Calc 3 Midterm 3 Study Guide

we are going to continue learning how to apply Calculus to 3D...

More on...

Derivatives in 3D

- we can find tangent lines on 3D objects but we can also find tangent planes. Just like we use tangent lines to make approximations on 2D graphs, we use tangent lines for Linear approximation on 3D.

Finding Tangent Planes

- to define a plane we need...
  - ☐ a point
  - ☐ a normal vector
    - ↳ and we know the ∇ is normal to a point...
- So the equation for a tangent plane...

$$F_x(a,b,c)(x-a)+F_y(a,b,c)(y-b)+F_z(a,b,c)(z-c)=\emptyset$$

where we have point $P_0(a,b,c)$ and normal vector $\nabla F(a,b,c)$

- can also write...

$$z=f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$$

Linear Approximation

- if we're at a point $(a,b,f(a,b))$ and we want to use this to approx. a point (x,y)...

$$L(x,y)=f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$$

- and we can expand this to more variables just by adding a term for it.
- we can also approximate the change in z...

$$\Delta z \approx dz = f_x(x,y)dx+f_y(x,y)dy$$

$dx = \text{new } x - \text{old } x$

- Just like in 2D we can use Calculus for word problems in finding Maximums and minimums.
- some vocab...

local maximum/minimum/extrema
- if $f(x,y)\le f(a,b)$ for all (x,y) in some open disk centered at (a,b) it's a local maximum.
- if $f(x,y)\ge f(a,b)$ it's a local minimum.
- $f_x(a,b)=f_y(a,b)=0$ at any local extrema

absolute maximum/minimum
- if $f(a,b)\ge f(x,y)$ for every (x,y) in R, then $f(a,b)$ is a absolute maximum
- if $f(a,b)\le f(x,y)$, then it's a absolute minimum

saddle point
- if in every open disk at (a,b) there are points (x,y) for which $f(x,y)>f(a,b)$ and points $f(x,y)<f(a,b)$.

saddle point

Critical Point
- a point where $f_x(a,b)=f_y(a,b)=0$ or where at least one of the partial derivatives don't exist.

Second Derivative Test

- just like in 2D
- we find the critical points and plug them into

$$D(x,y)=f_{xx}(x,y)f_{yy}(x,y)-(f_{xy}(x,y))^2$$

- if $D(x,y)>0$ and $f_{xx}(x,y)<0$ there is a local <u>maximum</u>
- if $D(x,y)>0$ and $f_{xx}(x,y)>0$ there is a local <u>minimum</u>
- if $D(x,y)<0$ there is a saddle point
- if $D(x,y)=0$, inconclusive

Finding Absolute Maxs & Mins!

1. determine the values of f for all critical points
2. use 2nd deriv test to find max and min
3. test all boundary points
4. whichever has the largest f is the abs. max and the lowest f is the abs. min.

now we do more calc in 3D... we do...

Integrals in 3D

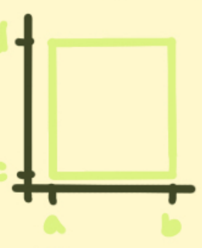
- we find integrals in 3D with double integrals. We then evaluate the double integral w/ iterated integrals.

double integral form $\iint_R f(x,y)\,dA$

- like in 2D, integral will find volume under $f(x,y)$

Rectangular Regions

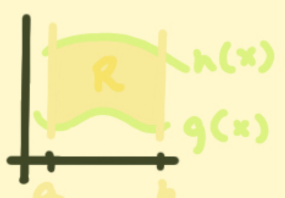
- $R=\{(x,y): a\le x\le b,\ c\le y\le d\}$



$$=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

General Regions

- $R=\{(x,y): a\le x\le b,\ g(x)\le y\le h(x)\}$



$$=\int_a^b\int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

- if we want to change the order of the integral we have to change the bounds, cause the function bounds need to be on the inside.
- if $f=1$ then the double integral finds the area of R.

Average Value of Function over Region

$$\bar{f}=\frac{1}{\text{area of } R}\iint_R f(x,y)\,dA$$

Volume Between 2 Surfaces

Volume $=\iint_R f(x,y)-g(x,y)\,dA$

Triple Integrals

- can be used to find volume if $f=1$
- can be used to find mass if f is a density function

$$\iiint_D f(x,y,z)\,dV = \iint_R\left(\int_{G(x,y)}^{H(x,y)} f\,dz\right)dA$$

first evaluate the 3D, then evaluate the 2D one

$$=\int_a^b\int_{h(x)}^{g(x)}\int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Average Value of Function over 3D Region

$$\bar{f}=\frac{1}{\text{volume of } D}\iiint_D f(x,y,z)\,dV$$

Polar Coordinates

r = radius

θ = angle from x axis

$z=f(r,\theta)$

$x=r\cos\theta \quad y=r\sin\theta$

- in polar rect. regions...
- $R=\{(r,\theta): 0\le a\le r\le b,\ \alpha\le\theta\le\beta\}$

$$=\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

- in general regions...
  $R=\{(r,\theta): g(\theta)\le r\le h(\theta),\ \alpha\le\theta\le\beta\}$

$$=\int_\alpha^\beta\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

- remember to add an r

Cylindrical Coordinates

- coordinates (r,θ,z)
- $r^2=x^2+y^2$ $\quad x=r\cos\theta$
  $\tan\theta=y/x$ $\quad y=r\sin\theta$
  $z=z$ $\quad z=z$
- $D=\{(r,\theta,z): 0\le g(\theta)\le r\le h(\theta),\ \alpha\le\theta\le\beta,\ G(x,y)\le z\le H(x,y)\}$

$$=\int_\alpha^\beta\int_{g(\theta)}^{h(\theta)}\int_{G(r\cos\theta,r\sin\theta)}^{H(r\cos\theta,r\sin\theta)} f(r\cos\theta,r\sin\theta)\,dz\,r\,dr\,d\theta$$

Spherical Coordinates

- coordinates (ρ,φ,θ)
- ρ → distance from origin
- φ → angle between z axis
- θ → angle on xy plane
- $\rho^2=x^2+y^2+z^2$ $\quad x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$
- $D=\{(\rho,\varphi,\theta): 0\le g(\varphi,\theta)\le\rho\le h(\varphi,\theta),\ a\le\varphi\le b,\ \alpha\le\theta\le\beta\}$

$$=\int_\alpha^\beta\int_a^b\int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\theta)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

finally we move onto Vector Fields and applying calculus to them...

Vector Fields

- vector field...

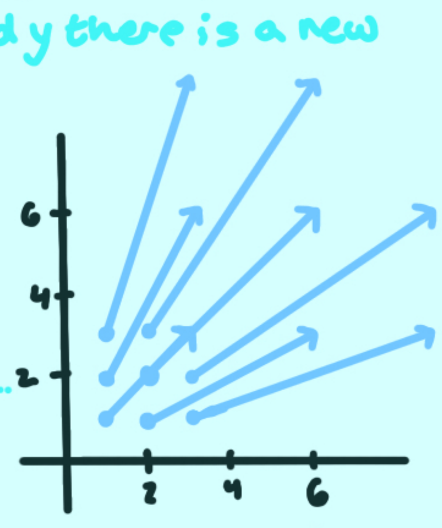
$$\vec{F}(x,y)=\langle f(x,y), g(x,y)\rangle$$

- magnitude of vector field...

$$|\vec{F}|=\sqrt{f(x,y)^2+g(x,y)^2}$$

- and this can all be expanded to 3D...

- A vector field is a function $\vec{F}$ for which we give the $\langle i,j\rangle$ i and j value a function, so that for each x and y there is a new vector...
- vector field $\vec{F}=\langle 2x,2y\rangle$
- magnitude... $2\sqrt{x^2+y^2}$



Radial Vector Fields

- all the vectors, except at the origin, point away from the origin.

$$\vec{F}(x,y)=\frac{\vec{r}}{|\vec{r}|^p}=\frac{\langle x,y\rangle}{|\vec{r}|^p}$$

$$|\vec{F}|=\frac{1}{|\vec{r}|^{p-1}}$$

Line Integrals

- evaluate an integral along curve C.

$$\int_C f(x(t),y(t))\,ds$$

- if $f(x,y)=1$ we get the length of C
- if $f(x,y)>0$ we get the are of one side of the "curtain" between f and C.
- if $f(x,y)$ is a density curve we get the mass of C.
- since we have $C: \vec{r}(t)=\langle x(t),y(t)\rangle$, $a\le t\le b$

$$\int_C f\,ds=\int_a^b f(x(t),y(t))\,|\vec{r}\,'(t)|\,dt$$

- easily extends to 3D

Line Integrals on Vector Fields

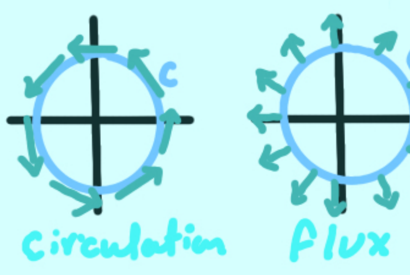
- a line integral of C on the vector field $\vec{F}$ is...

$$\int\vec{F}\cdot\vec{T}\,ds=\int_a^b\vec{F}\cdot\vec{r}\,'(t)\,dt=\int_C\vec{F}\cdot d\vec{r}$$

- this is used to find Work along a path in a vector field or Circulation of $\vec{F}$ on C.

$$\int_C\vec{F}\cdot\vec{n}\,ds=\int_a^b\vec{F}\cdot\vec{n}\,dt$$

- this finds Flux of $\vec{F}$ over C



Circulation/Work Integrals

$$\int_C\vec{F}\cdot\vec{T}=\int_C\vec{F}\cdot d\vec{r}=\int_C \underbrace{f\,dx+g\,dy}_{}$$ → divergence

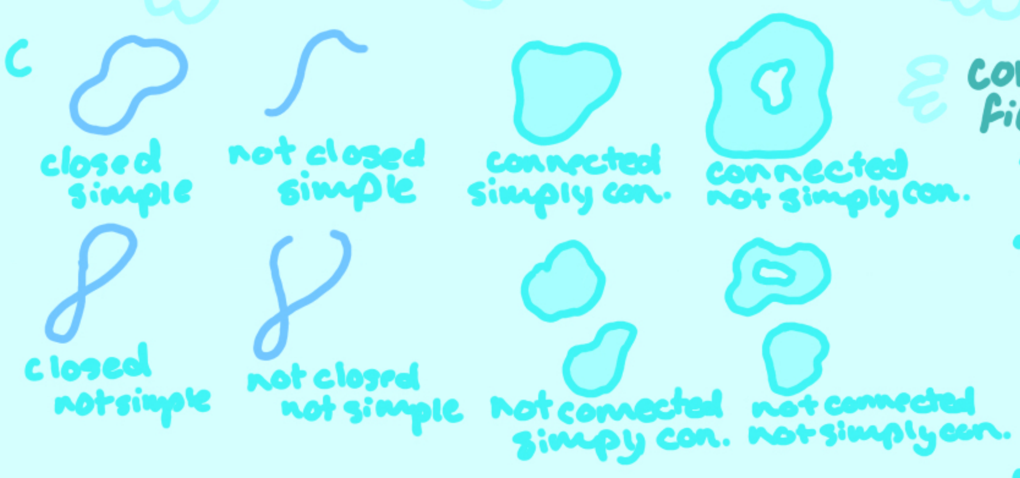
	C closed	C not closed
$\vec{F}$ conservative $\vec{F}=\nabla\varphi$	$\oint_C\vec{F}\cdot d\vec{r}=\emptyset$	$\int_C\vec{F}\cdot d\vec{r}=\varphi(B)-\varphi(A)$
$\vec{F}$ not conservative	green's theorem $\oint_C\vec{F}\cdot d\vec{r}=\iint_R g_x-f_y\,dA$	direct evaluation $\int_C\vec{F}\cdot d\vec{r}=\int_a^b fx'+gy'\,dt$

Flux Integrals

$$\int_C\vec{F}\cdot\vec{n}\,ds=\int_C \underbrace{f\,dy-g\,dx}_{}$$ → curl

	C closed	C not closed
$\vec{F}$ source free $f=\psi_y,\ g=-\psi_x$	$\oint_C\vec{F}\cdot\vec{n}\,ds=\emptyset$	$\int_C\vec{F}\cdot\vec{n}\,ds=\psi(B)-\psi(A)$
$\vec{F}$ not source free	green's theorem $\oint_C\vec{F}\cdot\vec{n}\,ds=\iint_R f_x+g_y\,dA$	direct evaluation $\int_C\vec{F}\cdot\vec{n}\,ds=\int_a^b fy'-gx'\,dt$

Curve & Region Definitions



independence of path
- if the work done on 2 dif paths between the same points is equal then those line integrals are independent of path.

Finding Potential Function

- find φ for $\vec{F}=\langle f,g,h\rangle$
1. $\int\varphi_x=\int f$ to get $\varphi+c(y,z)$
2. $\frac{d}{dy}\varphi+c(y,z)$ and compare to g to get $c(y,z)$
3. $\int c_y(y,z)$ to get $\varphi+d(z)$, repeat above steps to get $d(z)$

Vector Field Definitions

conservative vector field
- a vector field where $\vec{F}=\nabla\varphi$
- test by seeing if... $\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$ and if 3D also do... $\frac{\partial f}{\partial z}=\frac{\partial h}{\partial x}$; $\frac{\partial g}{\partial z}=\frac{\partial h}{\partial y}$
- any closed curve in a conservative VF will have... $\oint_C\vec{F}\cdot d\vec{r}=\emptyset$
- if conservative then... $\int_C\vec{F}\cdot\vec{T}=\varphi(B)-\varphi(A)$

independence of path
- if a line integral is independent of path then the vector field it's in is conservative.

source free vector fields
- a function where $\frac{\partial\psi}{\partial y}=f$ $\quad \frac{\partial\psi}{\partial x}=-g$
- $\emptyset$ flux on closed C

good luck even tho you didn't take this 1