CALC 3

STUDY GUIDE 2

14.1 & 14.2 VECTOR VALUED FUNCTION

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

3 dependent variables, independent t

3 parametric equations

think of t as time, as t increases the curve is drawn out

DOMAIN
largest set of values of t on which all of $x, y, \&\ z$ are defined

ORIENTATION
direction the curve is generated in. tangent vectors always point in direction of positive orientation.

Limits!

$\lim_{t \to a} \vec{r}(t) = \vec{L}$

(mostly same as norm. limits. same limit rules)

CONTINUOUS
continuous if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

CALCULUS (mostly the same)

DERIVATIVE

$\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$

↳ this is also the tangent vector

→ UNIT TANGENT VECTOR

$T(t) = \dfrac{r'(t)}{|r'(t)|}$

→ DERIVATIVE RULES (just the new ones...)

$\frac{d}{dt}(\vec{m}(t) \cdot \vec{v}(t)) = \vec{m}\,'(t) \cdot \vec{v}(t) + \vec{m}(t) \cdot \vec{v}\,'(t)$

$\frac{d}{dt}(\vec{m}(t) \times \vec{v}(t)) = \vec{m}\,'(t) \times \vec{v}(t) + \vec{m}(t) \times \vec{v}\,'(t)$

INTEGRALS

indefinite...

$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{c}$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

definite...

$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\vec{i} + \left(\int_a^b y(t)\right)\vec{j}$

14.3 & 14.4 APPLICATION OF VECTORS

POSITION
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
↳ time

VELOCITY
$v(t) = r'(t) = \langle x'(t), y'(t), z'(t) \rangle$

SPEED
$|v(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

ACCLERATION
$a(t) = v'(t) = r''(t)$

if $|\vec{r}|$ is a constant then $\vec{r} \cdot \vec{v} = 0$, so the position and velocity vectors are alway ⊥

$|\vec{r}|$ is constant when motion is circle or sphere centered at origin

we can tell if path is on circle/sphere centered @ orgin by seeing if $|\vec{r}|$ is constant

LENGTH OF CURVE

Arc length of vector func...

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

$$= \int_a^b |r'(t)|\,dt$$

For reasons we'll learn about later we want...

arc length $s = t$

So...

arclength...

$s(t) = \int_a^t |v(u)|\,du$

if $|v(t)| = 1$, then...

$s(t) = \int_a^t 1\,du = u\Big]_a^t = t - a$

$\underbrace{s(t)}_{\text{arclength}} = \underbrace{t - a}_{\text{parameter}}$

arclength = parameter

We want $|v(t)| = 1$

To make any func. parametrized by arc length we...

1. solve $s(t) = ct$
2. sub $\frac{s}{c} = t$ into equation

15.1 & 15.2 MULTIVARIABLE FUNCTIONS (graphs, continuity, & limits)

We can take what we know about vector val. func. and use it to gain intuition about Multivariable Functions

1 Variable	$y = f(x)$	$\mathbb{R}^2$
2 Variable	$z = f(x, y)$	$\mathbb{R}^3$
3 Variable	$w = f(x, y, z)$	$\mathbb{R}^4$
n Variable	$x_{n+1} = f(x_1, x_2, \ldots, x_n)$	$\mathbb{R}^{n+1}$

2 variable... is what we're gonna focus on

$\underbrace{Z}_{\text{range}} = f\underbrace{(x, y)}_{\text{each } (x,y) \text{ is a set... Domain of } f}$

graph as... $(x, y, f(x, y))$

can graph in 3d but can also...

LIMITS & CONTINUITY

written as...

$\lim_{(x,y) \to (a,b)} f(x, y) = L$

(same limit rules as 1 variable)

→ Boundry Point
> on edge of R
part in R, part out of R

→ Interior Point
> inside R
can make a disk entirely in R.

OPEN REGION
All interior points

CLOSED REGION
has all it's boundary points

(a limit exists even if (a,b) is a boundary point)

TWO PATH TEST

if $f(x, y)$ approaches 2 dif values as (x, y) approaches (a, b) along 2 dif paths, then the limit DNE

PROCEDURE →

1. set $y = mx$ and find $\lim_{x \to 0}$ (or vice versa)
2. if the limit contains m it DNE (cause the limit will change for every new constant)
3. set $x = my$ and find $\lim_{y \to 0}$ if this doesn't match limit from 1, the limit DNE.
4. can also set $x = my^2$, $x = y^3$, etc.

CONTINUITY

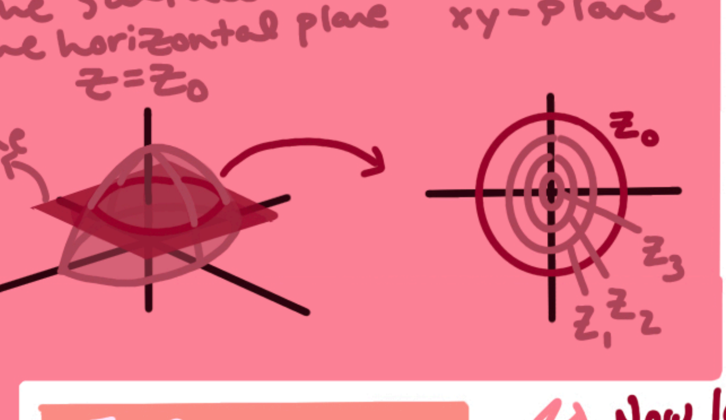
Continuous at (a, b) if...
- f is defined at (a, b)
- and, $\lim_{(x,y) \to (a,b)} f(x, y)$ exists
- and, $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$

also...
if $u = g(x, y)$ is cont. at (a, b) and $z = f(u)$ is cont. at $g(a, b)$ then $z = f(g(x, y))$ is cont. at (a, b).

LEVEL CURVES

CONTOUR CURVES
the intersection of the surface and the horizontal plane $z = z_0$

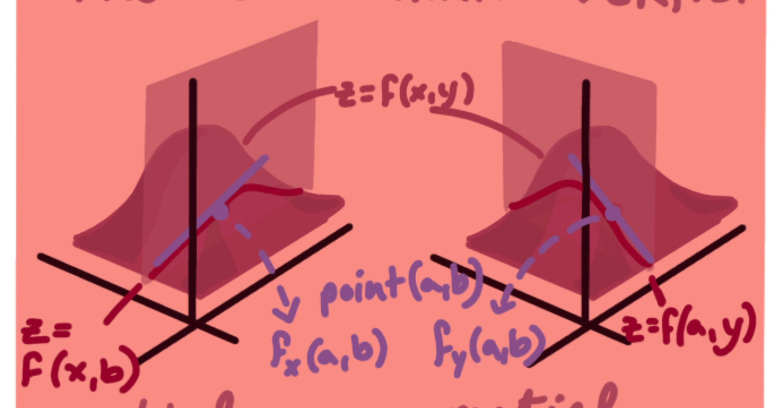
LEVEL CURVES
multiple countour curves projected onto xy-plane



15.3 PARTIAL DERIVATIVES

Now let's apply some Calculus to this stuff

since we can find multiple tangent lines in many directions on a 3d object we need a new way to find tangent at a point. The answer is PARTAL DERIVS.



partial derivative of f w/ respect to x: $f_x(a, b)$

partial derivative of f w/ respect to y: $f_y(a, b)$

we hold one component constant and find the deriv. w/ respect to the other comp.

So w/ f_x we take deriv. w/ respect to x and treat y like a constant and vice versa

HIGHER ORDER PART. DERIV

$\frac{\partial f}{\partial x} = f_x$ $\quad$ $\frac{\partial f}{\partial y} = f_y$

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$ $\quad$ $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

Notation

$\frac{\partial f}{\partial x}(a, b) = f_x(a, b)$

$\frac{\partial f}{\partial y}(a, b) = f_y(a, b)$

DIFFERENTIABLE

if f has partial derivs f_x and f_y defined on an open set containg (a, b) AND f_x and f_y are continuous at (a, b), Then f is differentiable at (a, b).

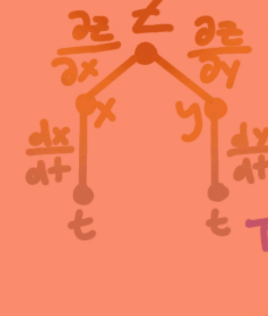
And differentiability implies continuity

15.4 CHAIN RULE

Here are some tools to solve these derivatives

1 INDEPENDENT VARIABLE

$z = f(x, y)$ $\quad x = f(t)$ $\quad y = f(t)$ or $z = f(x(t), y(t))$



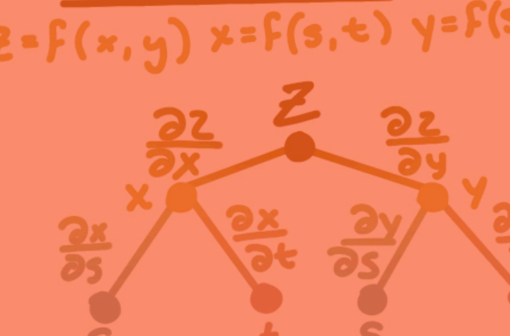
multiply down the branches

Then add the products

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2 INDEPENDENT VARIABLES

$z = f(x, y)$ $\quad x = f(s, t)$ $\quad y = f(s, t)$

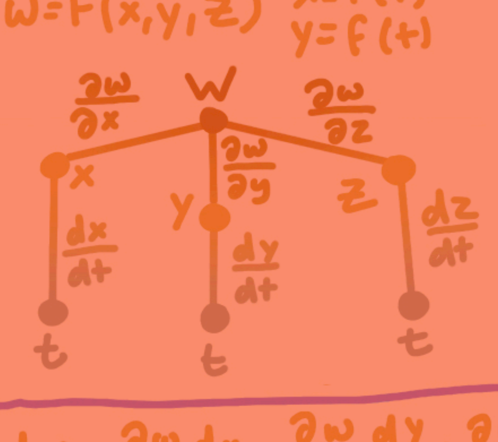


$$z_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$z_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

3 INTERMEDIATE VARIABLES

$w = f(x, y, z)$ $\quad x = f(t)$ $\quad z = f(t)$ $\quad y = f(t)$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

IMPLICIT DIFFERENTIATION

can't remember the original way but... NEW

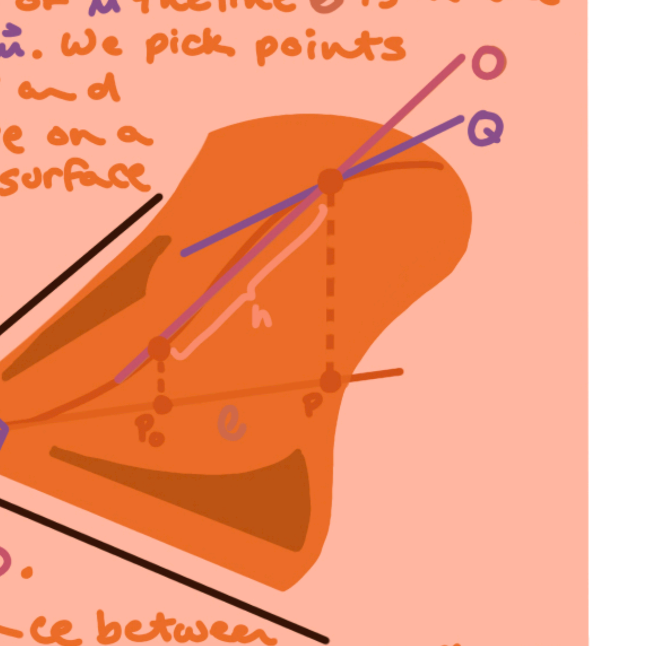
$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

Basically everything is the exact same when adding another variable

15.5 DIRECTIONAL DERIVATIVES & GRADIENT

Can we find partial derivatives in other directions?

We want to find the derivative in the direction of $\vec{u}$ the line ℓ is in the direction of $\vec{u}$. We pick points on ℓ, P_0 and P and project these on a splice of the surface above ℓ. We find the secant line between these two points, O. As the distance between these 2 points, h, approaches ∅, we get the tangent line Q, AKA The Derivative.



The Equation

$$D_u f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

which we turn into the simpler...

DIRECTIONAL DERIVATIVE

$$D_u f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

WOW! a dot product!

think of it as a weighted average of the partial derivs. and the direction is the weight

This part is special!

GRADIENT

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$$

what's special about it?

1. f has it's max rate of increase at (a, b) in the direction of $\nabla f(a, b)$ and it's max rate of decrease in the opposite direction of $\nabla f(a, b)$
2. the maximum rates of change in these directions is $|\nabla f(a, b)|$ and $-|\nabla f(a, b)|$ respectivly
3. the rate of change is ∅ in any direction orthogonal to $\nabla f(a, b)$.
4. the line tangent to the level curve at f is orthogonal to $\nabla f(a, b)$

HOW TO FIND ORTHOGONAL VECTOR

$\langle a, b \rangle \to \langle -b, a \rangle$ perpendicular

Good Luck ♡