

### 3. ASSIGNMENT 3

3.1. **Problem 1.** Let  $\mathcal{C}$  be a category with terminal object  $\star$ . Define  $\tilde{\mathcal{C}}$  to have objects  $(f, x_0) \in \mathcal{C}(X, X) \times \mathcal{C}(\star, X)$  with arrows  $(f, x_0) \rightarrow (g, y_0)$  given by  $\mathcal{C}$ -arrows  $\varphi : X \rightarrow Y$  such that the following two diagrams commute.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{\varphi} & Y \end{array} \qquad \begin{array}{ccc} \star & & \\ x_0 \downarrow & \searrow y_0 & \\ X & \xrightarrow{\varphi} & Y \end{array}$$

- (a) Prove that  $\tilde{\mathcal{C}}$  is a category.
- (b) Show that the objects of  $\mathbf{Set}$  can be seen as *discrete dynamical systems*

$$\begin{cases} x_{n+1} = f(x_n) \\ x_0 \end{cases}$$

- (c) Define the *successor* map as  $s : \mathbb{N} \rightarrow \mathbb{N} :: n \mapsto n + 1$ . Prove that  $(s, 0)$  is the initial object in  $\mathbf{Set}$ .

3.2. **Problem 2.** Characterize the terminal object of  $\mathcal{C}/c$  and initial object of  $c/\mathcal{C}$ .

3.3. **Problem 3.** Let  $\mathcal{C}$  be a category with products such that  $\mathcal{C}^{\text{op}} = \mathcal{C}$ .

- (a) Prove that  $\mathcal{C}$  has biproducts; i.e. that it has coproducts and that these are isomorphic to products.
- (b) Recall the category **Rel** consisting of objects sets and arrows  $X \rightarrow Y$  subsets  $R \subseteq X \times Y$ , i.e. maps  $X \times Y \rightarrow \mathbb{B}$ . Writing  $xRy$  for  $R(x, y) = \top$ , we compose relations  $R : X \rightarrow Y$  and  $Q : Y \rightarrow Z$  by defining  $x(QR)z$  if and only if there exists  $y$  such that  $xRy$  and  $yQz$ . Prove that the disjoint union  $X + Y$  gives a product in this category.
- (c) Prove that  $\mathbf{Rel} = \mathbf{Rel}^{\text{op}}$ . Conclude that  $+$  gives a biproduct in **Rel**.
- (d) This allows us to conceive of any relation as a matrix of relations between singletons  $\{x\}R\{y\}$ . What should the entries of this matrix look like?
- (e) Interpret what matrix multiplication should mean in this context.

3.4. **Problem 4.** Show that any equalizer is a monomorphism. Argue by duality that any coequalizer is an epimorphism.

3.5. **Problem 5.** Let  $\mathcal{C}$  be a category with pullbacks and a terminal object  $\star$ .

- (a) Let  $X, Y$  be objects. Show that their product  $X \times Y$  is isomorphic to the pullback of the following diagram.

$$\begin{array}{ccc} & X & \\ & \downarrow \exists! \varphi & \\ Y & \dashrightarrow \exists! \psi & \star \end{array}$$

- (b) Let  $f, g : X \rightarrow Y$  and consider the following pullback diagram

$$\begin{array}{ccccc} X \times_Y X & \xrightarrow{\sigma_1} & X & & \\ \sigma_2 \downarrow & \lrcorner & \downarrow f & & \\ X & \xrightarrow{g} & Y & & \end{array}$$

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Prove that the equalizer of  $f, g$  is the pullback of the following diagram.

$$\begin{array}{ccc} & X & \\ & \downarrow (\sigma_1, \sigma_2) & \\ X & \xrightarrow{(\mathbb{1}_X, \mathbb{1}_X)} & X \times X \end{array}$$

Conclude that the existence of pullbacks and a terminal object implies the existence of products and equalizers. Now suppose  $\mathcal{C}$  has products and equalizers.

(c) Consider the following diagram

$$\begin{array}{ccc} & X & \\ & \downarrow f & \\ Y & \xrightarrow{g} & Z \end{array}$$

Show that the pullback of this diagram is isomorphic to the equalizer of the following pair of parallel arrows.

$$X \times Y \begin{array}{c} \xrightarrow{\pi_X \parallel f} \\ \xrightarrow{\pi_Y \parallel g} \end{array} Z$$

- (d) Argue, as in assignment 1, why terminal objects should be nullary products.  
(e) Conclude that a category  $\mathcal{C}$  has pullbacks and a terminal object if and only if it has products and equalizers. Use duality to argue why  $\mathcal{C}$  has pushouts and an initial object if and only if it has coproducts and coequalizers.

**3.6. Problem 6.** This problem is a classic. Consider the following commutative rectangle, whose righthand square is a pullback. Prove that the left hand square is a pullback if and only if the whole rectangle is a pullback.

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

**3.7. Problem 7.** Let  $\overline{\mathbb{N}} = \mathbf{T}(\mathbb{N}, \leq)$ . Consider the diagram  $\mathcal{G} : \overline{\mathbb{N}} \rightarrow \mathbf{Set}$  defined by  $\mathcal{G}n = \mathbf{1} + X + X^2 + \cdots + X^n$  and  $\mathcal{G}[n \rightarrow n+1]$  the canonical inclusion map  $\mathbf{1} + X + X^2 + \cdots + X^n \rightarrow \mathbf{1} + X + X^2 + \cdots + X^n + X^{n+1}$ . We show  $\text{colim } \mathcal{G} = \text{List } X$ .

(a) Show that the maps

$$\iota_n : \mathbf{1} + X + \cdots + X^n \rightarrow \text{List } X :: (x_1, \dots, x_n) \mapsto [x_1, \dots, x_n],$$

where  $\iota_0 : \mathbf{1} \rightarrow \text{List } X :: 0 \mapsto []$ , define a cocone to  $\text{List } X$ . In other words, check the necessary commutativity requirements.

- (b) Now suppose there is some other cocone with components  $\varphi_n : \mathbf{1} + \cdots + X^n \rightarrow Z$ . Define a map  $\Phi : \text{List } X \rightarrow Z$  for which  $\iota_n \parallel \Phi = \varphi_n$ .

**3.8. Problem 8.** Let  $(P, \leq)$  be a preorder whose associated thin category is denoted by  $\overline{P}$ . Let  $x \in P$  and consider the hom functor  $\overline{P}(x, -) : P \rightarrow \mathbf{Set}$ . This takes values either  $\star$  or  $\emptyset$  and hence can be reconceptualized as a monotone map:

$$\overline{P}(x, -) : P \rightarrow \mathbb{B}.$$

- (a) Show that  $\mathcal{U}_x = \overline{P}(x, -)^*(\top)$ .

(b) In this context, the Yoneda Lemma states:

$$\mathbf{Pre}(\overline{P}(x, -), \overline{P}(y, -)) \cong \overline{P}(y, x)$$

Use this, in conjunction with (a), to prove

$$\mathcal{U}_x \subseteq \mathcal{U}_y \Rightarrow y \preceq x.$$

(c) Now suppose  $\overline{P}(x, -) \cong \overline{P}(y, -)$ . Show that this implies  $x \cong y$ .