good luck

optimization

midterm 2

the garden

We've seen the things we can do with LPs in the flower shop now lets look at some of the processes behind those examples!

Linear Algebra Review

> we know about matrices and how to perform operations on them, now lets look at using matrices to solve systems of linear equations

Gauss-Jordan Elimination

> we have a system of linear equations we've turned into a matrix·vector problem...

vector of variables $A\vec{x} = \vec{b}$ numbers

> we want to assign each variable in $\vec{x}$ to a numerical value! we do this by reducing A to an identity matrix with gauss elimination...

> 3 ways to reduce A...

① switch two rows location ② multiply a row by a constant ③ add a multiple of a row to another row

Ex. Gauss-Jordan Elimination

$A\vec{x} = \vec{b} \rightarrow \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 1 & 0 & 5 \end{array}\right]$ ①R_2 $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 2 & 3 & 4 \end{array}\right]$ ③R_2 $R_2 - 2R_1$ $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 3 & -6 \end{array}\right]$ ②R_2 $\frac{1}{3}R_2$ $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right]$

so... $x_1 = 0$ $x_2 = -2$

Inverse of A

> we define the inverse of A as... inverse of A $A \cdot A^{-1} = I$

> if we treat A^{-1} as an unknown we can solve this equation w/ Gauss Jordan elimination as well!

Ex. Inverse of A

$A \cdot A^{-1} = I \rightarrow \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$ ①R_2 $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right]$ ③R_2 $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & -2 \end{array}\right]$ ②R_2 $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \end{array}\right]$

so... $A^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

notice, same opperations as previous example!

Possible LP Outcomes

> we know from solving LPs previously that there are 3 general outcomes. Lets take a closer look at each...

Infeasible

the constraints don't overlap

> an LP is infeasible if there is no feasible area.

> we can tell from the graph that this LP is infeasible but how can we tell for larger LPs?

> we need a **Certificate of Infeasibility**

- given an LP max $\vec{c}^T\vec{x}$ s.t. $A\vec{x} = \vec{b}$, $\vec{x} \geq 0$
- suppose we find a vector $\vec{y}$ such that... $\vec{y}^T A \geq 0$, $\vec{y}^T\vec{b} < 0$
- if an LP is infeasible than a $\vec{y}$ that meets the above requirements exists.
- find $\vec{y}$ by solving... min $\vec{y}^T\vec{b}$ s.t. $\vec{y}^T A \geq 0$ — if > 0 we know LP is infeas.

Unbounded

the feasible region goes on forever

> an LP is unbounded if the feasible area goes on forever in 1 direction.

> again, its easy to see its unbounded in the graph, but how do we tell for bigger LPs?

> we need a **certificate of unboundedness**

- lets say we have a standard LP and a feasible solution $\vec{x}$.
- we want a $\vec{d}$ such that... $\vec{x} + t\vec{d}$ is feasible for $t \geq 0$, $\lim_{t\to+\infty} \vec{c}^T(\vec{x} + t\vec{d}) \to +\infty$
- if the LP is unbounded such a $\vec{d}$ exists and... $\vec{d} \geq 0$, $\vec{c}^T\vec{d} > 0$, $A\vec{d} = 0$
- we can find $\vec{d}$ by solving... $A\vec{d} = \vec{0}$

Ex. Infeasible LP

$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

we think this is infeasible, so lets find a $\vec{y}$ to prove it...

- we can solve... min $\vec{y}^T\begin{bmatrix} -3 \\ 1 \end{bmatrix} = -3y_1 + y_2$ s.t. $\vec{y}^T\begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} = [y_1,\ \frac{1}{2}y_2 - y_1,\ 2y_2 - y_1] \geq 0$
- we could solve it with simplex, or we can just guess, we need... $-3y_1 + y_2 < 0$ and $[y_1,\ \frac{1}{2}y_2 - y_1,\ 2y_2 - y_1] \geq 0$

so... $y_1 = 1$ and $y_2 = 2$ should work!

$\vec{y}^T = [1\ 2] \Rightarrow \vec{y}^T\vec{b} = -1 < 0$

$\Rightarrow \vec{y}^T A = [1, 0, 3] \geq 0$

so $\vec{y}^T$ is a certificate of infeasibility!

Ex. Unbounded LP

$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\vec{c}^T = [0\ 1\ 0\ 0]$

we think this LP is unbounded, lets find a $\vec{d}$ to prove it...

$A\vec{d} = \vec{0}$

$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \end{array}\right]$ $R_2 + R_1$ $\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$ $R_1 - R_2$ $\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

$\Rightarrow d_1 - d_2 - d_4 = 0$, $d_3 + d_4 = 0$

$\Rightarrow d_1 = d_2 + d_4$, $d_3 = -d_4$

$\vec{c}^T\vec{d} = d_2 > 0$, $\vec{d} \geq 0$

working w/ these restrictions we can set... $\vec{d} = [1\ 1\ 0\ 0]$

if we wanted to show the t we would find... $\vec{x} = [2\ 0\ 2\ 2]$ and write... $\vec{x} + t\vec{d}$

Optimal Solution

a bounded feasible region

> an LP has an optimal solution if the feasible region is bounded.

> Again, how do we solve without a graph...

> The Simplex Method!

Standard Equality Form

> before we can learn the simplex algorithm we need to add a couple tools to our toolkit, the first is Standard Equality Form...

SEF

> An LP is in SEF if it looks like...

max $\vec{c}^T\vec{x} + \bar{z}$ — $\bar{z}$ is a scalar constant, can be 0

s.t. $A\vec{x} = \vec{b}$

$\vec{x} \geq 0$

also max and =

> every linear program can be brought into SEF...

Steps for putting in SEF

① "what if we have min and not max?"

min $\vec{c}^T\vec{x} \Rightarrow$ max $(-\vec{c}^T)\vec{x}$

② "what if there are inequalities"

$\vec{a}^T\vec{x} \leq \beta \Rightarrow \vec{a}^T\vec{x} + s = \beta$

s is called a slack variable, we need a seperate slack variable for each constraint.

③ "what if there are inequalities"

$\vec{a}^T\vec{x} \geq \beta \Rightarrow \vec{a}^T\vec{x} - s = \beta$

this is a surplus variable

④ "what if x_i is a free variable"

$x_i \Rightarrow x_i^+ - x_i^-$, $x_i^+ \geq 0$ $x_i^- \geq 0$

this works since any # can be written as the dif. of 2 nonneg. #s.

Ex. Rewriting in SEF

LP → min $x_2 - 2x_3$

s.t. $x_1 + x_2 - 2x_3 \leq 1$

$2x_1 - x_2 = 0$

$x_1 + x_3 \geq 0$

$x_2 \leq 5$

$x_1, x_2 \geq 0$

lets rewrite in SEF!

① min $x_2 - 2x_3 \Rightarrow$ max $-x_2 + 2x_3$

② create slack variables...

$x_1 + x_2 - 2x_3 \leq 1 \Rightarrow x_1 + x_2 - 2x_3 + s_1 = 1$

$x_2 \leq 5 \Rightarrow x_2 + s_2 = 5$

③ create surplus variables

$x_1 + x_3 \geq 0 \Rightarrow x_1 + x_3 + s_3 = 0$

④ fix free variables

$x_3 \Rightarrow x_3^+ - x_3^-$, $x_3^+ \geq 0$ $x_3^- \geq 0$

max $-x_2 + 2x_3 \Rightarrow$ max $-x_2 + 2x_3^+ - 2x_3^-$

$x_1 + x_2 - 2x_3 + s_1 = 1 \Rightarrow x_1 + x_2 - 2x_3^+ + 2x_3^- + s_1 = 1$

$x_1 + x_3 + s_3 = 0 \Rightarrow x_1 + x_3^+ - x_3^- + s_3 = 0$

So our new LP is...

min $-x_2 + 2x_3^+ - x_3^-$

s.t. $x_1 + x_2 - 2x_3^+ + 2x_3^- + s_1 = 1$

$2x_1 - x_2 = 0$

$x_1 + x_3^+ - x_3^- + s_3 = 0$

$x_2 + s_2 = 5$

$x_1, x_2, x_3^+, x_3^-, s_1, s_2, s_3 \geq 0$

Basis and Canonical Form

> the second tool we need to add is the concepts of basis and canonical form. Canonical form will allow us to more easily identify a solution...

Basis

> we say B is a set of numbers and A_B is the submatrix of A w/ rows corresponding to the #s in B.

> we say B is a **Basis** if...

A_B is square and,

A_B^{-1} exists

> we say x_j is a **basic variable** if $j \in B$ and $\vec{x}_B$ is the subvector of $\vec{x}$ w/ all basic variables.

> N is a set made of the #s not in B, A_N is all the columns in A that aren't in A_B, similarly $\vec{x}_N$ is all variables not in $\vec{x}_B$.

> we say $\bar{\vec{x}}$ is a **basic solution** if...

$A\bar{\vec{x}} = \vec{b}$ and,

all $\bar{\vec{x}}_N = 0$

Canonical Form

> given an LP and a basis B, the LP is in **canonical form** for B if...

$A_B = I$

$\vec{c}_B = \vec{0}$

> given a basis B we can put any LP into canonical form...

Ex. Basis

$A = \begin{bmatrix} 2 & 3 & 7 & 0 & 0 & -2 & 1 \\ 0 & 2 & 4 & 4 & 5 & 4 & 0 \\ 0 & 4 & 1 & 0 & 1 & -1 & 2 \end{bmatrix}$

if we take $B = \{2, 3\}$ we get...

$A_B = \begin{bmatrix} 3 & 7 \\ 2 & 4 \\ 4 & 1 \end{bmatrix}$

this is not a basis because it's not square.

but if we take $B = \{2, 3, 5\}$...

$A_B = \begin{bmatrix} 3 & 7 & 0 \\ 2 & 4 & 5 \\ 4 & 1 & 1 \end{bmatrix}$

this is a basis because A_B is a square and it's invertible.

Converting to Canonical Form

> given a basis B we can put any LP into an equivalent canonical form...

① lets start by achieving $A_B = I$

lets say...

$A\vec{x} = \vec{b} \Rightarrow \underbrace{A_B^{-1}A}_{\text{"new A"}}\vec{x} = \underbrace{A_B^{-1}\vec{b}}_{\text{"new B"}}$

we know these two equations are equivalent since A_B^{-1} is invertible

And we know this will make $A_B = I$ since $A_B A_B^{-1} = I$

② now we need $\vec{c}_B = \vec{0}$

we have $z(x) = \vec{c}^T\vec{x}$ and $0 = \vec{b} - A\vec{x}$, we can combine these without changing the value of $z(x)$...

$z(x) + \vec{y}^T(\vec{b} - A\vec{x}) = \vec{y}^T\vec{b} - \vec{y}^T A\vec{x} + \vec{c}^T\vec{x}$

$z(x) + \vec{y}^T(0) = \vec{y}^T\vec{b} + (\vec{c}^T - \vec{y}^T A)\vec{x}$

$z(x) = \vec{y}^T\vec{b} + \underbrace{(\vec{c}^T - \vec{y}^T A)}_{\text{"new c"}}\vec{x}$

now we need to choose this $\vec{y}$ to ensure $\vec{c}_B = \vec{0}$

$\underbrace{\vec{c}_B^T - \vec{y}^T A_B}_{\text{"new } c_B\text{"}} = 0 \Rightarrow \vec{c}_B^T A_B^{-1} = \vec{y}^T$

Ex. Converting to Canonical Form

$A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\vec{c}^T = [1\ 1\ 0\ 0]$

LP → max $\vec{c}^T\vec{x}$ s.t. $A\vec{x} = \vec{b}$, $\vec{x} \geq 0$

lets convert this to canonical form for $B = \{1, 3\}$

① $A_B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ $A_B^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

$A_B^{-1}A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{7}{2} & 1 & -\frac{1}{2} \end{bmatrix}$

$A_B^{-1}\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

so we have...

$\underbrace{\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{7}{2} & 1 & -\frac{1}{2} \end{bmatrix}}_{\text{"new A"}}\vec{x} = \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\text{"new B"}}$

② $\vec{c}_B^T = [1\ 0]$

$\vec{c}_B^T A_B^{-1} = \vec{y}^T = [0\ \frac{1}{2}]$

$\vec{c}^T - \vec{y}^T A = [0\ \frac{1}{2}\ 0\ \frac{1}{2}]$

$\vec{y}^T\vec{b} = [0\ \frac{1}{2}]\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1$

So we have...

$1 + \underbrace{[0\ \frac{1}{2}\ 0\ \frac{1}{2}]}_{\text{"new c"}}\vec{x}$

Canonical Form

max $1 + [0\ \frac{1}{2}\ 0\ \frac{1}{2}]\vec{x}$

s.t. $\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{7}{2} & 1 & -\frac{1}{2} \end{bmatrix}\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{x} \geq 0$

Simplex Method

> but why do we want an LP in the canonical form? It will be more apparent how useful canonical form is when we use the **Simplex Method** for solving LP's...

Steps of Simplex

① we want to make sure the LP we are solving is in SEF... max $z(x) = \bar{z} + \vec{c}^T\vec{x}$ s.t. $A\vec{x} = \vec{b}$, $\vec{x} \geq 0$

select a basis B.

② if the basis is not yet in canonical form convert it to canonical form...

$A_B = I$ and $\vec{c}_B = \vec{0}$

Find a basic feasible solution $\bar{\vec{x}}$ such that...

$\bar{\vec{x}}_B = \vec{b}$ and $\bar{\vec{x}}_N = \vec{0}$

if $\vec{c}_N \leq 0$ go to step 4

if $\vec{c}_N \nleq 0$ and there's a column $k \in N$ where $c_k > 0$ and A_k has at least 1 positive element go to step 3.

if $\vec{c}_N \nleq 0$ and every column $k \in N$ where $c_k > 0$ has $A_k \leq 0$ go to step 5.

③ we choose one variable k to enter our basis B, and solve...

$\vec{x}_B I + x_k A_k = \vec{b} \rightsquigarrow x_{B_i} = b_i - A_{k,i}x_k \Rightarrow x_k \leq \frac{b_i}{A_{k,i}}$ ("new A", "new B")

For all i in B. If $A_{k,i} < 0 \Rightarrow x \leq +\infty$, and we can ignore those. we should end up with a couple upper bounds for x_k, we should choose the largest value that satisfies them all.

x_k will enter the basis and $x_{B_i} = 0$ will exit.

Repeat step 2 with new basis B.

④ if $\vec{c}_N \leq 0$, then $\bar{\vec{x}}$ is the optimal solution and $\vec{y}^T\vec{b}$ is the optimal value.

The Simplex Method ends here.

⑤ if $A_k \leq 0$ then the LP is unbounded. We construct a certificate of unboundedness $\vec{d}$ to prove this by solving...

$A\vec{d} = \vec{0}$

The Simplex Method ends here.

Ex. Simplex Method

LP → max $\vec{c}^T\vec{x}$ s.t. $A\vec{x} = \vec{b}$, $\vec{x} \geq 0$

$A = \begin{bmatrix} 1 & 1 & 4 & 0 & 3 \\ -1 & 0 & 2 & 1 & 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 12 \\ 2 \end{bmatrix}$

$\vec{c}^T = [1\ 0\ 10\ 0\ 7]$

lets use the simplex method to solve this LP...

① the LP is in SEF ✓

lets start w/ basis $B = \{2, 4\}$

② the LP is already in canonical form! lets say...

$\bar{\vec{x}} = [0\ 12\ 0\ 2\ 0]$

and lets select from $\vec{c}^T$ 1 as the new entering var.

③ $x_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \end{bmatrix}$

$\Rightarrow x_2 = 12 - x_1 \geq 0 \rightsquigarrow x_1 \leq 12$

$x_4 = 2 + x_1 \geq 0 \rightsquigarrow$ unbounded

new basis...

$B = \{1, 4\}$ $\bar{\vec{x}} = [12\ 0\ 0\ 14\ 0]^T$

② put in canonical form for $B = \{1, 4\}$...

→ max $12 + [0\ -1\ 6\ 0\ 4]\vec{x}$

s.t. $\begin{bmatrix} 1 & 1 & 4 & 0 & 3 \\ 0 & 1 & 6 & 1 & 5 \end{bmatrix}\vec{x} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$

lets select 3 as the new entering variable

③ $x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$

$\Rightarrow x_1 = 12 - 4x_3 \geq 0 \rightsquigarrow x_3 \leq 3$

$x_4 = 14 - 6x_3 \geq 0 \rightsquigarrow x_3 \leq \frac{7}{3}$

we set $x_3 = \frac{7}{3}$

new basis...

$B = \{1, 3\}$ $\bar{\vec{x}} = [\frac{8}{3}\ 0\ \frac{7}{3}\ 0\ 0]$

② put in canonical form for $B = \{1, 3\}$...

→ max $26 + [0\ -2\ 0\ -1\ -1]\vec{x}$

s.t. $\begin{bmatrix} 1 & \frac{1}{3} & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{6} & 1 & \frac{1}{6} & \frac{5}{6} \end{bmatrix}\vec{x} = \begin{bmatrix} \frac{8}{3} \\ \frac{7}{3} \end{bmatrix}$

in "new c" $\vec{c}_N \leq 0$!

so...

④ since $\vec{c}_N \leq 0$

$\bar{\vec{x}} = [\frac{8}{3}\ 0\ \frac{7}{3}\ 0\ 0]^T$ is the optimal solution.

and $z(x) = 26$ is the optimal value!