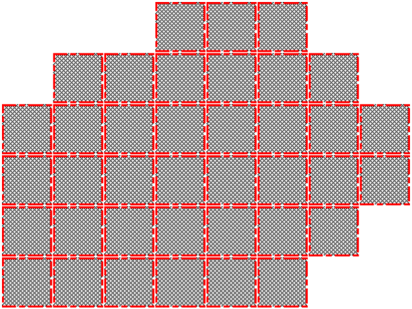
Answer: Yes

Bijectivity

Proposition

A digitized rigid transformation is bijective if it is a composition of a rotation by an angle defined by a twin Pythagorean triple (a, b, c) *such that* $b = c - 1$*, a translation* $\mathbf{t}$ *such that* $\mathbf{t} - \mathcal{D}(\mathbf{t}) \in \left(-\frac{1}{2c}, \frac{1}{2c}\right)^2$.

Quasi-bijective digitized rigid motions



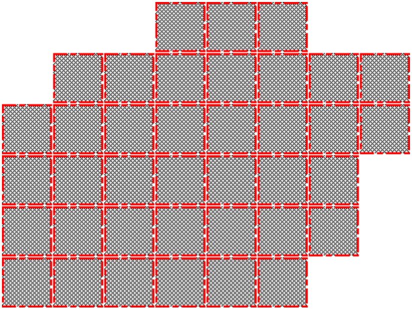
Is U bijective when restricted to a finite subset of $\mathbb{Z}^2$?

Quasi-bijective digitized rigid motions

Data: A finite set $S \subset \mathbb{Z}^2$, a digitized rigid transformation U.
Result: The subset $\bar{B} \subseteq S$ whose points are not bijective under U.

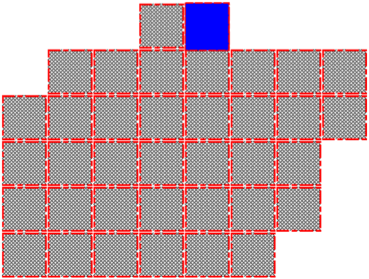
$\bar{B} = \emptyset$;
foreach $\mathbf{p} \in S$ **do**
    foreach $* \in \{\uparrow, \rightarrow, \downarrow, \leftarrow\}$ **do**
        if $\mathbf{p} + \mathbf{d}_* \in S$ *and* $\rho(\mathbf{p}) \in f_*^2$ **then**
            $\bar{B} \leftarrow \bar{B} \cup \{\mathbf{p}\}$;

Quasi-bijective digitized rigid motions



OK

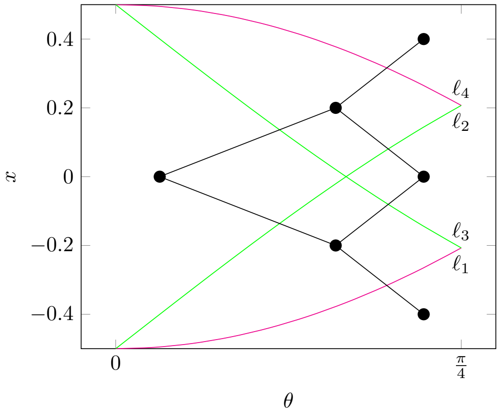
Quasi-bijective digitized rigid motions

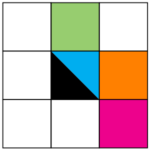


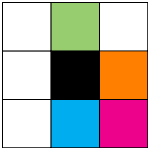
Non-injective

Neighbourhood motion map graph

Neighbourhood motion map graph

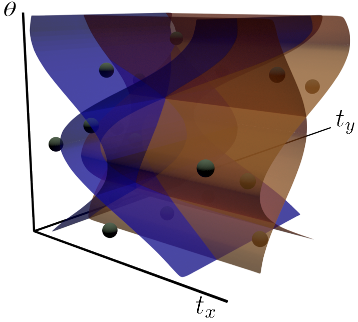






Neighbourhood motion map graph

Link with previously proposed by Ngo *et al.* '13, discrete rigid transformation graph.



$$t_x + u\cos\theta - v\sin\theta = k_x + \frac{1}{2}$$
$$t_y + u\sin\theta + v\cos\theta = k_y + \frac{1}{2}$$

Conclusions and perspectives

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Conclusions

- Extension of the previous result obtained by Nouvel and Rémila
- Characterization of bijective digitized rigid transformations on $\mathbb{Z}^2$
- Algorithm for quasi-bijective digitized rigid transformation on $\mathbb{Z}^2$
- We have shown that under some assumption our framework is equivalent to one proposed by Ngo *et al.*

Perspectives

Recently, we have been working on the extension of the presented framework to $\mathbb{Z}^3$.

Thank you for your attention!