Bijective rigid motions of the 2D Cartesian grid

K. Pluta[1,2], P. Romon[2], Y. Kenmochi[1], N. Passat[3]

[1] Université Paris-Est, LIGM
[2] Université Paris-Est, LAMA
[3] Université de Reims, CReSTIC

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Agenda

Introduction to digitized rigid motions

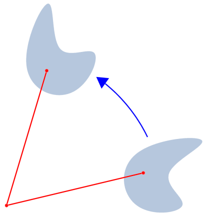
Rigid motions in $\mathbb{R}^2$

$$\left|\begin{array}{lll}\mathcal{U}: \mathbb{R}^2 & \to & \mathbb{R}^2 \\ \mathbf{x} & \mapsto & \mathbf{R}\mathbf{x} + \mathbf{t}\end{array}\right.$$

$\mathbf{x}$ – point

$\mathbf{R}$ – rotation matrix

$\mathbf{t}$ – translation vector



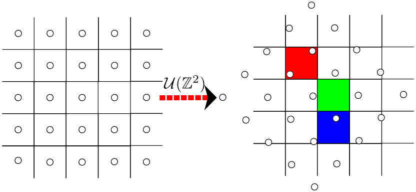
Properties:

- distance and angle preserving maps
- bijective
- inverse: $\mathcal{T} = \mathcal{U}^{-1}$ is also a rigid motion

Rigid motions in $\mathbb{Z}^2$

$$U = \mathcal{D} \circ \mathcal{U}_{|\mathbb{Z}^2}$$

where $\mathcal{D}$ is a digitization e.g. rounding function



Properties:

- in general, does not preserve distances and angles
- not always injective
- not always surjective $U(\mathbb{Z}^2) \subsetneq \mathbb{Z}^2$

Related work

Digitized rotations

- Bertrand Nouvel and Eric Rémila. Characterization of Bijective Discretized Rotations.
- Bertrand Nouvel and Eric Rémila. Configurations induced by discrete rotations: Periodicity and quasi-periodicity properties.
- Tristan Roussillon and David Cœurjolly. Characterization of bijective discretized rotations by Gaussian integers.

Digitized rigid motions

- Phuc Ngo, Yukiko Kenmochi, Nicolas Passat, and Hugues Talbot.

  Combinatorial structure of rigid transformations in 2D digital images.

Contributions

- Local configurations induced by digitized rigid motions in 2D
- Bijective digitized rigid motions on $\mathbb{Z}^2$ and its subsets.

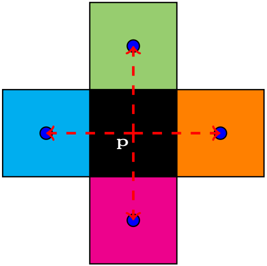
Neighbourhood motion maps

Neighbourhood

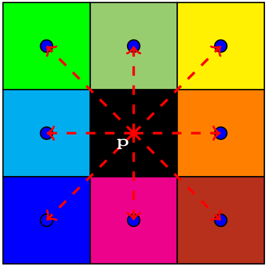
Definition

The *neighbourhood of* $\mathbf{p} \in \mathbb{Z}^2$ *(of squared radius* $r \in \mathbb{R}_+$*)*

$$\mathcal{N}_r(\mathbf{p}) = \{\mathbf{p} + \mathbf{d} \in \mathbb{Z}^2 \mid \|\mathbf{d}\|^2 \leq r\}.$$



$\mathcal{N}_1(\mathbf{p})$ a.k.a.
4-neighbourhood



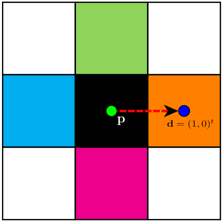
$\mathcal{N}_2(\mathbf{p})$ a.k.a.
8-neighbourhood

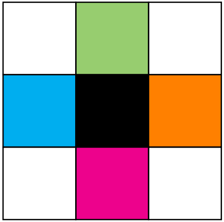
Neighbourhood motion maps

Definition

The *neighbourhood motion map* of $\mathbf{p} \in \mathbb{Z}^2$ with respect to U and $r \in \mathbb{R}_+$ is the function

$$\left| \begin{array}{cccccc} \mathcal{G}_r^U & : & \mathcal{N}_r(\mathbf{0}) & \to & \mathcal{N}_{r'}(\mathbf{0}) \\ & & \mathbf{d} = (u, v)^t & \mapsto & U(\mathbf{p} + \mathbf{d}) - U(\mathbf{p}). \end{array} \right.$$





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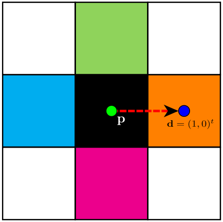

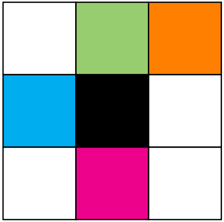

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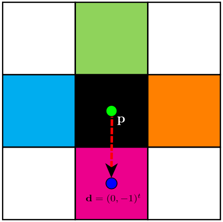
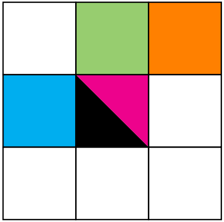
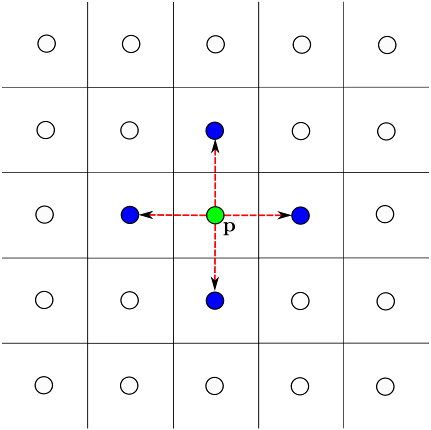
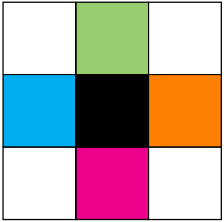




Remainder map step-by-step





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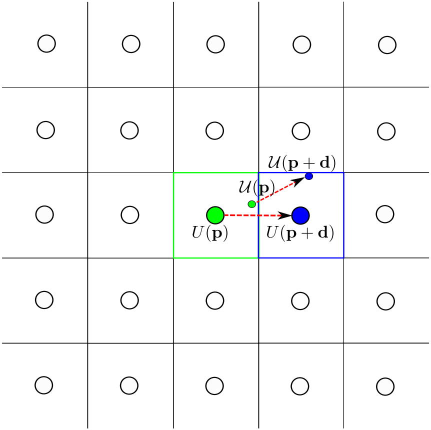

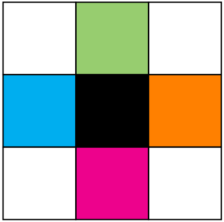

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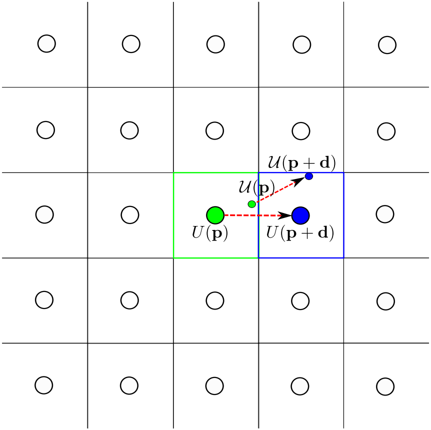
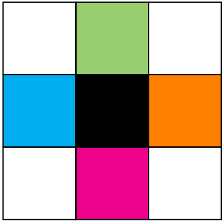
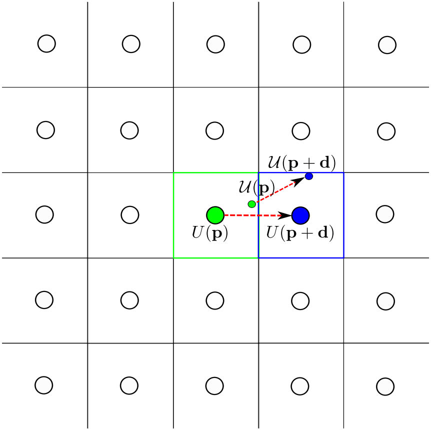
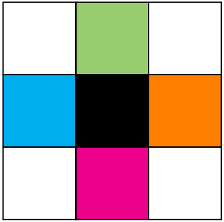




$$\mathcal{U}(\mathbf{p}+\mathbf{d}) = \mathbf{Rd} + \mathcal{U}(\mathbf{p})$$

Remainder map step-by-step

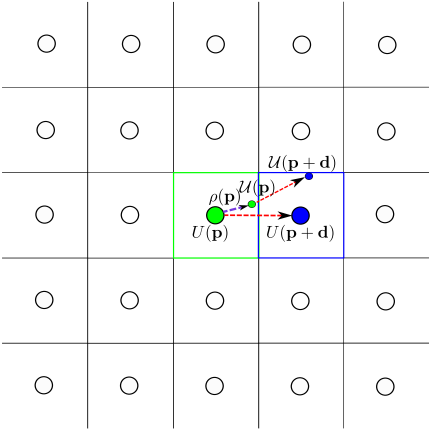


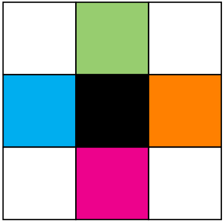


Without loss of generality $U(\mathbf{p})$ is an origin then

$$\mathcal{U}(\mathbf{d}) = \mathbf{Rd} + \mathcal{U}(\mathbf{p}) - U(\mathbf{p}).$$

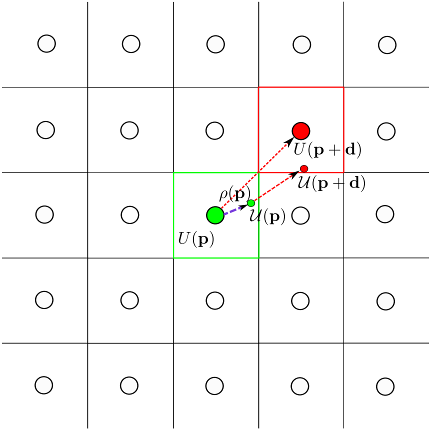
Remainder map step-by-step

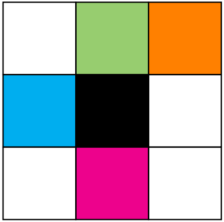




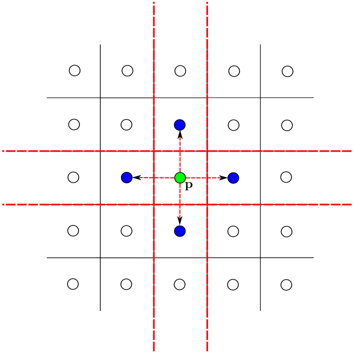
Remainder map defined as $\rho(\mathbf{p}) = \mathcal{U}(\mathbf{p}) - U(\mathbf{p}) \in \left[-\frac{1}{2}, \frac{1}{2}\right)^2$, where the range $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$ is called the remainder range.

Remainder map step-by-step





Remainder map and critical rigid motions



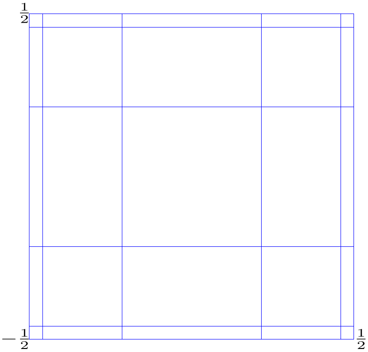
More generally,

$$x + u\cos\theta - v\sin\theta = k_x + \frac{1}{2}$$

$$y + u\sin\theta + v\cos\theta = k_y + \frac{1}{2}$$

where $\rho(\mathbf{p}) = (x, y)^t, \mathbf{d} = (u, v)^t$ and $k_x, k_y \in \mathbb{Z}$.

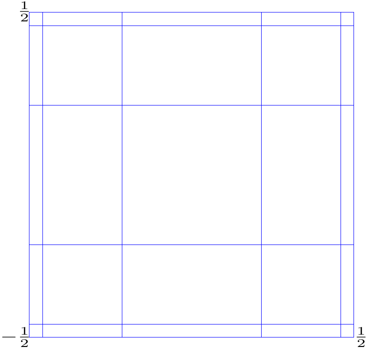
Critical lines in remainder range



$$x + u\cos\theta - v\sin\theta = k_x + \frac{1}{2}$$
$$y + u\sin\theta + v\cos\theta = k_y + \frac{1}{2}$$

For $\mathcal{N}_1$, $\theta \in \left[0, \frac{\pi}{4}\right]$ and $k_x, k_y \in \{-1, 0\}$

Frames



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Each region bounded by critical lines is called a frame.

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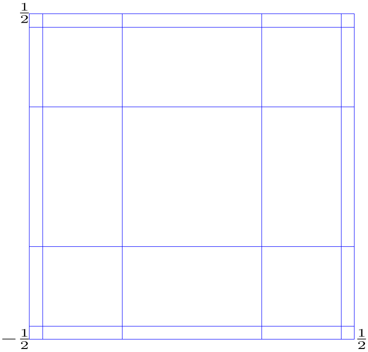


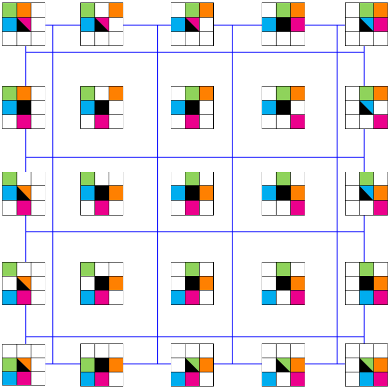
Each region bounded by critical lines is called a frame.

For $\mathcal{N}_1$, $\theta \in \left[0, \frac{\pi}{4}\right]$ and $k_x, k_y \in \{-1, 0\}$

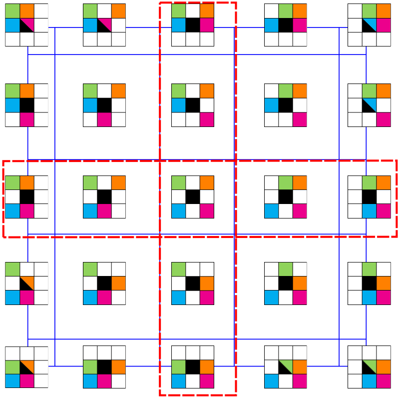
Proposition

For any $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^2$, $\mathcal{G}_r^U(\mathbf{p}) = \mathcal{G}_r^U(\mathbf{q})$ iff $\rho(\mathbf{p})$ and $\rho(\mathbf{q})$ are in the same frame.

Frames and neighbourhood motion maps



$0 < \theta < \frac{\pi}{6}$

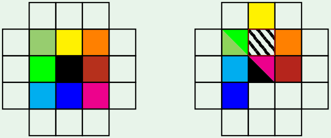


$\frac{\pi}{6} < \theta < \frac{\pi}{4}$

Non-injective and non-surjective rigid motions

Non-surjectivity

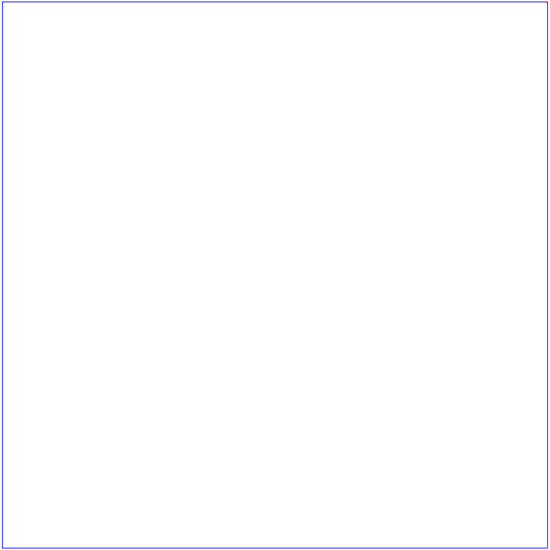
Example of non-surjectivity



Neighbourhood motion map which corresponds to the frame
$f_{\uparrow}^{0} = (1/2 - \cos\theta, \sin\theta - 1/2) \times (3/2 - \cos\theta - \sin\theta, 1/2)$.

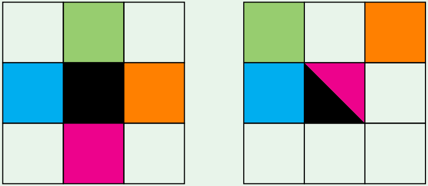
In particular, there exists $\mathbf{q}' = U(\mathbf{p}) + (0,1)^t$ with no preimages.

Non-surjectivity



Non-injectivity

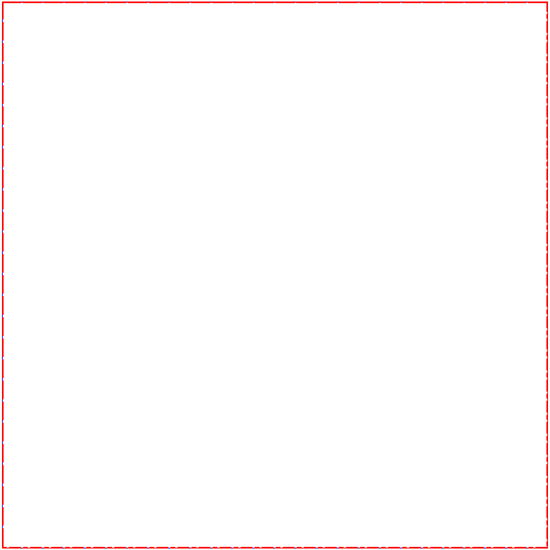
Example of non-injectivity



Neighbourhood motion maps which corresponds to a non-injective frame $f_{\downarrow}^{2} = (-1/2, 1/2 - \sin\theta) \times (\cos\theta - 1/2, 1/2)$.

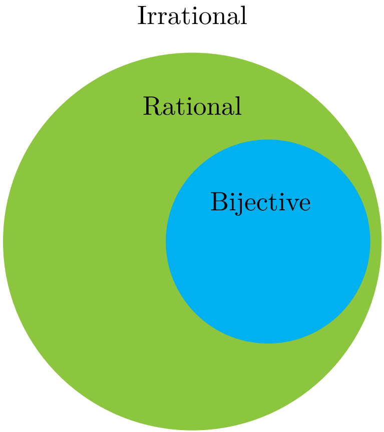
A point $\mathbf{q}'$ will have two preimages $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^2$, such that $|\mathbf{p} - \mathbf{q}| = 1$, and $\mathbf{q}' = U(\mathbf{p}) = U(\mathbf{q})$.

Non-injectivity

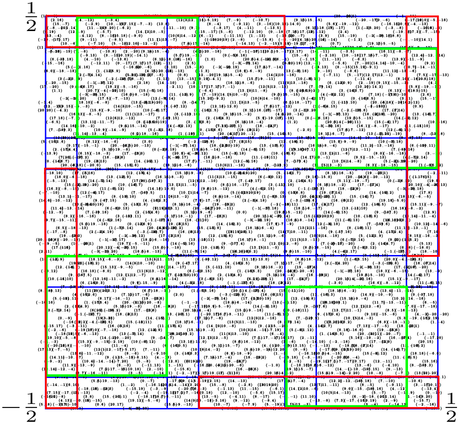


Which digitized rigid motions are bijective?

Sets of rigid motions



Irrational rigid motions

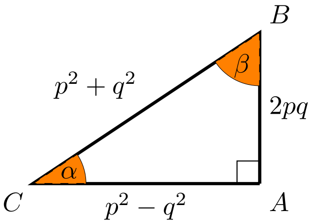


The image of $\mathbb{Z}^2$ by ρ is dense.

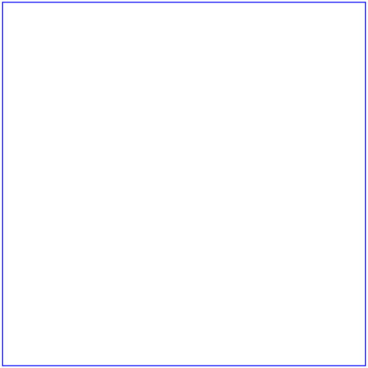
Rational rigid motions

Definition

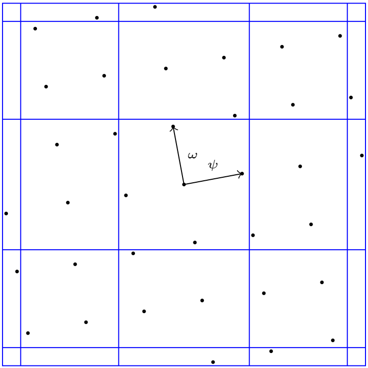
Rational rotations are defined by *primitive Pythagorean triples* $(p^2 - q^2, 2pq, p^2 + q^2)$ with $p, q \in \mathbb{Z}, p > q$ and $p - q$ is odd, such that they are pairwise coprime. Then $\cos\alpha = \frac{p^2-q^2}{p^2+q^2}$ and $\sin\alpha = \frac{2pq}{p^2+q^2}$.



Rational rigid motions



Rational rigid motions



Property (Nouvel and Remila 06)

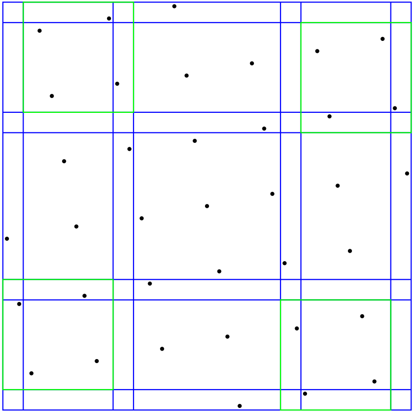
The image of $\mathbb{Z}^2$ by ρ corresponds to a cyclic group $\mathcal{G}$ which is generated by $\psi = \left(\frac{p}{p^2+q^2}, \frac{q}{p^2+q^2}\right)$ and $\omega = \left(-\frac{q}{p^2+q^2}, \frac{p}{p^2+q^2}\right)$.

Rational rotations followed by translations

Theorem (Nouvel and Rémila '06)

Digitized rotation is one-to-one iff is onto. Thus bijectivity is equivalent to surjectivity or injectivity.

Rational rotations followed by translations



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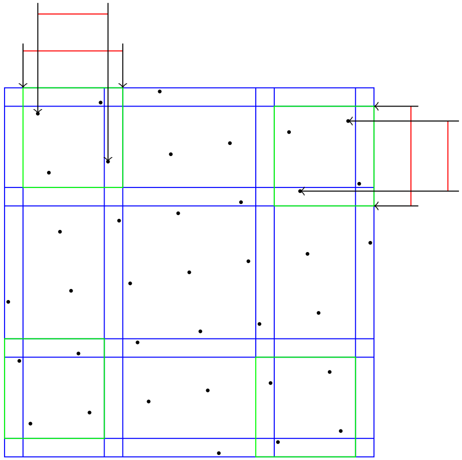

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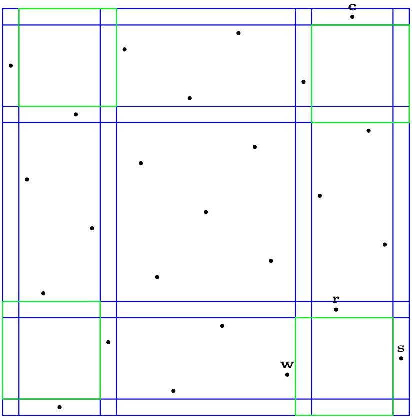
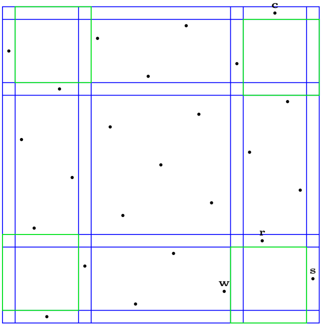
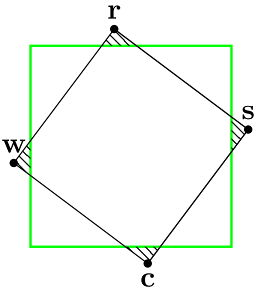


Theorem (Nouvel and Remila 06, Roussillon and Cœurjolly 14)

Digitized rotation is bijective if and only if its cosine and sine are defined by so-called Pythagorean twin triple such that $p = q + 1$.

Rational rotations followed by translations

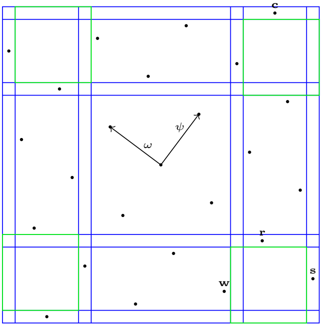


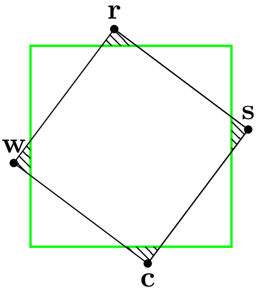


Proposition

A digitized rigid motion is bijective if and only if it is composed of a rotation by an angle defined by a twin Pythagorean triple and a translation $\mathbf{t} = \mathbf{t}' + \mathbb{Z}\boldsymbol{\psi} + \mathbb{Z}\boldsymbol{\omega}$, *where* $\mathbf{t}' \in \left(-\frac{1}{2(p^2+q^2)}, \frac{1}{2(p^2+q^2)}\right)^2$.

Rational rotations followed by translations





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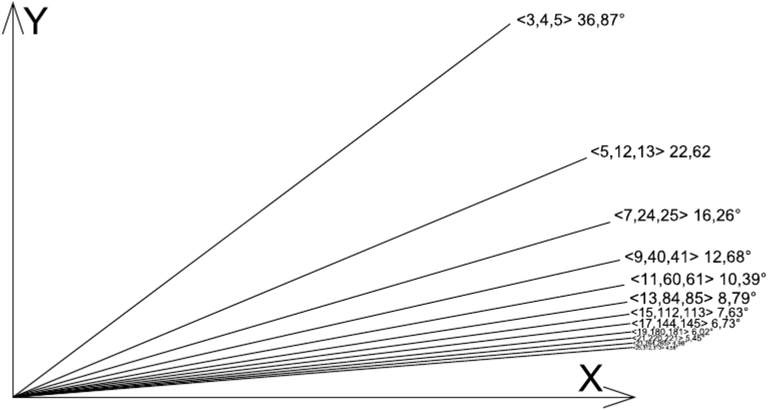
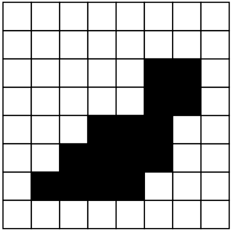
Is U bijective when restricted to a finite subset of $\mathbb{Z}^2$?

Bijective digitized rigid motions are not dense

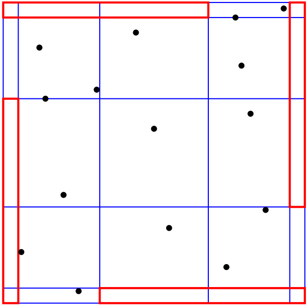


Y.Thibault 09'

Forward algorithm

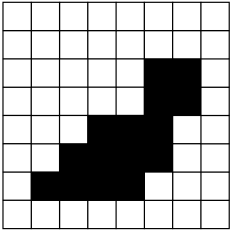


a Input: $S \subset \mathbb{Z}^2$



b $\rho(S)$

Forward algorithm



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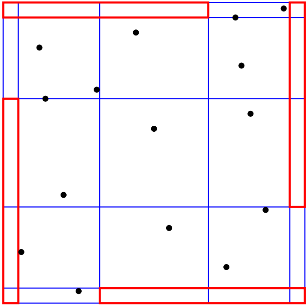
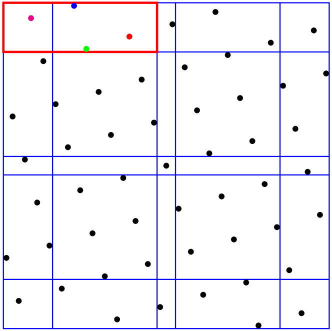


b $\rho(S)$

Complexity $\mathcal{O}(|S|)$

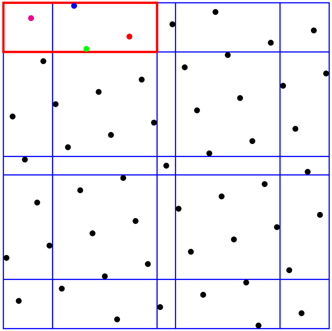
Backward algorithm

Step 1 – identify non-injective elements of $\mathcal{G} + \{\mathbf{t}\}$



Backward algorithm

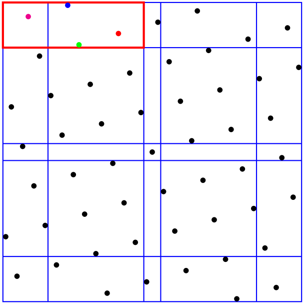
Step 1 – identify non-injective elements of $\mathcal{G} + \{\mathbf{t}\}$

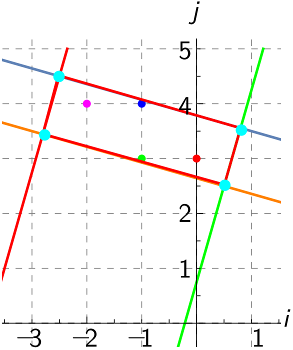


$$\left\{ (i,j) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} -\frac{1}{2} < \frac{p}{p^2+q^2} i - \frac{q}{p^2+q^2} j + \{t_x\} < \frac{1}{2} - \frac{2pq}{p^2+q^2}, \\ \frac{p^2-q^2}{p^2+q^2} - \frac{1}{2} < \frac{q}{p^2+q^2} i + \frac{p}{p^2+q^2} j + \{t_y\} < \frac{1}{2} \end{array} \right\}$$

Backward algorithm

Step 1 – identify non-injective elements of $\mathcal{G} + \{\mathbf{t}\}$

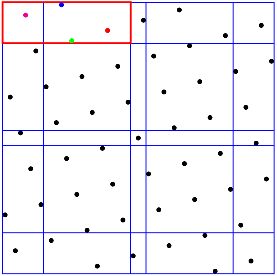


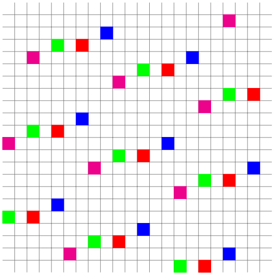


$$\left\{ (i,j) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} -\frac{1}{2} < \frac{p}{p^2+q^2} i - \frac{q}{p^2+q^2} j + \{t_x\} < \frac{1}{2} - \frac{2pq}{p^2+q^2}, \\ \frac{p^2-q^2}{p^2+q^2} - \frac{1}{2} < \frac{q}{p^2+q^2} i + \frac{p}{p^2+q^2} j + \{t_y\} < \frac{1}{2} \end{array} \right\}$$

Backward algorithm

Step 2 – find all the non-injective points



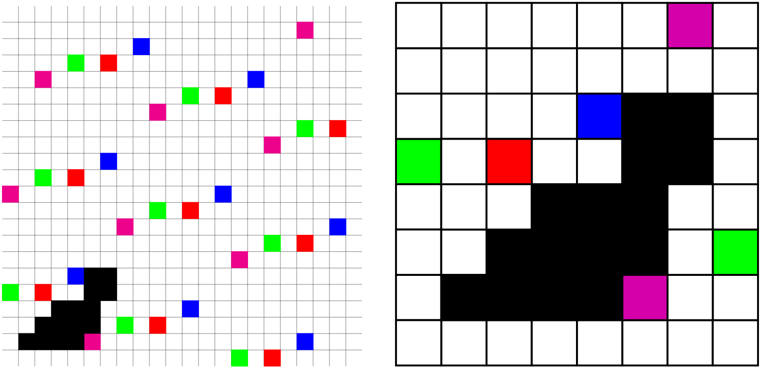


$$\mathbb{T}(i,j) = p\frac{\mu - v}{2}\begin{pmatrix} i \\ j \end{pmatrix} + \mathbb{Z}\begin{pmatrix} p^2 - q^2 \\ -2pq \end{pmatrix} + \mathbb{Z}\begin{pmatrix} (p^2+q^2)\sigma \\ (p^2+q^2)\tau \end{pmatrix}$$

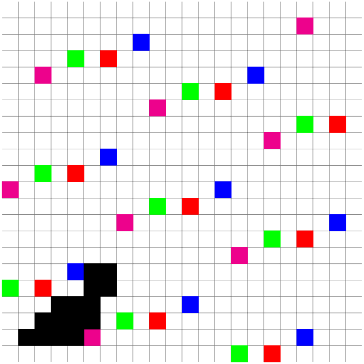
where μ, v and σ, τ are the Bézout coefficients satisfying $\mu p^2 + vq^2 = 1$, and $\sigma a + \tau b = 1$, respectively.

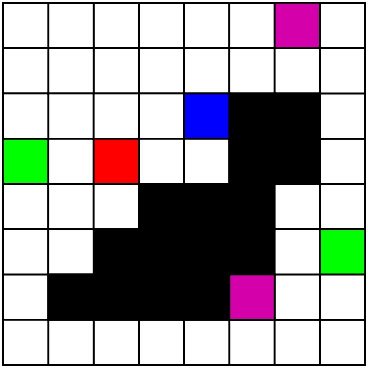
Backward algorithm

Step 3 – find all the non-injective points in a set



Step 3 – find all the non-injective points in a set





Complexity: $\underbrace{\mathcal{O}(q)}_{Step\ 1} + \underbrace{\mathcal{O}(\log\min(a, b))}_{Step\ 2} + \underbrace{\mathcal{O}(\sqrt{|S|})}_{Step\ 3}$

Conclusions and perspectives

Conclusions and perspectives

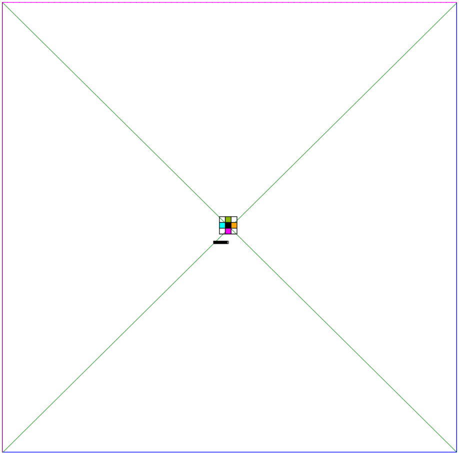
Conclusions

- Extension of the previous result obtained by Nouvel and Rémila
- Characterization of bijective digitized rigid motions on $\mathbb{Z}^2$
- Algorithms for bijective when restricted digitized rigid motions on a finite subset of $\mathbb{Z}^2$

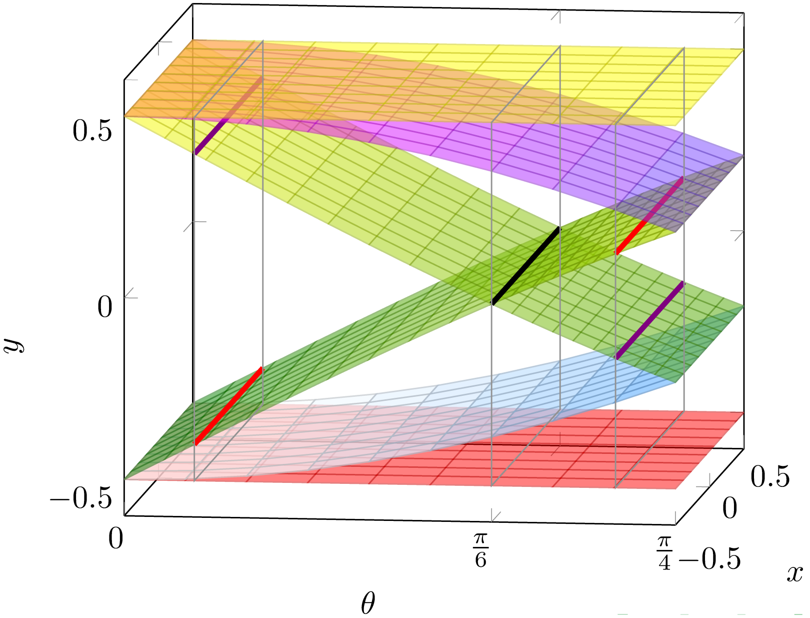
Perspectives

- We work on the extension of the presented framework to $\mathbb{Z}^3$.
- K. Pluta, P. Romon, Y. Kenmochi, and N. Passat. Bijectivity certification of 3D digitized rotations.
  In *CTIC*. Springer, Forthcoming 2016

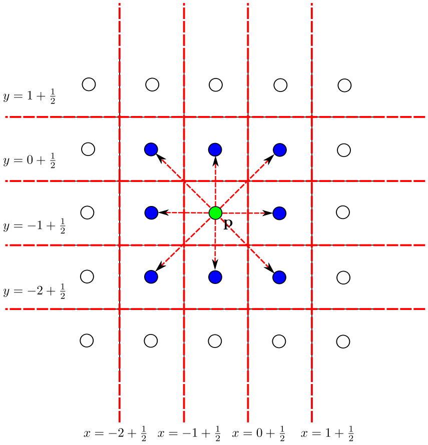
Thank you for your attention!



4-neighbourhood

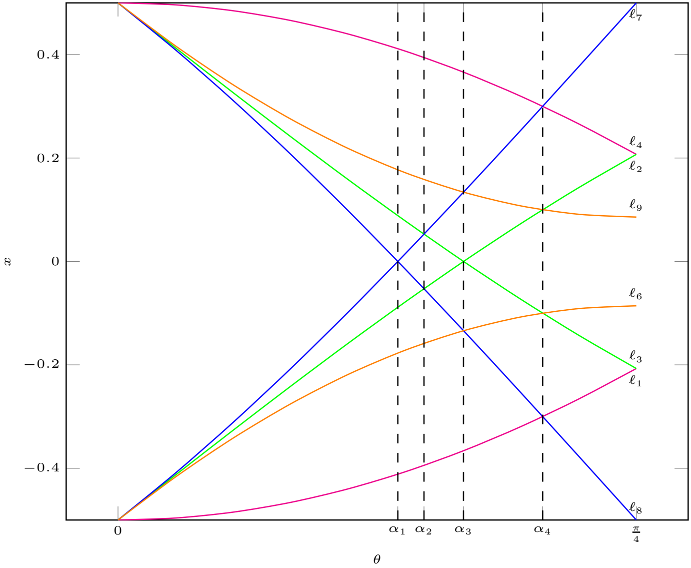


8-neighbourhood case: $\mathcal{G}_2^U$



For a given θ we have at most 81 frames.

8-neighbourhood case: $\mathcal{G}_2^U$



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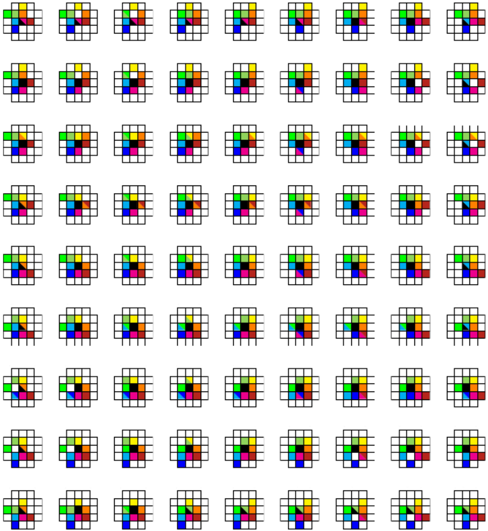

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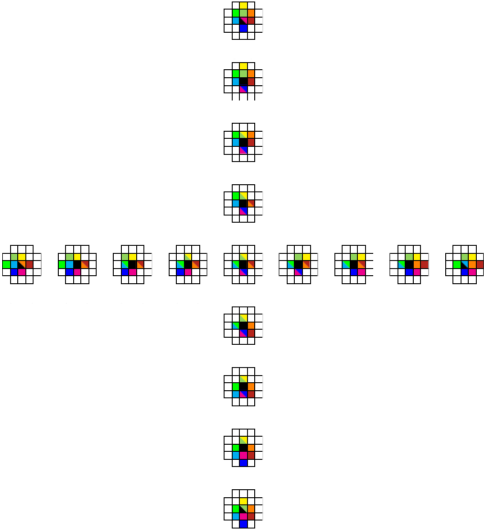

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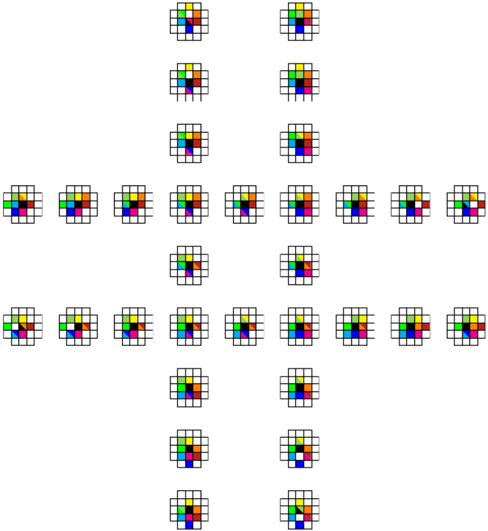
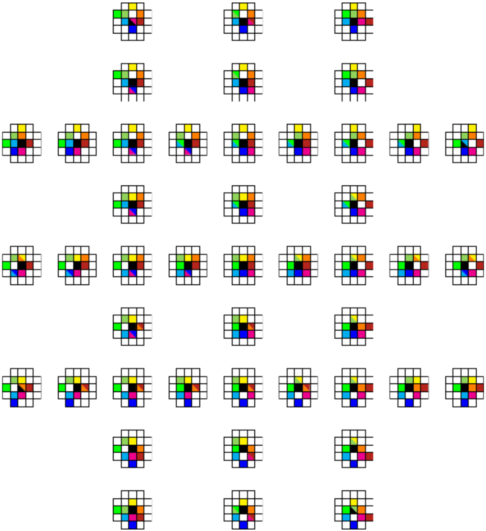


8-neighbourhood case: $\mathcal{G}_2^U$



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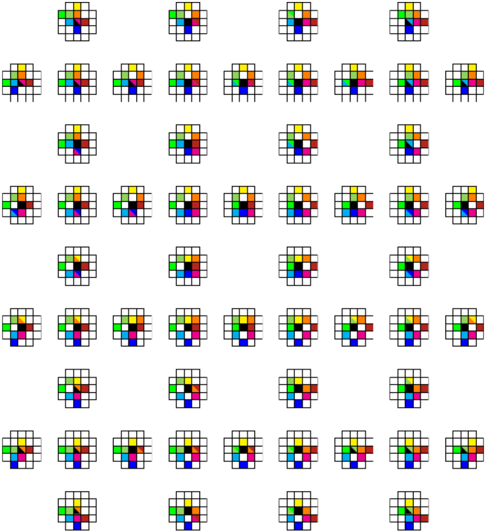

Four and eight neighbourhood motion maps in numbers

4-neighbourhood

In total 34 different neighbourhood motion maps.

8-neighbourhood

In total 231 different neighbourhood motion maps.