3. Assignment 3

3.1. **Problem 1.** Let \mathcal{C} be a category with terminal object \star . Define $\widetilde{\mathcal{C}}$ to have objects $(f, x_0) \in \mathcal{C}(X, X) \times \mathcal{C}(\star, X)$ with arrows $(f, x_0) \to (g, y_0)$ given by \mathcal{C} -arrows $\varphi : X \to Y$ such that the following two diagrams commute.



- (a) Prove that $\widetilde{\mathcal{C}}$ is a category.
- (b) Show that the objects of **Set** can be seen as discrete dynamical systems

$$\begin{cases} x_{n+1} = f(x_n) \\ x_0 \end{cases}$$

- (c) Define the *successor* map as $s : \mathbb{N} \to \mathbb{N} :: n \mapsto n+1$. Prove that (s,0) is the initial object in $\widetilde{\mathbf{Set}}$.
- 3.2. **Problem 2.** Characterize the terminal object of \mathcal{C}/c and initial object of c/\mathcal{C} .
- 3.3. **Problem 3.** Let \mathcal{C} be a category with products such that $\mathcal{C}^{op} = \mathcal{C}$.
- (a) Prove that $\mathcal C$ has biproducts; i.e. that it has coproducts and that these are isomorphic to products.
- (b) Recall the category **Rel** consisting of objects sets and arrows $X \to Y$ subsets $R \subseteq X \times Y$, i.e. maps $X \times Y \to \mathbb{B}$. Writing xRy for $R(x,y) = \top$, we compose relations $R: X \to Y$ and $Q: Y \to Z$ by defining x(QR)z if and only if there exists y such that xRy and yQz. Prove that the disjoint union X + Y gives a product in this category.
- (c) Prove that $Rel = Rel^{op}$. Conclude that + gives a biproduct in Rel.
- (d) This allows us to conceive of any relation as a matrix of relations between singletons $\{x\}R\{y\}$. What should the entries of this matrix look like?
- (e) Interpret what matrix multiplication should mean in this context.
- 3.4. **Problem 4.** Show that any equalizer is a monomorphism. Argue by duality that any coequalizer is an epimorphism.
- 3.5. **Problem 5.** Let \mathcal{C} be a category with pullbacks and a terminal object \star .
- (a) Let X,Y be objects. Show that their product $X\times Y$ is isomorphic to the pullback of the following diagram.



(b) Let $f, g: X \to Y$ and consider the following pullback diagram

$$\begin{array}{c|c} X \times_Y X & \xrightarrow{\sigma_1} & X \\ \downarrow \sigma_2 & \downarrow & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

Prove that the equalizer of f, g is the pullback of the following diagram.

$$X \\ \downarrow^{(\sigma_1,\sigma_2)} \\ X \underset{(\overline{1_X},\overline{1_X})}{\longrightarrow} X \times X$$

Conclude that the existence of pullbacks and a terminal object implies the existence of products and equalizers. Now suppose \mathcal{C} has products and equalizers.

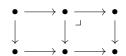
(c) Consider the following diagram

$$\begin{array}{c}
X \\
\downarrow f \\
Y \xrightarrow{g} Z
\end{array}$$

Show that the pullback of this diagram is isomorphic to the equalizer of the following pair of parallel arrows.

$$X \times Y \xrightarrow{\pi_X /\!\!/ f} Z$$

- (d) Argue, as in assignment 1, why terminal objects should be nullary products.
- (e) Conclude that a category \mathcal{C} has pullbacks and a terminal object if and only if it has products and equalizers. Use duality to argue why \mathcal{C} has pushouts and an initial object if and only if it has coproducts and coequalizers.
- 3.6. **Problem 6.** This problem is a classic. Consider the following commutative rectangle, whose righthand square is a pullback. Prove that the left hand square is a pullback if and only if the whole rectangle is a pullback.



- 3.7. **Problem 7.** Let $\overline{\mathbb{N}} = \mathbf{T}(\mathbb{N}, \leq)$. Consider the diagram $\mathcal{G} : \overline{\mathbb{N}} \to \mathbf{Set}$ defined by $\mathcal{G}n = \underline{1} + X + X^2 + \cdots + X^n$ and $\mathcal{G}[n \to n+1]$ the canonical inclusion map $1 + X + X^2 + \cdots + X^n \to 1 + X + X^2 + \cdots + X^n + X^{n+1}$. We show colim $G = \mathrm{List} X$.
- (a) Show that the maps

$$\iota_n: \mathbf{1} + X + \dots + X^n \to \operatorname{List} X :: (x_1, \dots, x_n) \mapsto [x_1, \dots, x_n],$$

where $\iota_0: \mathbf{1} \to \operatorname{List} X :: 0 \mapsto [\,]$, define a cocone to $\operatorname{List} X$. In other words, check the necessary commutativity requirements.

- (b) Now suppose there is some other cocone with components $\varphi_n : \mathbf{1} + \dots + X^n \to Z$. Define a map $\Phi : \text{List } X \to Z$ for which $\iota_n /\!\!/ \Phi = \varphi_n$.
- 3.8. **Problem 8.** Let (P, \preceq) be a preorder whose associated thin category is denoted by \overline{P} . Let $x \in P$ and consider the hom functor $\overline{P}(x, -) : P \to \mathbf{Set}$. This takes values either \star or \varnothing and hence can be reconceptualized as a monotone map:

$$\overline{P}(x,-): P \to \mathbb{B}.$$

(a) Show that $\mathcal{U}_x = \overline{P}(x, -)^*(\top)$.

(b) In this context, the Yoneda Lemma states:

$$\mathbf{Pre}(\overline{P}(x,-),\overline{P}(y,-))\cong \overline{P}(y,x)$$

Use this, in conjunction with (a), to prove

$$\mathcal{U}_x \subseteq \mathcal{U}_y \Rightarrow y \preceq x.$$

(c) Now suppose $\overline{P}(x,-)\cong \overline{P}(y,-)$. Show that this implies $x\cong y$.