1 Presentation of the rule of l'Hopital

L'Hôpital's rule is generally used to find the limit of fraction of undetermined form. For example, given $lim_{x\to+\infty}f(x)=0$ and $lim_{x\to+\infty}g(x)=0$, l'hopital is used to determine $lim_{x\to+\infty}(f/g)(x)$ by successive derivations of f and g. This theorem can be generalized to a lot of cases : $f(x)\to+\infty$, $g(x)\to+\infty$, etc

A general statement of this rule is the following :
Given $(b1,b2)\in\{(+\infty,+\infty),(+\infty,-\infty),(-\infty,+\infty),(-\infty,-\infty)(0,0)\}$ and $L\in\mathbb{R}\cup\{+\infty,-\infty\}$, $\forall a\in\mathbb{R}\cup+\infty,-\infty$, f and g are 2 continuous and derivable functions on a neighbourood of a (g is not 0 on this neighborood, g' as well), if $lim_{x\to a}f(x)=b1$, $lim_{x\to a}g(x)=b2$ and $lim_{x\to a}(f'/g')(x)=L$ then $lim_{x\to a}(f/g)(x)=L$.

2 Implementation of the rule

The general statement is not directy implementable in Coq as we want to use it in the Coq standard library of Reals. This library does not support using $+\infty$ and $-\infty$ as other Reals in what could be an extension of the $\mathbb{R}$ with new axioms for the infinities. So, we are obligated to define divergences in $-\infty$ and $+\infty$. As if it was not sufficient, we also need to distinguish $lim_{x\to a+}$ and $lim_{x\to a-}$ as these 2 limits can be sligthly different. For example, the inverse function on 0. We are also obligated to explicitly define the "neighborood".

So we need new definitions limits in $+\infty$ and $-\infty$. These definitions are exactly the one that can be found in a manual of maths.

Definition : $lim_{x\to a+}f(x)=+\infty$
$\forall a,b,m\in\mathbb{R},a<b,m>0\to$
$\exists\alpha\in\mathbb{R},\alpha>0$ and
$\forall x,x\in]a;b[\to|x-a|<\alpha\to m<f(x)$
Definition : $lim_{x\to b-}f(x)=+\infty$
$\forall a,b,m\in\mathbb{R},a<b,m>0\to$
$\exists\alpha\in\mathbb{R},\alpha>0$ and
$\forall x,x\in]a;b[\to|x-b|<\alpha\to m<f(x)$
Definition : $lim_{x\to a+}f(x)=-\infty$
$\forall a,b,m\in\mathbb{R},a<b,m>0\to$
$\exists\alpha\in\mathbb{R},\alpha>0$ and
$\forall x,x\in]a;b[\to|x-a|<\alpha\to -m>f(x)$
Definition : $lim_{x\to b-}f(x)=-\infty$
$\forall a,b,m\in\mathbb{R},a<b,m>0\to$
$\exists\alpha\in\mathbb{R},\alpha>0$ and
$\forall x,x\in]a;b[\to|x-b|<\alpha\to -m>f(x)$

3 Unnecessary hypotheses

While proving this rule, we found out that some hypotheses were useless : In the case of $g\to+\infty$, we did not need the hypotheses $f\to+\infty$. The l'Hopital rule would be true even if

the limit of f is finite which seems to be natural. But in Coq, this would avoid to the user to prove this statement.
The hypotheses "g′ is not 0 on a neighborood" is sometimes as well not needed. For example when $g \to +\infty$, it is clear that there exists a region near infinity where g is not going to be 0. We tried to avoid the user some work and avoid this hypothesis when possible.

4 Cases implemented

The file Hopital.v contains a proof of l'Hôpital's rule.

These tables shows all the cases implemented yet :

when $x \to a+$	L is finite	$L = +\infty$	$L = -\infty$
$g(x) \to 0$	Hopital_g0_Lfin_right	Hopital_g0_Lpinf_right	Hopital_g0_Lninf_right
$g(x) \to +\infty$	Hopital_gpinf_Lfin_right	Hopital_gpinf_Lpinf_right	Hopital_gpinf_Lninf_right
$g(x) \to -\infty$	Hopital_gninf_Lfin_right	Hopital_gninf_Lpinf_right	Hopital_gninf_Lninf_right

when $x \to a-$	L is finite	$L = +\infty$	$L = -\infty$
$g(x) \to 0$	Hopital_g0_Lfin_left	Hopital_g0_Lpinf_left	Hopital_g0_Lninf_left
$g(x) \to +\infty$	Hopital_gpinf_Lfin_left	Hopital_gpinf_Lpinf_left	Hopital_gpinf_Lninf_left
$g(x) \to -\infty$	Hopital_gninf_Lfin_left	Hopital_gninf_Lpinf_left	Hopital_gninf_Lninf_left