Stochastic Modeling

Midterm 2 Study Guide

RV Review

Exponential RVs →

Exponential

- density ↓ pdf: $f(x)=\lambda e^{-\lambda x}$
- distribution ↓ cdf: $P(X\le x) = F(x) = 1-e^{-\lambda x}$
  - ↓ reverse: $P(X>x) = 1-F(x) = e^{-\lambda x}$
- rv ↓ has cdf + pdf and properties… $E[X]=1/\lambda$, $Var(X)=1/\lambda^2$; write as… $\sim exp(\lambda)$

Erlang

- density ↓ pdf: $f(x)=\lambda e^{-\lambda x}\frac{(\lambda x)^{k-1}}{(k-1)!}$
- distribution ↓ cdf: $F(x)= 1-\sum_{r=0}^{k-1} e^{-\lambda x}\frac{(\lambda x)^r}{r!}$
- rv ↓ has cdf + pdf + properties… $E[X]=\frac{k}{\lambda}$, $Var(X)=\frac{k}{\lambda^2}$; write as… $\sim erl(k,\lambda)$

more **Properties of Exponential**

- memoryless property → $P(X>t+s \mid X>s) = P(X>t)$
- $exp(\lambda)$ rv $\iff$ memoryless
- Sum of $exp(\lambda)$… $X_i \sim exp(\lambda)$ iid $i\in 1,\dots,n$; $Z_n = X_1+X_2+\dots+X_n$ $\Rightarrow Z_n \sim Erl(n,\lambda)$
- Min of $exp(\lambda_i)$… $X_i \sim exp(\lambda_i)$ iid $i\in 1\dots k$; $X=\min\{X_1,X_2,\dots,X_k\}$ $\Rightarrow X\sim exp(\lambda)$, $\lambda=\sum_{i=1}^{k}\lambda_i$
- still thinking about min… $X_i = X$, i = time first X occurs $= Z$; Z and X are indp, and… $P(Z=i, X>x) = P(Z=i)P(X>x) = \frac{\lambda_i}{\lambda}e^{-\lambda x}$

Possion RVs →

Poisson

- distribution ↓ pmf: $p_k = e^{-\lambda}\frac{\lambda^k}{k!}$
- rv ↓ has the pmf and properties… $E[X]=\lambda$, $Var(X)=\lambda$; write as… $\sim P_o(\lambda)$

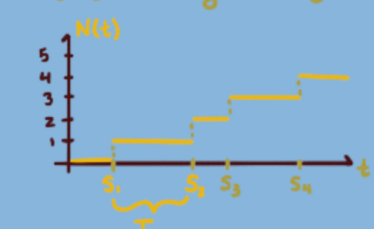
more **Properties of Poissons**

- a poisson is a binomial rv where $n\to\infty$ and $p\to 0$… $\lim_{\substack{n\to\infty\\ p\to 0}} \binom{n}{k}p^k(1-p)^{n-k}$
- Sum of $P_o(\lambda_i)$… $X_i\sim P_o(\lambda_i)$ iid $i\in 1\dots n$; $Z_n = X_1+X_2+\dots+X_n$ $\Rightarrow Z_n\sim P_o(\lambda)$, $\lambda=\sum_{i=1}^{n}\lambda_i$

Building Blocks of CTMCs

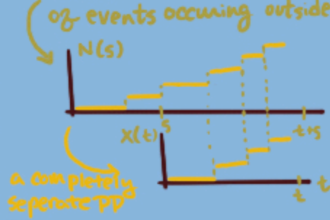
Poisson Process →

- we're tracking times of occurences in a system…



t = time
S_n = occurence time of n^{th} event
$T_n = S_n - S_{n-1}$ = time between n^{th} and $n-1^{th}$ event
$N(t) = \max\{n\ge 0 \mid S_n \le t\}$ = total # of events during interval $(0,t]$

- what does this have to do w/ Poisson… if $T_n\sim exp(\lambda)$ iid… then $N(t)$ is a Poisson Process aka $\{N(t), t\ge 0\}\sim PP(\lambda)$
- Poisson Process → if $N(t)$ is $PP(\lambda)$ that means for a given t $N(t)\sim P_o(\lambda t)$
  and… $P(N(t)=k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$, $k\ge 0$
  and… $E[N(t)]=\lambda t$ $Var(N(t))=\lambda t$
- Markov Property → $\{N(t), t\ge 0\}\sim PP(\lambda)$, then it has markov property at each time t.
  > $P(N(t+s)=k \mid N(s)=j, N(u), 0\le u\le s) = P(N(t+s)=k \mid N(s)=j)$
  aka the distribution over any interval $(s,t+s]$ is indp. of events occuring outside the interval (ie $(0,s]$)



independent! $N(t+s)-N(s)\sim P_o(\lambda t)$
"stationary + indp. increments"

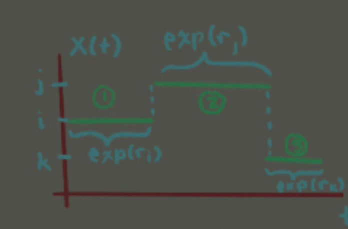
Thinning →

- we have a $N(t)$ defined by a $PP(\lambda)$. Some events are special, w/ prob. p
- Thinning → $R(t)$ = # of special events over $(0,t]$; $\{R(t), t\ge 0\}\sim PP(\lambda p)$
- Splitting → just the unspecial events; indp. of $R(t)$; $\{N(t)-R(t), t\ge 0\}\sim PP(\lambda(1-p))$

CTMCs

Contiuous-Time Markov Chains →

- we want to look at a system like a Discrete Time Markov Chain but observed continuously…
  $X(t)$ = state at time t → $\{X(t), t\ge 0\}$ → on state space S → Continuous Time Markov Chain
  if… $P(X(s+t)=j \mid X(s)=i, X(u); 0\le u\le s) = P(X(s+t)=j \mid X(s)=i)$
  time homogeneous if… $P(X(s+t)=j \mid X(s)=i) = P(X(t)=j \mid X(0)=i)$
  Transition Prob. $p_{i,j}(t) = P(X(t)=j \mid X(0)=i)$
  Transition Prob. Matrix $P(t) = \begin{bmatrix} p_{1,1}(t) & \cdots & p_{1,N}(t)\\ \vdots & & \vdots \\ p_{N,1}(t) & \cdots & p_{N,N}(t)\end{bmatrix}$ notice, a function of t
  $P(t+s) = P(t)P(s) = P(s)P(t)$ → transition prob. matrices commute!
- lets explore transition prob. a bit more… $\{X(t), t\ge 0\}$
  1. starts in state i; we measure time between transitions so $p_{i,i}=0$; stays in state i for $exp(r_i)$ amount of time; amount of time called sojourn time in state i
  2. at end of sojourn time system transitions to state $j\ne i$ w/ prob $p_{i,j}$; $p_{i,j}$ = prob that system moves to state j after leaving i. $\ne p_{ij}(t)$
  3. stays in state j for an $exp(r_j)$ amount of time then moves to state $k\ne j$. and so on…



- lets describe CTMC's w/ these r_i's and $p_{i,j}$'s, since $p_{ij}(t)$ is too messy…
  r_i = parameter in $exp(r_i)$ descr. amount of time in state i
  $p_{i,j}$ = prob. of a system moving to state j after it leaves state i
  $r_{i,j} = r_i\cdot p_{i,j}$ = transition rate from i to j; $r_i=\sum_{j=1}^{N} r_{i,j}$; $p_{i,j}=\frac{r_{i,j}}{r_i}$, if $r_i\ne 0$
  $\rightsquigarrow R = \begin{bmatrix} r_{1,1} & \cdots & r_{1,N}\\ \vdots & & \vdots\\ r_{N,1} & \cdots & r_{N,N}\end{bmatrix}$ Rate Matrix
- Poisson Processes come into the mix since they are used to describe entrances and exits from a system

Common CTMC's →

- Two state machine → either operating or failed
- Single Server Queue → arrival $\sim PP(\lambda)$, service $\sim exp(\mu)$, capacity $=K$
- Finite Birth + Death Process → births → transitions from i to $i+1$ → λ_i param. deaths → transitions from i to $i-1$ → μ_i param.

CTMC Applications

Transient Analysis →

- we have our rate matrix, but what if we want $P(X(t)=j)$/a pmf of $X(t)$? or $P(t)$?
- Transient Distribution $= P(X(t)=j) = \sum_{i=1}^{N} P(X(t)=j \mid X(0)=i)P(X(0)=i)$; $p_{i,j}(t)$ → we need to find this…
- how to find Matrix $P(t)$ →
  1) define $r\ge\max\{r_1,r_2,\dots,r_N\}$
  $\hat{p}_{i,j} = \begin{cases} 1-\frac{r_i}{r}, & \text{if } i=j\\ \frac{r_{i,j}}{r}, & \text{if } i\ne j\end{cases}$ $\hat{P} = [\hat{p}_{i,j}]$ ↳ stochastic!
  2) solve… $P(t) = \sum_{k=0}^{\infty} e^{-rt}\frac{(rt)^k}{k!}\hat{P}^k$
  (typically use a computer)

Occupancy Times →

- When we're looking at a CTMC over a finite interval $(0,T]$, we might want to know how long the system is in a state j…
- $m_{i,j}(T)$ = expected amount of time spent in j over $[0,T]$ = *occupancy time* of state j until time T starting in i $= \int_0^T p_{i,j}(t)\,dt$ → since we typically don't know $P(t)$ we can use the ∞ series… ↳ we compute w/ Matlab.
- $M(T) = [m_{i,j}(T)]$

Limiting Behavior →

- We know of limiting distributions in DTMCs, like in DTMC's we need to define *irreducibility* → A CTMC is irreducible if its corresponding DTMC is irrd.
- but we don't need to worry about periodicity w/ CTMC's
- Limiting Distribution →
  > $p_j = \lim_{t\to\infty} P(X(t)=j)$ ← if this limit exists it's the *limiting probability*
  $\vec{p} = [p_1, \dots p_j \dots, p_N]$ ← if this vector exists it's the *limiting distribution*
  > an irreducible CTMC has a *unique limiting distribution* $\vec{p}$
  > we can find it by solving…
  $\underbrace{p_j r_j}_{\text{rate leaving } j} = \underbrace{\sum_{i=1}^{N} p_i r_{i,j}}_{\text{rate entering } j}$, $1\le j\le N$ & $1=\sum_{i=1}^{N}p_i$
- Stationary Distribution → if the limiting distribution $\vec{p}$ exists for a CTMC then that is also the *stationary distribution*. If the stationary distribution is chosen for the initial dist. the transient dist. will be $\vec{p}$ for all time.
- Occupancy Distribution → if the limiting dist. $\vec{p}$ exists for a CTMC than that is also the *occupancy distribution* aka the long run fraction of time a CTMC stays in a certain state.

Cost Models →

- lets define… $c(i)$ → when a ctmc is in state i it incurs costs at rate $c(i)$.
- Expected Total Cost up to finite time T → $\vec{g}(T) = M(T)\vec{c}$; $\vec{c} = \begin{bmatrix} c(1)\\ \vdots\\ c(N)\end{bmatrix}$; $M(T)$: occupancy matrix; $\vec{g}(T) = \begin{bmatrix} g(1,T)\\ \vdots \\ g(N,T)\end{bmatrix}$ starting state
- Long Run Cost Rate per unit time → $g = \sum_{j=1}^{N} p_j c(j) = \vec{p}\,\vec{c}$ → cost vector; $\vec{p}$ → limiting dist.; does not depend on initial state

First Passage Time →

- we want to know the first time a model passes into a set of states A…
- First Passage Time → m_i = expected first passage time, find by solving system of eq…
  $r_i m_i(A) = 1 + \sum_{j\notin A} r_{i,j} m_j(A)$, $i\notin A$

good luck ♡