

Honeycomb Geometry

Rigid Motions on the
Hexagonal Grid

by Kacper Pluta, Pascal Romon,
Yukiko Kenmochi and Nicolas Passat



The figure comes from "Insects The Yearbook of Agriculture 1952" United States Dept. of Agriculture." Published by the US Government Printing Office. Deemed to be in the Public Domain under US Law.

Motivations

Digitized rigid motions defined on the square grid are burdened with an incompatibility between rotations and the geometry of the grid.

Agenda



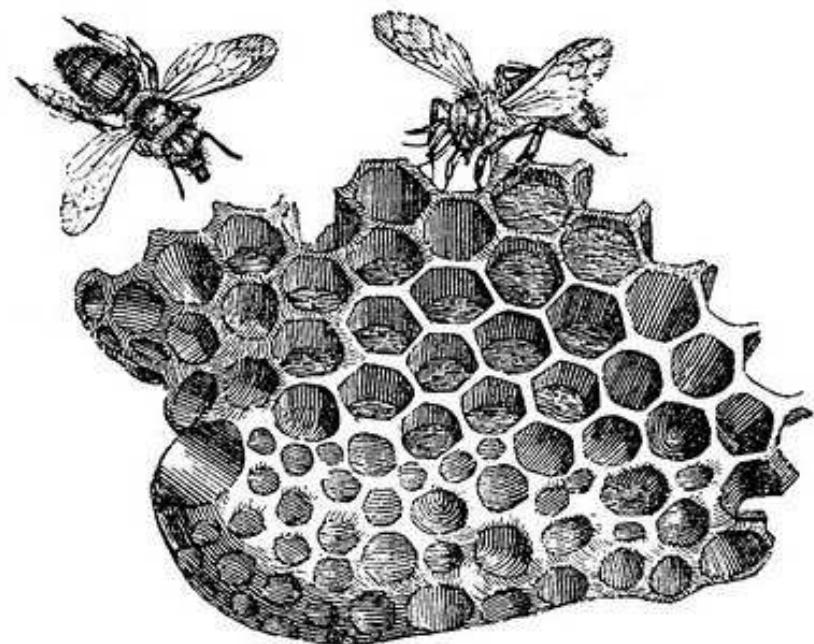
- Introduction to the Bees' Point of View
- Quick Introduction to Rigid Motions
- Neighborhood Motion Maps
- Contributions
- Conclusions & Perspectives



The beehive figure's source and author unknown (if you recognize it, please let me know). The image of the bee comes from <http://karenswhimsy.com/public-domain-images> (public domain)

Introduction to the Bees' Point of View

Or why bees are right



Pros and Cons

Square grid

- + Memory addressing
- + Sampling is easy to define
- Sampling is not optimal (ask bees)
- Neighbors are not equidistant
- Connectivity paradox

Hexagonal grid

- + Uniform connectivity
- + Equidistant neighbors
- + Sampling is optimal
- Memory addressing is not trivial
- Sampling is difficult to define

Pros and Cons

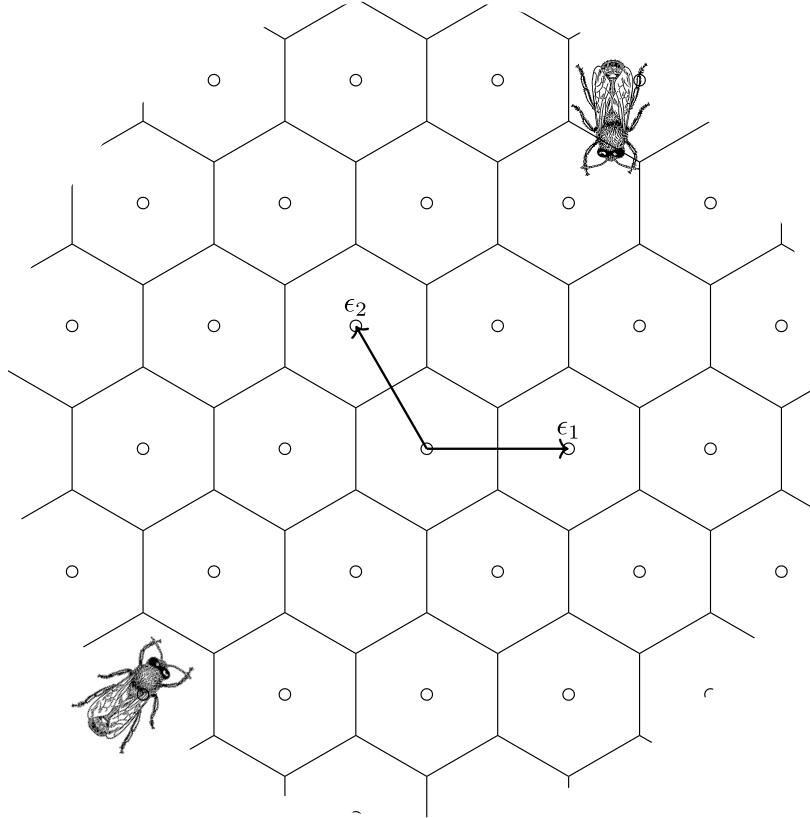
Square grid

- + Memory addressing
- + Sampling is easy to define
- Sampling is not optimal (ask bees)
- Neighbors are not equidistant
- ~ Connectivity paradox

Hexagonal grid

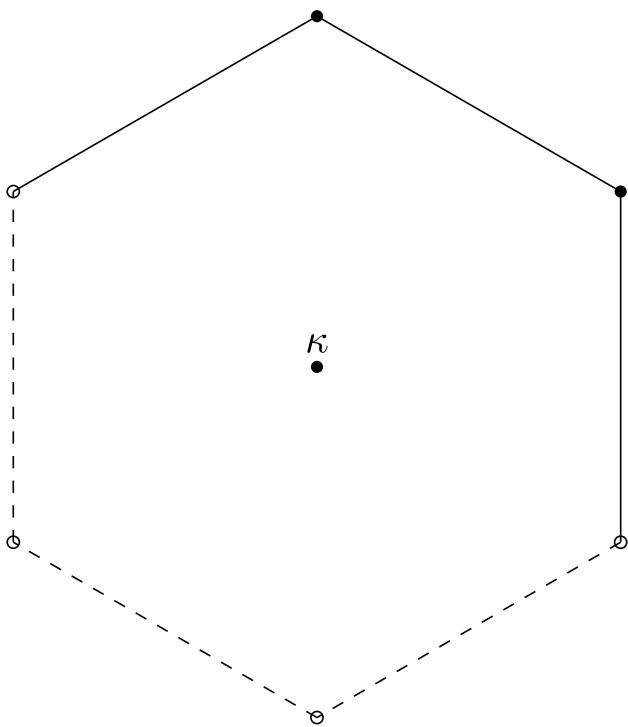
- + Uniform connectivity
- + Equidistant neighbors
- + Sampling is optimal
- ~ Memory addressing is not trivial
- ~ Sampling is difficult to define

Hexagonal Grid



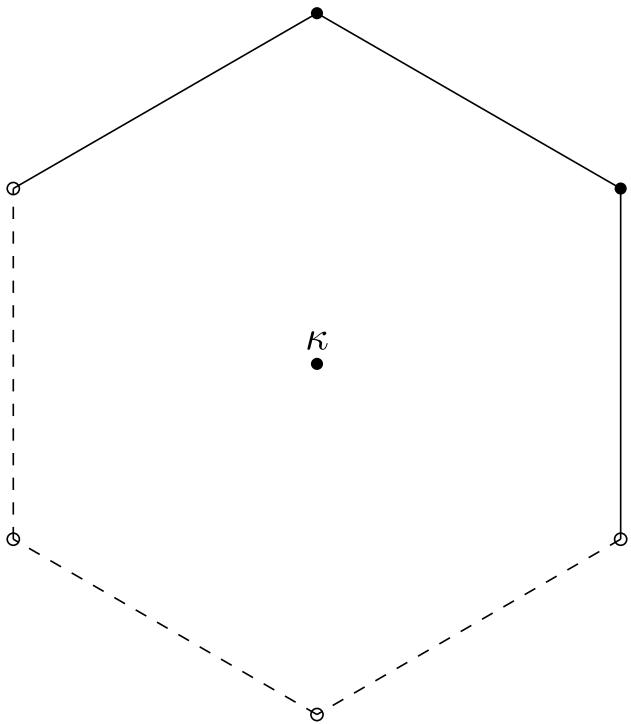
The hexagonal lattice: $\Lambda = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ and the hexagonal grid \mathcal{H}

Digitization Model



The digitization cell of κ denoted by $\mathcal{C}(\kappa)$.

Digitization Model



The digitization operator is defined as

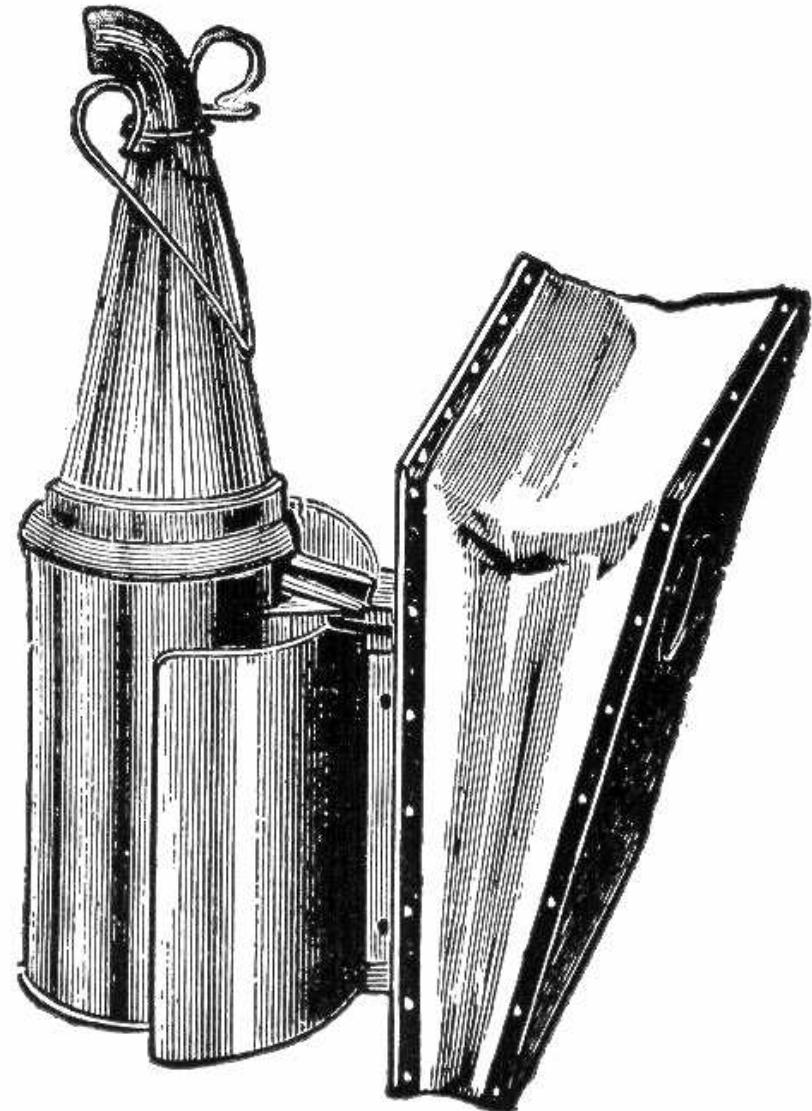
$$\mathcal{D} : \mathbb{R}^2 \rightarrow \Lambda$$

such that

$$\forall \mathbf{x} \in \mathbb{R}^2, \exists! \mathcal{D}(\mathbf{x}) \in \Lambda \text{ and } \mathbf{x} \in \mathcal{C}(\mathcal{D}(\mathbf{x})).$$

Quick Lesson on Rigid Motions

Or how to become a beekeeper. Part I - Equipment



Rigid Motions on \mathbb{R}^2

Properties

$$\begin{array}{lcl} \mathcal{U} : \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ \mathbf{x} & \mapsto & \mathbf{Rx} + \mathbf{t} \end{array} \quad \begin{array}{l} \bullet \text{ Isometry} \\ \bullet \text{ Bijective} \end{array}$$

R - rotation matrix

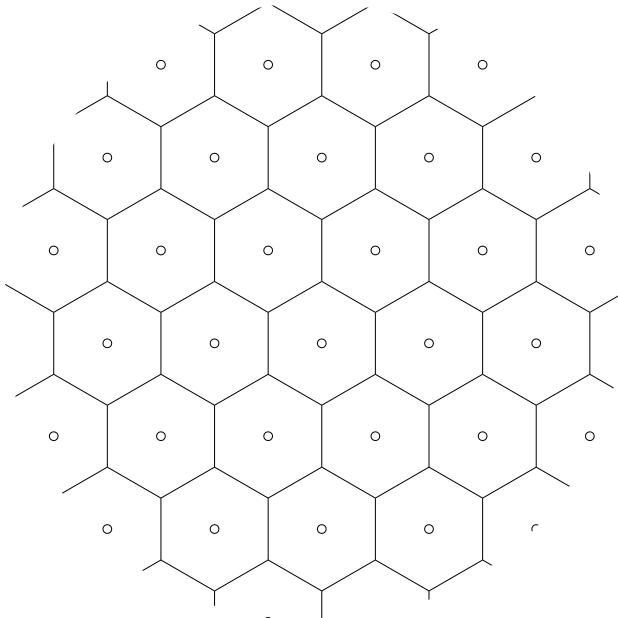
t - translation vector

Rigid Motions on Λ

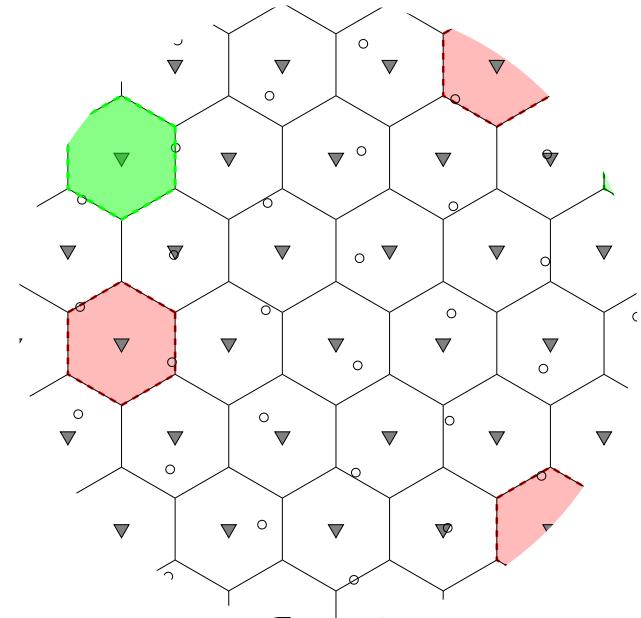
$$U = \mathcal{D} \circ \mathcal{U}_{|\Lambda}$$

Properties

- Do not preserve distances
- Non-injective
- Non-surjective



$\mathcal{U}(\Lambda)$



Related Studies

- Nouvel, B., Rémy, E.: On colorations induced by discrete rotations. In: DGCI, Proceedings. *Volume 2886 of Lecture Notes in Computer Science.*, Springer (2003) 174–183
- Pluta, K., Romon, P., Kenmochi, Y., Passat, N.: Bijective digitized rigid motions on subsets of the plane. *Journal of Mathematical Imaging and Vision* (2017)

Contributions in Short

Pure extracted honey

- Extension of the former framework to the hexagonal grid
- Comparison of the loss of information between the hexagonal and square grids
- Complete set of neighborhood motion maps
- Source code of a tool to study digitized rigid motions on the hexagonal grid



Neighborhood Motion Maps

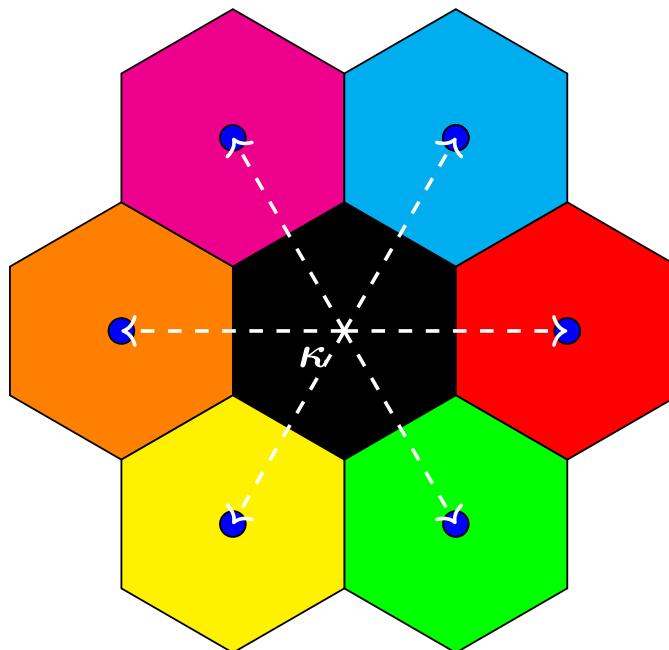
Or a manual of instructions in
apiculture



Neighborhood

The *neighbourhood* of $\kappa \in \Lambda$ (of squared radius $r \in \mathbb{R}_+$):

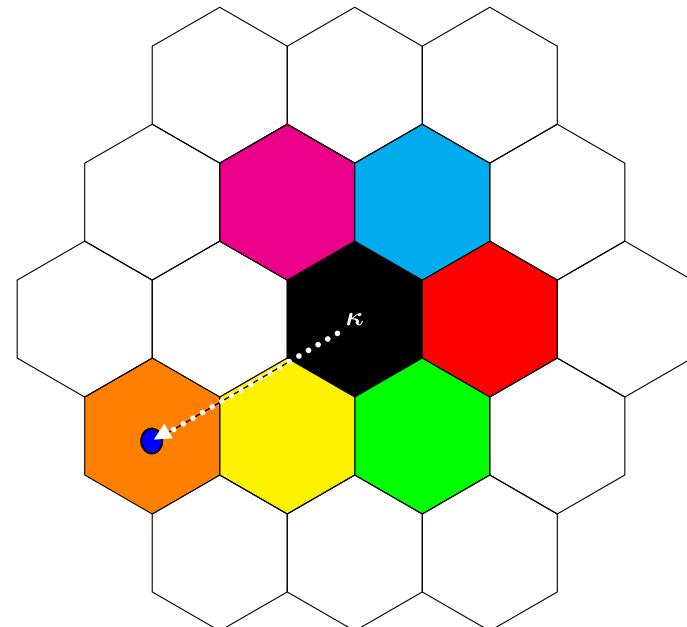
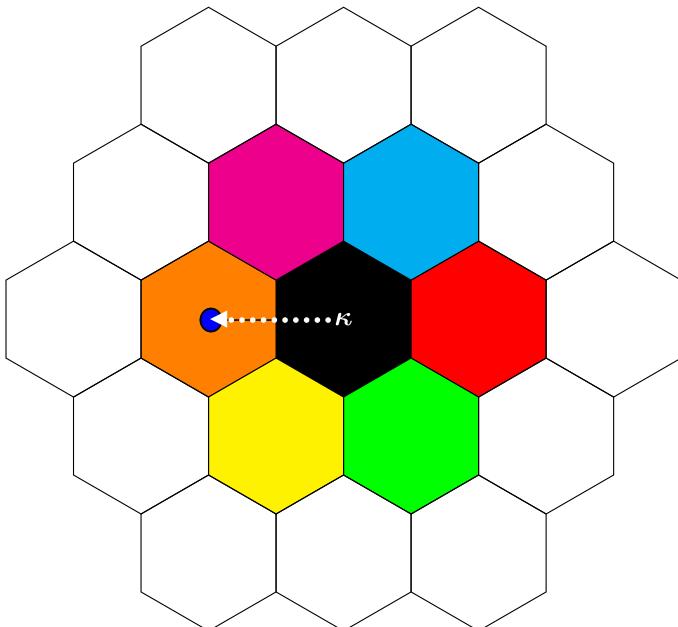
$$\mathcal{N}_r(\kappa) = \{\kappa + \delta \in \Lambda \mid \|\delta\|^2 \leq r\}$$



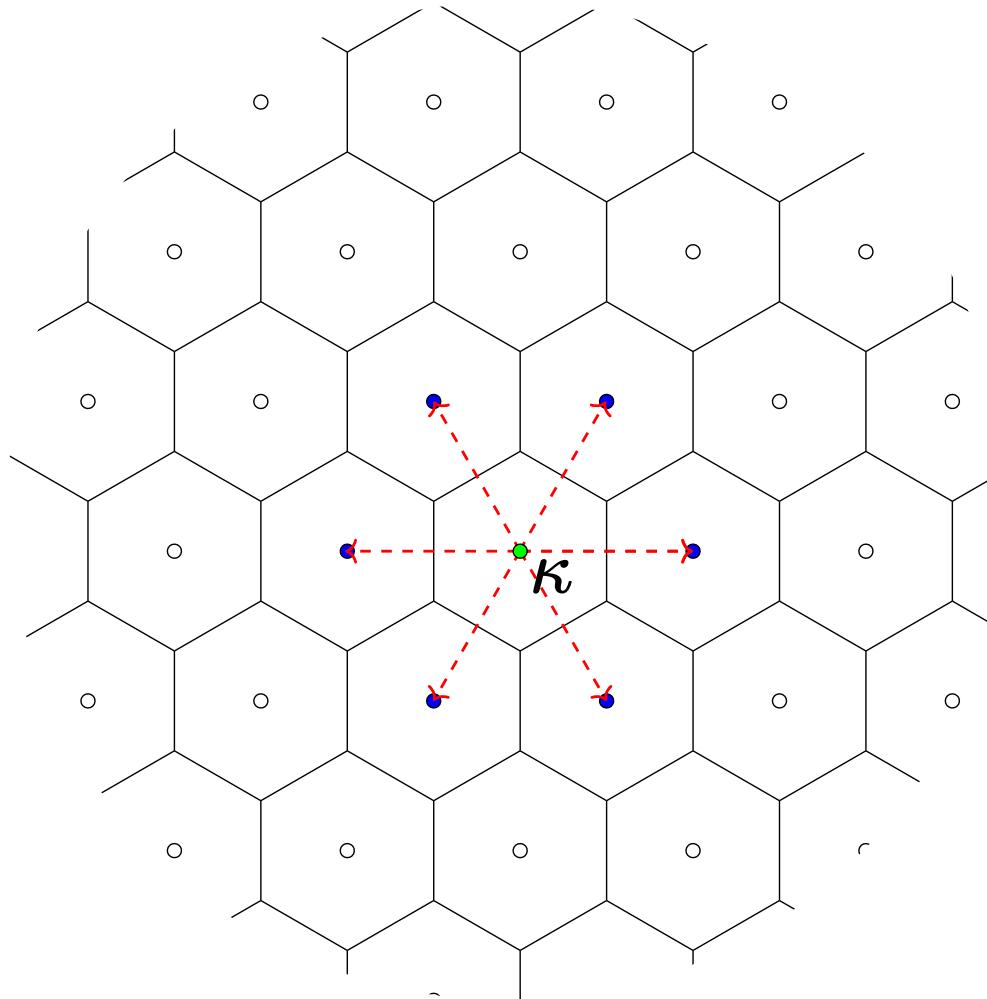
Neighborhood Motion Maps

The *neighbourhood motion map* of $\kappa \in \Lambda$ for given a rigid motion U and $r \in \mathbb{R}_+$

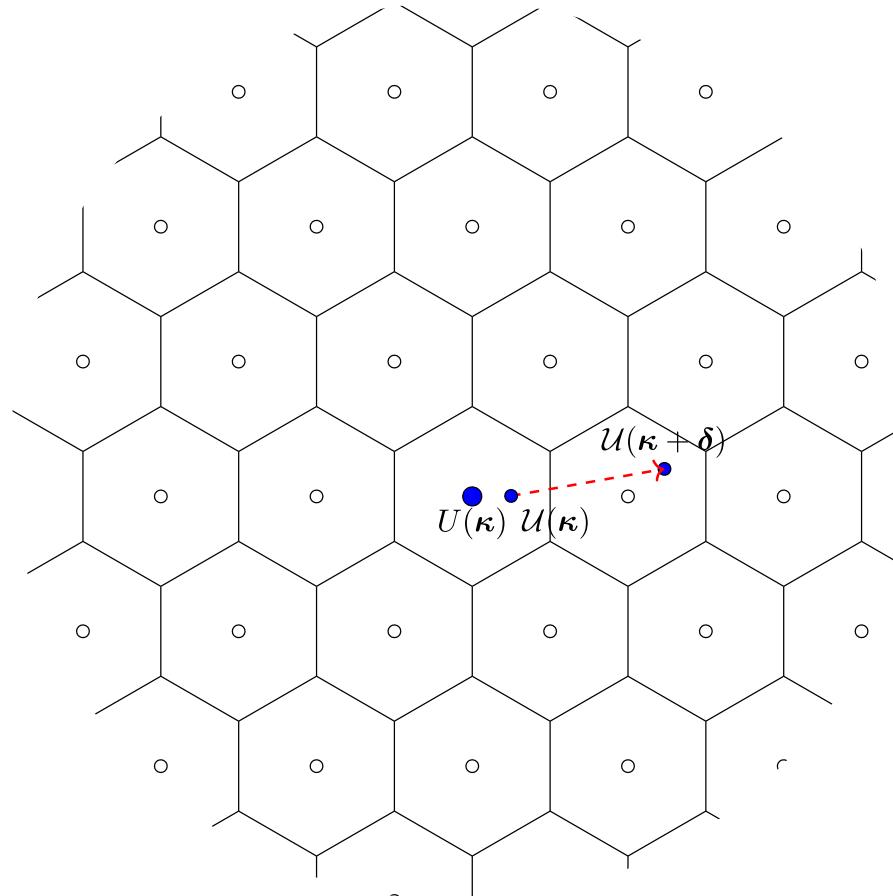
$$\begin{array}{rccc} \mathcal{G}_r^U & : & \mathcal{N}_r(0) & \rightarrow \mathcal{N}_{r'}(0) \\ & & \delta & \mapsto U(\kappa + \delta) - U(\kappa). \end{array}$$



Remainder Map step-by-step

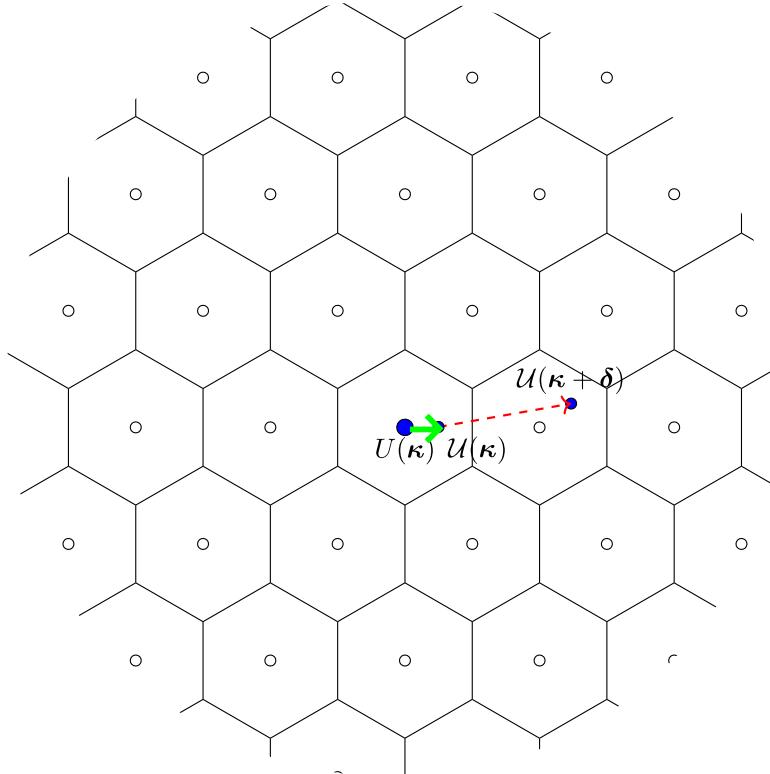


Remainder Map step-by-step



$$U(\kappa + \delta) = \mathbf{R}\delta + U(\kappa)$$

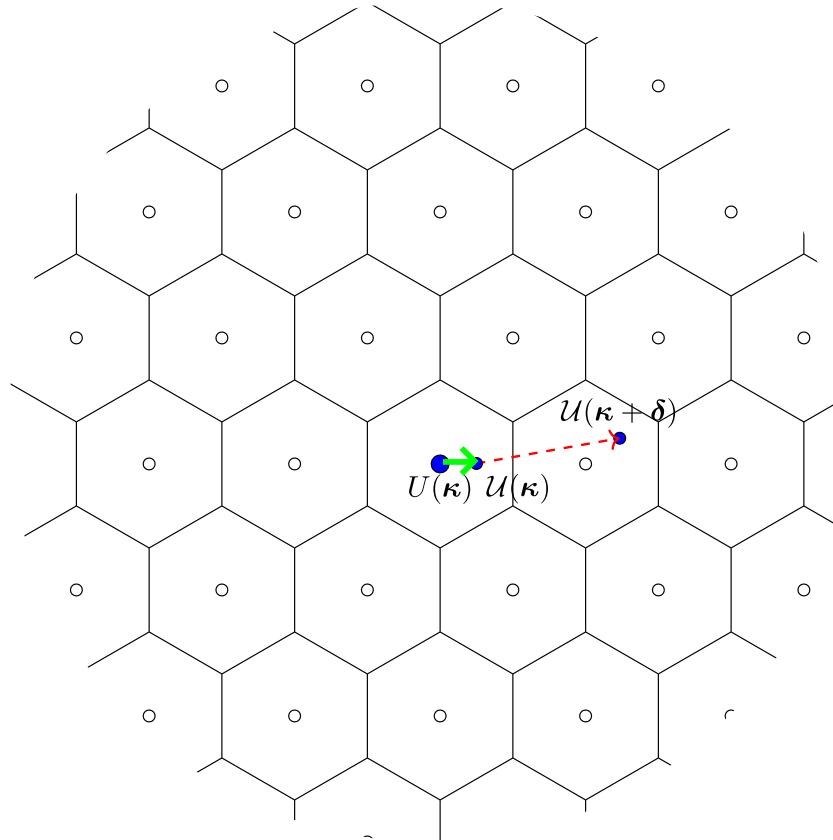
Remainder Map step-by-step



Without loss of generality, $U(\kappa)$ is the origin, and then

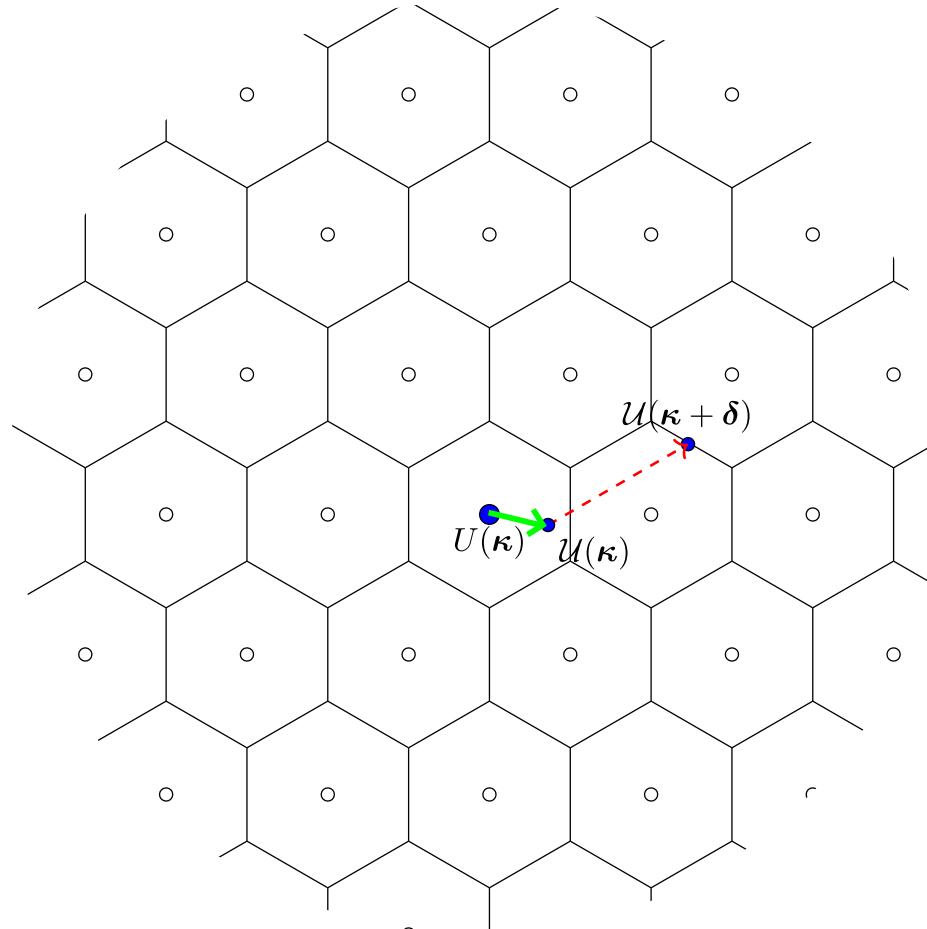
$$\mathcal{U}(\delta) = \mathbf{R}\delta + \mathcal{U}(\kappa) - U(\kappa)$$

Remainder Map step-by-step

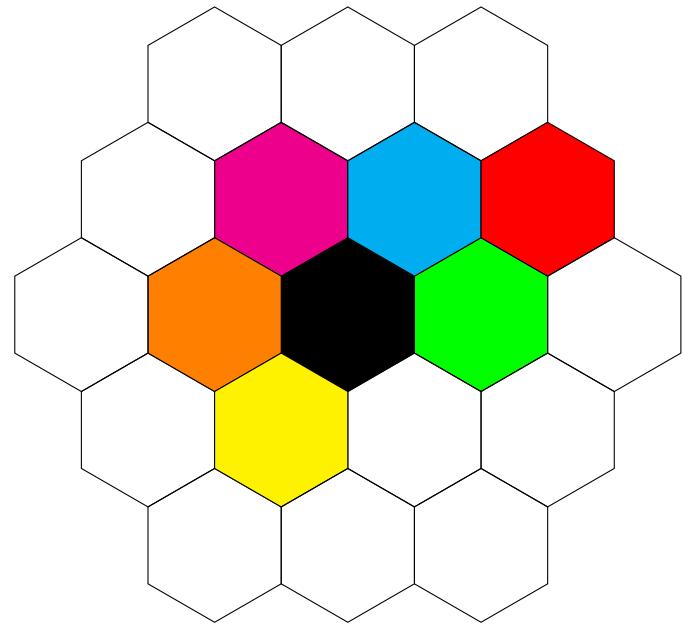
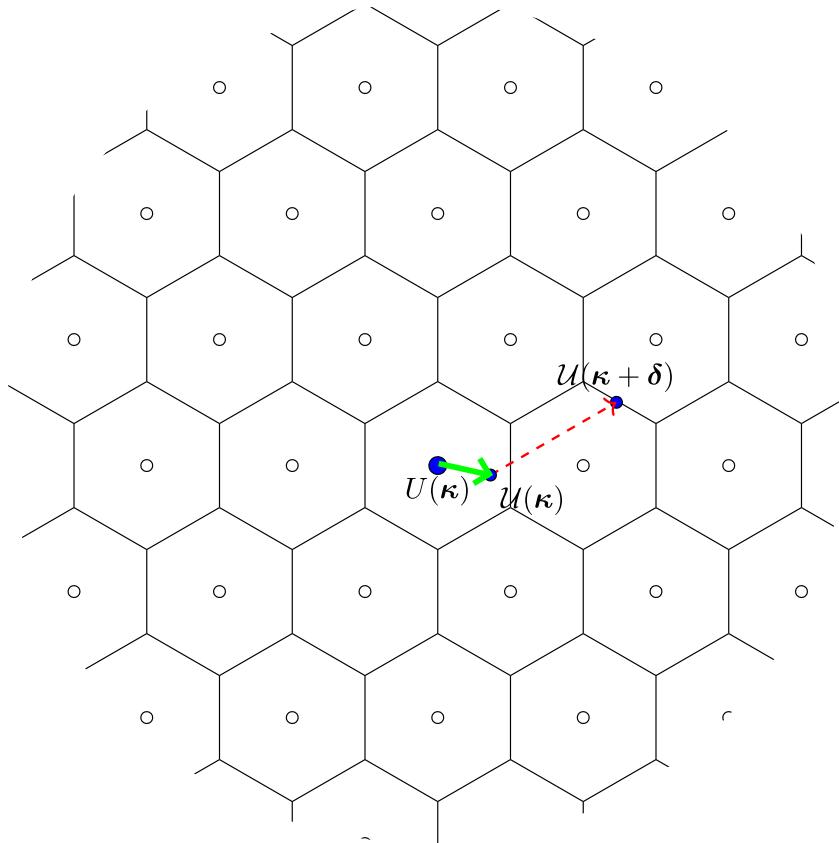


The remainder map defined as $\mathcal{F}(\kappa) = U(\kappa) - U(\kappa + \delta) \in \mathcal{C}(\mathbf{0})$
where the range $\mathcal{C}(\mathbf{0})$ is called the remainder range.

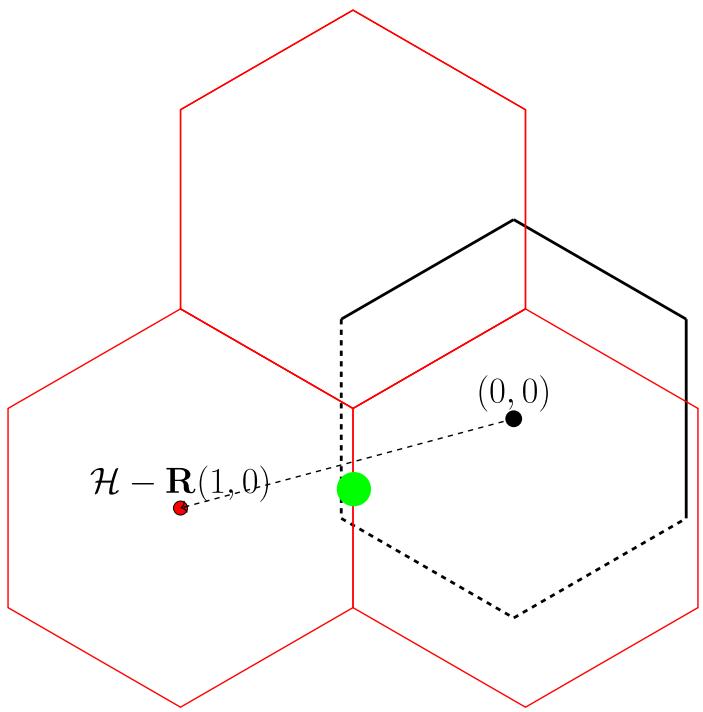
Remainder Map and Critical Rigid Motions



Remainder Map and Critical Rigid Motions

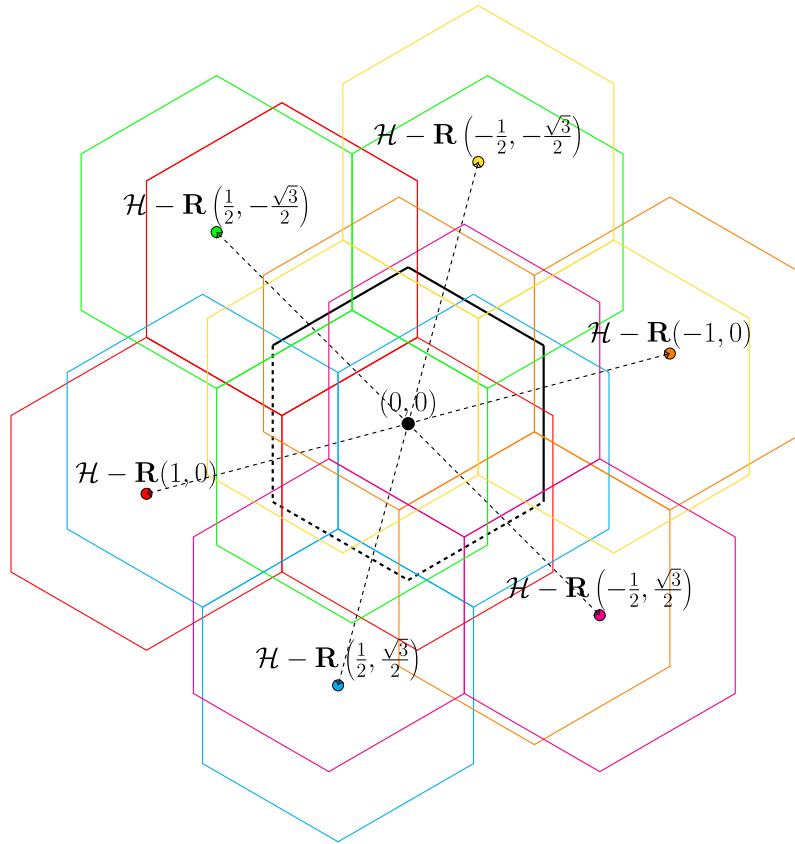


Remainder Map and Critical Rigid Motions



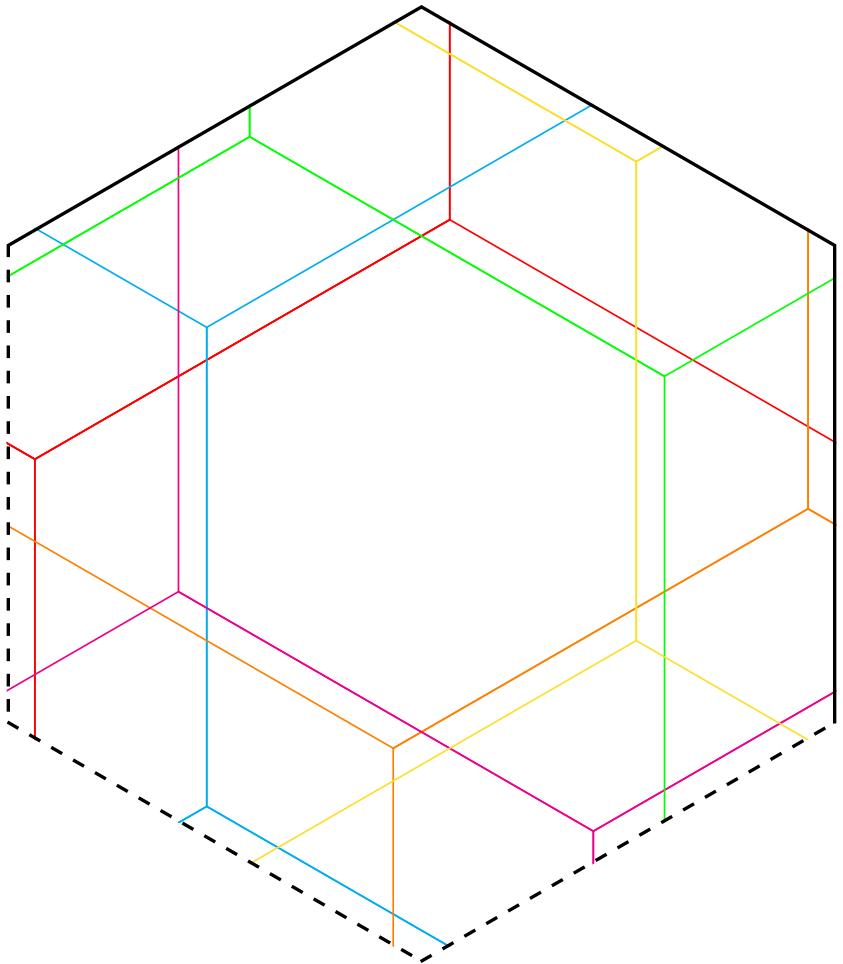
Such critical cases can be observed via the relative positions of $\mathcal{F}(\kappa)$, and are formulated as the translation $\mathcal{H} - \mathbf{R}\delta$. That is to say $\mathcal{C}(\mathbf{0}) \cap (\mathcal{H} - \mathbf{R}\delta)$.

Critical line segments



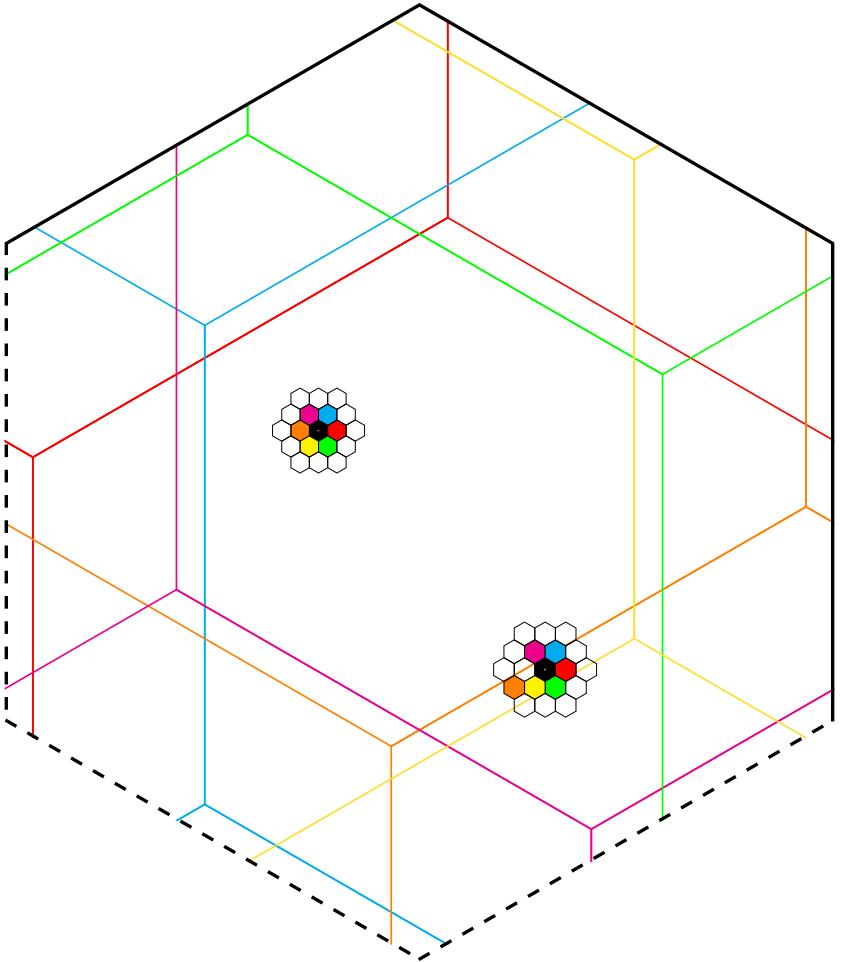
$$\mathcal{H} = \bigcup_{\delta \in \mathcal{N}_r(\mathbf{0})} (\mathcal{H} - \mathbf{R}\delta) \cap \mathcal{C}(\mathbf{0})$$

Frames



Each region bounded by the critical line segments is called a frame.

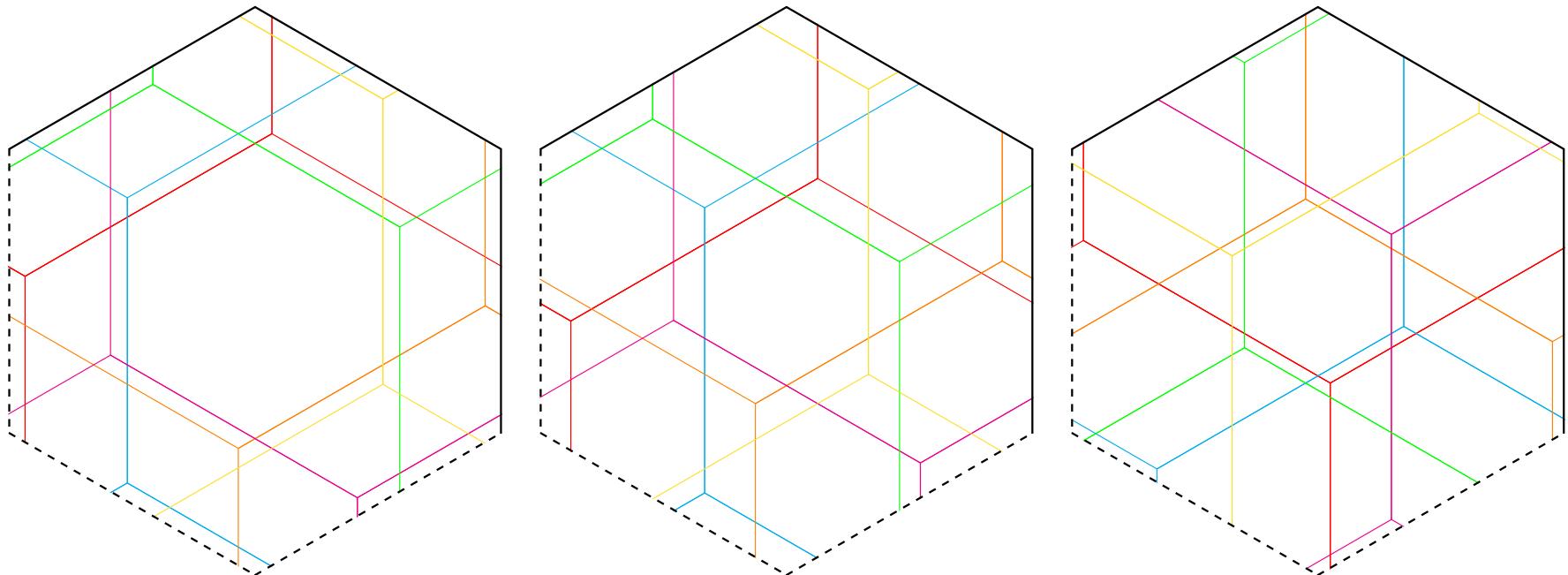
Frames



Proposition

For any $\kappa, \lambda \in \Lambda$, $\mathcal{G}_r^U(\kappa) = \mathcal{G}_r^U(\lambda)$ if and only if $\mathcal{F}(\kappa)$ and $\mathcal{F}(\lambda)$ are in the same frame.

Remainder Range Partitioning

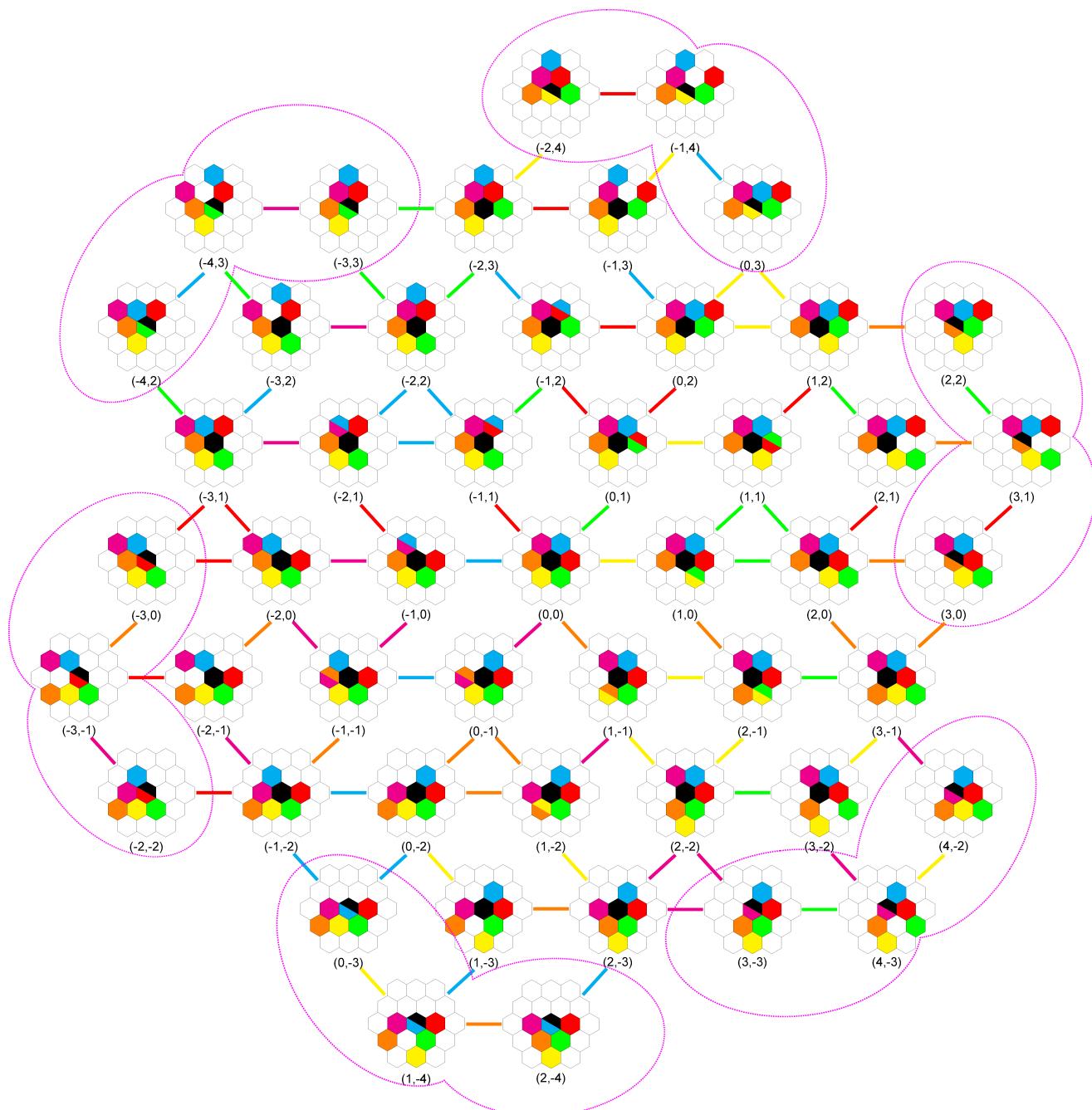


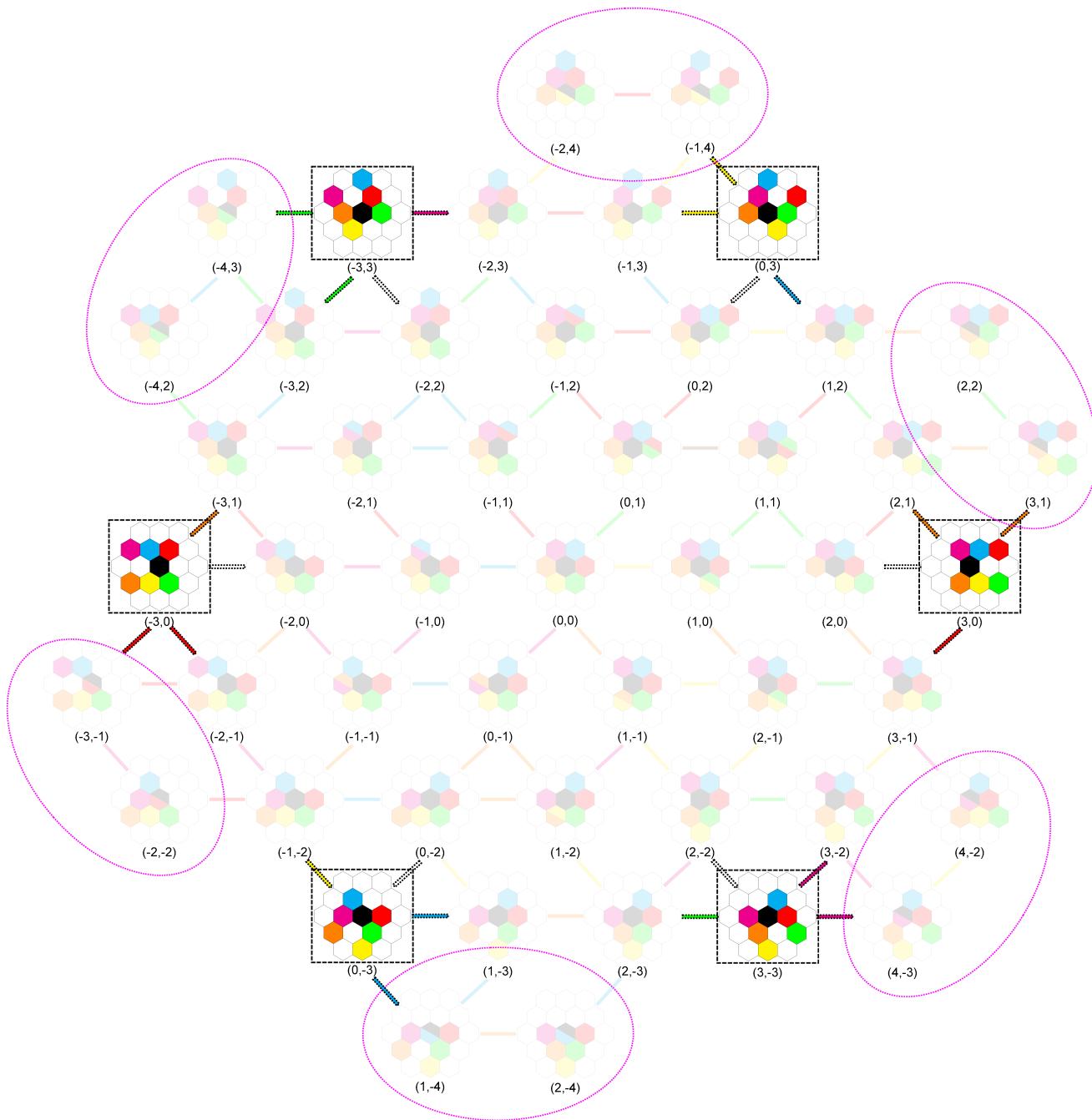
At most 49 frames per partitioning.

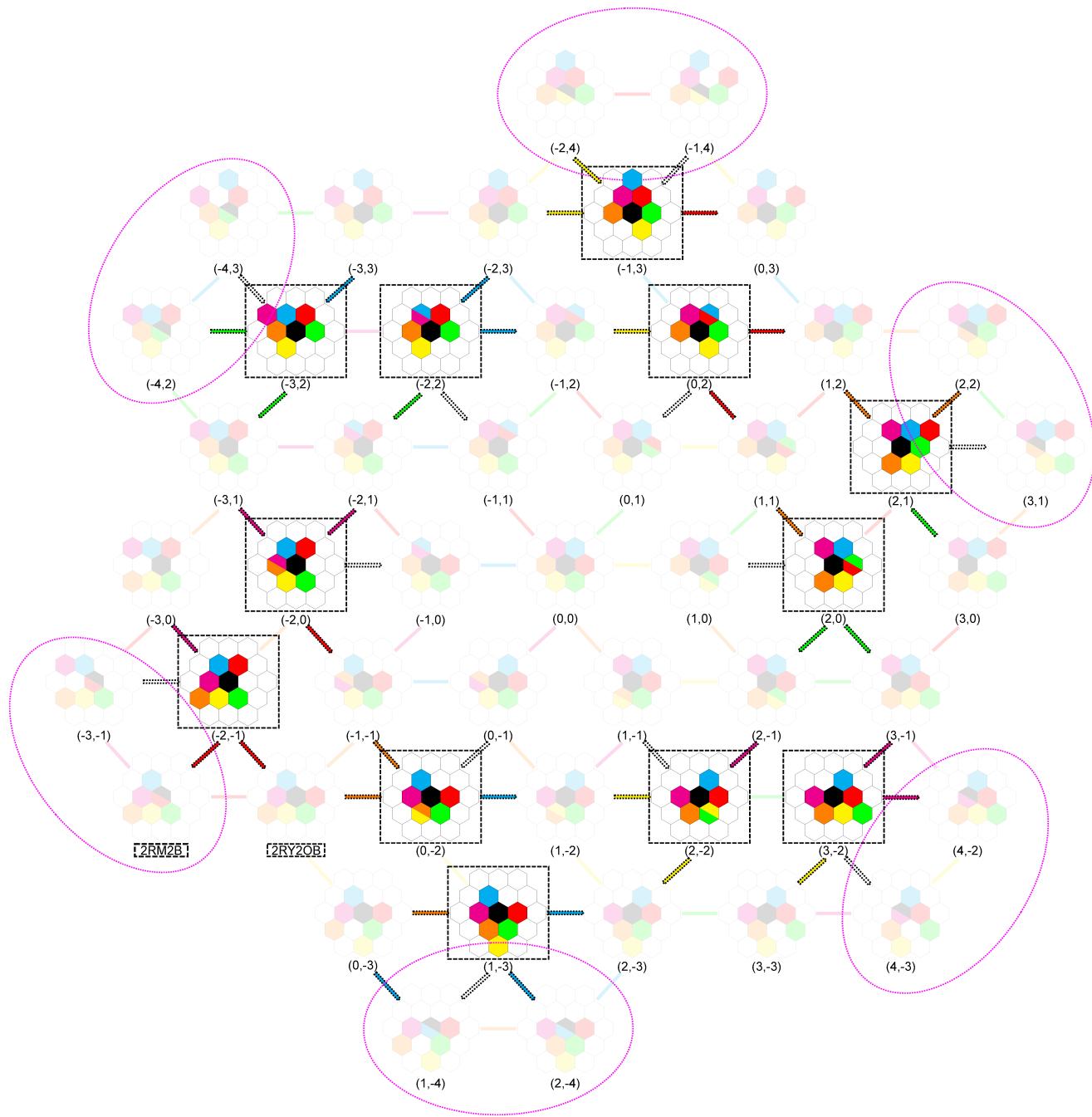
Contributions

Or extracting the pure,
organic honey

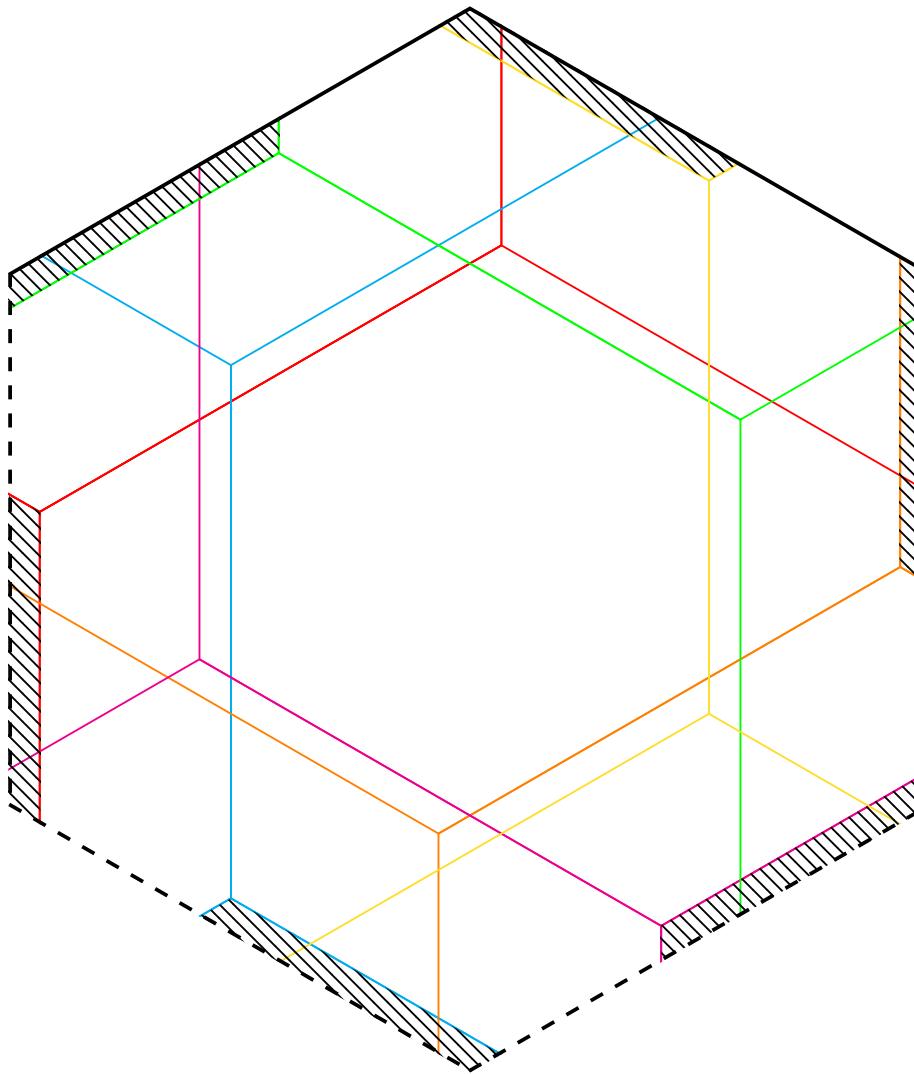




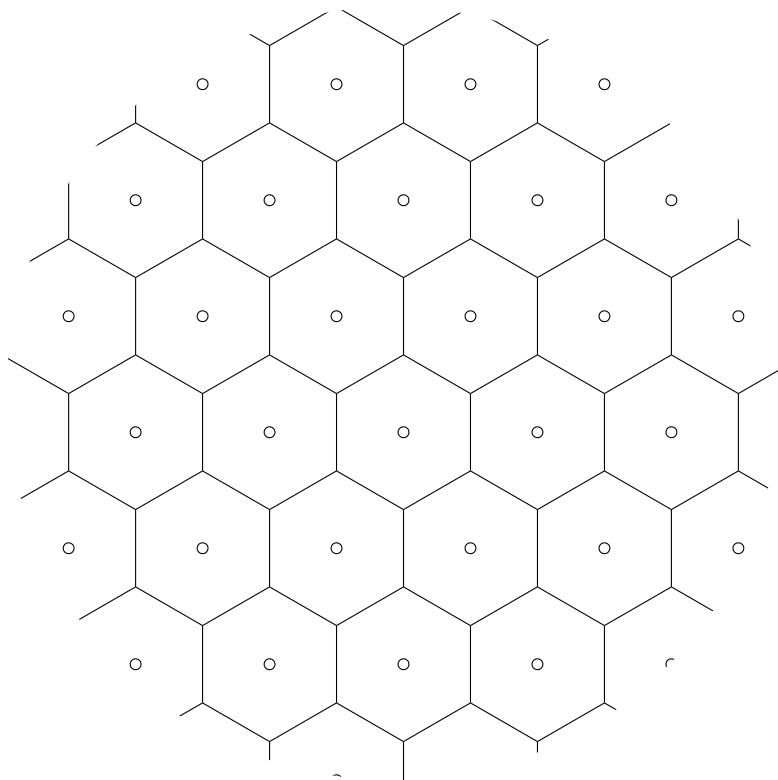




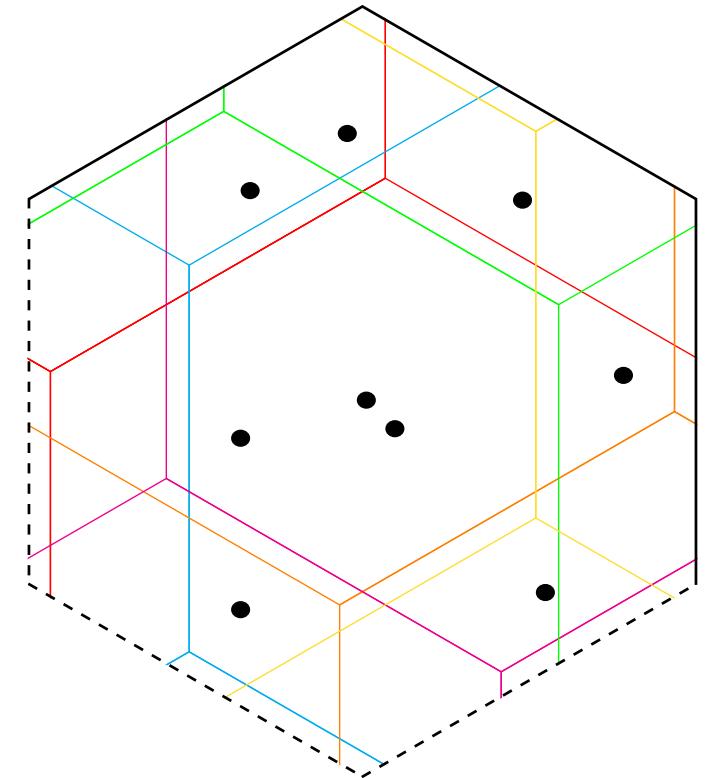
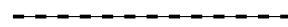
Non-injective Digitized Rigid Motions



Eisenstein Rational Rotations

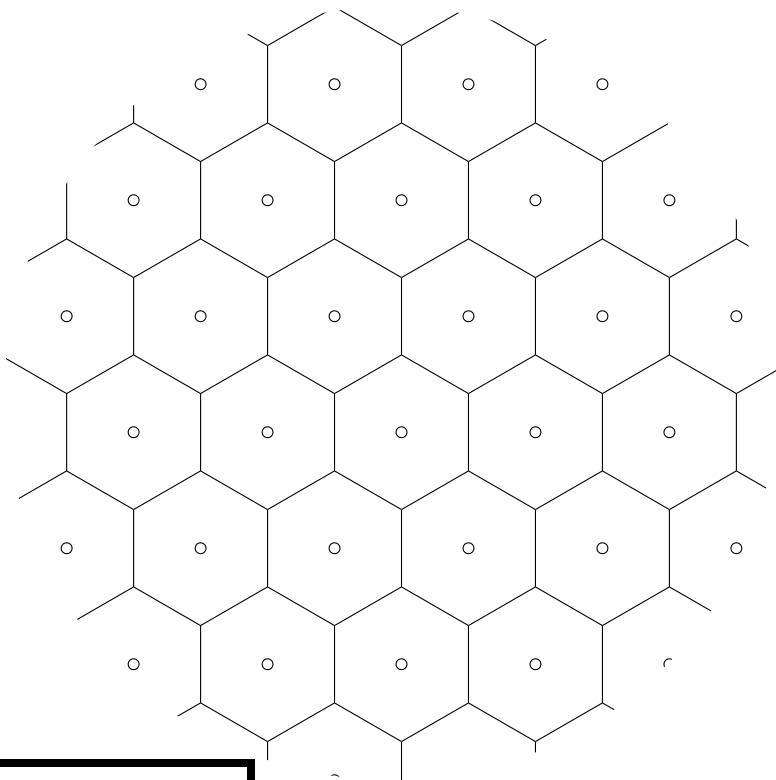


$$\mathcal{F}(\Lambda)$$

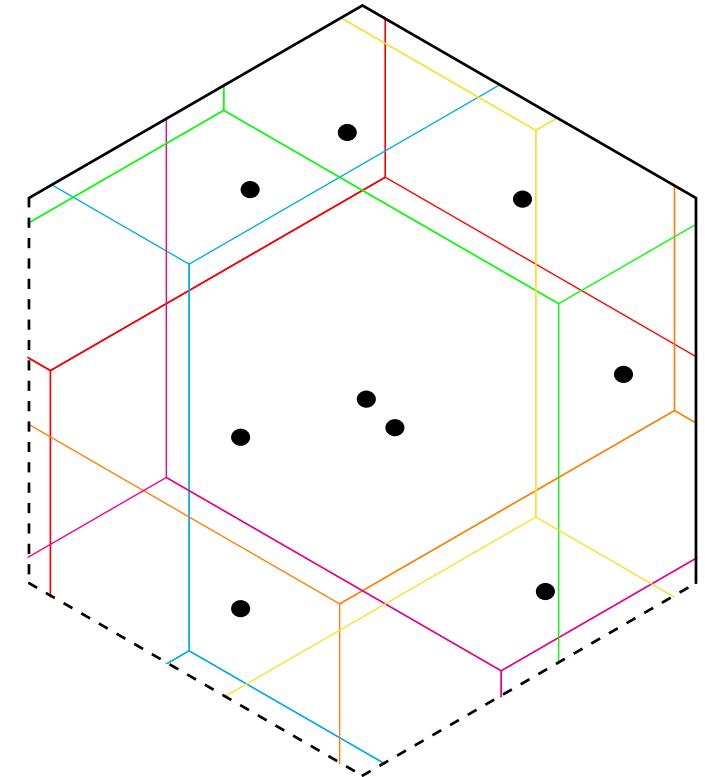
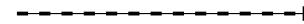


For what kind of parameters has the remainder map a finite number of images?

Eisenstein Rational Rotations



$\mathcal{F}(\Lambda)$

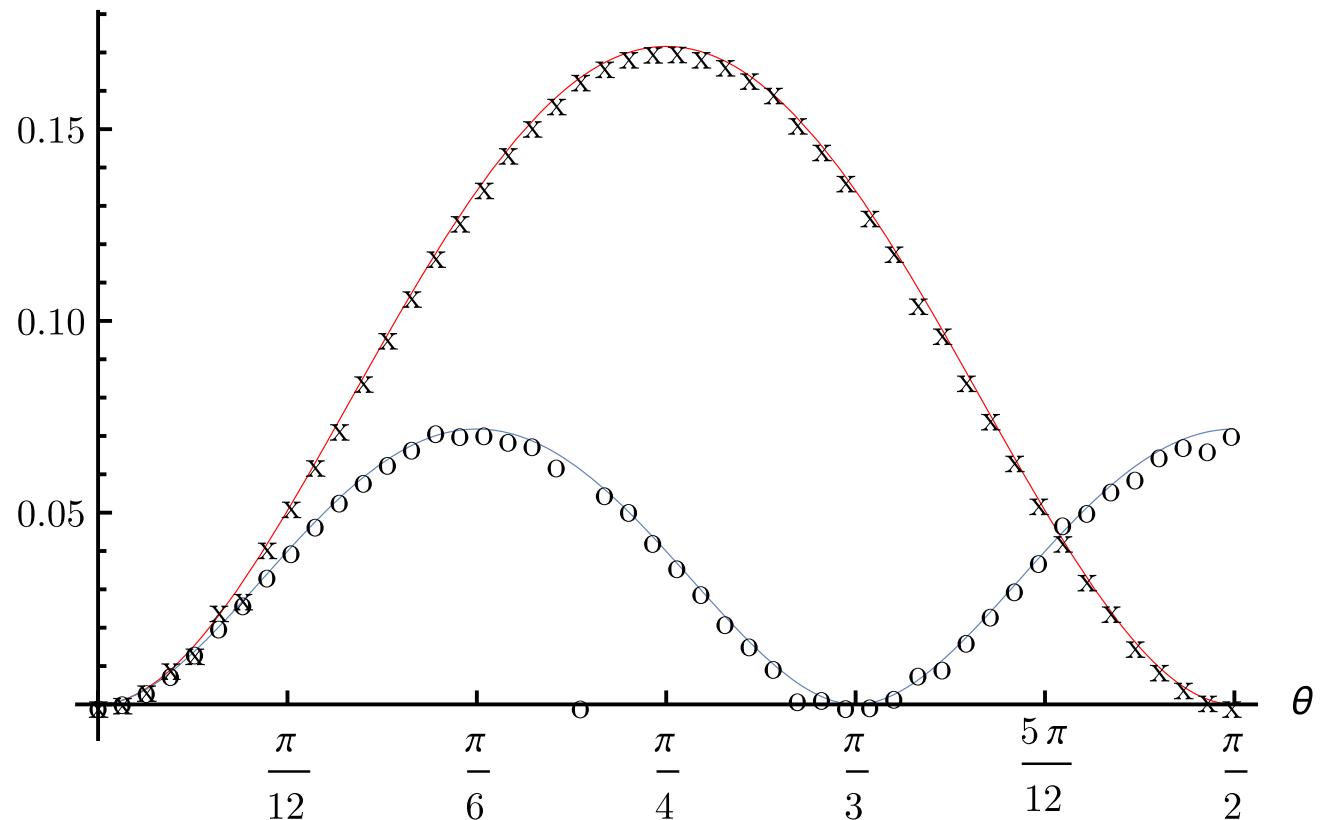


Corollary

If $\cos \theta = \frac{2a-b}{2c}$ and $\sin \theta = \frac{\sqrt{3}b}{2c}$ where $(a, b, c) \in \mathbb{Z}^3$, $\gcd(a, b, c) = 1$ and $0 < a < c < b$, then the remainder map has a finite number of images.

Loss of Information

Information loss rate



Conclusions & Perspectives

- An extension of a framework to study digitized rigid motions
- Characterization of rational rotations
- We have showed that the loss of information is relatively lower for digitized rigid motions defined on the hexagonal grid
- Our tools on BSD-3 license:
<https://github.com/copyme/NeighborhoodMotionMapsTools>

hal.archives-ouvertes.fr/hal-01540772

The screenshot shows the HAL (archives-ouvertes.fr) website interface. At the top, there is a navigation bar with links to CCSD, HAL, Episciences.org, Sciencesconf.org, Support, and language options (fr, en). A "Sign in" button is also present. The main header features the HAL logo and a stylized illustration of an owl surrounded by geometric shapes like cubes and spheres.

The page title is "Characterization of bijective digitized rotations on the hexagonal grid". Below the title, the authors listed are Kacper Pluta^{1,2}, Tristan Roussillon³, David Cœurjolly³, Pascal Romon², Yukiko Kenmochi¹, Victor Ostromoukhov⁴. There are four numbered footnotes below the authors:

- 1 LIGM - Laboratoire d'Informatique Gaspard-Monge
- 2 LAMA - Laboratoire d'Analyse et de Mathématiques Appliquées
- 3 M2DisCo - Geometry Processing and Constrained Optimization
- 4 R3AM - Rendu Réalist pour la Réalité Augmentée Mobile

The abstract discusses digitized rotations on discrete spaces, noting they are generally not bijective but can be for specific angles on the square grid. It then focuses on the hexagonal grid, using arithmetical properties of Eisenstein integers to characterize bijective digitized rotations.

Keywords listed include digital topology, honeycomb geometry, digital geometry, bijective transformations, digitized rotations, and hexagonal grid.

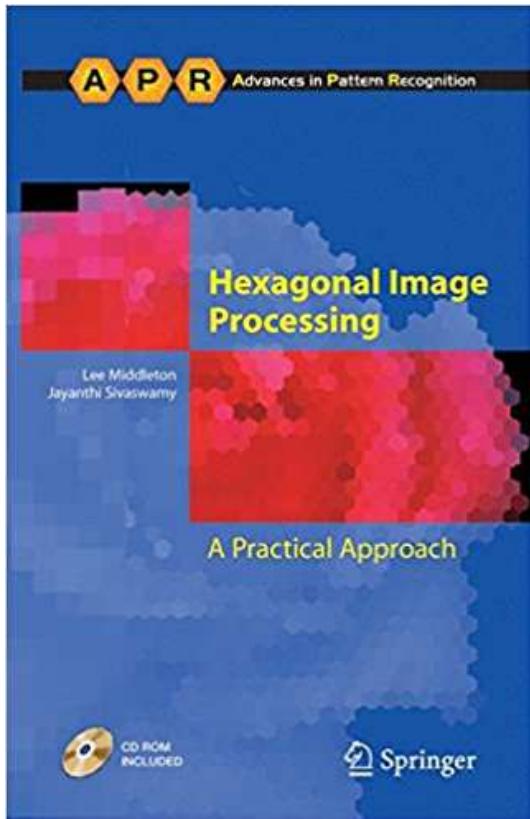
Document type: Preprints, Working Papers, ...

Domain: Computer Science [cs] / Discrete Mathematics [cs.DM]

Submitted to Journal of Mathematical Imaging and Vision. 2017

The right side of the page contains several sections: FILE (with a PDF icon and the file name article.pdf), IDENTIFIERS (with the HAL ID), COLLECTIONS (listing EC-LYON, TDS-MACS, UPEC-UPEM, UNIV-LYON2, LIRIS, PARISTECH, LIGM, ENPC, LIGM_A3SI, LAMA_PLC, and LAMA_UMR8050), and CITATION (listing the authors and their work).

Homework



If you want to get into the honey business,
then this book is an obligatory lecture:
Middleton, Lee, and Jayanthi Sivaswamy.
*Hexagonal image processing: A practical
approach*. Springer Science & Business Media,
2006.