good luck ♡

Time Series Data Analysis

midterm 2

new models

ARMA(p,q)

stationary!

- a stationary process X_t is ARMA(p,q) w/ mean $\emptyset$ if...

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$$

$\hookrightarrow Z_t \sim WN(0,\sigma^2)$

- if it has mean μ we subtract μ from every X_t (ie $\phi_p(X_{t-p}-\mu)$)
- MA(1) process $\Rightarrow$ ARMA(0,1) $\Rightarrow X_t = Z_t + \theta Z_{t-1}$
- AR(1) process $\Rightarrow$ ARMA(1,0) $\Rightarrow X_t - \phi X_{t-1} = Z_t$
- We can also define w/ a backward shift opperator

$$\phi(B)(X_t-\mu) = \theta(B)Z_t$$

Linear Process

stationary!

- X_t is a linear process if..

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

$\hookrightarrow Z_t \sim WN(0,\sigma^2)$

- Or rewrite w/ backword shift opperator as...

$$X_t = \psi(B) Z_t$$

- becomes an MA(∞) process when a linear process has $\psi_j = 0$ for all $j<0$
- ACVF of linear process $\rightarrow$

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Z(h+k-j)$$

if $Z \sim WN(0,\sigma^2)$...

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j-h} \sigma^2$$

model features

Process Mean

- we like when $E[X_t] = \emptyset$, we can get our series to have mean $\emptyset$ by estimating the mean and subtracting it from the rest of the observations.
- Sample Mean $\Rightarrow$ $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$
- Central Limit Theorem for Stationary Series $\rightarrow$

$$E[\bar{X}_n] = \mu, \ Var(\bar{X}_n) \approx \frac{v}{n}$$

$$v = \sum_{h=-\infty}^{\infty} \gamma_X(h)$$

and...

$$\frac{\bar{X}_n - \mu}{\sqrt{v/n}} \sim N(0,1)$$

- 95% Confidence Interval for μ...

$$\bar{x}_n \pm 1.96\sqrt{v/n}$$

for MA(1)...

$$v = \sum_{h=-\infty}^{\infty} \gamma_X(h) = (1+\theta^2+2\theta)\sigma^2 = (1+\theta)^2\sigma^2$$

$$\bar{x}_n \pm \frac{1.96(1+\theta)\sigma}{\sqrt{n}}$$

for AR(1)

$$\sum_{h=1}^{\infty} \gamma_X(h) = \frac{\sigma^2}{1-\phi^2}(\phi+\phi^2+\dots) = \frac{\sigma^2}{1-\phi^2}\frac{\phi}{1-\phi}$$

$$v = \frac{\sigma^2}{(1-\phi)^2}$$

$$\bar{x}_n \pm \frac{1.96\sigma}{\sqrt{n}(1-\phi)}$$

Nonnegative Definiteness

- K is a real-valued function defined on integers. We say K is a nonnegative definite if for all $n \geq 1$ for all $a_1, \dots a_n$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j K(i-j) \geq \emptyset$$

- K is the ACVF of a stationary process $\Leftrightarrow$ K is an even function and a non-negative definite.
- So we can use K = ACVF to verify stationary!
- for MA(1) $\rightarrow$ stationary if and only if $|\rho| \leq \frac{1}{2}$

Causality

(by Chappell Roan)

- if $\psi_j = 0$ for all $j<0$, we call the process casual. Aka Future independent
- $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$
- We always use casual processes because we can't predict the future if it's reliant on the future.
- for AR(1) $\rightarrow$ casual if $|\phi|<1$. casual condition or stationary condition
- for MA(q) it's easy to find casual by setting $\psi_0 = 1$ and $\psi_j = \theta_j$

Identifiability

- if θ is replaced by $\frac{1}{\theta}$ and is identical to itself then the process is unidentifiable since it cannot be distinguished by ACF.
- but if restrict θ we can make it identifiable $\rightarrow$ this is an identifiability condition
- MA(1) is identifiable for $|\theta| \leq 1$.

Invertibility

- What if we wanted to inverse MA(q)...

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

$\hookrightarrow$ ie $X_t = Z_t + \theta Z_{t-1}$

$$Z_t = \sum_{j=0}^{\infty}(-\theta)^j X_{t-j}$$

$$\pi_j = \begin{cases} (-\theta)^j & j \geq 0 \\ 0 & j<0 \end{cases}$$

- convergent only for $|\theta|<1 \rightarrow$ invertibility condition
- We want casual, identifiable, + invertible.
- Many times if $\{X_t\}$ is noncasual/noninvertible we can redefine (find a new WN( ) to make it so.

prediction

On Infinite Past

- for our Linear Process that meets all 3 predictions we have our equivalent representations...

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

- $\tilde{P}_t X_{t+h} \rightarrow$ optimal linear predictor given obs. $-\infty < s < t$

$P_t X_{t+h} = \hat{X}_{t+h} \rightarrow$ optimal linear predictor given obsv. $X_1, \dots, X_t$

- we can see from a first step process $\tilde{P}_t X_{t+1} = -\sum_{j=1}^{\infty} \pi_j X_{t+1-j}$
- for $h>1$... $\tilde{P}_t X_{t+h} = -\sum_{j=1}^{h-1} \pi_j \tilde{P}_t X_{t+h-j} - \sum_{j=h}^{\infty} \pi_j X_{t+h-j}$ but this is hard to calculate
- we can measure performance w/ mean squared pred. error...

$$MSPE = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$$

- more specifically for AR(1)...

$$\tilde{P}_t X_{t+h} = \phi^h X_t$$

$$MSPE = \sigma^2 \sum_{j=0}^{h-1} \phi^{2j}$$

(0 mean, $|\phi|<1$)

- or for MA(1)...

$$\tilde{P}_t X_{t+h} = \emptyset \quad \text{for } h>1$$

$$MSPE = \begin{cases} \sigma^2, & h=1 \\ \sigma^2(1+\theta^2), & h>1 \end{cases}$$

(0 mean, $|\theta|<1$)

On finite Past

- suppose we want to predict Y on obs. $W_1 \dots W_n$

$$\Gamma = [\Gamma_{i,j}] = [Cov(W_i, W_j)] \qquad \vec{\gamma} = [\gamma_i] = [Cov(W_i, Y)]$$

- we want optimal linear predictor $\hat{Y}$, which is the $\hat{Y}$ w/ $\vec{a}$ that solves...

$$\vec{\gamma} = \Gamma \vec{a}$$

$$\hookrightarrow \vec{a} = \Gamma^{-1}\vec{\gamma}$$

- MSPE is... $MSPE = Var(Y) - \vec{a}^T\vec{\gamma}$
- more specifically for AR(1)...

$$\Gamma = \frac{\sigma^2}{1-\phi^2}\begin{bmatrix} 1 & \phi & \cdots & \phi^{n-1} \\ \phi & 1 & & \vdots \\ \vdots & & \ddots & \\ \phi^{n-1} & \phi^{n-2} & \cdots & 1 \end{bmatrix} \quad \vec{\gamma} = \frac{\sigma^2}{1-\phi^2}\begin{bmatrix} \phi^n \\ \phi^{n-1} \\ \vdots \\ \phi \end{bmatrix} \quad \vec{a} = [0, \dots, 0, \phi]^T$$

$$\hat{Y} = \hat{X}_{n+1} = \mu + \phi^h(X_n - \mu) \quad MSPE = \sigma^2 \sum_{j=0}^{h-1} \phi^{2j}$$

- predicting missing values in AR(1)...

$$W = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \quad Y = X_2 \quad \Gamma = \frac{\sigma^2}{1-\phi^2}\begin{bmatrix} 1 & \phi^2 \\ \phi^2 & 1 \end{bmatrix} \quad \vec{\gamma} = \frac{\sigma^2}{1-\phi^2}\begin{bmatrix} \phi \\ \phi \end{bmatrix} \quad \vec{a} = \left[\frac{\phi}{1+\phi^2}, \frac{\phi}{1+\phi^2}\right]$$

$$MSPE = Var(X_2) - \vec{a}^T\vec{\gamma}$$

important examples

ARMA(1,1)

- ARMA(1,1) ...

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

w/ $Z_t \sim WN(0,\sigma^2)$

- aka

$$(1-\phi B)X_t = (1+\theta B)Z_t$$

- Linear Process...
  rewrite as...

$$X_t = \frac{1+\theta B}{1-\phi B} Z_t$$

- Casual Representation...

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

$$\psi_j = \begin{cases} 1, & j=0 \\ (\theta+\phi)\phi^{j-1}, & j \geq 1 \end{cases}$$

if $|\phi|<1$

- Invertibility...
  rewrite as...

$$Z_t = \frac{1-\phi B}{1+\theta B} X_t$$

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

$$\pi_j = \begin{cases} 1 & j=0 \\ -(\theta+\phi)(-\theta)^{j-1} & j \geq 1 \end{cases}$$

if $|\theta|<1$

- Identifiability...
  if $\theta + \phi = \emptyset \Rightarrow$ not identifiable

- Conditions
  - ☐ $|\phi|<1$ casual
  - ☐ $|\theta|<1$ invertible
  - ☐ $\theta + \phi \neq \emptyset$ identifiable
- ACVF...

$$\gamma_X(h) = \begin{cases} \sigma^2\left(1+\frac{(\phi+\theta)^2}{1-\phi^2}\right), & h=0 \\ (\theta+\phi)\phi^{|h|-1}\sigma^2\left(1+\frac{(\theta+\phi)\phi}{1-\phi^2}\right), & |h| \geq 1 \end{cases}$$

$$\gamma_X(h+1) = \phi\gamma_X(h) \quad \text{for } h \geq 1$$