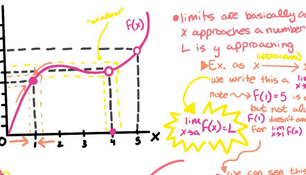
Limits

what's a limit?



• limits are basically asking, as x approaches a number a, what number L is y approaching (approaches)

▶ Ex. as $x \to 1$, $y \to 5$

we write this as $\lim_{x \to 1} y = 5$

note ⇝ $f(1) = 5$ is also true here but not always. infact, $f(1)$ doesn't even have to exist for $\lim_{x \to a} f(x)$ to exist.

$\lim_{x \to a} f(x) = L$

we can see this with...

$\lim_{x \to 4} f(x) = 6$

but...

$f(4) = \emptyset$

also...

$\lim_{x \to 5} f(x) = 8$

but

$f(5) =$ Does Not Exist (DNE)

Computing Limits

$\lim_{x \to 4} f(x) =$ DNE

Ways the Limit DNE...

* when $\lim_{x \to a^-} \neq \lim_{x \to a^+}$
* vertical asymptotes
* wild behavior

short answer: the limit Does Not Exist

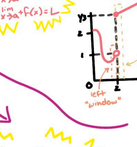
We know how to take limits by thinking about a "window" around the x value, but what if the "window" is empty on one side?

Long Answer: we take the limit from the left and from the right

notation

left ↳ $\lim_{x \to a^-} f(x) = L$

right ↳ $\lim_{x \to a^+} f(x) = L$



$\lim_{x \to 2^-} f(x) = 1$

$\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

So...

$\lim_{x \to 2} f(x) =$ DNE

THE TRICKS

1. factor + reduce
2. make 1 fraction
3. rationalize using $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$, $(a+b)(a-b) = a^2 - b^2$, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
4. Expand $(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 - 2ab + b^2$

Some times we've gotta have to use alternate tricks to rewrite $\frac{0}{0}$ forms

Some more tricks to find limits

Rewrite absolute value as piecewise

THE RULES

But what if we don't have a graph?

1. Sum / Difference: $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
2. Constant Multiple: $\lim_{x \to a}(f(x)c) = c\lim_{x \to a} f(x)$
3. Product: $\lim_{x \to a}(f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
4. Quotient: $\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, if $\lim_{x \to a} g(x) \neq 0$
5. Power / Root: $\lim_{x \to a}(f(x))^n = (\lim_{x \to a} f(x))^n$, if $f(x) > 0$ and n is even

we can find limits of a function w/ limit rules!

AND... we can solve linear functions, polynomials and rational functions with direct substitution.

Squeeze Theorem

> functions $f(x)$ & $g(x)$ have the same limit L at a
> $f(x), g(x)$, and $h(x)$ satisfy $f(x) \le h(x) \le g(x)$
> then $\lim_{x \to a} h(x) = L$

visually

useful for trig functions

Ex.

$\lim_{x \to 4} p(x)$ $\quad p(x) = 7x^3 + 3x^2 + 4x + 2$

$\lim_{x \to 4}(7x^3 + 3x^2 + 4x + 2)$

$7\lim_{x \to 4}(x^3) + 3\lim_{x \to 4}(x^2) + \lim_{x \to 4}(4x + 2)$ rule 1

$7\lim_{x \to 4}(x^3) + 3\lim_{x \to 4}(x^2) + \lim_{x \to 4}(4x) + \lim_{x \to 4} 2$ rule 2

$7(4^3) + 3(4^2) + 4(4) + 2$ rule 3

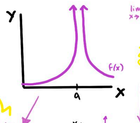
$7 \cdot 64 + 3 \cdot 16 + 16 + 2$

$448 + 48 + 16 + 2$

$= 514$

Limits of and at ±∞

limits of ∞ or infinite limits

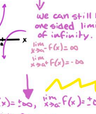


$\lim_{x \to a} f(x) = \infty$

↳ x dependent variable becomes an infinitely large number

↳ technically the limit DNE

if a function is $\frac{\#}{0}$ where # is a non zero number, then the $\lim \frac{\#}{0} = \pm\infty$



we can still have one sided limits of infinity.

$\lim_{x \to a^-} f(x) = \infty$

$\lim_{x \to a^+} f(x) = -\infty$

if $\lim_{x \to a^-} f(x) = \infty$, $\lim_{x \to a^+} f(x) = \pm\infty$, then $\lim_{x \to a} f(x) = \pm\infty$

then there is a vertical asymptote at $x = a$.

Evaluating limits at ∞

limits at ∞

Rules

1. if n is even $\lim_{x \to \pm\infty} x^n = \infty$
2. if n is odd $\lim_{x \to \infty} x^n = \infty$, $\lim_{x \to -\infty} x^n = -\infty$
3. $\lim_{x \to \pm\infty} \frac{1}{x^n} = \emptyset$
4. $\lim_{x \to \pm\infty} f(x) = \pm\infty$

as x approaches ∞ y approaches L or...

$\lim_{x \to \infty} f(x) = L$

$f(x)$ can equal ∞ too...

$\lim_{x \to \infty} f(x) = \infty$ (y is unbounded)

if $\lim_{x \to \infty} f(x) = L$ then there is a horizontal asymptote at $y = L$.

Intuitively finding the limit of f(x) at ∞

note $\lim f(x) = \frac{p(x)}{q(x)}$

* if leading degree of $p(x) > q(x)$ then $\lim_{x \to \pm\infty} f(x) = \pm\infty$ (rule 4) ; there no horiz. asymptote
* if leading degree of $p(x) < q(x)$ then $\lim_{x \to \pm\infty} f(x) = \emptyset$; there is a horiz. asymptote at $y = \emptyset$
* if leading degree of $p(x) = q(x)$ then $\lim_{x \to \pm\infty} f(x) =$ ratio of the leading coefficients

End Behavior for e and ln

* $\lim_{x \to \infty} e^x = \infty$; $\lim_{x \to -\infty} e^x = \emptyset$
* $\lim_{x \to \infty} e^{-x} = \emptyset$; $\lim_{x \to -\infty} e^{-x} = \infty$
* $\lim_{x \to \infty} \ln(x) = \infty$; $\lim_{x \to \infty} \ln x = \infty$

Continuity

what is continuity

A function is continuous at a if $\lim_{x \to a} f(x) = f(a)$

↳ checklist

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

a function is continuous from the right if... $\lim_{x \to a^+} f(x) = f(a)$

a function is continuous from the right if... $\lim_{x \to a^-} f(x) = f(a)$

Functions that are continuous

If $f(x)$ & $g(x)$ are continuous, then...

* $f(x) + g(x)$
* $f(x) - g(x)$
* $f(g(x))$
* $f(x) / g(x)$
* $c(f(x))$

are continuous

polynomials + rational functions are cont.

All of the following are continuous in their domain

trig: $\sin x$, $\cos x$, $\cos x$, $\csc x$, $\tan x$, $\cot x$

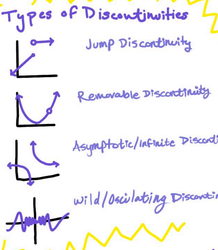
inverse trig: $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\csc^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$

exponential: b^x, e^x

logarithmic: $\log_b x$, $\ln x$

Intermediate Value Theorem

Types of Discontinuities



Jump Discontinuity

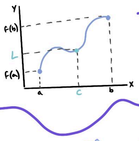
Removable Discontinuity

Asymptotic/Infinite Discontinuity

wild/Oscillating Discontinuity

Intermediate Value Theorem

* f is continuous on the interval $[a, b]$
* L is a number strictly between $f(a)$ and $f(b)$
* then there exists at least one number c in (a, b) satisfying $f(c) = L$



DERIVATIVES

defining Derivatives

↳ secant line

aka a line that passes through 2 points

aka average velocity

aka $\frac{f(x) - f(a)}{x - a} = \frac{\Delta y}{\Delta x}$

* ↳ tangent line

aka a line that passes through 1 point

aka instantaneous velocity

aka the lim of the secant line

$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

↳ the derivative can also be a function if we do...

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Derivative Rules

Graphing Derivatives

Derivative Rules

1) Derivative of a constant
if $f(x) = c$
then $f'(x) = \emptyset$

2) Derivative of $f(x) = x$
if $f(x) = x$
then $f'(x) = 1$

3) Power Rule
if $f(x) = x^n$ $n \neq \emptyset$
then $f'(x) = nx^{n-1}$

4) Derivative of a constant multiple
if c is a real # and $f(x)$ is differentiable

$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{df}{dx} = c(f'(x))$

5) Derivative sum and DIFFERENCE Rule
if $f(x)$ and $g(x)$ are differentiable...

$\frac{d}{dx}(f(x) \pm g(x)) =$

and...

$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

6) e^x rule

$\frac{d}{dx}(e^x) = e^x$

that's crazy nice ok

we find the graph of the derivative by making a series of assumptions

1 ↳ at curves the tangent will be $\emptyset$

2 ↳ we can find where the slope of the tangent is pos. or neg.

3 ↳ at a vertical asymptot the slope often will be approaching ∞ or −∞

4 ↳ at a horizontal asymptot the slope of the tan will be approaching $\emptyset$

There are some connections between continuity and differentiability that we can see here

* if f is differentiable @ a, then f is continuous @ a
* if f is not continuous @ a, then f is not differentiable @ a
* BUT f can be continuous @ a and not differentiable @ a

if one of these is true, f is not differentiable

1) f is not cont. @ a
2) f has a corner @ a
3) f has a vertical tangent @ a

Higher Order Derivatives

Function → 1st Deriv. → 2nd Deriv. → n^{th} Deriv.

$f(x) \to f'(x) \to f''(x) \to f^{(n)}(x)$

$\to \frac{df}{dx} \to \frac{d^2f}{dx^2} \to \frac{d^nf}{dx^n}$