

Honeycomb Geometry

Rigid Motions on the
Hexagonal Grid

by Kacper Pluta, Pascal Romon,
Yukiko Kenmochi and Nicolas Passat



The figure comes from "Insects The Yearbook of Agriculture 1952" United States Dept. of Agriculture." Published by the US Government Printing Office. Deemed to be in the Public Domain under US Law.

Motivations

We came to agree with Nouvel & Rémila that digitized rigid motions defined on the square grid are burdened with a fundamental incompatibility between rotations and the geometry of the grid.

Agenda

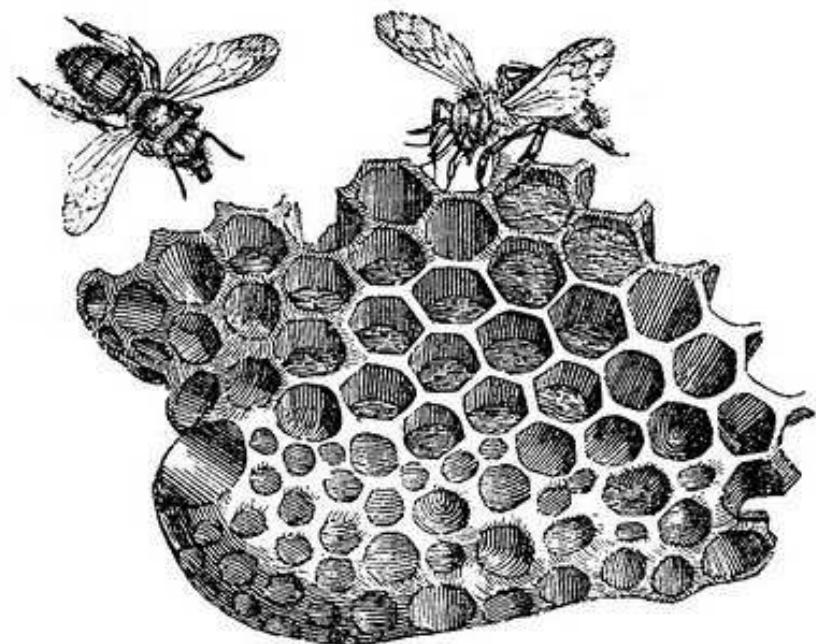


- Introduction to the Bees' Point of View
- Quick Introduction to Rigid Motions
- Neighborhood Motion Maps
- Contributions
- Conclusions & Perspectives



Introduction to the Bees' Point of View

Or why bees are right



Pros and Cons

Square grid

- + Memory addressing
- + Sampling is easy to define
- Sampling is not optimal (ask bees)
- Neighbors are not equidistant
- Connectivity paradox

Hexagonal grid

- + Uniform connectivity
- + Equidistant neighbors
- + Sampling is optimal
- Memory addressing is not trivial
- Sampling is difficult to define

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Howdy vision lads and gals! These problems seem to be somehow solved.

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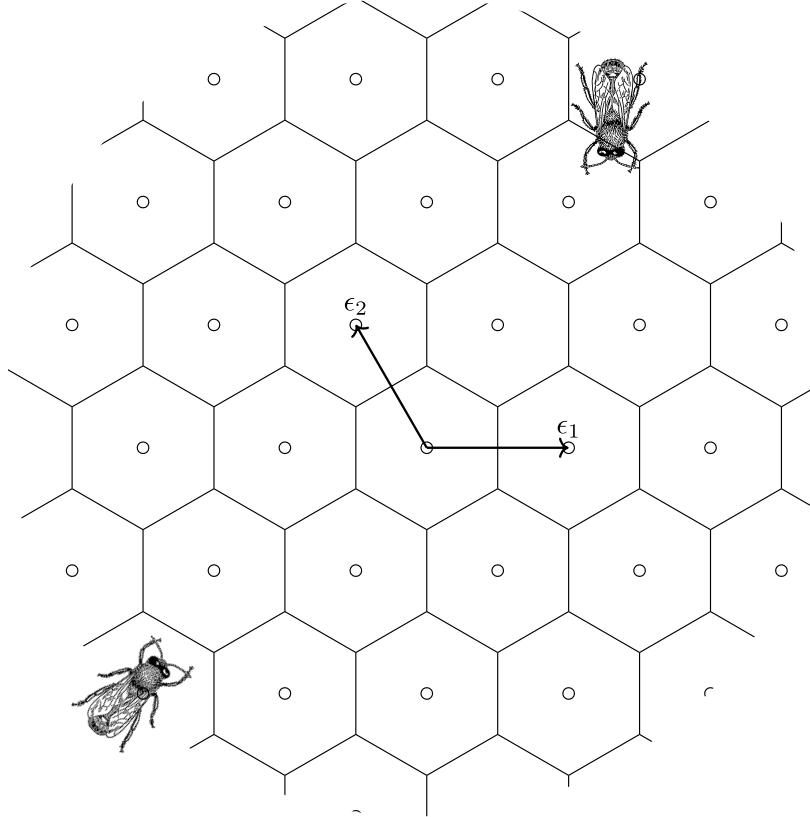
Hexagonal grid

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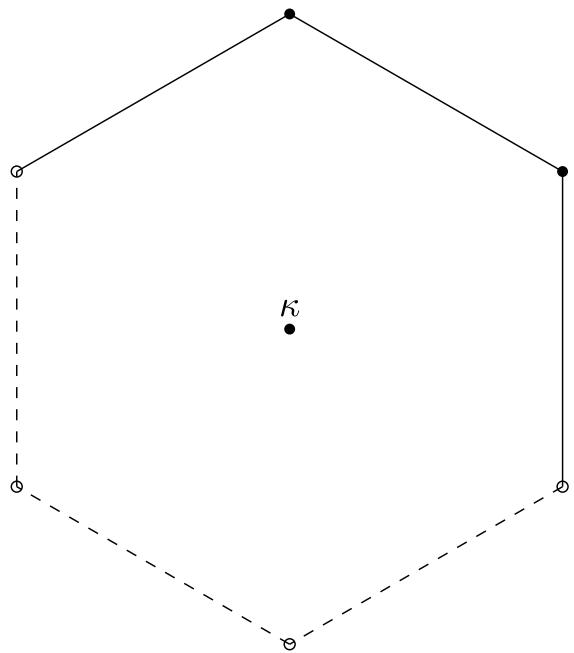
You may think: "Hold your horses! It's not a bug, it's a feature..."

Hexagonal Grid



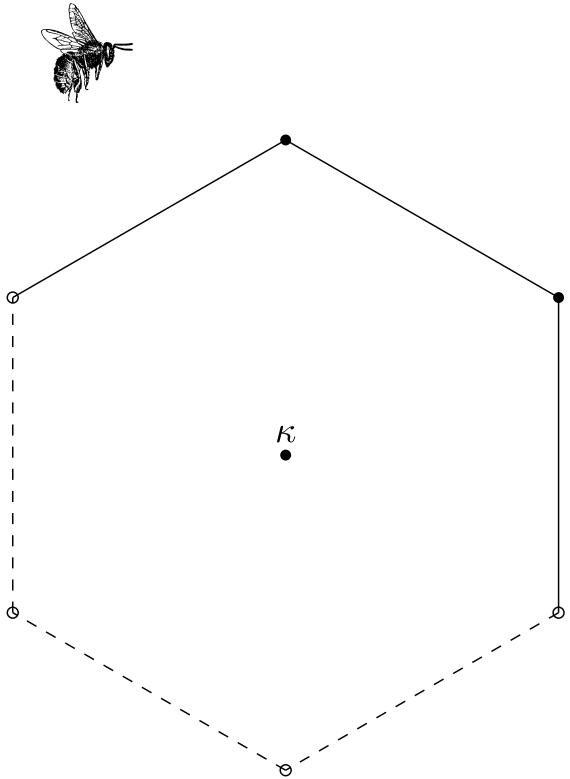
The hexagonal lattice: $\Lambda = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ and the hexagonal grid \mathcal{H}

Digitization Model



The digitization operator is defined as a function $\mathcal{D} : \mathbb{R}^2 \rightarrow \Lambda$ such that $\forall \mathbf{x} \in \mathbb{R}^2, \exists! \mathcal{D}(\mathbf{x}) \in \Lambda$ and $\mathbf{x} \in \mathcal{C}(\mathcal{D}(\mathbf{x}))$.

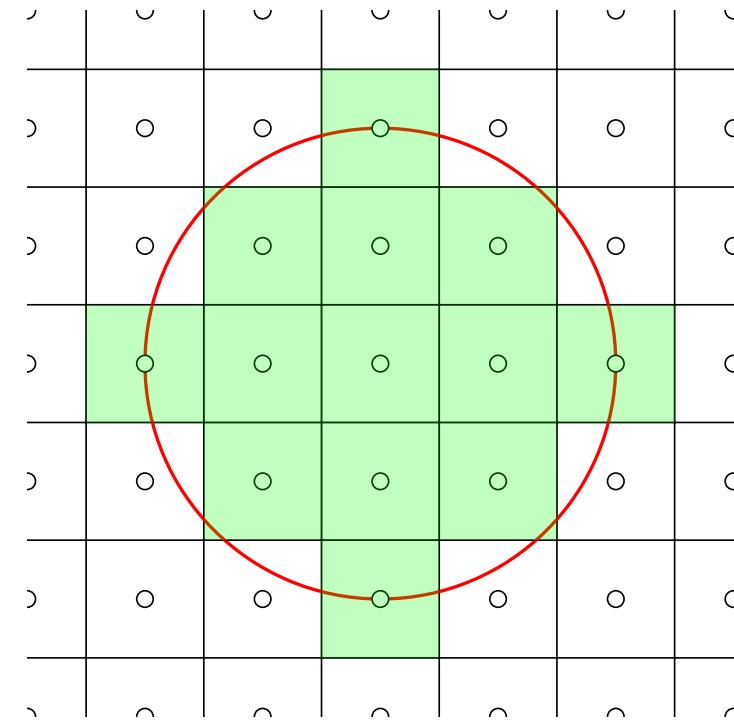
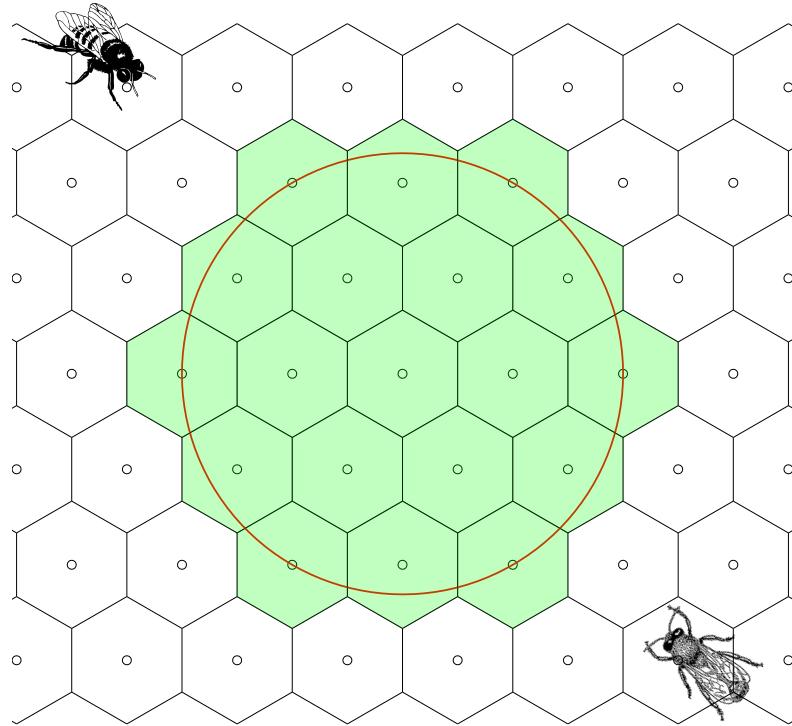
Digitization Model



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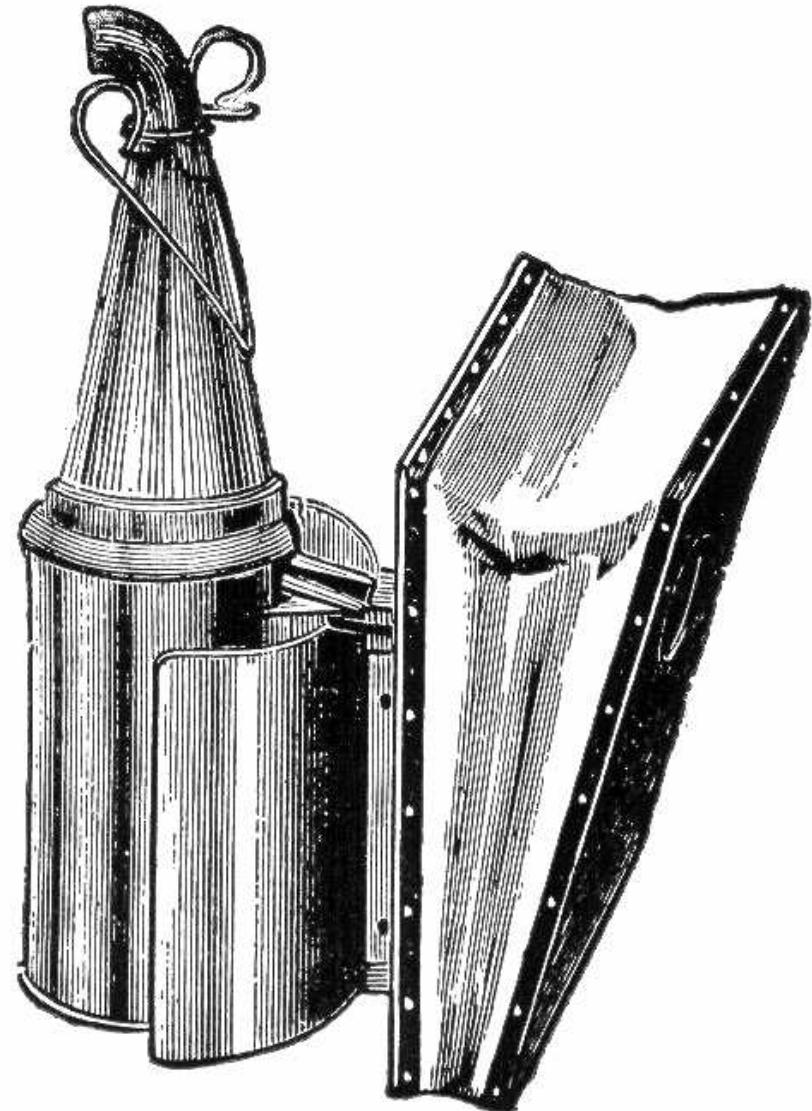
This is a definition for digital geometers not for computer vision guys...

How many digital balls do you see?



Quick Lesson on Rigid Motions

Or how to become a beekeeper. Part I - Equipment



Rigid Motions on \mathbb{R}^2

Properties

$$\begin{array}{lcl} \mathcal{U} & : & \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ & & \mathbf{x} \mapsto \mathbf{Rx} + \mathbf{t} \end{array}$$

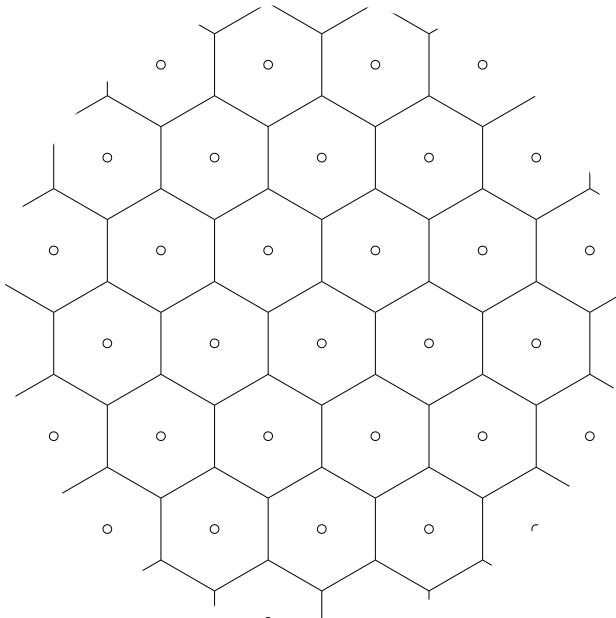
- Isometry map - distance preserving map
- Bijective

Rigid Motions on Λ

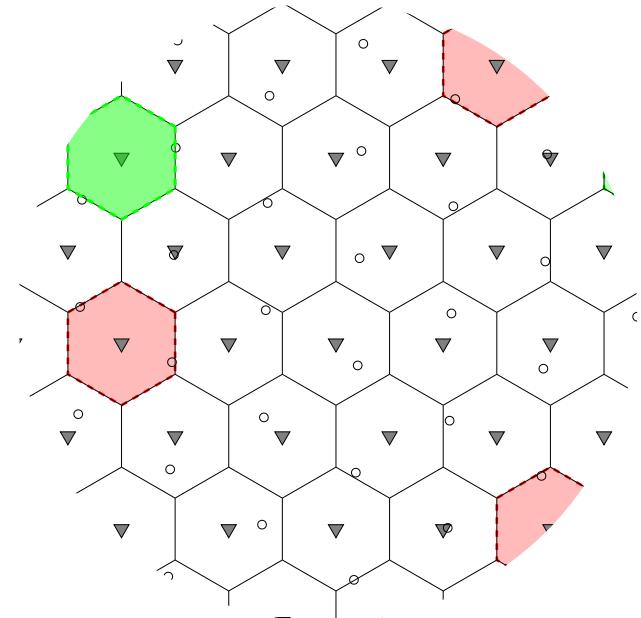
$$U = \mathcal{D} \circ \mathcal{U}_{|\Lambda}$$

Properties

- Do not preserve distances
- Non-injective
- Non-surjective



$\mathcal{U}(\Lambda)$



Related Studies

- Nouvel, B., Rémy, E.: On colorations induced by discrete rotations. In: DGCI, Proceedings. *Volume 2886 of Lecture Notes in Computer Science.*, Springer (2003) 174–183
- Pluta, K., Romon, P., Kenmochi, Y., Passat, N.: Bijective digitized rigid motions on subsets of the plane. *Journal of Mathematical Imaging and Vision* (2017)

Contributions in Short

Pure extracted honey

- Extension of the former framework to the hexagonal grid
- Comparison of the loss of information between the hexagonal and square grids
- Complete set of neighborhood motion maps
- Source code of a tool to study digitized rigid motions on the hexagonal grid



Neighborhood Motion Maps

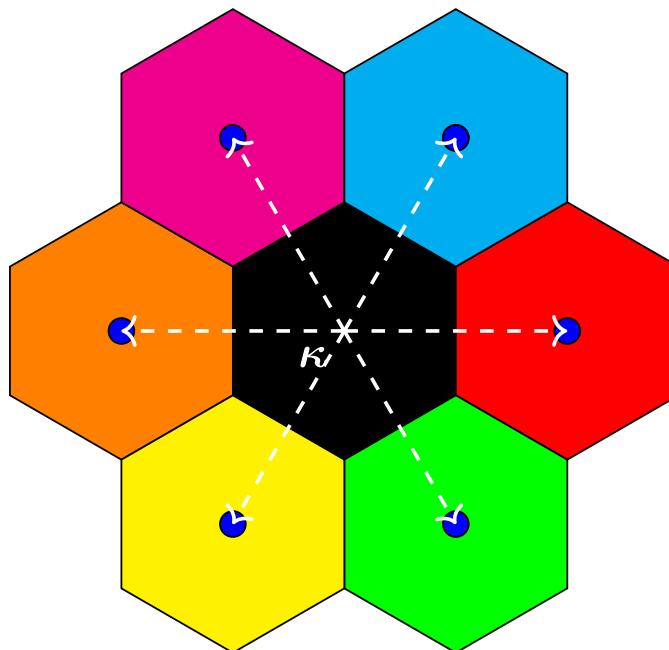
Or a manual of instructions in
apiculture



Neighborhood

The *neighbourhood* of $\kappa \in \Lambda$ (of squared radius $r \in \mathbb{R}_+$)

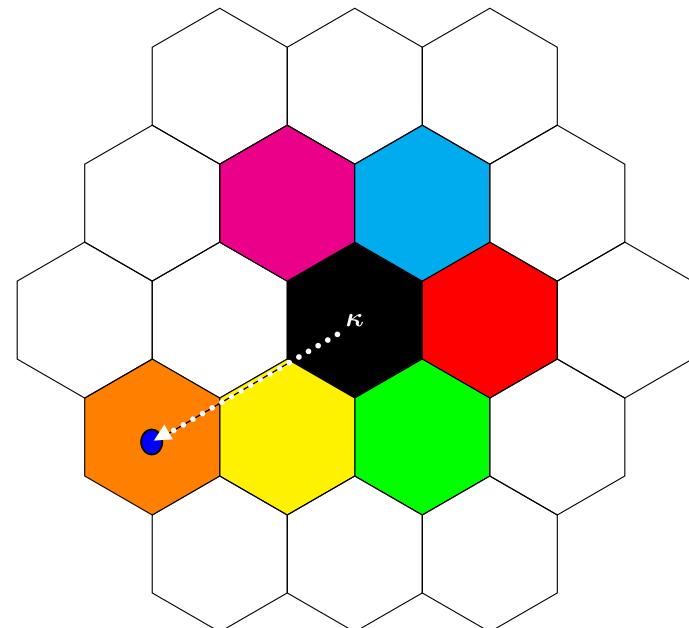
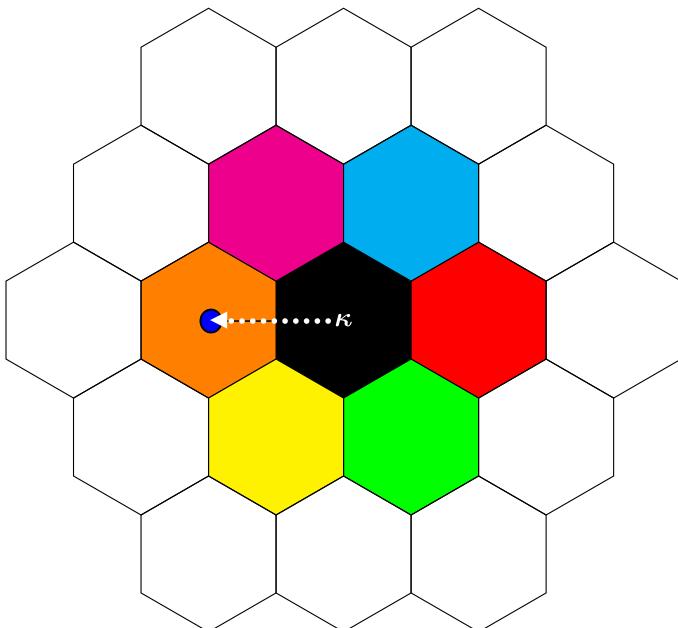
$$\mathcal{N}_r(\kappa) = \{\kappa + \delta \in \Lambda \mid \|\delta\|^2 \leq r\}$$



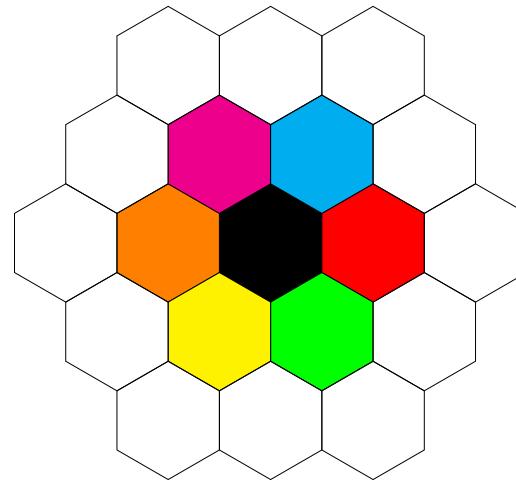
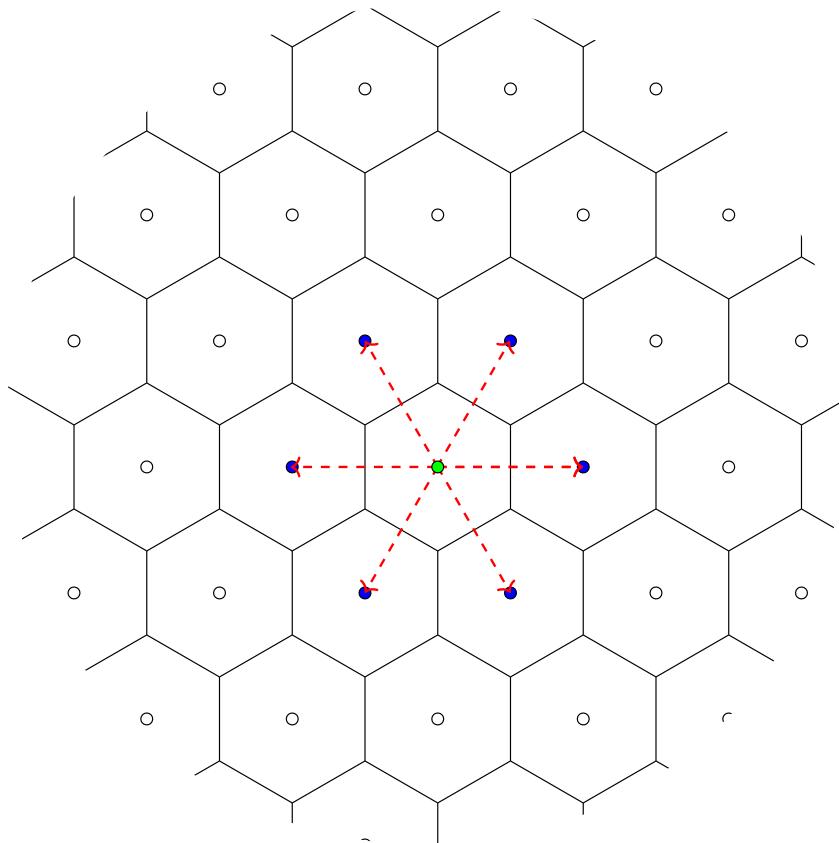
Neighborhood Motion Maps

The *neighbourhood motion map* of $\kappa \in \Lambda$ with respect to U and $r \in \mathbb{R}_+$ is the function

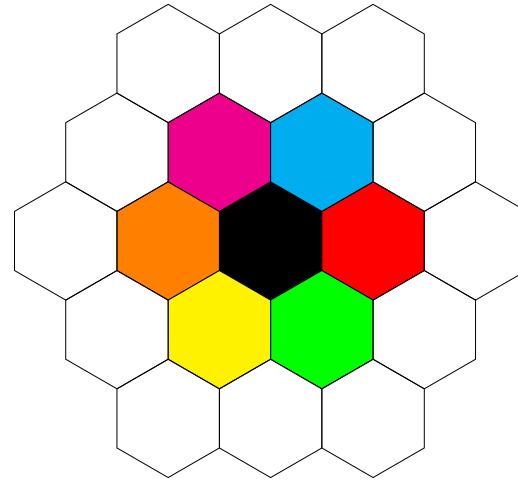
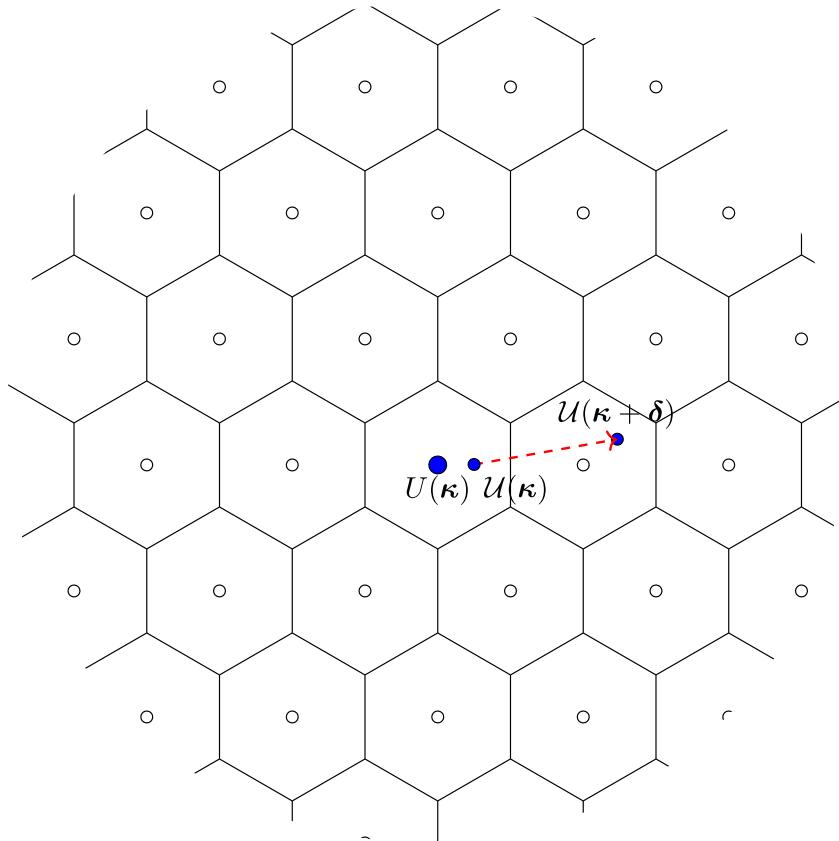
$$\begin{array}{rccc} \mathcal{G}_r^U & : & \mathcal{N}_r(0) & \rightarrow \mathcal{N}_{r'}(0) \\ & & \delta & \mapsto U(\kappa + \delta) - U(\kappa). \end{array}$$



Remainder Map step-by-step

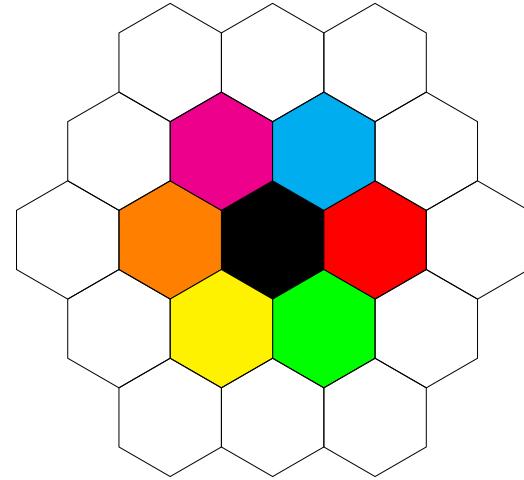
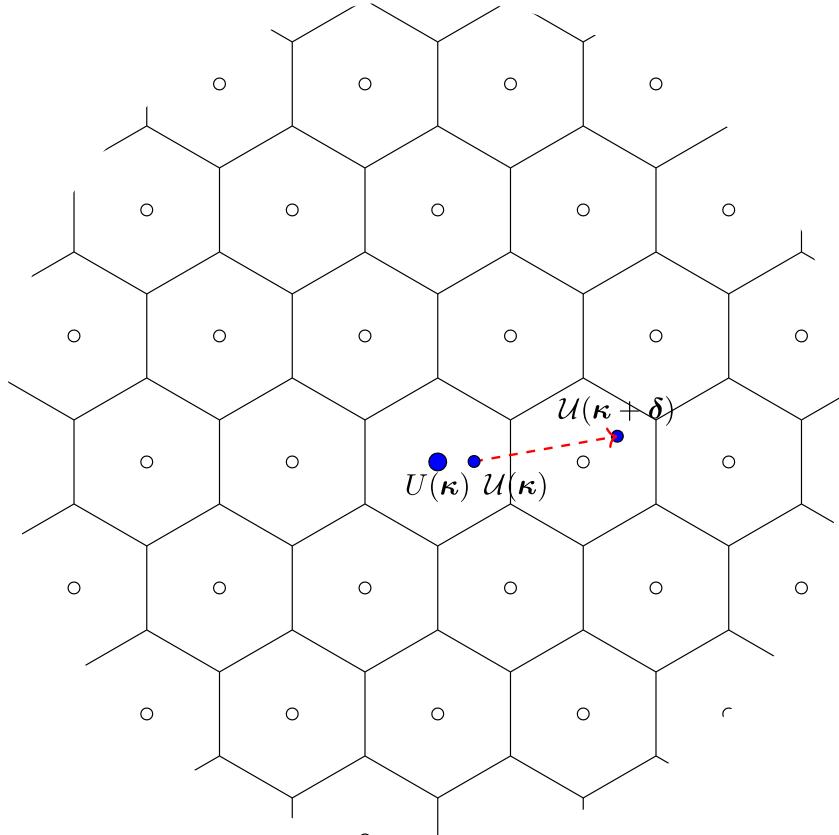


Remainder Map step-by-step



$$\mathcal{U}(\kappa + \delta) = \mathbf{R}\delta + \mathcal{U}(\kappa)$$

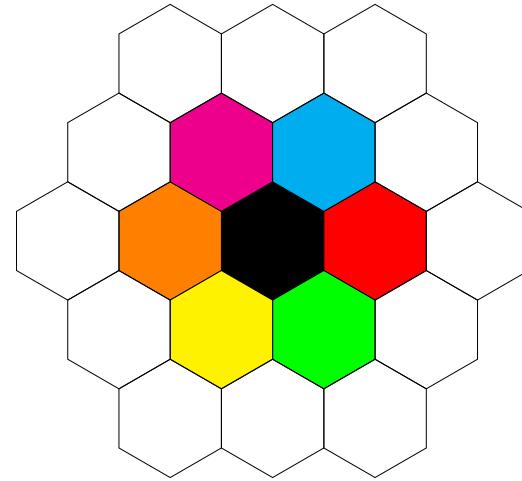
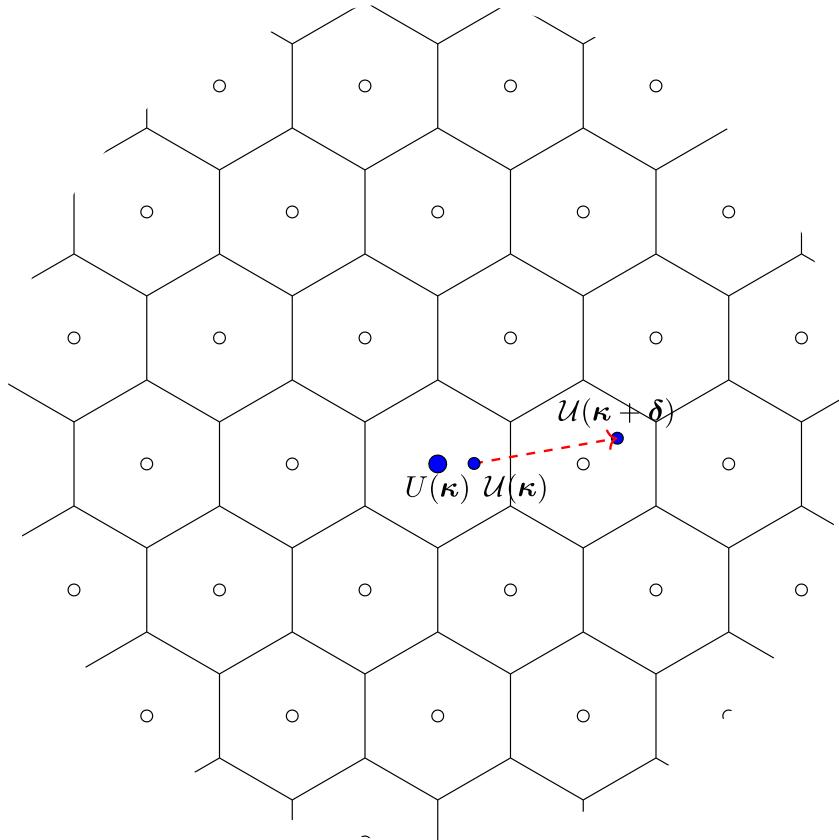
Remainder Map step-by-step



Without loss of generality, $U(\kappa)$ is an origin, then

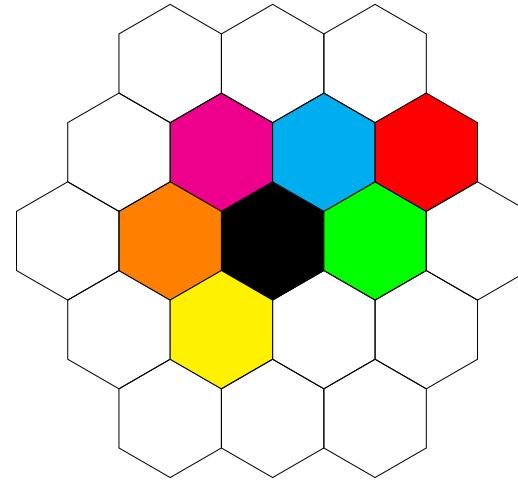
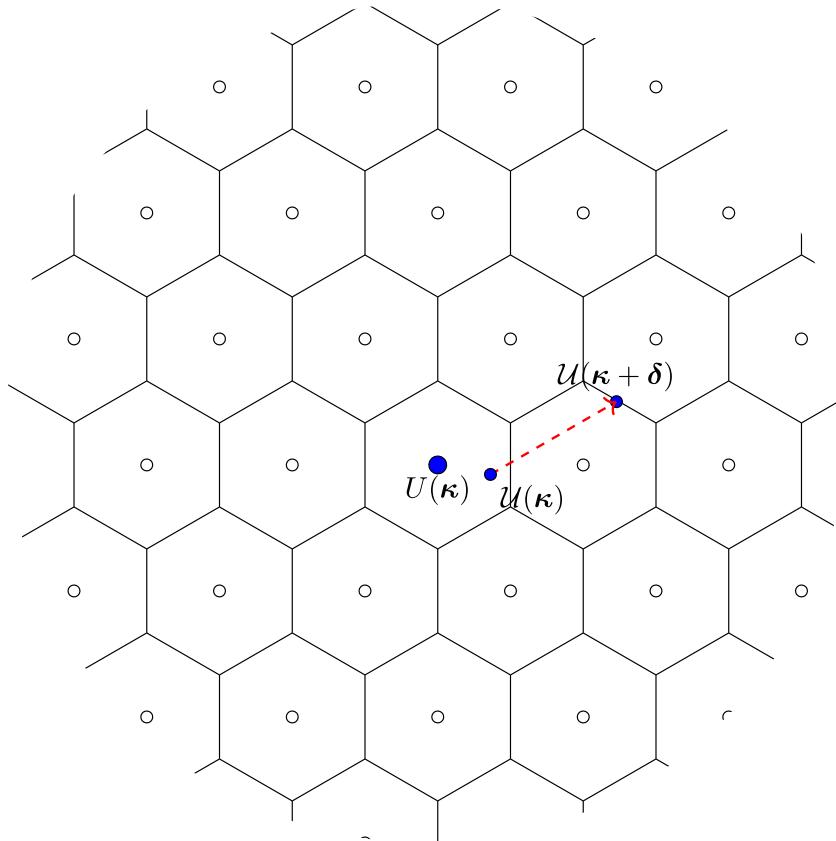
$$U(\delta) = \mathbf{R}\delta + U(\kappa) - U(\kappa)$$

Remainder Map step-by-step

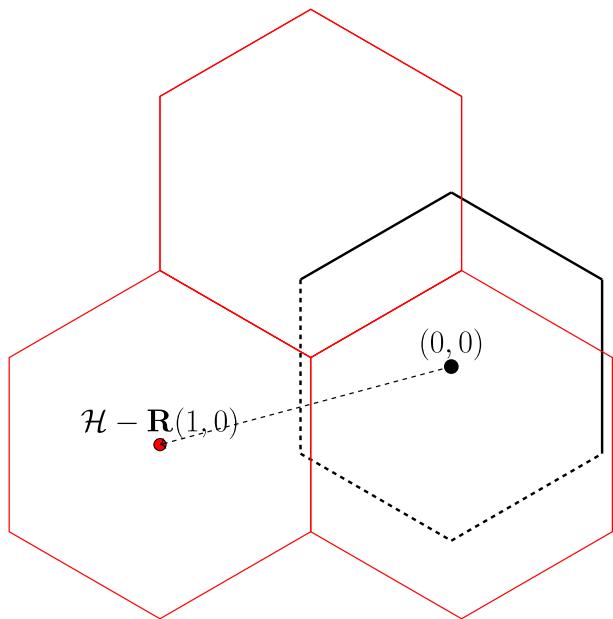


Remainder map defined as $\mathcal{F}(\kappa) = \mathcal{U}(\kappa) - U(\kappa) \in \mathcal{C}(\mathbf{0})$
where the range $\mathcal{C}(\mathbf{0})$ is called the remainder range.

Remainder Map step-by-step

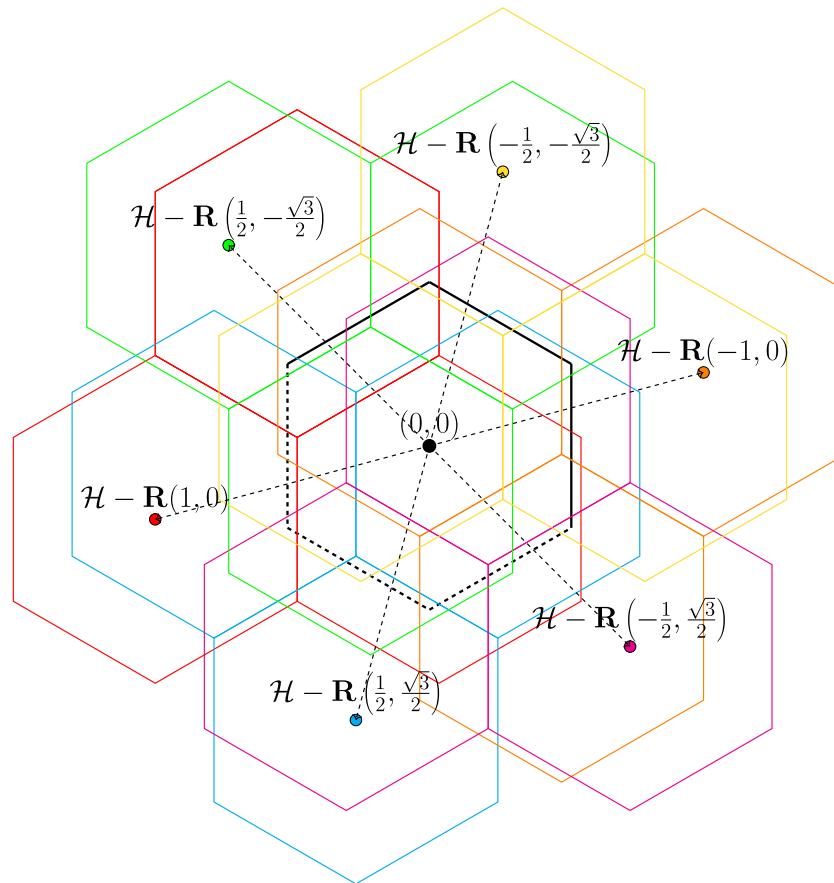


Remainder Map and Critical Rigid Motions



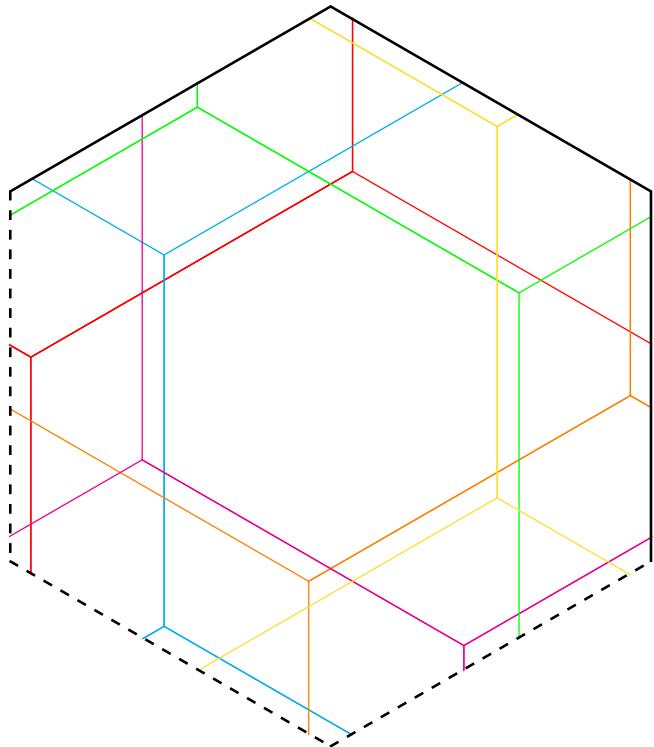
Critical cases can be observed via the relative positions of $\mathcal{F}(\kappa)$ which are formulated by the translation $\mathcal{H} - \mathbf{R}\delta$ that is to say $\mathcal{C}(\mathbf{0}) \cap (\mathcal{H} - \mathbf{R}\delta)$.

Remainder Map and Critical Rigid Motions



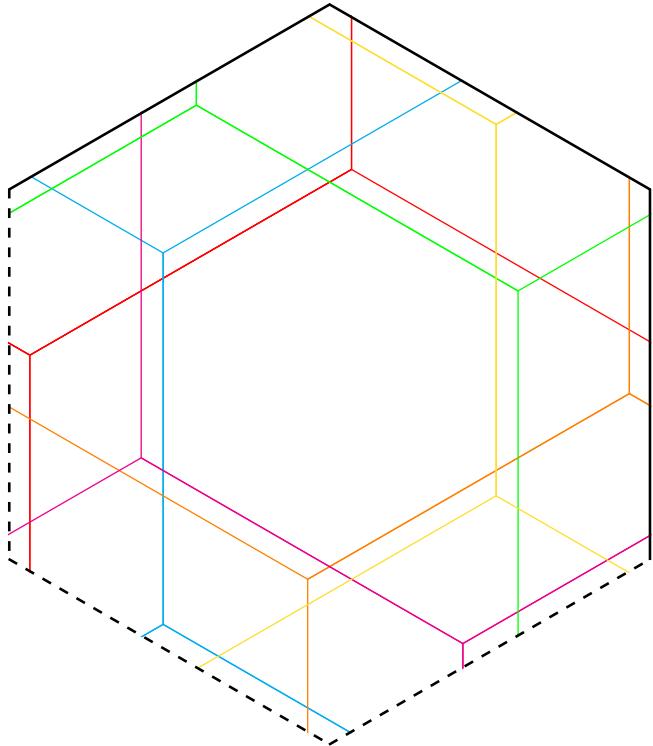
$$\mathcal{H} = \bigcup_{\delta \in \mathcal{N}_r(\mathbf{0})} (\mathcal{H} - \mathbf{R}\delta) \cap \mathcal{C}(\mathbf{0})$$

Frames



Each region bounded by critical lines is called a frame.

Frames

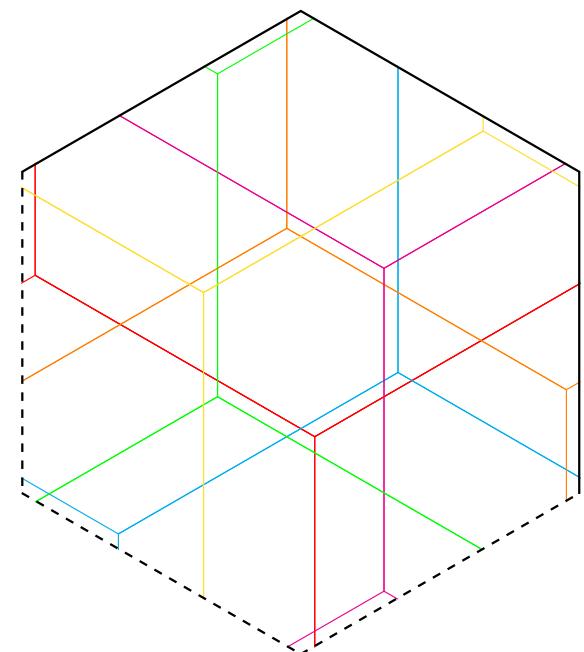
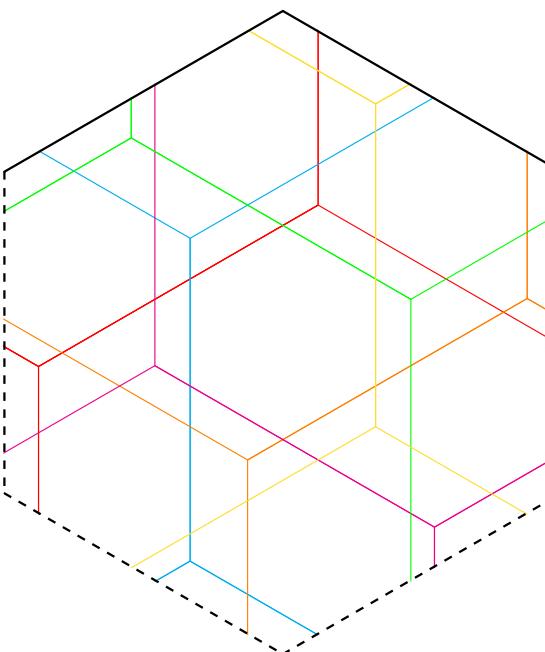
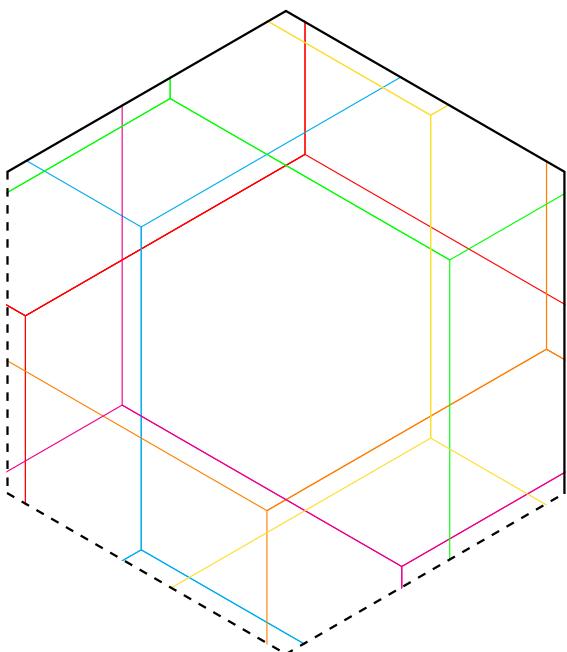


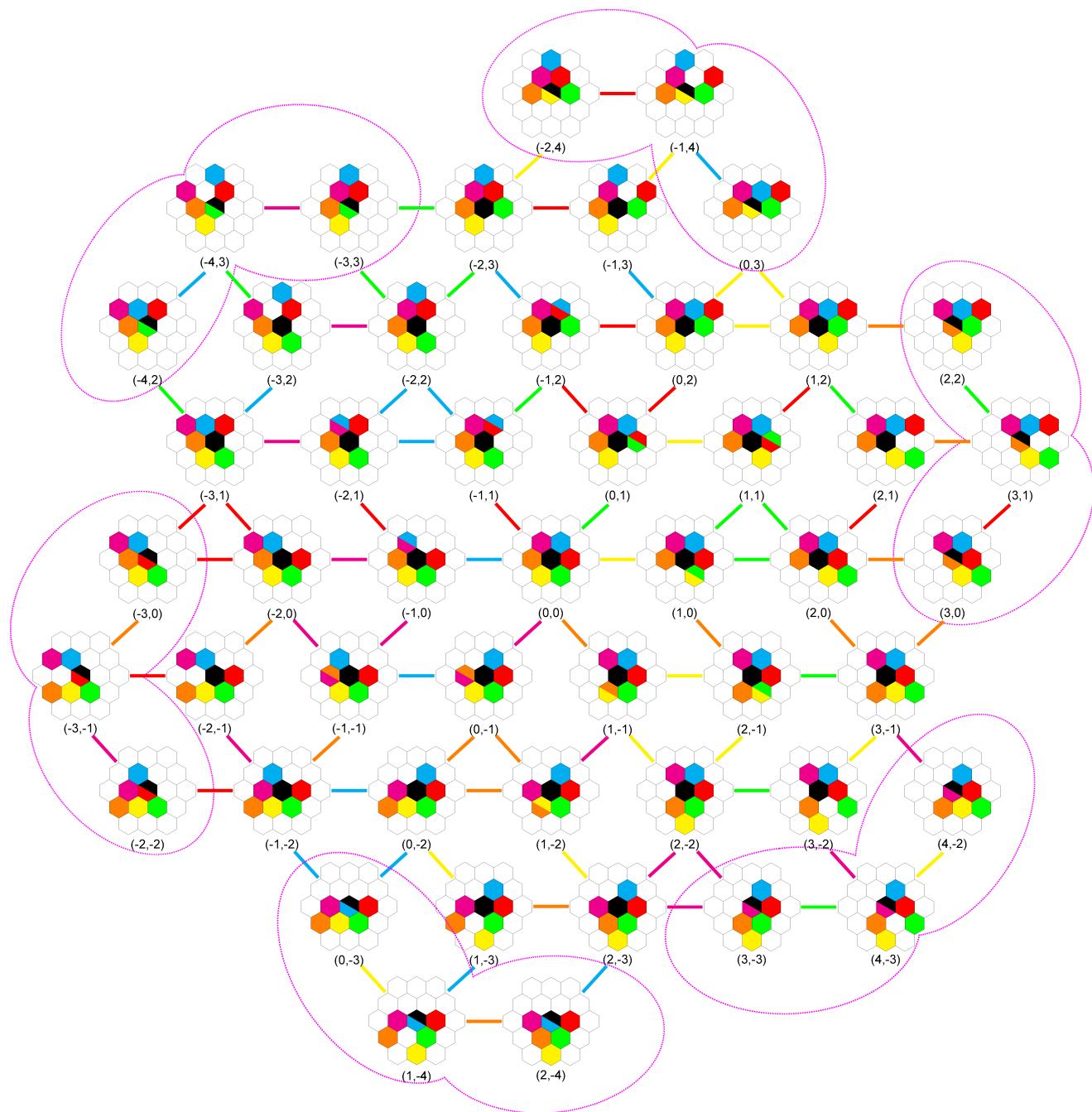
Each region bounded by critical lines is called a frame.

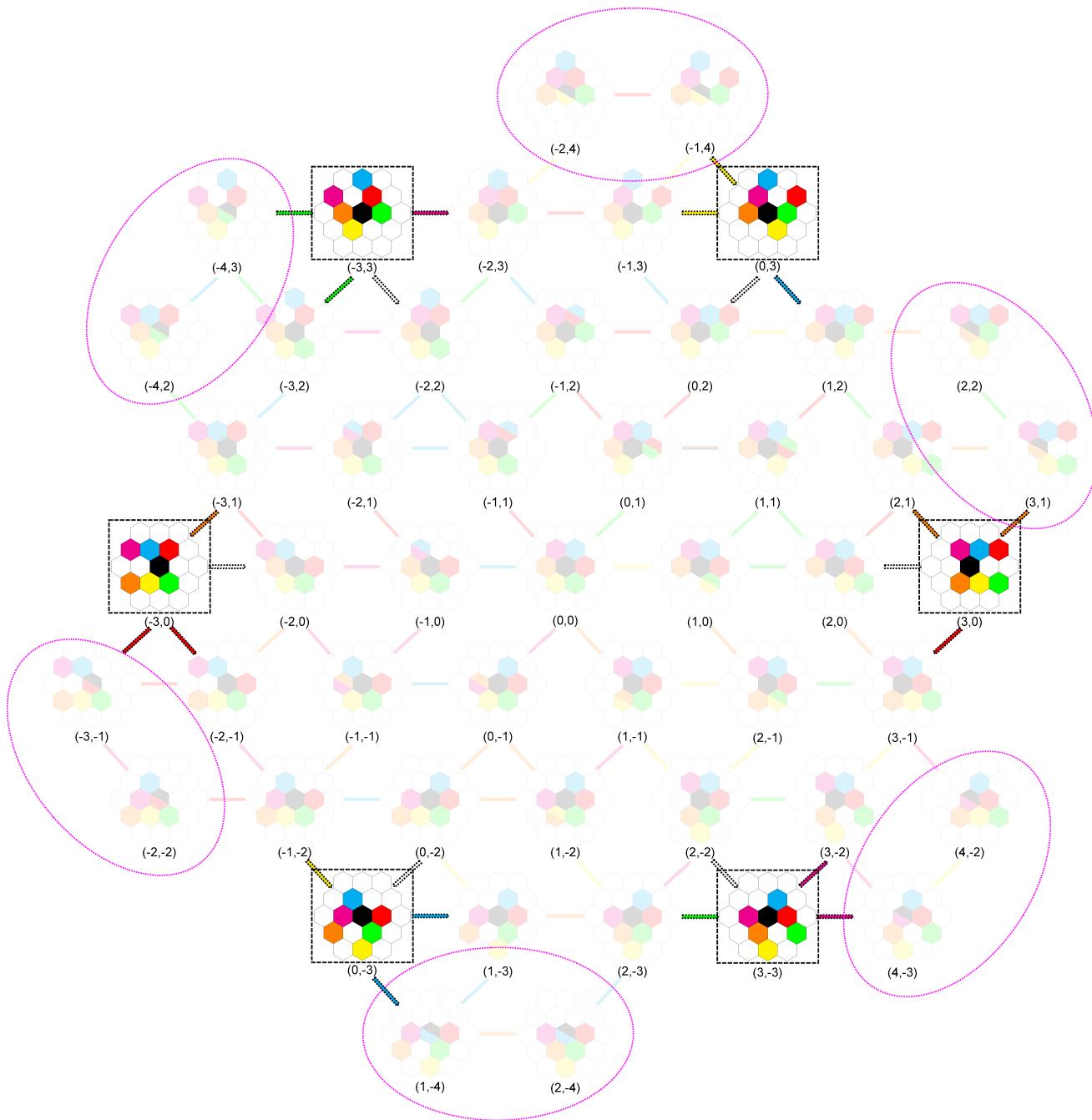
Proposition

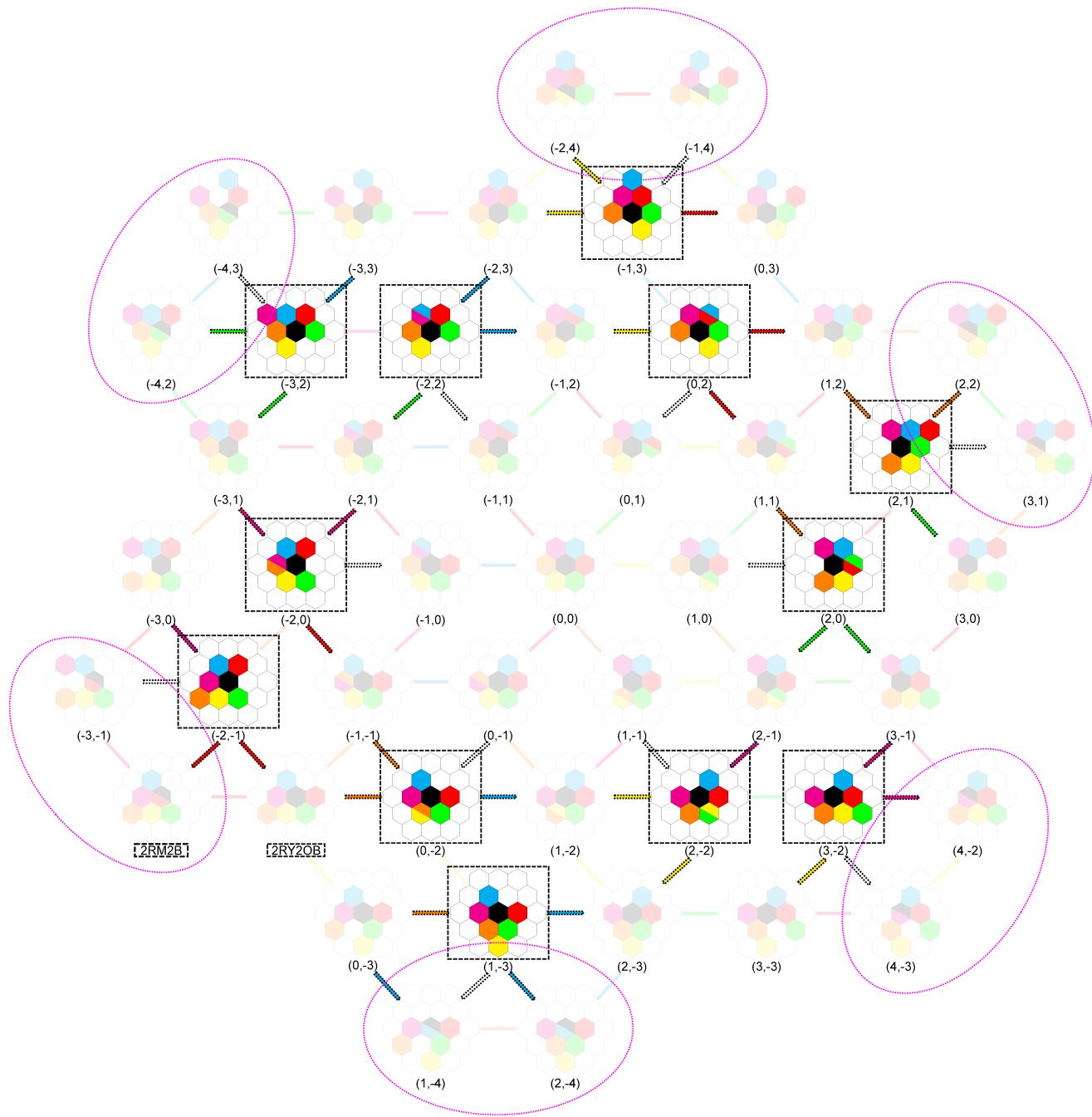
For any $\kappa, \lambda \in \Lambda$, $\mathcal{G}_r^U(\kappa) = \mathcal{G}_r^U(\lambda)$ if and only if $\mathcal{F}(\kappa)$ and $\mathcal{F}(\lambda)$ are in the same frame.

Remainder Range Partitioning







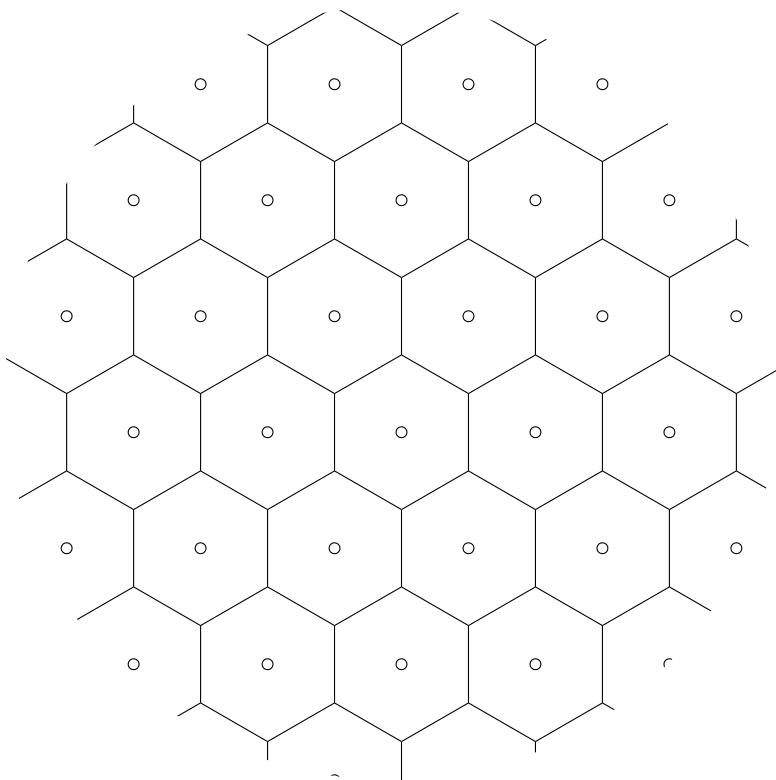


Contributions

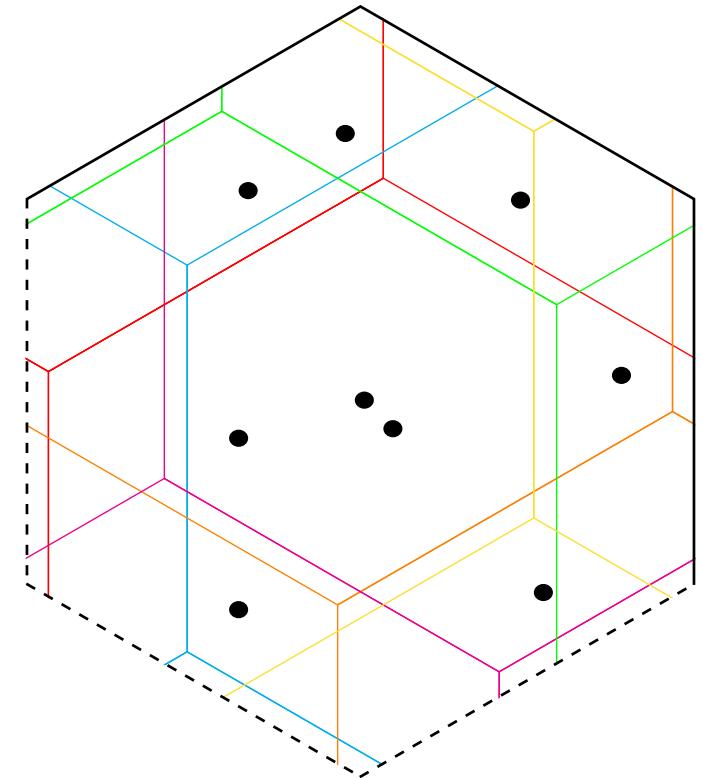
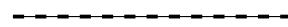
Or extracting the pure,
organic honey



Rational Rotations

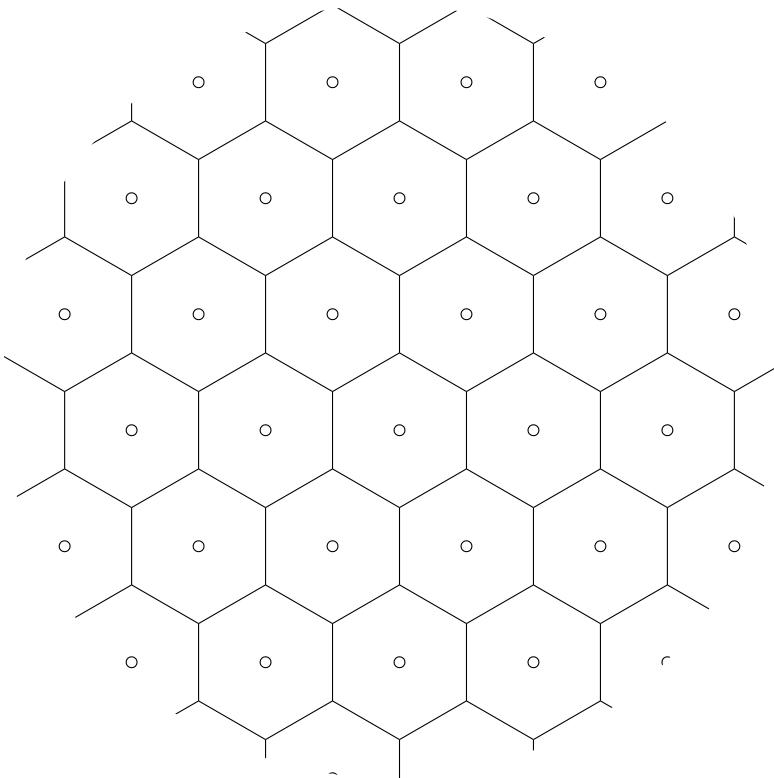


$$\mathcal{F}(\Lambda)$$

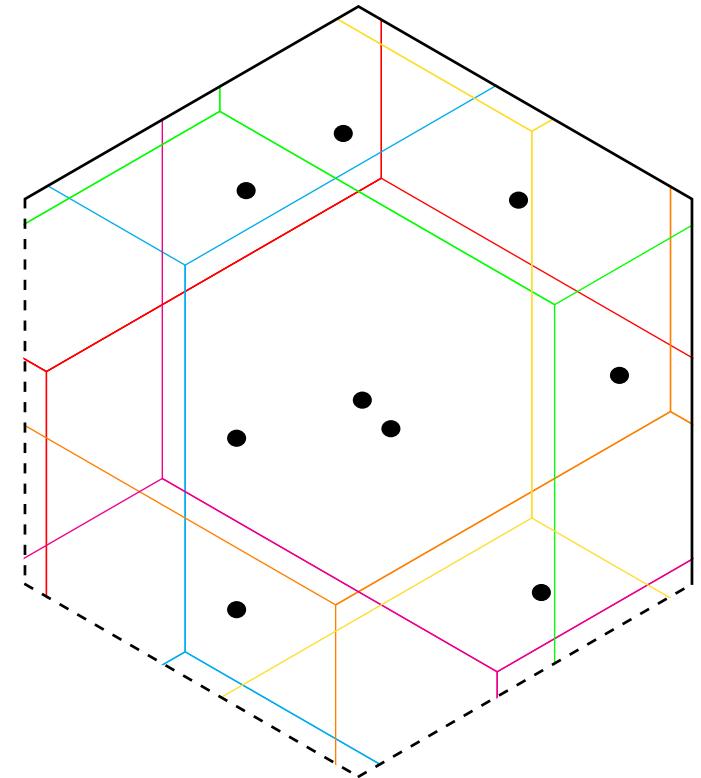
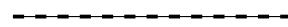


For what kind of parameters has the mapping a finite number of images?

Rational Rotations



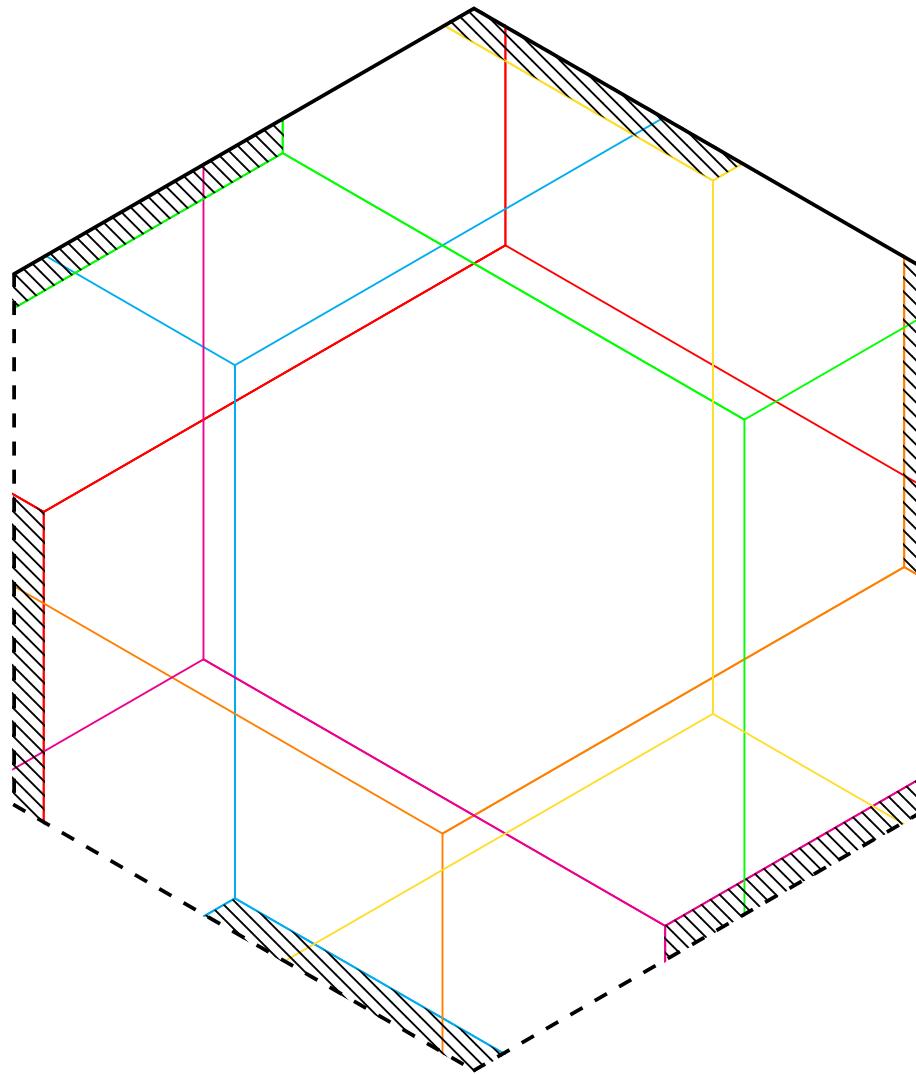
$\mathcal{F}(\Lambda)$



Corollary

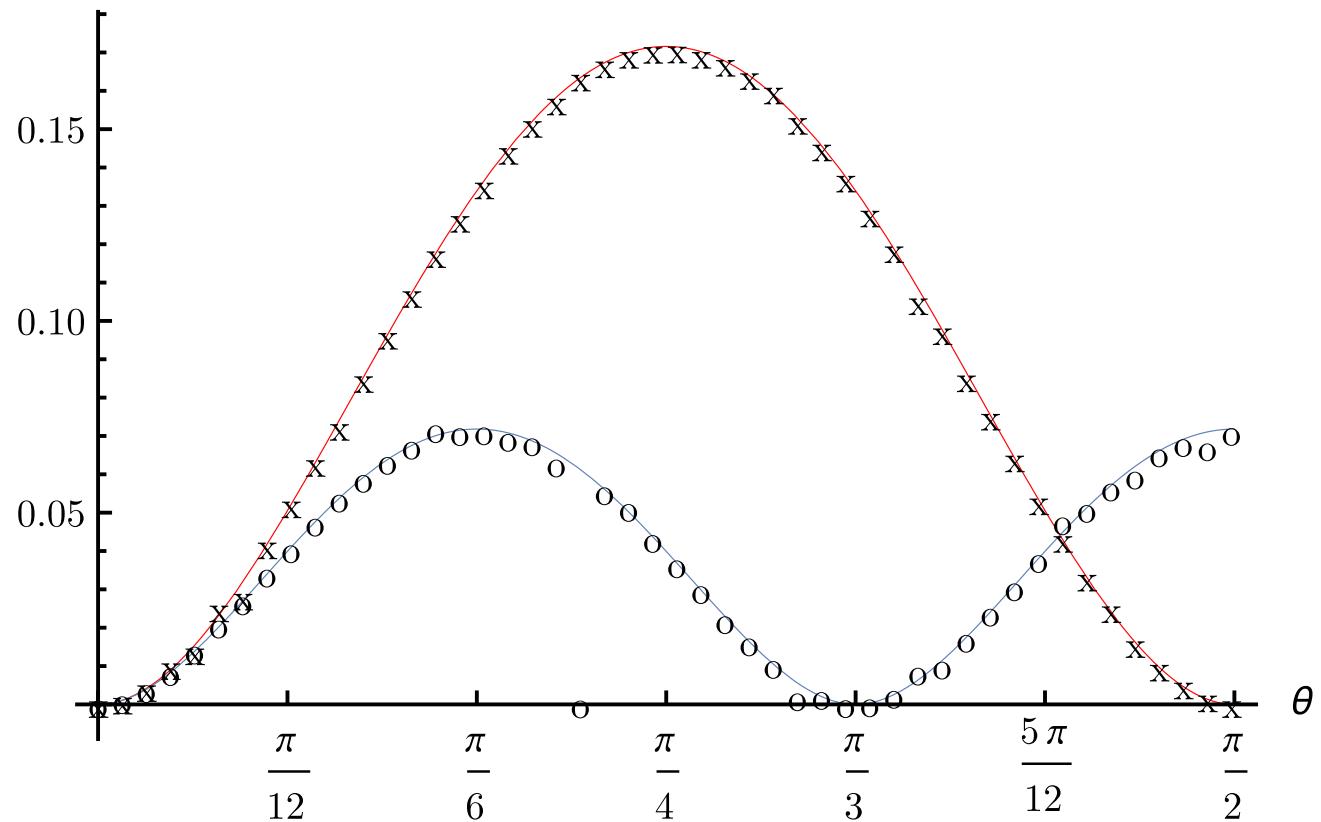
If $\cos \theta = \frac{2a-b}{2c}$ and $\sin \theta = \frac{\sqrt{3}b}{2c}$ where $(a, b, c) \in \mathbb{Z}^3$, $\gcd(a, b, c) = 1$ and $0 < a < c < b$, then the mapping has a finite number of images.

Non-injective Digitized Rigid Motions



Loss of Information

Information loss rate

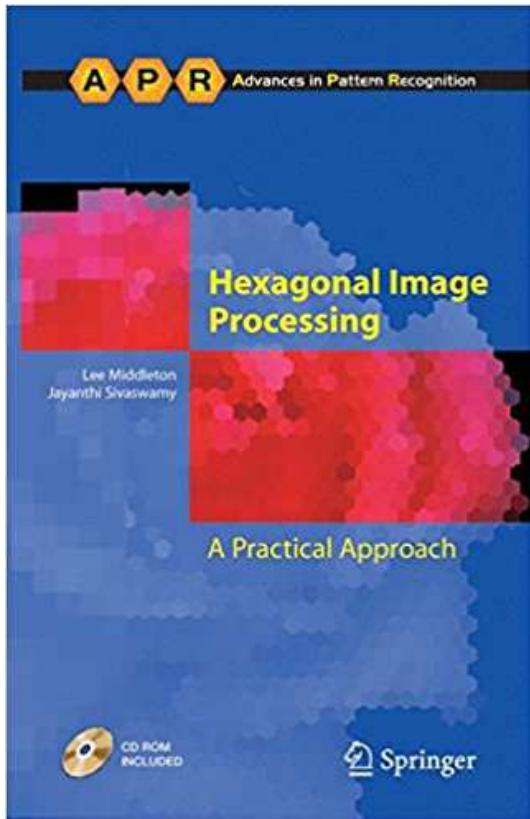


Conclusions & Perspectives

- An extension of a framework to study digitized rigid motions
- Characterization of rational rotations
- We have showed that the loss of information is relatively lower for digitized rigid motions defined on the hexagonal grid
- Our tools on BSD-3 license:
<https://github.com/copyme/NeighborhoodMotionMapsTools>

The humble bees have been working with David Cœurjolly, Tristan Roussillon and Victor Ostromoukhov of University Lyon 1, LIRIS on some new exciting results. Stay tuned...

Homework



If you want to get into the honey business,
then this book is an obligatory lecture:
Middleton, Lee, and Jayanthi Sivaswamy.
*Hexagonal image processing: A practical
approach*. Springer Science & Business Media,
2006.