Discrete

Midterm 2 Study Guide

we start by finishing up on proofs...

some strategies to begin thinking about probability...

chapters 4.1 - 4.3

Induction

we know how to prove by...

$p \Rightarrow q$

$\neg q \Rightarrow \neg p$

$p \wedge \neg q \Rightarrow p_n$
↳ false

...now we have 1 more type of proof

Inductive Proof

- helpful when we want to prove $p \Rightarrow q$ for all $P(n)$, especially when n is very large or infinite
- the idea...

$$P(1) \wedge (\forall_n \geq 1, P(n) \Rightarrow P(n+1)) \equiv \forall_n \in \mathbb{N}, P(n)$$

↳ if 1 is true and all #s > 1 are true, then all n is true

- the process...
  1. prove $P(1)$ is true (or $P(m)$ in a more general case)
  2. prove for any $n \geq 1$, if $P(n)$ is true then $P(n+1)$ is true.
- Induction is really great for proving sums

Strong Induction

- very similar to induction, but in the second step instead of just proving that $P(n) \Rightarrow P(n+1)$ is true, we prove that every # leading up to $P(n)$ is true thus $P(n+1)$ is true.
- (2) prove $P(1) \wedge P(2) \cdots \wedge P(n) \Rightarrow P(n+1)$ for any $n \geq 1$

Recursion

- we also introduce another way of defining a sequence...

Recursively Defined

- A sequence is recursively defined if...

f_0 is defined and f_n is defined in terms of $f_0, f_1, \ldots, f_{n-1}$, for $n \geq 1$

- Ex. →

$f_0 = 0! = 1 \quad f_n = n! = n(n-1)!, n \geq 1$

↑ factorial! many functions we've already seen can also be defined recursively

Multistep Recursion

- Just like recursion but where the n^{th} term is defined by the $(n-1)$st and $(n-2)$nd term (possibly more)
- The most common multistep Recursion...

Fibonacci

$F_n, n \geq 0$

$F_0 = 1 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2}, n \geq 2$

- many interesting counting applications

chapters 6.1 - 6.4, 7.1 - 7.4 & 7.6

Basic Counting

- many times in probability we need ways to "count" large #s of items or occurances, lets see how we can do that...

Product Rule

Start — Stage 1 — Stage 2 — End

multiply!

- if a task has n ind. stages and K_i ways to perform each stage then...

$$\prod_{i=1}^{n} K_i = K_1 K_2 \cdots K_n$$

... total ways to perform task.

Sum Rule

Start — Method 1 / Method 2 — End

add!

- n metods w/ K_i ways to perform each method

$$\sum_{i=1}^{n} K_i = K_1 + K_2 \cdots + K_n$$

... ways of performing task

Both!

Start — Method 1 (stage 1_1 — stage 2_1) / Method 2 (stage 1_2 — stage 2_2) — End

- Multiply for stages the sum for methods!

- A couple more principles to help with counting...

Inclusion-Exclusion

- for the cardinality of a union of 2 sets we know...

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

- we can extend this to n sets... $A_1, A_2, \ldots A_n$ ⟶

$$N\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i N(A_i) - \sum_{i<j} N(A_i \cap A_j) + \sum_{i<j<k} N(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} N\left(\bigcap_{i=1}^{n} A_i\right)$$

- More likely we will just run into an $n=3$ type...

$$N(A_1 \cup A_2 \cup A_3) = N(A_1) + N(A_2) + N(A_3) - N(A_1 \cap A_2) - N(A_1 \cap A_3) - N(A_2 \cap A_3) + N(A_1 \cap A_2 \cap A_3)$$

Pigeonhole

- if m objects are placed into n boxes ther is atleast one box contains $\lceil m/n \rceil$ or more objects.
- $\lceil x \rceil$ is the smallest integer $\geq x$.

Combinatorics

Permutations of distinct obj.

- The # of ordered arrangments (permutations) of length m from n distinct objects is...

$$P(n,m) = \frac{n!}{(n-m)!}, \quad n \geq m \geq 0$$

Circular Permutations

- There are $(n-1)!$ circular permutations (ordered loops) of n distinct objects.

Permutations of non-dist. obj.

- The # of permutations of n objects, m_i of which are of type i $(1 \leq i \leq r)$ is...

$$\frac{n!}{m_1! m_2! \cdots m_r!}$$

Combinations

- The # of combinations if the order of the items doesn't matter is...

$$C(n,m) = \frac{n!}{m!(n-m)!}$$

$n \geq m \geq 0$

- # of ways of dividing a group of n distinct obj. into $r \geq 2$ subgroups w/ m_i obj. in group i is...

$$C(n; m_1, m_2, \cdots m_r) = \frac{n!}{m_1! m_2! \cdots m_r!}$$

chapters 8.1 - 8.3 & 8.5

Binomials

- if you are familiar w/ $(x+y)^n$ and various expansions as n increases the combinations may look familiar...

$(x+y)^1 = x + y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3xy^2 + 3x^2y + y^3$

these are...

Binomial Coefficients

- these are calculated in the same way as combinations...

$$C(n,m) = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

- we use binomial coefs. to expand...

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}$$

Pascal's Triangle

- 1 interesting application / way to remember the binomial coefs is Pascal's triangle...

mth entry of nth row

$$P_{n,m} = \binom{n}{m}, \quad 0 \leq m \leq n$$

find next # by summing

row $n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$n=6$	1 6 15 20 15 6 1
	...

- Many interesting properties...

Other Applications

- if $K_1 + K_2 + \cdots + K_r = n$, and $n \geq 0$ and $r \geq 1$, then the # of nonneg. solutions is...

$$\binom{n+r-1}{r-1}$$

- There are some useful binomial identities...

$$\sum_{m=0}^{n} \binom{n}{m} = 2^n \qquad \sum_{m=0}^{n} m\binom{n}{m} = n \cdot 2^{n-1} \qquad \sum_{k=0}^{m} \binom{n}{k}\binom{r}{m-k} = \binom{n+r}{m}$$

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

GOOD LUCK!