STOR 215

DISCRETE
Midterm 1 study guide

Review of Basic Math Concepts
ch.1 & ch.3

Number Systems
ch. 1

Natural
- 1 < 2 < 3 < ...
- give us... **addition** > a+b=b+a > (a+b)+c = a+(b+c)
  multiplication > ab=ba > (ab)c=a(bc) > a(b+c) = ab+ac
  exponents > $(a^b)\cdot(a^c) = a^{b+c}$ > $(a^b)^c = a^{bc}$ > $a^c \cdot b^c = (ab)^c$

Integers
- but what if want to subtract 3-5? we need to add integers
- ... < -3 < -2 < -1 < 0 < 1 < 2 < 3 < ...
- give us.... **subtraction** > same laws as addition

Rational
- subtraction is inverse of add. what's inverse of multiplication for that we need rationals...
- where a, b are integers, b≠0, a rational number is $x = \frac{a}{b}$
- gives us... **division** > $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ > $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$

Real & Irrational
- what about a^b where b is not an integer?
- we get numbers that cannot be fractions... irrational
- with irrational numbers we now have all the Real #s.

Complex
- what about sqr root of -1? no real solution, so imaginary! which is a complex #.
- a + ib

Sets
ch.3.1

SET → an unordered collection of objects called elements

x ∈ S — x "is in" sets; x ∉ S — "is not in"

S = {x ∈ Ω : P(x)} — S is, x in universal set, such that, P(x) is true

Common Sets
- N natural #s {1, 2, 3, ...}
- Z_+ non-neg integers {0, 1, 2, 3, ...}
- Z integers {..., -2, -1, 0, 1, ...}
- Q rational #s {p/q : p∈Z, q∈N}
- R real #s (-∞, ∞)
- [a,b] intervals {x : a ≤ x ≤ b}
- (a,b) {x : a < x < b}
- ∅ empty set { }

Set Operations
- A∪B Union
- A∩B Intersection
- A^c Complement
- A−B Difference
- A△B Symmetric dif.
- A∩B=∅ disjoint
- A⊆B subset
- A=B Equal

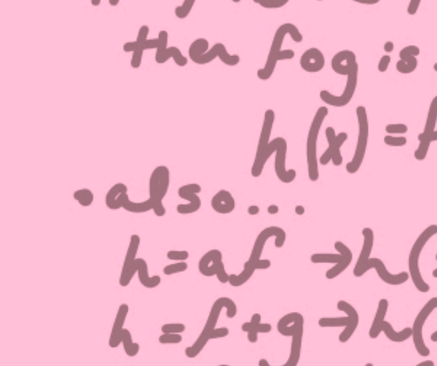
Properties
- double complement → $(A^c)^c = A$
- communative laws → A∩B = B∩A
- assoc. laws → (A∩B)∩C = A∩(B∩C)
- distributive laws → A∪(B∩C) = A∪B ∩ A∪C
- demorgans laws → $(A\cap B)^c = A^c \cup B^c$, $(A\cup B)^c = A^c \cap B^c$

(very similar)

Functions
ch 3.2-3.4

f : A → B — function, domain, codomain aka target

{f(a) : a∈A} — range of function



mostly written as... $f(x) = y_{x\in A,\, y\in B}$

Combining Functions
- if f : B → C, g : A → C then f∘g is... h : A → C
  h(x) = f∘g(x) = f(g(x))
- also...
  h = af → h(x) = af(x)
  h = f+g → h(x) = f(x)+g(x)
  h = fg → h(x) = f(x)g(x)

Definitions
- **one-to-one** → f : A→B if f(a)=f(b) ⇒ a=b
- **onto** → f : A→B if each b ∈ B there is a ∈ A that f(a)=b
- **bijection** → a function that is one to one & onto
- **inverse** → if bijection, it's inver-sable. $g(b) = a \Leftrightarrow f(a) = b$, f^{-1}
- **increasing** → x ≥ y ⇒ f(x) ≥ f(y), ∀x, y∈A
- **strictly increasing** → x > y ⇒ f(x) > f(y), " "
- **decreasing** → x ≥ y ⇒ f(x) ≤ f(y), " "
- **strictly increasing** → x > y ⇒ f(x) < f(y), " "

Binary Functions

f : A×A → {0,1} — almost like a true or false

↳ x and y are elements in A and f(x,y) can be 1 or ∅.

More Definitions
- **reflexive** → if f(x,x)=1 for all x∈A
- **symmetric** → if f(x,y) = f(y,x) for all (x,y) ∈ A×A
- **transitive** → if f(x,y)=1 and f(y,z)=1 ⇒ f(x,z)=1

Sequences & Sums
ch.3.5-3.6

f : N → S or f : Z_+ → S (set)
- $f(n) = a_n$
- or f : N → S $\{a_1, a_2, a_3, \dots\}$
- or $\{a_n, n \geq 1\}$

Common Sequences
- $a_n = n$ nonneg. integers {0, 1, 2, 3, ...}
- $a_n = 2n$ nonneg even integers {0, 2, 4, 6, ...}
- $a_n = 2n+1$ nonneg odd integers {1, 3, 5, 7, 9, ...}
- $a_n = n^2$ square #s {1, 4, 9, 16, 25, ...}
- $a_n = \frac{n(n+1)}{2}$ triangular #s {1, 3, 6, 10, 15, ...}
- $a_n = n^3$ cubic #s {1, 8, 27, 64, ...}
- $a_n = 2^n$ powers of 2 {1, 2, 4, 8, 16, ...}
- $a_n = n!$ factorials {1, 1, 2, 6, 24}

Sums
- if we add every element in a given sequence together we get a sum.

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \dots + a_n$$

"sum of a_j over j=m to n" — sequence

Common Sums

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{r} a_i + \sum_{i=r+1}^{n} a_i$$ where r is between 1 and n

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} c = nc$$

$$\sum_{i=m}^{n} ax_i + by_i = a\sum_{i=m}^{n} x_i + b\sum_{i=m}^{n} y_i$$

$$\sum_{i=1}^{n} b_n - b_{n-1} = b_n - b_0$$ telescoping sequence

$$\sum_{i=1}^{n} a_i = \frac{n(n+1)}{2}$$

$$\sum_{i=0}^{n} r^i = \begin{cases} \frac{r^{n+1}-1}{r-1} & \text{if } r \neq 1 \\ n+1 & \text{if } r = 1 \end{cases}$$

Logic
ch.2.1-2.5

Compound Propositions
ch.2.1-2.2

- **Proposition** → statement that is true or false

Propositions	NOT Propositions
"1+1=2"	"please sit down"
"1+2=2"	"will you marry me?"
"All apples are green"	"This sentence is false."
"it will rain tommorow"	

- **Compound Proposition**
  - combining propositions
  - Negation → ¬p, "not p"

p	¬p
T	F
F	T

  - Conjuction → p∧q "p and q". only true if both are true

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

  - Disjunction → p∨q "p or q". true if either p or q is true or both r true

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

  - Exclusive Or → p⊕q only true if p or q is true and not both.

p	q	p⊕q
T	T	F
T	F	T
F	T	T
F	F	F

- **Implications** more compound proposition
  - Implication → p⇒q "p implies q", or "if p then q". if p is true then q must be true.
    - if p is false we can't imply anything about q, so we can't disprove anything. so false implies anything.

p	q	p⇒q
T	T	T
T	F	F
F	T	T
F	F	T

the rest of these we derive from impl.
  - Converse → q⇒p switch!
  - Inverse → ¬p⇒¬q negation of original
  - Contraposition → ¬q⇒¬p it is NOT possible for implication to be true and contraposition to be false

p	q	¬p	¬q	p⇒q	q⇒p	¬p⇒¬q	¬q⇒¬p
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

matching

Logical Equivalences
ch. 2.3-2.4

Logically Equivalent → when 2 compound have the same truth tables, they are logically the same.
- Tautology → a proposition that is always true
- Contradiction → a proposition that is always false

- **Equivalences**
  - ¬(¬p) ≡ p double negation
  - p∧q ≡ q∧p commutative
  - (p∧q)∧r ≡ p∧(q∧r) associative
  - p∨(q∧r) ≡ p∨q ∧ p∨r distributive
  - ¬(p∧q) ≡ ¬p∨¬q demorgan's
  - p⇒q ≡ ¬q⇒¬p contrapositive law
  - p∧p ≡ p idempotent
  - p∨p ≡ p

all of these can be proven w/ truth tables!
- We can use these logical equivalences to prove other logical equivalences!
  ↳ this idea is based on...

(p≡q)∧(q≡r)⇒(p≡r)

or

$(p\equiv p_1)\wedge\cdots\wedge(p_n\equiv p_{n+1})\wedge\cdots\wedge(p_{k+1}\equiv q)\Rightarrow(p\equiv q)$

(this is better)

Proofs
ch. 2.6-2.7

Predicates & Quantifiers
ch. 2.5

- Predicate → a proposition with a variable (i.e. "x<8") write as P(x). Predicates also need a domain D
- Domain → a set of values a variable x can take, D
- Quantifiers
  - ∀ → Universal Quantifier: ∀xP(x) "for all x in D P(x)"
  - ∃ → Existensial Quantifier: ∃xP(x) "at least one x in D for which P(x)"

When D is finite...

$\forall x P(x) \equiv P(x_1)\wedge P(x_2)\wedge\cdots\wedge P(x_n)$

$\exists x P(x) \equiv P(x_1)\vee P(x_2)\vee\cdots\vee P(x_n)$

Extension of De Morgan's...

¬(∀xP(x)) ≡ ∃x¬P(x)

¬(∃xP(x)) ≡ ∀x¬P(x)

Deductions
ch. 2.6

- throughout this course we will encounter mathematical "truths" they will be a logical proposition in the form p⇒q (hypothesis ⇒ conclusion)
- a **proof** is an argument showing p⇒q is always true.
- if p⇔q just prove (p⇒q)∧(q⇒p)

Types of Proofs

Direct Proof

assume p is true and construct a series of true proposition until you get to q.

$p\Rightarrow p_1,\ p_1\Rightarrow p_2,\ \dots,\ p_{n-1}\Rightarrow p_n,\ p_n\Rightarrow q$

Proof by Contraposition

we assume q is false and use that to show p is false.

p⇒q ≡ ¬q⇒¬p

Proof by Contradiction

assume p is true and q is false. construct a series of propositions $p_1, p_2, \dots, p_{n-1}, p_n$, so all the propositions but p_n are true. This shows p∧¬q is false, and that p being true and q being false will never happen. So every other combo allows p⇒q to be true, so it must be true.

$p\wedge\neg q\Rightarrow p_1,\ p_1\Rightarrow p_2,\ \dots,\ p_{n-1}\Rightarrow p_n\equiv F$

aka Reductio Ad Absurdum

Good Luck