MIDTERM 2

PROBABILITY

PROPERTIES OF RANDOM VARIABIALS

Discrete RV Review

$$p_X(x) = P(\{X = x\})$$

"the prob. of X being x"

PMF → probability mass function

Expection, Mean, & Variance

Expectation → the weighted average of the possible values of X. Aka the **Mean**

$$E[X] = \sum_x x\, p_X(x)$$

2nd Moment → the expected value of the rv X^2.

$$E[X^2] = \sum_x x^2 p_X(x)$$

Variance → the spread of an X around the mean.

$$Var(X) = E[(X - E[X])^2]$$

$$Var(X) = E[X^2] - (E[X])^2$$ ← easiest to use

Standard Deviation → a measure of spread in the orig. units.

$$\sigma_X = \sqrt{Var(X)}$$

Mean & Variance of Brand Name RV

Bernoulli

$E[X] = p$ $E[X^2] = p$

$Var(X) = p(1-p)$

Discrete Uniform

$E[X] = \frac{a+b}{2}$ $E[X^2] = \frac{1}{b}\sum_{k=a}^{b} k^2$

$Var(X) = (\frac{1}{b}\sum_{k=a}^{b} k^2) - (\frac{1+b}{2})^2$

Poisson

$E[X] = \lambda$ $Var(X) = \lambda$

Binomial

$E[X] = np$ $Var(X) = np(1-p)$

Geometric

$E[X] = \frac{1}{p}$ $E[X^2] = \frac{2}{p^2} - \frac{1}{p}$

$Var(X) = \frac{1-p}{p^2}$

Brand Name RV's

Bernoulli RV

- an experimet w/ prob. of success p

$p_X(1) = p$ $p_X(0) = 1-p$

Binomial RV

- an experiment w/ prob. of sucess p, w/ n trials

$p_X(x) = (x$ sucesses in n trials$)$
$= \binom{n}{x} p^x (1-p)^{n-x}$

Geometric RV

- an experiment w/ prob of sucess p, where X = # of trials until the first sucess

$p_X(x) = P(x-1$ fails followed by a sucess$)$
$= (1-p)^{x-1}(p)$

Negative Binomial RV

- an experiment w/ prob of sucess p, we run the experiment until we get r trials that end in sucess

$p_X(x) = P(x$ fails in $x+r-1$ trials and then a sucess$)$
$= \binom{x+r-1}{x}(1-p)^x p^r$

Hypergeometric RV

- a container w/ W desired objects and B other objects, we draw K objects w/o looking X = # of desired objects in K

$$p_X(x) = \frac{\binom{W}{x}\binom{B}{k-x}}{\binom{W+B}{k}}$$

Uniform RV

- There are n outcomes that occur w/ equal probability X = outcome of 1 trial

$$p_X(x) = \frac{1}{n}$$

Poisson RV

- somewhat like a binomial w/ a very small x and large n

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Some Important Properties...

$$E[g(X)] = \sum_x g(x) p_X(x)$$

if $Y = aX + b$...
$E[Y] = aE[X] + b$ $Var(Y) = a^2 Var(X)$

EXPANDING ON RANDOM VARIABIALS

Applying Expections

- we can use expectation to make decisions when we have X be a monetary value and P(x) be the probability of getting that amount.

Joint PMF's

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

we can extend all the stuff below to 3+ rv's

Joint PMF → the probability of X being x and Y being y

$p_X(x) = \sum_y p_{XY}(x,y)$ $p_Y(y) = \sum_x p_{XY}(x,y)$

marginal PMF's

the individual PMF's of X and Y. Can calculate using above formula or tabular method...

y \ x	1	2	3	
1	a	b	c	→ a+b+c
2	d	e	f	→ d+e+f
3	g	h	i	→ g+h+i
	a+d+g	b+e+h	c+f+i	

$p_{XY}(3,1)$; rows → $p_Y(y)$; columns → $p_X(x)$

We can get marginal pmf's from joint, but we can't get joint from marginals.

functions of multiple rv's...

$$p_Z(z) = \sum_{\{x,y \mid g(x,y) = z\}} p_{X,Y}(x,y)$$

extending!

Expected Value Rule

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{XY}(x,y)$$

extending!

When g is in Linear Form

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Conditioning

$$p_{X|A}(x) = P(X = x \mid A) = \boxed{\frac{P(\{X = x\} \cap A)}{P(A)}}$$

condition on event

$$p_{X|Y}(x|y) = P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \boxed{\frac{p_{X,Y}(x,y)}{p_Y(y)}}$$

conditioning on another RV

Use conditional to calculate joint pmf...

$$\boxed{\begin{aligned} p_{X,Y}(x,y) &= p_Y(y) p_{X|Y}(x|y) \\ p_{X,Y}(x,y) &= p_X(x) p_{Y|X}(y|x) \end{aligned}}$$

and marginals...

$$p_X(x) = \sum_y p_Y(y) p_{X|Y}(x|y)$$

Conditional Expectation

$$E[X|A] = \sum_x x\, p_{X|A}(x)$$

$$E[X|Y=y] = \sum_x x\, p_{X|Y}(x|y)$$

total expectation theorem

> the unconditional average can be obtained by averaging the conditional averages.

Independence

X is independent of A if... → $P(X = x \cap A) = P(X = x)P(A) = p_X(x)P(A)$

or... $p_{X|A}(x) = p_X(x)$

two RV's are independent if... → $p_{X,Y}(x,y) = p_X(x)p_Y(y)$

or... $p_{X|Y}(x|y) = p_X(x)$

this all can be extended to 3+ rv's

two RV's are conditionally independent if... → $P(X = x, Y = y \mid A) = P(X = x \mid A)P(Y = y \mid A)$
$= p_{X,Y|A}(x,y) = p_{X|A}(x) p_{Y|A}(y)$

If X and Y are Independent

$$E[X \cdot Y] = E[X]E[Y]$$

$$Var(X + Y) = Var(X) + Var(Y)$$

CONTINUOUS RANDOM VARIABLES

Probability Density Function

- we use a continuous rv when a rv X can take infinitly many possible values

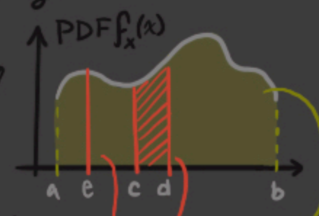
$$P(a \le X \le b) = \int_a^b \underbrace{f_X(x)}_{\text{PDF}}\, dx$$

probability that X is between a and b

PDF → probability density function

since calculated as an integral we know that prob. of exactly a = 0

$$P(X = a) = \int_a^a f_X(x)\,dx = 0$$



- entire area under pdf = 1
- area between c and d = $P(c \le X \le d) = \int_c^d f_X(x)\,dx$
- probability of e = $P(X = e) = \int_e^e f_X(x)\,dx = 0$

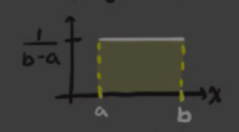
To Qualify as a PDF

- nonnegative $f_X(x) \ge 0$, for every x
- normalization $\int_{-\infty}^{\infty} f_X(x)\,dx = P(-\infty < X < \infty) = 1$

Brand Name CRV's

Continuous Uniform RV

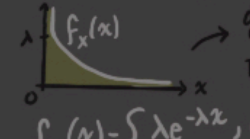
- an experiment where every number between a and b are equally likely, X = # selected



$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Exponential RV

- the prob. that X exceeds a certain value decreases exponentially



decrease exponentially $P(X \ge a) = e^{-\lambda a}$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

There is one more very important RV that we'll cover in the next part...

Expectation

Expectation → $E[X] = \int_{-\infty}^{\infty} x f_X(x)$

Variance → $Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$

if $Y = aX + b$, then...

$E[Y] = aE[X] + b$ $Var(Y) = a^2 Var(X)$

Mean & Variance of Brand Name RV's

Continuous Uniform

$E[X] = \frac{a+b}{2}$ $E[X^2] = \frac{a^2 + ab + b^2}{3}$

$Var(X) = \frac{(b-a)^2}{12}$

Exponential

$E[X] = \frac{1}{\lambda}$ $E[X^2] = \frac{2}{\lambda^2}$

$Var(X) = \frac{1}{\lambda^2}$

CUMULATIVE DISTRIBUTION FUNCTION

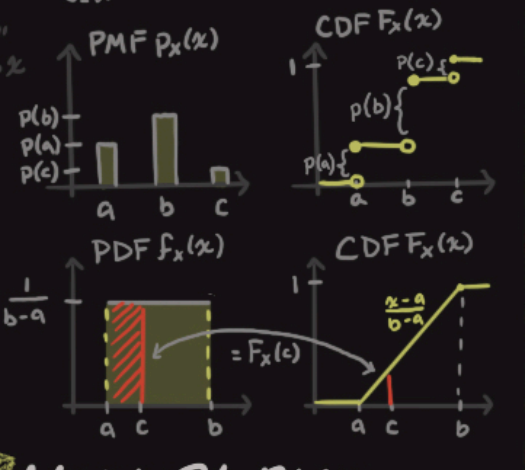
Cumulative Distribution Functions

- We want one mathmatical concept to describe continuous & discrete rv's...

$$F_X(x) = P(X \le x) = \begin{cases} \sum_{k \le x} p_X(k), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt, & \text{if } X \text{ is continuous} \end{cases}$$

CDF → cumulative distribution function

"accumulates" probability up to x



Properties of CDF

- tends to 0 as $x \to -\infty$ and 1 as $x \to \infty$
- find PMF from CDF & vice versa...

$$F_X(k) = \sum_{i=-\infty}^{k} p_X(i)$$

$$p_X(k) = F_X(k) - F_X(k-1)$$

- find PDF from CDF & vice versa...

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$f_X(x) = \frac{dF_X}{dx}(x)$$

Brand Name CDF's

Geometric

- geometric rv w/ parameter p...

$$F_{geo}(n) = 1 - (1-p)^n$$

Exponential

- exponential rv w/ λ...

$$F_{exp}(x) = 1 - e^{-\lambda x}$$

The geometric and exp. cdf's are equal when...

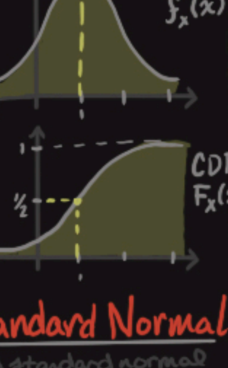
$\delta = -\ln(1-p)/\lambda$ $x = n\delta$

$$F_{exp}(n\delta) = F_{geo}(n)$$

Normal RV's

$$f_X(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}$$

Normal/Gaussian PDF



Properties

- if X is normal and Y is linear... then Y is also normal w/ mean + var...

$E[Y] = a\mu + b$
$var(Y) = a^2\sigma^2$

- we can find the CDF from the standard normal...

once we have a standard normal consult a sn table to get cdf

Standard Normal

- A standard normal has $\mu = 0$ and $\sigma^2 = 1$

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\,dt$$

- w/ property...

$\Phi(-y) = 1 - \Phi(y)$

- We can standardize any X by defining Y...

$\boxed{Y = \frac{X - \mu}{\sigma}}$ $E[Y] = 0$ $Var(Y) = 1$

GOOD LUCK