Derivation of the 3 Governing Equations

Satyam Sharma (2023MS10199) and Arnab Bhakta (2023MS10739)

(Part 1 of the course project APL104)

1. Equilibrium Equations

Body forces have been considered absent in each direction.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0,$$
$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$
$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0.$$

2. Stress-Strain Relations

The stress-strain relations (Hooke's law involving Lame's constants) are:

$$\sigma_{xx} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{xx},$$
$$\sigma_{yy} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{yy},$$
$$\sigma_{zz} = \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{zz},$$
$$\tau_{xy} = 2\mu\epsilon_{xy}, \quad \tau_{xz} = 2\mu\epsilon_{xz}, \quad \tau_{yz} = 2\mu\epsilon_{yz}.$$

3. Strain-Displacement Relations

The strain-displacement relations are:

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z},$$
$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),$$
$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right),$$
$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right).$$

4. Substitution and Simplification

Substitute the stress-strain and strain-displacement relations into the equilibrium equations. Consider the equation in the x-direction:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0.$$

Substituting for σ_{xx} and τ_{xy} using the stress-strain relations:

$$\frac{\partial}{\partial x}\left(\lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) + 2\mu\epsilon_{xx}\right) + \frac{\partial}{\partial y}(2\mu\epsilon_{xy}) + \frac{\partial}{\partial z}(2\mu\epsilon_{xz}) = 0.$$

Next, substitute the strain-displacement relations:

$$\frac{\partial}{\partial x}\left(\lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\right) = 0.$$

5. Governing Equations in Component Form

After expanding and simplifying, we get the equations in component form. For the x-direction, this becomes:

$$(\lambda + 2\mu)\frac{\partial^2 u_x}{\partial x^2} + \mu\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right) = 0.$$

Similar equations can be derived for the y- and z-directions.

$$(\lambda + 2\mu)\frac{\partial^2 u_y}{\partial y^2} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial y \partial z}\right) = 0.$$

$$(\lambda + 2\mu)\frac{\partial^2 u_z}{\partial z^2} + \mu\left(\frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial x^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z}\right) = 0.$$

These equations are known as the **Navier-Cauchy equations**

5. Considering a 2-D Linear Elastic Plate

$$(\lambda + 2\mu)\frac{\partial^2 u_x}{\partial x^2} + \mu\left(\frac{\partial^2 u_x}{\partial y^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u_y}{\partial x \partial y}\right) = 0.$$

$$(\lambda + 2\mu)\frac{\partial^2 u_y}{\partial y^2} + \mu\left(\frac{\partial^2 u_y}{\partial x^2}\right) + (\lambda + \mu)\left(\frac{\partial^2 u_x}{\partial x \partial y}\right) = 0.$$

These equations will be used further in the project for optimization.

Finding the Values of λ and μ using Optimization Algorithm

Satyam Sharma (2023MS10199) and Arnab Bhakta (2023MS10739)

(Part 2 of the course project APL104)

Algorithm

Algorithm 1 Optimization Algorithm for Finding Lame's Constants λ and μ

Require: Dataset with the values of x, y, u_x, and u_y
Ensure: Optimal Lame's constants λ and μ

1: **Initialize the values:**

- Represent u_x and u_y as polynomial functions of degree 3 using the dataset.
- Define boundary conditions σ_{xx} and σ_{yy} based on the reaction force at the edges of the plate.

2: **Calculate Derivatives:**

- Compute first derivatives:
$$\frac{\partial u_x}{\partial x}, \quad \frac{\partial u_y}{\partial y}, \quad \frac{\partial u_x}{\partial y}, \quad \frac{\partial u_y}{\partial x}$$
- Compute second derivatives:
$$\frac{\partial^2 u_x}{\partial x^2}, \quad \frac{\partial^2 u_y}{\partial y^2}, \quad \frac{\partial^2 u_x}{\partial y^2}, \quad \frac{\partial^2 u_y}{\partial x^2}, \quad \frac{\partial^2 u_x}{\partial x \partial y}, \quad \frac{\partial^2 u_y}{\partial y \partial x}$$

3: **Define the Objective Function(λ, μ):**

- Compute equilibrium equation residuals using the second derivatives in the governing equations.
- Compute stress residuals based on the boundary stress at $x = 1$ and $y = 1$. This is done by calculating the difference of the predicted and the experimental value.
- Return the sum the squares of residuals to obtain the cost function. This is the mean squared cost function.

4: **Optimize λ and μ:**

- Use the **Differential Evolution model** to minimize the cost function.
- Ensure that λ and μ are within the specified bounds, $[10^9, 10^{12}]$.
- Iteratively update λ and μ to minimize the cost function until convergence or maximum iterations.

Results

The values of λ and μ obtained for the different datasets provided are:

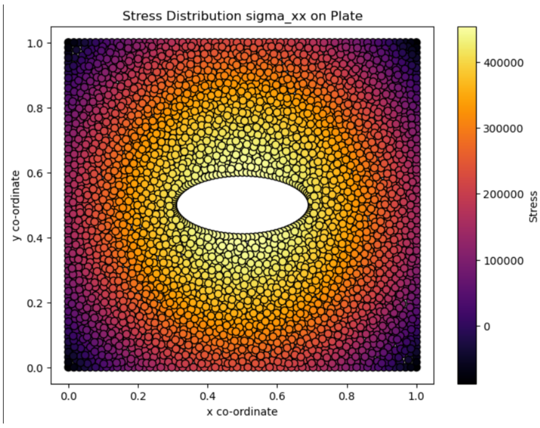
1. **Dataset 1**
   - $\lambda = 4541886348.864563$
   - $\mu = 15703673174.252502$

- residual = 1.2544330247099709

2. **Dataset 2**
   - $\lambda = 121933862225.64502$
   - $\mu = 25805948782.59845$
   - residual = 111.06089055607922
3. **Dataset 3**
   - $\lambda = 30425373432.63977$
   - $\mu = 5262632591.866699$
   - residual = 14.568533838443177
4. **Dataset 4**
   - $\lambda = 1080009497.5234375$
   - $\mu = 1021665510.3303223$
   - residual = 1.285803607306849
5. **Dataset 5**
   - $\lambda = 10990620982.570984$
   - $\mu = 5625329288.85437$
   - residual = 8.43108239846309

Heat Maps (can be accessed in the python notebooks)

November 15, 2024



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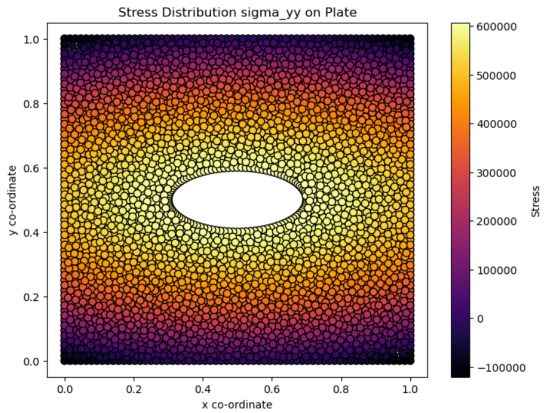

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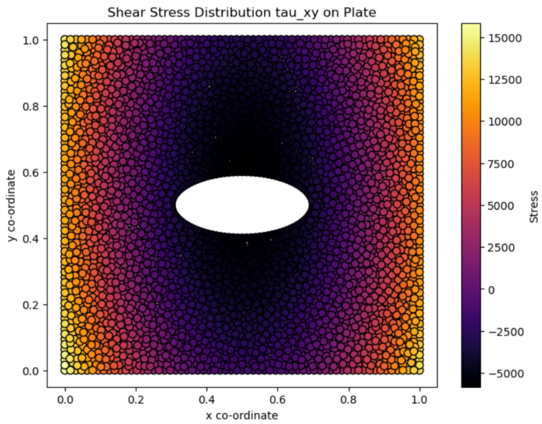

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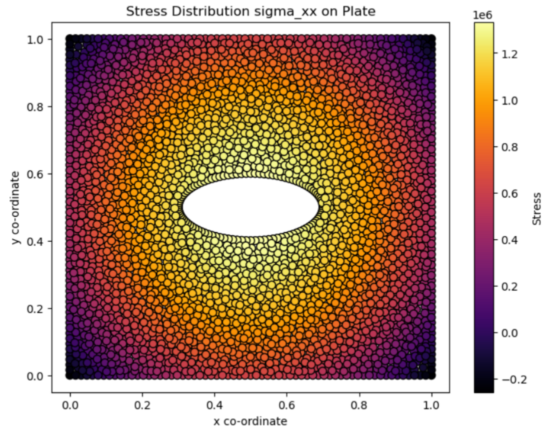

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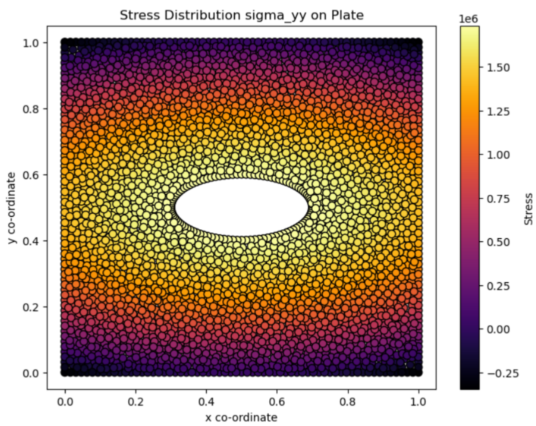

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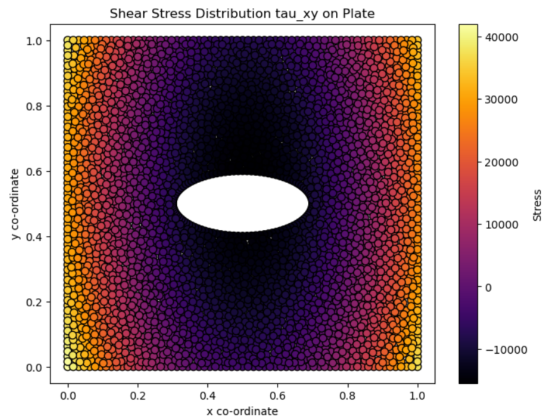

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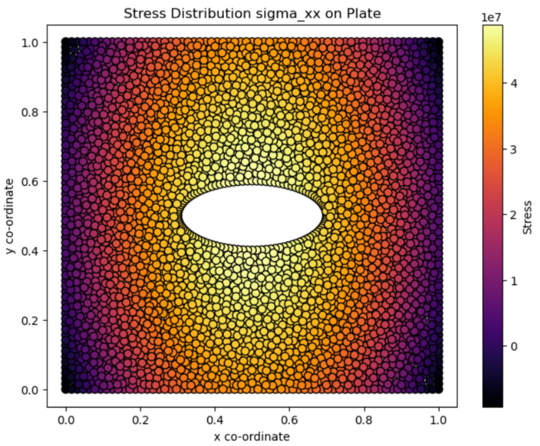

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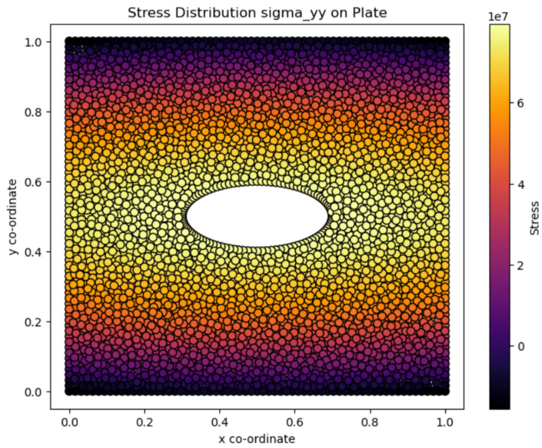

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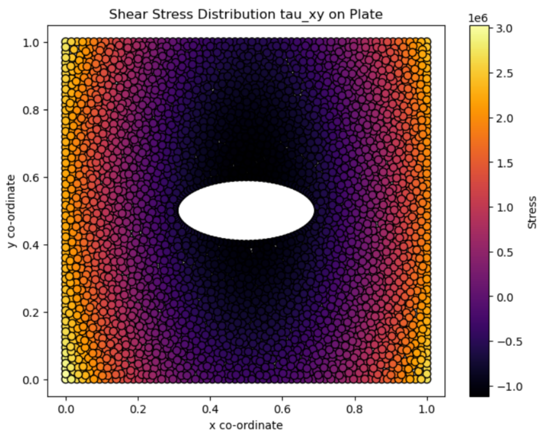

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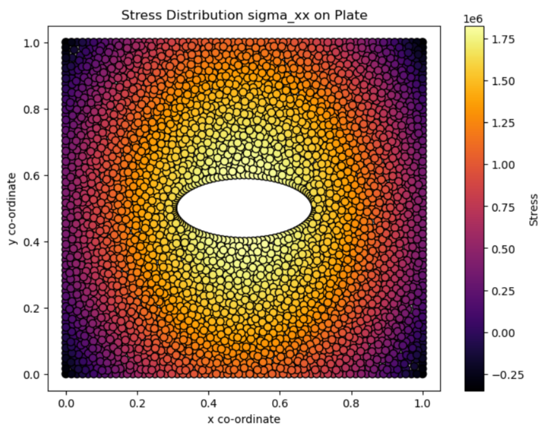

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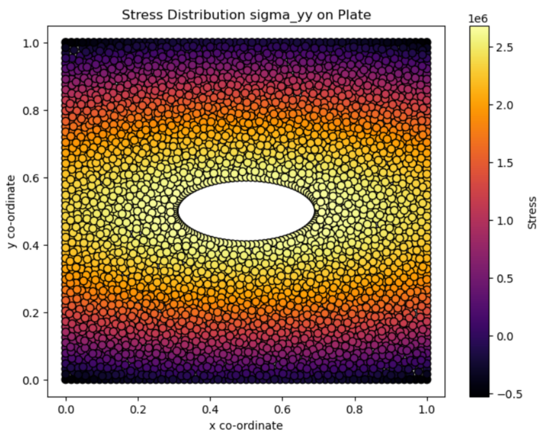

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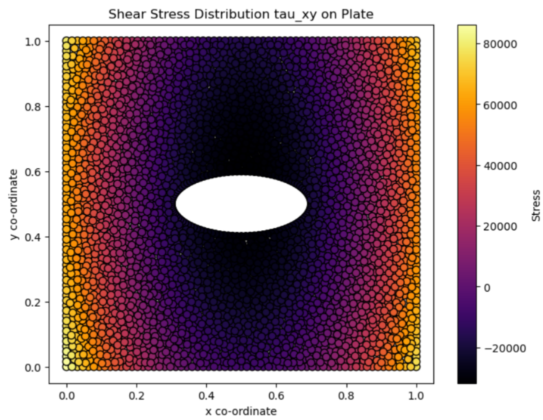

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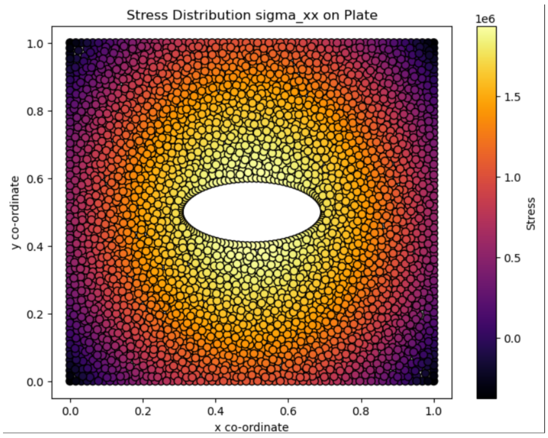
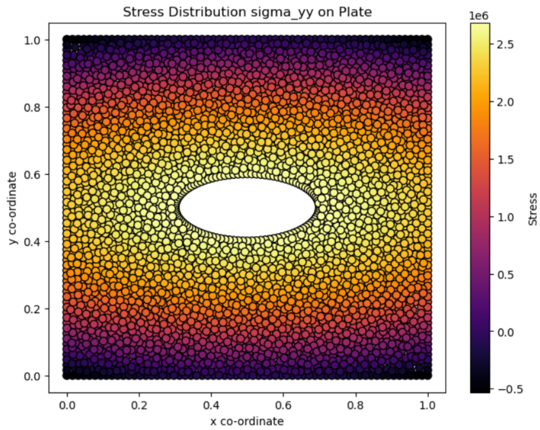
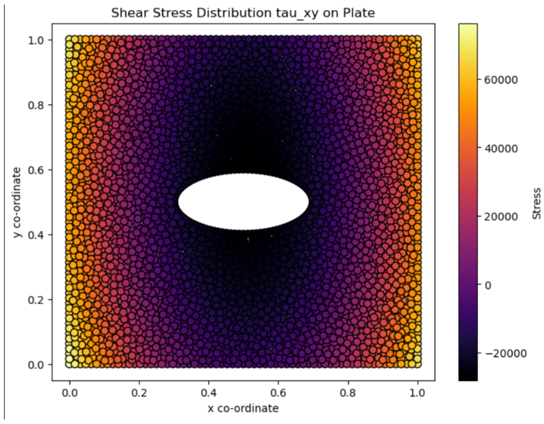

Stress Distribution sigma_xx on Plate
1e6
1.5
1.0
0.5
0.0
Stress
1.0
0.8
0.6
0.4
0.2
0.0
y co-ordinate
0.0
0.2
0.4
0.6
0.8
1.0
x co-ordinate

*L5


Stress Distribution sigma_yy on Plate
1e6
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
Stress
1.0
0.8
0.6
0.4
0.2
0.0
y co-ordinate
0.0
0.2
0.4
0.6
0.8
1.0
x co-ordinate

*L5


Shear Stress Distribution tau_xy on Plate
60000
40000
20000
0
−20000
Stress
1.0
0.8
0.6
0.4
0.2
0.0
y co-ordinate
0.0
0.2
0.4
0.6
0.8
1.0
x co-ordinate

*L5