NOTE: if outlined in this color, need to memorize.

CALCULUS 2 FINAL

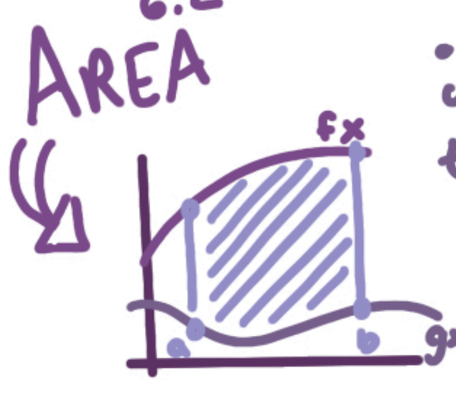
study guide

INTEGRALS

6.2 - 6.5 WHAT WE DO WITH INTEGRALS

what can we do w/ integrals...

6.2 AREA



- we find the area w/ reimann sums, which then we turn into an integral...

$$A=\int_a^b f(x)-g(x)\,dx$$

↳ can also be done w/ respect to y

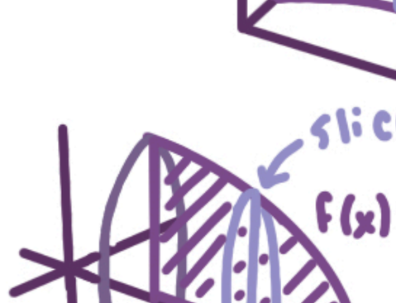
$$A=\int_c^d f(y)-g(y)\,dy$$

6.3 VOLUME

- to find the volume we use a "slice" of the obj.

slice area

$$V=\int_a^b A(x)\,dx$$

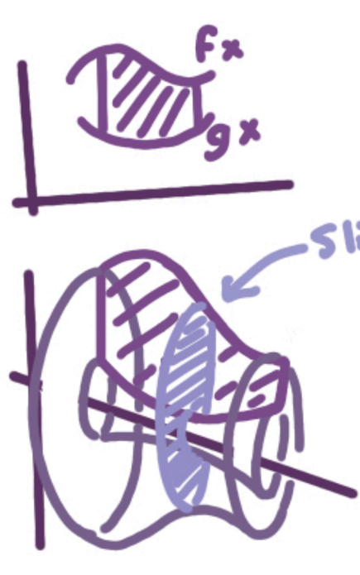


- Disk Method takes circular cross sections of shapes

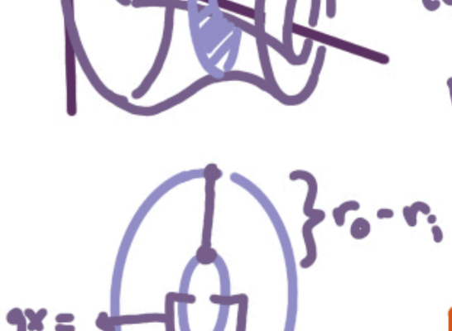
$$V=\int_a^b \pi f(x)^2\,dx$$

about y axis

↳ $V=\int_c^d \pi\, p(y)^2\,dy$



- Washer Method takes a slice of the cross section that has a circular piece of the middle missing



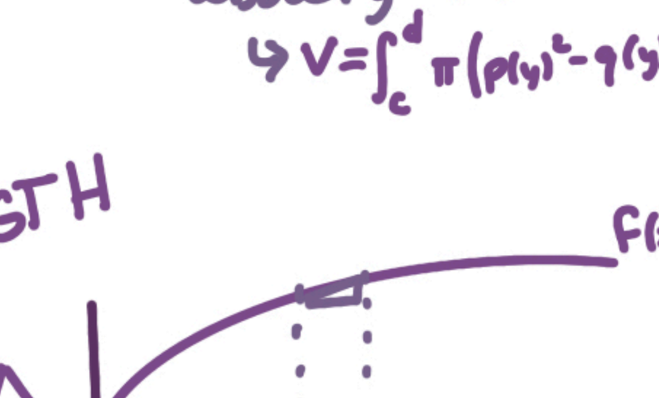
$A(x)=\pi(r_o^2-r_i^2)$
$=\pi(f(x)^2-g(x)^2)$

$$V=\int_a^b \pi\left(f(x)^2-g(x)^2\right)dx$$

about y axis

↳ $V=\int_c^d \pi\left(p(y)^2-q(y)^2\right)dy$

6.5 ARC LENGTH



use a ton of cross sections and Reimann sums (again) to find arc length.

$$L=\int_a^b \sqrt{1+f'(x)^2}\,dx$$

We can also use integrals to determine the convergence of series

Great! but what can we do w/ all this?

8.1 - 8.9 APPROACHES TO EVALUATING INTEGRALS

8.6 STEPS

1. Simplify
2. look for obvious solutions
3. Classify integrand according to form...
   - ↳ Trig Function — Product of Powers
   - ↳ Rational Function — Partial Fractions
   - ↳ Product of Polynomial w/ Trig, Log, or Exp. — Integration by Parts
   - ↳ Radicals – sum or dif of squares inside square root — Trig Sub
   - ↳ Radicals — Substitution
4. Try again ¯_(ツ)_/¯

Substitution

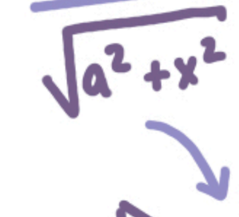
- normal u substitution
- have u equal part of the integrand then sub in u and the derivative of u, du.

8.4 Trig Sub

- To make integrands w/ hard radicals easier to solve we substitute trig equations into the integrand

Exp.	Sub.	Idnt.
$\sqrt{a^2-x^2}$	$x=a\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$, $0\le\theta\le\frac{\pi}{2}$, $\pi\le\theta<\frac{3\pi}{2}$	$\sec^2\theta-1=\tan^2\theta$

$\sin\theta=\frac{o}{h}=\frac{x}{a}$, $a\sin\theta=x$



$\tan\theta=\frac{o}{a}=\frac{x}{a}$, $a\tan\theta=x$

$\sec\theta=\frac{h}{a}=\frac{x}{a}$, $a\sec\theta=x$

- make sure to sub x back in, do not leave θ.

8.2 Integration by Parts

- reverse of product rule

$\frac{d}{dx}uv=u'v+uv'$

integrate each side

$uv=\int u'v+uv'$

rearrange

$\int uv'=uv-\int u'v$ (dv, du)

memorize

$$\int u\,dv=uv-\int v\,du$$

8.5 Partial Fractions

- rewriting rational functions as partial fractions makes them easier to integrate!

$\frac{3x}{(x-2)(x+4)}=\frac{A}{x-2}+\frac{B}{x+4}$ ← (1) what we want the final frac to look like

$3x=\frac{A(x-2)(x+4)}{x-2}+\frac{B(x-2)(x+4)}{x+4}$ ← (2) mult. both sides

$3x=Ax+4A+Bx-2B$

$0+3x=x(A+B)+(4A-2B)$ ← (3) collect like powers of x

$3x=x(A+B)$ $\quad 0=4A-2B$ ← (4) set = to

$3=A+B$

$A=1$ $B=2$ ← (5) Solve!

- for repeated linear factors $(x-r)^m$, you have to include a fraction for each power up to and including m.
  ↳ ex. $\frac{5x+32}{x^2}=\frac{A}{x}+\frac{B}{x^2}$

8.3 Product of Powers

- when integrating trig products of powers we typically use half angle formulas or pythagorean identities

Strategies for $\int \sin^m x\cos^n x\,dx$

$\int \sin^m x\cos^n x\,dx$	Strategy
m = odd, pos. n = real	split off sin x, rewrite even power of sin x in terms of cos x. u sub u = cos x
m = real n = odd, pos.	split of cos x, rewrite even power of cos x in terms of sin x. u sub u = sin x
m = even, pos. n = even, pos.	use half angle formulas to transfer the integrand into polynomial in cos 2x. then cont. to reapply until no powers > 1.

Strategies for $\int \tan^m x\sec^n x\,dx$

$\int \tan^m x\sec^n x\,dx$	Strategy
m = real n = even, pos.	split off $\sec^2 x$, rewrite even power in terms of tan x, u sub, u = tan x
m = odd, pos. n = real	split off sec x tan x rewrite even power of tan x in terms of sec x, u sub u = sec x
m = even, pos. n = odd, pos.	rewrite even power of tan x in terms of sec x to make polynomial in sec x, $\int \sec^n x\,dx=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}\int \sec^{n-2}x\,dx$

Useful trig identities

Memorize!

pythagorean

$\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

halfangle

$\sin^2 x=\frac{1-\cos 2x}{2}$

$\cos^2 x=\frac{1+\cos 2x}{2}$

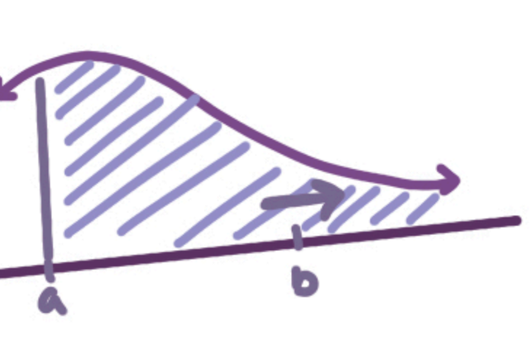
8.9

But what if the integral is improper?

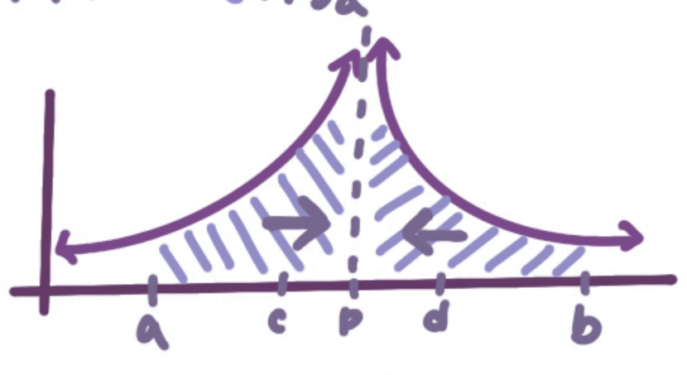
an integral is improper if the interval of integration is ±∞ or if there is a discontinuity in the interval of integration

Use limits

$$\int_a^\infty f(x)\,dx=\lim_{b\to\infty}\int_a^b f(x)\,dx$$



$$\int_a^b f(x)\,dx=\lim_{c\to p^-}\int_a^c f(x)\,dx+\lim_{d\to p^+}\int_d^b f(x)\,dx$$



8.1 Basic Integration Formulas

Memorize

$\int k\,dx=kx+C$	$\int \frac{dx}{x}=\ln\lvert x\rvert+C$
$\int x^p\,dx=\frac{x^{p+1}}{p+1}+C$	$\int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$
$\int \cos ax\,dx=\frac{1}{a}\sin ax+C$	$\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C$
$\int \sin ax\,dx=-\frac{1}{a}\cos ax+C$	$\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\lvert\frac{x}{a}\rvert+C$
$\int \sec^2 ax\,dx=\frac{1}{a}\tan ax+C$	$\int \tan ax\,dx=\frac{1}{a}\ln\lvert\sec ax\rvert+C$
$\int \csc^2 ax\,dx=-\frac{1}{a}\cot ax+C$	$\int \cot ax\,dx=\frac{1}{a}\ln\lvert\sin ax\rvert+C$
$\int \sec ax\tan ax\,dx=\frac{1}{a}\sec ax+C$	$\int \sec ax\,dx=\frac{1}{a}\ln\lvert\sec ax+\tan ax\rvert+C$
$\int \csc ax\cot ax\,dx=-\frac{1}{a}\csc ax+C$	$\int \csc ax\,dx=-\frac{1}{a}\ln\lvert\csc ax+\cot ax\rvert+C$
$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+C$	$\int b^x\,dx=\frac{b^x}{\ln b}+C$

SERIES

but first... 10.1 - 10.2 SEQUENCES

10.1

$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ (term)

$\{a_n\}$ → SEQUENCE: an ordered list of numbers

$a_n=f(n)$ ↳ explicit formula

10.2

- We can find if a sequence converges by finding the limit of the sequence.
- For a geometric sequence $\{r^n\}$ we can find the convergence based on r...

$$\lim_{n\to\infty} r^n=\begin{cases}0, & \text{if } \lvert r\rvert<1\\ 1, & \text{if } r=1\\ \text{dne}, & \text{if } r\le -1 \text{ or } r>1\end{cases}$$

- We can also use squeeze theorem

10.3 - 10.8 Infinite SERIES

is an infinite series just the sum of a sequence

10.3

- Partial sums are the sums of certain number of terms...

partial sum:
$S_1=a_1$
$S_2=a_1+a_2$
$S_3=a_1+a_2+a_3$
$S_n=\sum_{k=1}^{n}a_k$

- So an infinite series is $\lim_{n\to\infty}\sum_{k=1}^{n}a_k$
- And with w/ series what we really want is to find if it Converges. (and occasionally what it converges to)

Convergence Tests

Memorize!

	Series/ Test	Form	Converges	Diverges
10.3	geometric series	$\sum_{k=0}^{\infty} ar^k$, $a\neq 0$	$\lvert r\rvert<1$	$\lvert r\rvert\ge 1$
10.4	divergence test	$\sum_{k=1}^{\infty} a_k$	DNE	$\lim_{k\to\infty}a_k\neq 0$
10.4	integral test	$\sum_{k=1}^{\infty} a_k$, $a_k=f(x)$ and f is... □ cont. □ pos. □ decreases	$\int_1^\infty f(x)\,dx$ converges	$\int_1^\infty f(x)\,dx$ diverge
10.4	p series	$\sum_{k=1}^{\infty}\frac{1}{k^p}$	$p>1$	$p\le 1$
10.7	ratio test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty}\left\lvert\frac{a_{k+1}}{a_k}\right\rvert<1$	$\lim_{k\to\infty}\left\lvert\frac{a_{k+1}}{a_k}\right\rvert>1$
10.7	root test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty}\sqrt[k]{\lvert a_k\rvert}<1$	$\lim_{k\to\infty}\sqrt[k]{\lvert a_k\rvert}>1$
10.5	comparison test	$\sum_{k=1}^{\infty} a_k$, $a_k>0$	$a_k\le b_k$; $\sum_{k=1}^{\infty} b_k$ converges	$b_k\le a_k$; $\sum_{k=1}^{\infty} b_k$ diverges
10.5	limit comparison test	$\sum_{k=1}^{\infty} a_k$ □ $a_k>0$ □ $b_k>0$	$0\le\lim_{k\to\infty}\frac{a_k}{b_k}<\infty$; $\sum_{k=1}^{\infty} b_k$ converges	$\lim_{k\to\infty}\frac{a_k}{b_k}>0$; $\sum_{k=1}^{\infty} b_k$ diverges
10.6	alternating series test	$\sum_{k=1}^{\infty}(-1)^k a_k$ □ $a_k>0$	$\lim_{k\to\infty}a_k=0$; $0<a_{k+1}\le a_k$	$\lim_{k\to\infty}a_k\neq 0$
10.6	absolut convergence	$\sum_{k=1}^{\infty} a_k$	$\sum_{k=1}^{\infty}\lvert a_k\rvert$ converges	
10.3	telescoping series	a sequence where the middle "folds in on itself", where the second part of one term cancles out the first part of the next term.		

bonus: we know if $\lvert r\rvert<1$ then $\sum_{k=0}^{\infty} ar^k=\frac{a}{1-r}$

since we typically can't find where a series converges to, we try to estimate it, and we do this we want to know how correct our estimates are...

Remainder

10.4

- we bound the remainder w/ ...

$$R_n<\int_n^\infty f(x)\,dx$$

10.6

- the remainder of an alternating series can be found w/...

$$\lvert R_n\rvert\le a_{n+1}$$

but what if I want to know the sum of an infinite series?

that is one use of...

11.1 - 11.4 TAYLOR SERIES

11.1

- An infinite series of the form...

$$\sum_{k=0}^{\infty}c_k(x-a)^k=c_0+c_1(x-a)+c_2(x-a)^2+\ldots c_n(x-a)^n+\ldots$$

(coefficient) (center)

is a Power Series

- A polynomial approx. is an infinite series that stops at some degree n...

$$P_n(x)=c_0+c_1(x-a)+c_2(x-a)^2+\ldots c_n(x-a)^n=\sum_{k=0}^{n}c_k(x-a)^k$$

11.2

- much like w/ infinite series we want to know the convergence of power series. We find convergence with the Ratio or Root test and we find...
  - > the interval of convergence > set of values for which x converges
  - > radius of convergence (R) > the distance from the center to the boundry of the interval of convergence.
- we can also Differentiate & Integrate power series

$$f'(x)=\sum_{k=1}k c_k(x-a)^{k-1}$$

$$\int f(x)\,dx=\sum c_k\frac{(x-a)^{k+1}}{k+1}+C$$

11.1

- A polynomial approximation where the coefficients are...

$$c_k=\frac{f^{(k)}(a)}{k!}$$

and the full polynomial is...

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called a Taylor Polynomial.

- The remainder of a Taylor Polynomial can be seen as...

$$R_n(x)=f(x)-p_n(x)$$

and can be found w/ ...

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

- But what if we want a Taylor polynomial where n is ∞...

11.3

- A Taylor Series is an infinite series of terms that are expressed in terms of the functions derivatives at a single point.
- A Taylor Series is an infinite Taylor Polynomial

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots$$

note: a Taylor Series centered at 0 is a Maclaurin Series

- A Taylor Series also has the property that they can converge to a certain function. Modifying these functions can be an easy way to find a Taylor Polynomial

11.3 Common Maclaurin Series

Function	Series	Radius of Convergence
$\frac{1}{1-x}$	$=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\ldots$	R=1
e^x	$=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$	R=∞
$\sin x$	$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	R=∞
$\cos x$	$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	R=∞
$\tan^{-1}x$	$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	R=1
$\ln(1+x)$	$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$	R=1
$(1+x)^k$	$=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\ldots$	R=1

this chart is included on the test!

Binomial Series

- this series converges for $\lvert x\rvert<1$ and it's endpoints depend on k...
  - $p\ge 0$, both endpoints included [,]
  - $-1<p<0$, only x=1 endpoint included (,]
  - $p\le -1$, no endpoint included (,)
- important for stats

11.4 Some Taylor Series Uses...

- Evaluate Limits
- Rep functions as power series
- Differentiate those power series
- Integrate those power series
- Rep real numbers
- Prove L'Hôpitals Rule
- Find sum of Infinite Series

Note: I did not include 12.1 & 12.2 on the study guide cause Miranda said they wouldn't be on the test.

Note: You got this!!! Good Luck