optimization

final study guide

the garden what are the ingredients for optimization

Linear Algebra

- we have matrices and vectors. We can multiply matrices by vectors and matrices by matrices...

$A\vec{x} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + \cdots + A_{1n}x_n \\ \vdots \\ A_{n1}x_1 + \cdots + A_{nn}x_n \end{bmatrix}$ produces a vector; i^{th} row is $= \sum_{j=1}^{n} A_{i,j} x_j$

$AB = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots & B_{1K} \\ \vdots & & \vdots \\ B_{n1} & \cdots & B_{nK} \end{bmatrix} = C$

element $C_{ij} = \sum_{l=1}^{n} A_{il} B_{lj}$

- if we have a collection of column vectors, $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_K$, they are **linearly indp.** if $\lambda_1 \vec{v}_1 + \cdots + \lambda_K \vec{v}_K = \vec{0}$ is only true if $\lambda_1 = \cdots = \lambda_K = 0$.
- if the columns of A are linearly indp. that also implies... ① A^{-1} exists ② $(A^T)^{-1}$ exists ③ there is a unique solution to $A\vec{x} = \vec{b}$ for all $\vec{b}$.

Linear Programing

- linear programing just means that we will only be using linear equations
- a **Linear Program** maximizes/minimizes a linear function subject to linear constraints. parts of an LP are...
  - **variables** → what we have control over, what can change?
  - **objective** → what we want to maximize or minimize?
  - **constraints** → what rules do we have to follow? what is bounding us?
- To write LP's more consicily we will often put all variables into a vector and all data related to a constraint into a matrix.
- lets define a standard way to write LP's...

Standard Equality Form

- An LP is in SEF if it looks like...

$$\max\ \vec{c}^T\vec{x} + \bar{z} \quad \text{s.t. } A\vec{x} = \vec{b},\ \vec{x} \geq 0$$

- every linear program can be brought into SEF...
  - $\min \vec{c}^T\vec{x} \Rightarrow \max (-\vec{c}^T)\vec{x}$
  - $\vec{a}^T\vec{x} \leq \beta \Rightarrow \vec{a}^T\vec{x} + s_1 = \beta$
  - $\vec{a}^T\vec{x} \geq \beta \Rightarrow \vec{a}^T\vec{x} - s_2 = \beta$
  - x_i (free variable) $\Rightarrow x_i^+ - x_i^-,\ x_i^+ \geq 0,\ x_i^- \geq 0$

Solutions

- Now that we have our LP in the correct form lets think about solving it. There are 3 types of solutions... **infeasible**, **unbounded**, and **optimal**.

infeasible
> no feasible area.

to prove we need a **certificate of infeasibility**...
> given an LP in SEF we need to find a vector $\vec{y}$ s.t...
$\vec{y}^T A \geq 0$
$\vec{y}^T \vec{b} < 0$
> if the LP is infeasible such a $\vec{y}$ exists and we can find by solving... $\min \vec{y}^T\vec{b}$ s.t. $\vec{y}^T A \geq 0$

unbounded
> the feasible region goes on forever.

to prove we need a **certificate of unboundedness**
> given an LP in SEF and a feasible solution $\vec{x}$, we want a $\vec{d}$ s.t...
$\vec{d} \geq 0$
$\vec{c}^T\vec{d} > 0$
$A\vec{d} = 0$
> if the LP is unbounded such a $\vec{d}$ exists and can be found by solving $A\vec{d} = 0$.

optimal
> an optimal solution exists when the feasible region is bounded. We can solve LP's w/ optimal solutions using the simplex method...

Basis & Canonical Form

- Before we can learn simplex we need to know some tools.
- **Basis**
  > if B is a set of #'s and A_B is a submatrix of A w/ the rows corresponding to the #s in B, then B is a Basis if A_B is square & A_B^{-1} exists.
  > N is a set of all the #s not in B.
  > $\vec{x}_B$ is a subvector of $\vec{x}$ w/ elements corresponding to B. Similarly w/ $\vec{x}_N$.
  > $\vec{x}$ is a **basic solution** if... $A\vec{x} = \vec{b}$ and $\forall\ \vec{x}_N = 0$
- **Canonical Form**
  > given an LP and a basis B the LP is in **canonical form** for B if...
  $A_B = I$
  $\vec{c}_B = \vec{0}$
  > given a basis B we can put any LP into canonical form

① lets start by achieving $A_B = I$ by...
$A\vec{x} = \vec{b} \Rightarrow \underbrace{A_B^{-1}A}_{\text{"new A"}}\vec{x} = \underbrace{A_B^{-1}\vec{b}}_{\text{"new B"}}$

② now lets get $\vec{c}_B = \vec{0}$, we get this by combining $z(x) = \vec{c}^T\vec{x}$ and $0 = \vec{b} - A\vec{x}$...
$z(x) \Rightarrow z(x) = \vec{y}^T\vec{b} + \underbrace{(\vec{c}^T - \vec{y}^T A)}_{\text{"new c"}}\vec{x}$

now lets choose $\vec{y}$ to ensure $\vec{c}_B = \vec{0}$...
$\vec{y}^T = \vec{c}_B^T A_B^{-1}$

Simplex Method

- steps for Simplex Method

① convert the LP you're solving into SEF. Select a basis B.

② convert the basis to canonical form and find a basic feasible solution s.t. $\vec{x}_B = \vec{b}$ and $\vec{x}_N = \vec{0}$
> if $\vec{c}_N \nleq 0$ and theres a column $K \in N$ w/ $c_K > 0$ & A_K has atleast 1 pos. element goto 3
> if $\vec{c}_N \leq 0$ go to step 4
> if $\vec{c}_N \nleq 0$ and every column $K \in N$ w/ $c_K > 0$ has $A_K \leq 0$ goto step 5

③ we choose one variable K to enter the basis B and solve
$\vec{x}_B I + x_K A_K = \vec{b} \Rightarrow x_K \leq b_i / A_{K,i}$
> if $A_{K,i} < 0 \Rightarrow x_K \leq +\infty$ and we can ignore those
> if mult. $A_{K,i} > 0$ these all serve as upper bounds for x_K and we choose the largest value that satisfies them all for x_K
x_K enters the basis and $x_{B_i} = 0$ exits. Repeat step 2 w/ new basis.

④ if $\vec{c}_N \leq 0$ then $\vec{x}$ is the **optimal solution** and $\vec{y}^T\vec{b}$ is the optimal value. End

⑤ if all $A_K \leq 0$ then the LP is **unbounded** and we can construct a certificate of unboundedness $\vec{d}$ as proof. End

the shop Lets use what we grew in the garden

A Linear Program

- lets apply these concepts we've learned to some actual problems. Lets look at this table w/ info about Lulu + Cici's flower shop...

	1 pink bouquet	1 green bouquet	avail.
flowers	12	8	200
work (hr)	1	1.5	24
profit ($)	20	25	
demand	∞	20	

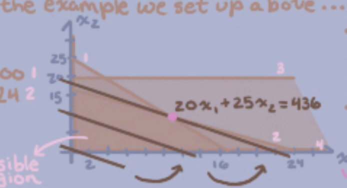
maximize profit!

LP → var → x_1 = pink b's to make, x_2 = green b's to make
obj → $\max\ 20x_1 + 25x_2$
cons → s.t. $12x_1 + 8x_2 \leq 200$
$x_1 + (1.5)x_2 \leq 24$
$x_2 \leq 20$
$x_1, x_2 \geq 0$

Solving Graphically

- if we don't want to use the simplex method we can also solve 2 var. LP's graphically. Lets use the example we set up above...

$\max\ 20x_1 + 25x_2$
s.t. $12x_1 + 8x_2 \leq 200$ 1
$x_1 + (1.5)x_2 \leq 24$ 2
$x_2 \leq 20$ 3
$x_1, x_2 \geq 0$ 4



- to solve using this graph we graph the **isoprofit line** based on the objective and push it to the edge of the feasible region.
- the point where the region and isoprofit line intersect is the optimal solution.
→ $x_1 = 10.8$ $x_2 = 8.8$

- now lets look at some types of LP's we could see.

Blending Problem

- Lulu and Cici mix their own special flower soil. They buy 3 dif. types of gardening soil and combine them to make 3 dif. types of flower soil...

soil	fertilizer %	silt %	cost $	avail.
type1	.5	30	25	50
type2	2	10	35	50
type3	3	20	45	50

soil	fert. min	silt max	flowers	$ per flower
type1	2	15	2500	4
type2	3	15	5000	4
type3	1	25	900	3

lets make an LP to max Lulu + Cici's profits...

LP → var → g_i = amount of garden soil to buy
f_j = amount of flower soil to make
x_{ij} = amount of soil i used for flower soil j
$i, j \in \{1, 2, 3\}$
obj. → $4 \cdot 2500 \cdot f_1 + 4 \cdot 5000 \cdot f_2 + 3 \cdot 900 \cdot f_3 - 25 \cdot g_1 - 35 \cdot g_2 - 45 \cdot g_3$
cons. → (there will be a lot)

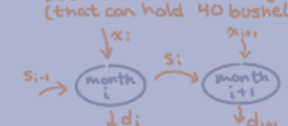
$\underbrace{0 \leq g_1, g_2, g_3 \leq 50,\ 0 \leq f_1, f_2, f_3,\ g_i = x_{i1} + x_{i2} + x_{i3},\ f_j = x_{1j} + x_{2j} + x_{3j}}_{\text{basics}}$

fert: $.005x_{11} + .02x_{21} + .03x_{31} \leq .02f_1$
$.005x_{12} + .02x_{22} + .03x_{32} \leq .03f_1$
$.005x_{13} + .02x_{23} + .03x_{33} \leq .01f_1$

silt: $.3x_{11} + .1x_{21} + .2x_{31} \geq .15f_1$
$.3x_{12} + .1x_{22} + .2x_{32} \geq .15f_1$
$.3x_{13} + .1x_{23} + .2x_{33} \geq .25f_1$

Multiperiod Problem

- Lulu and Cici want to grow flowers in good growing seasons and hold them in their fridge (that can hold 40 bushels) to minimize cost. How many flowers should they grow permonth?



parameters →
d_i = demand in month i
p_i = price to grow flowers in moth i

LP → var → x_i = flowers grown in month i
s_i = flowers leftover and in storage after demand is satisfied in month i.
obj. → $\min \sum_{i=1}^{n} p_i \cdot x_i$
const. → $x_i + s_{i-1} = d_i + s_i$ $x_i \geq 0$, $0 \leq s_i \leq 40$
nice and simple

LP's in Disguise

- we obviously can't put non linear functions into an LP, but what if we could find a way to include versions of these functions...

Minimum
$\min\{x_1, x_2\} \Rightarrow \max z$ s.t. $z \leq x_1$, $z \leq x_2$

Maximum
$\max\{x_1, x_2\} \Rightarrow \min z$ s.t. $z \geq x_1$, $z \geq x_2$

absolute value
$|x| = \max\{x, -x\} \Rightarrow \min z$ s.t. $z \geq x$, $z \geq -x$
(this can also apply to sums of abs. values!)

Portfolio Optimization

- We could use LPs to maximize profit from a stock portfolio. Lets look at some special vocab used in this scenario...
- **Return Matrix** → return for every $1 invested in stock j if scenario i occurs. $R = [r_{i,j}]$
- **probability** of scenario $i = p_i$
- **return in scenario** $i = r_i = \sum_{j=1}^{n} R_{ij} x_j$
- **expected return** $= r_p = \sum_{i=1}^{m} p_i r_i$
- **downside risk in s.** $i = d_i = \begin{cases} 0, & \text{if } r_i \geq 0 \\ -r_i, & \text{if } r_i \leq 0 \end{cases}$
- **avg. downside risk** $= ADR = \sum_{i=1}^{m} p_i d_i$

- Now lets use what we learned to help Lulu + Cici w/ their portfolio...
$R = \begin{bmatrix} 2 & 0 \\ -1 & .1 \end{bmatrix}$ $\vec{p} = [.7\ \ .3]$

Lets maximize expected return...
LP → var → $\vec{x}$ = amount invested in each stock
$\vec{r}$ = return in each s.
r_p = expected return
obj → $\max r_p$
const → $r_p = .7r_1 + .3r_1$
$r_1 = 2x_1$, $r_2 = -x_1 + .1x_2$
$x_1, x_2 \geq 0$, $x_1 + x_2 \leq 1$

what if we wanted to min risk?
obj → min ADR
const → $ADR = .7d_1 + .3d_2$
$d_1 = \max\{0, -r_1\} \rightarrow d_1 \geq 0,\ d_1 \geq -r_1$
$d_2 = \max\{0, -r_2\} \rightarrow d_2 \geq 0,\ d_2 \geq -r_2$

the franchise lets expand on what we have

Integer Programing

- An integer program is like an LP but all of the variables are restricted to be 0 or 1. These binary variables can represent a yes or no decision.
- Lets use an IP to find the best spots for a Lulu + Cici flower shop franchise...
> we have 5 cities to cover and 4 places to build a shop...

$$\begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline & 1 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 2 \\ A = & 1 & 1 & 0 & 0 & 3 \\ & 1 & 1 & 1 & 0 & 4 \\ & 0 & 0 & 1 & 1 & 5 \end{array}$$

$A_{i,j} = \begin{cases} 1, & \text{if shop j covers city i.} \\ 0, & \text{otherwise} \end{cases}$

IP → var → $x_j = \begin{cases} 1, & \text{if build shop in local j} \\ 0, & \text{else} \end{cases}$
obj → $\min x_1 + x_2 + x_3 + x_4$ (min # of shops)
const → $x_1 + x_3 \geq 1$
$x_2 + x_3 \geq 1$
$x_1 + x_2 \geq 1$
$x_1 + x_2 + x_3 \geq 1$
$x_3 + x_4 \geq 1$
aka... $A\vec{x} \geq \vec{e}$

IP Bigger Problem

- Lulu and Cici made 3 big bouquets of flowers, they have 5 people interested in buying a bouquet, how much each person will pay is below. Lets maximize Lulu + Cici's profits...

	Abby	Bob	Cara	Dom	Erin
bouquet 1	$60	$40	—	$65	$55
b2	$50	$70	$55	$90	$80
b3	—	$80	$75	—	$70

$A = \begin{bmatrix} 60 & 40 & 0 & 65 & 55 \\ 50 & 70 & 55 & 90 & 80 \\ 0 & 80 & 75 & 0 & 70 \end{bmatrix}$

IP → var → $x_{i,j} = \begin{cases} 1, & \text{if they sell bouquet i to person j} \\ 0, & \text{otherwise} \end{cases}$
obj → $\max \sum_{i=1}^{3}\sum_{j=1}^{5} A_{ij} x_{ij}$
const → $\sum_{j=1}^{5} x_{ij} = 1,\ \forall i$
$\sum_{i=1}^{n} x_{ij} \leq 1,\ \forall j$

Mixed Integer Programing

- Mixed integer programing is when you have a normal LP plus some of the variables are restricted to be binary.
- A common MIP is adding a fixed cost to a Multiperiod problem.
- Lets look at Lulu and Cici saving flowers during poor growing periods, lets say they incur a fixed cost of $c if they have to turn their fridge on to store any flowers (so if $f_i > 0$). Add this factor to the LP...

old LP → var → x_i = flowers grown in month i
s_i = flowers leftover and in storage after demand is satisfied in month i.
obj. → $\min \sum_{i=1}^{n} p_i \cdot x_i$
const. → $x_i + s_{i-1} = d_i + s_i$, $x_i \geq 0$, $0 \leq s_i \leq 40$

add → $y_i \in \{0, 1\}$
obj → $\min \sum_{i=1}^{n} p_i \cdot x_i + c \cdot y_i$
const → $0 \leq x_i \leq M \cdot y_i$
choose M so that x will never exceed it in an optimal sol.

Modeling Tricks

- when creating MIP we'll probably want to use binary variables to model various logical conditions. Here are a couple we might see...

① At least 1 of $x_1, \dots x_n$ is 1
$x_1 + x_2 + \cdots + x_n \geq 1$

② At most 1 of $x_1 \cdots x_n$ is 1
$x_1 + x_2 + \cdots + x_n \leq 1$

③ Exactly 1 of $x_1 \cdots x_n$ is 1
$x_1 + x_2 + \cdots + x_n = 1$

④ if $x_1 = 1$, then $x_2 = 1$
$x_1 \leq x_2$

⑤ if $x_1 = 0$, then $x_2 = 1$
$1 - x_1 \leq x_2$

⑥ if $x_1 = 1$, then $x_2 = 1$ or $x_3 = 1$
$x_1 \leq x_2 + x_3$

⑦ if $x_1 = 1$, then $x_2 = 1$ and $x_3 = 1$
$2x_1 \leq x_2 + x_3$

⑧ if $x_1 = 0$, then $x_2 = 1$ and $x_3 = 1$
$1 - x_1 \leq x_2$, $1 - x_1 \leq x_3$

⑨ if $x_1 = 0$, then $x_2 = 0$ or $x_3 = 1$
$x_1 \geq x_2 - x_3$

⑩ if $x_1 = 0$, then $x_2 = 0$ and $x_3 = 1$
$x_1 \geq x_2$, $(1 - x_1) \leq x_3$

⑪ at least 2 of the following are true...
$x_2 = 1$ $x_3 = 0$ $x_4 = 1$ $x_5 = 0$
$x_2 - x_3 + x_4 - x_5 \geq 0$

⑫ if $x_1 = 1$ then at least 2 of the following are true...
$x_2 = 1$ $x_3 = 1$ $x_4 = 1$ $x_5 = 1$
$2x_1 \leq x_2 + x_3 + x_4 + x_5$

⑬ if $x_1 = 1$ then at least 2 of the following are true...
$x_2 = 1$ $x_3 = 0$ $x_4 = 1$ $x_5 = 0$
$2(x_1 - 1) \leq x_2 - x_3 + x_4 - x_5$

good luck