good luck ♡♡♡♡

LINEAR ALGEBRA

Final Study Guide

Basics

Vectors & Matrices

- The main building blocks of linear algebra are Vectors and Matrices...

vector ↴ $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$

dot product: $\vec{x}\cdot\vec{y} = x_1y_1 + \cdots + x_ny_n$

if $\vec{x}\cdot\vec{y} = \emptyset$, then $\perp$.

$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} = A$

↑ Matrix (can be seen as a collection of vectors)

- Some matrix opperations...
  - Sum → sum each entry indv.
  - Scalar multiplication → multiply each entry by the scalar
  - Matrix multiplication → to get the ij^{th} entry of the product take the dot product of i^{th} row of A and j^{th} column of B...

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = C$

$C_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$

↑ to multiply must be... $A_{m\times n}B_{n\times k}$ match

↑ NOT COMMUNATIVE! $AB \neq BA$

- Another matrix opperation is Inversion. Sort of like Matrix Division.

$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

- we find the inverse by...

$\left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & 0 \cdots & 0 \\ \vdots & & \vdots & 0 & 1 & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{array}\right]$

A ↗ ↑ augmented w/ identity matrix

Row Reduce A → $\left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & A^{-1} \\ 0 & & 1 & \end{array}\right]$

- A can only be inverted if it's square...

Independence

- A set of vectors is linearly independent if no linear combo of the vectors $= \emptyset$. (other than $\emptyset$ times everything)
- A matrix is linearly indp. if it's vectors are indp. aka if...
  - every column has a pivot position
  - there are no free variables
- Free Variables →
  - a variable in $\vec{x}A$ whose corresponding column is not a pivot column.
  - means there is ∞ solutions

Determinants

- A function of a square matrix that produces a real #.
  - We use rules about row opperations to find the determinant (or QR factorization).
- Rules...
  - row replacement doesn't change det(A)
  - scaling a row scales det(A) by c ($c\cdot\det(A)$)
  - swapping 2 rows changes det(A) → $(-1\det(A))$
  - the det of identity matrix is 1.
- Properties
  - a square matrix is invertable if + only if $\det(A) \neq \emptyset$
  - if A and B same size then $\det(AB) = \det(A)\det(B)$
  - $\det(A) = \det(A^T)$

Now we'll look at 4 ways to break down Matrices and their uses...

A=LU

Gause Elimination & RREF

RREF → reduced row echelon form

- We need a way to solve linear equations like $A\vec{x} = \vec{b}$...
- We can solve w/ Row Reduction...
  1. create an augmented matrix w/ A and b...

  $[A|b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & & b_1 \\ \vdots & & \vdots & \vdots \\ & \cdots & a_{nn} & b_n \end{array}\right]$

  2. Use scale, swap and replacement to convert A to REF.
     - REF → all $\emptyset$s at the bottom & first nonzero entry to right and below row above
     - scale → multiply a row by a nonzero # ex. $2R_1$
     - swap → swap 2 rows ex. $R_1 \leftrightarrows R_2$
     - replacement → add a multiple of 1 row to another ex. $R_3 - 2R_1$
  3. Continue until in RREF → all pivots are 1 and each pivot is only non$\emptyset$
- To make Elimination a non destructive process we use **A=LU**
  - U → upper triangular $\begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{bmatrix}$
  - L → lower triangular $\begin{bmatrix} 1 & & 0 \\ \vdots & \ddots & \\ * & \cdots & 1 \end{bmatrix}$
  - U = REF of A
  - L → Since we know U and A we can solve backward for L.

Solutions

- to find solutions to $A\vec{x} = \vec{b}$ we can turn a RREF A back into a parametric form...

$\left[\begin{array}{ccc|c} 1 & 0 & e & a \\ 0 & 1 & f & b \end{array}\right] \to \begin{cases} x + ez = a \\ y + fz = b \end{cases} \to \begin{cases} x = a - ez \\ y = b - fz \\ z = z \end{cases}$ z is a free variable

- The form of the RREF tells us something about the # of solutions...
  1. if $\vec{b}$ column is a pivot column...
  $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \rightsquigarrow 0 = 1$ ... no solution
  2. some columns (not $\vec{b}$) are not pivot columns...
  $\left[\begin{array}{cccc|c} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \end{array}\right]$ free variables ... ∞ solutions
  3. every column except $\vec{b}$ is pivot column...
  $\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$ ... unique solutions

  only this one is linearly independent

Subspaces

- a subspace is a subset satisfying...
  1. non-emptiness
  2. closure under addition
  3. closure under multiplication
- Our favorite subspaces are the Column Space and the Null Space for matrix A.
- Column Space → spaned by the columns of A. Find by...
  1. find RREF of A. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
  2. take pivot col. of the RREF and find corresponding col in A. Those make up col space of A. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

  $\mathrm{Col}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$
- Null Space → spaned by solutions of $A\vec{x} = \emptyset$. Find by...
  1. find RREF of A.
  2. find parametric form of solutions.
  3. the vectors attatched to free variables make up the Nul(A).

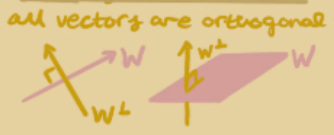
  RREF $\begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{cases} w = 8y + 7z \\ x = -4y - 3z \\ y = y \\ z = z \end{cases}$

  $\mathrm{Nul}(A) = \left\{ \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$
- Dimension → the # of vectors in a subspace
- Rank → dimension of the column space
- Nullity → dimension of the null space
- if A has n collums then rank(A) + nullity(A) = n

A=QR

Orthogonality

- Two vectors are orthogonal when $\vec{y}\cdot\vec{x} = \emptyset$
- We can use orthogonality for various applications such as minimizing distance, but first we need some definitions...
- Distance → $\|y - x\| = \sqrt{(y_y - x_y)^2 + (y_x - x_x)^2}$
- Orthogonal Compliment → $W^\perp$, a subspace where all vectors are orthogonal to all vectors in W. Find by..



  1. find col(A) = W
  2. $W^\perp = \mathrm{Nul}(A^T)$
- Orthogonal Projection → $\vec{x}_w$ or "$\hat{b}$", the closest vector in W to a vector $\vec{x}$. It will be orthogonal to $\vec{x}_{w^\perp}$. Find by...

$\vec{x}_{w^\perp} = \vec{x} - \vec{x}_w$

  1. $W = \mathrm{Col}(A)$, $\vec{x}$ is a given vector. define $\vec{x}_w = A\vec{c}$ for any solution c.
  2. say... $A^TA\vec{c} = A^T\vec{x}$
  3. solve... $\vec{x}_w = A(A^TA)^{-1}A^T\vec{x}$
- Orthogonal Basis → a basis where every vector is orthogonal to every other vector
- Orthonormal Basis → an orthogonal basis where every vector is also normal (unit distance.

Projection Formula

- We can also use a formula to find $\vec{x}_w$...
- W is a subspace, $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is an orthogonal basis for W. The orthogonal project of $\vec{x}$ is...

$$\vec{x}_w = \frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{x}\cdot\vec{u}_m}{\vec{u}_m\cdot\vec{u}_m}\vec{v}_m$$

the division normalizes it.

- But how do we find an orthogonal basis for W?

Gram Schmidt

- The gram schmidt allows us to find an orthogonal basis for any subspace...
  - Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a basis for W, define...
  1. $\vec{u}_1 = \vec{v}_1$
  2. $\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1$
  3. $\vec{u}_3 = \vec{v}_3 - \frac{\vec{v}_3\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 - \frac{\vec{v}_3\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$

  and so on till $\vec{u}_m$
- We can find an orthonormal basis...
  $\vec{e}_1 = \frac{\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}, \vec{e}_2 = \frac{\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}, \ldots, \vec{e}_m = \frac{\vec{u}_m}{\vec{u}_m\cdot\vec{u}_m}$
- Alot of this info can be found in the qr decomposition... **A=QR**
  - Col(A) = W
  - Q → orthonormal matrix... $[\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m]$
  - R → upper triangular matrix... $R = Q^TA$

Applications

- The QR decomp. can be used to find the det(A), since det(A) will = the product of the diagnol of R.
- We can use orthogonal projections to find the closest point in a plane to a vector in $\mathbb{R}^3$.
- We can also minimize the distance between $\vec{b}$ and $A\vec{x}$ in $A\vec{x} = \vec{b}$ by solving...

$A^TA\vec{x} = A^T\vec{b}$ ... for $\vec{x}$.

- But the most common application is the best fit line...
  1. we have n points points that are not linear but we want to find the best $y = mx + b$ line to model them.
  2. we make a system of n linear equations, one for each point.
  3. we put the linear equations into matrix form $A\vec{z} = \vec{b}$ w/ $\vec{z} = \begin{bmatrix} m \\ b \end{bmatrix}$, solve for $\vec{z}$ and plug back into $y = mx + b$ for a best fit line!

A=SΣS⁻¹

Eigenvectors

- We want to find eigenvectors and eigenvalues so we can find a way to make diagnol matrices that are easier to use in application.
- Eigenvector → A nonzero vector $\vec{v}$ such that a square A matrix will have $A\vec{v} = \lambda\vec{v}$ for some scalar λ.
- Eigenvalue → The scalar λ that makes $A\vec{v} = \lambda\vec{v}$.
- notes →
  - eigenvectors that don't share eigenvalues are linearly indp.
  - there can be at most n indp. eigenvectors of an n×n matrix.
  - 0 is an eigenvalue of A if and only if A is not invertable.
- Trace → sum of the diagnol of a matrix
- Finding all Eigenvectors of A...
  1. find all possible eigenvalues by solving $f(\lambda) = \det(A - \lambda I_n)$ aka... $f(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}Tr(A)\lambda^{n-1} + \cdots + \det(A)$
  2. find the nontrivial solutions to $\mathrm{Nul}(A - \lambda I_n)$.
- Algebraic Multiplicity → a # defining λ, the # of times λ shows up as a root of the characteristic polynomial.
- Geometric Multiplicity → of a λ is the dimension of the λ-eigenspace.

Diagonalization

- Now to get our diagonal matrix and eigen decomposition...

$$A = S\Sigma S^{-1}$$

S → A matrix made up of linearly indp. eigenvectors of A.

Σ → A Diagonal Matrix w/ eigenvalues on it's diagonal.

- eigenvalues and corresponding eigenvectors have to be listed in same order
- An n×n matrix can only be diagonalized if it has n eigenvectors.
- Matrix A is diagonalizable if and only if...
  - sum of geometric multiplicities of all eigenvalues of A = n
  - sum of algebraic multiplicities of all eigenvalues of A = 1
  - for each eigen value a multiplicity = g multiplicity.

Jordan Form

- For matrices that aren't diagonalizable we can put them in jordan form, which is like a diagnol matrix but there are 1's above repeated λ's.

$J = \begin{bmatrix} \lambda_1 & 1 & & \emptyset \\ & \lambda_1 & 1 & \\ & & \lambda_1 & \\ \emptyset & & & \lambda_2 \end{bmatrix}$

Applications

- Most common application is Difference Equations.
  - Dif Eq. → an equation of form...

  $\vec{v}_{t+1} = A\vec{v}_t$ A is a change of state matrice.
- To make solving these equations easier when we have something like $\vec{v}_{50}$ we can change A into a diagnol matrix... $A = S\Sigma S^{-1}$
- Then we need a new starting vector $\vec{v}_0$. Let's say it's... $\vec{w}_0 = S^{-1}\vec{v}_0$
- Now raise Σ to the power of t to find any itteration in the future...

$\vec{w}_t = \Sigma^t\vec{w}_0$

A=UΣVᵀ

Singular Value Decomp.

- often used for compression.
- Breaks A into...

$$A = U\Sigma V^T$$

U, V → columns are singular vectors

Σ → diagonal is singular value

- How to find singular values...
  1. A is an m×n matrix. We find A^TA to get an n×n matrix
  2. Find eigenvalues of A^TA.

  $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix}$
  3. Find eigenvectors of A^TA. V will be the eigenvectors of A^TA. Trans. it to get V^T.
  4. Solve backwards to get U.

Invertible Matrix Theorem

- all of the following are equivalent. If 1 is true, all are true for $A_{n\times n}$...
  1. A is invertible
  2. The linear system $A\vec{x} = \vec{b}$ has a unique solution
  3. rref of $A = I_n$
  4. rank A = n
  5. im $A = \mathbb{R}^n$
  6. $\mathrm{Ker}(A) = \{\vec{0}\}$
  7. column vectors of A form a basis of $\mathbb{R}^n$.
  8. column vectors of A span $\mathbb{R}^n$.
  9. column vectors of A are lin. indp.
  10. $\det(A) \neq \emptyset$
  11. $\emptyset$ is not an eigenvalue of A.