LINEAR ALGEBRA.

MIDTERM 2

WE ARE GOING TO CONTINUE OUR DISCUSSION OF MATRICES BY EXPLORING 3 MORE CONCEPTS AND THEIR APPLICATIONS...

II.VI - II.VII

THE BASIS.

Subspace

- A **subspace** is any collection of points in $\mathbb{R}^n$ satisfying...
  1. Non-emptyness > $\emptyset$ vector in it
  2. Closure under addition > if $\vec{u}$ and $\vec{v}$ are in it the $\vec{u}+\vec{v}$ is in it
  3. Closure under scalar mult. > if $\vec{v}$ is in it, then $c\vec{v}$ is in it.
- If $V = \text{span}\{\}$ we say V is the subspace is spaned by the vectors in the set.

the 4 fund sub-spaces

- **Column Space** > subspace of $\mathbb{R}^m$ spaned by columns of A. $\perp$ to left null space $\to Nul(A^T)$
- **Null Space** > subspace of $\mathbb{R}^n$ consiting of all solutions to $A\vec{x}=\emptyset$. $\perp$ to row space $\to Col(A^T)$

Basis

- A **Basis** is any set of vectors V such that $V = \text{Span}\{\}$ and the vectors are linearly independent.
- The **Dimension** of a subspace is the # of vectors in the basis.
- **Basis for Column Space**
  - The pivot columns of A form the basis for Col(A).

ex. $\to$ $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Pivot col.

$Col(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$

- **Basis for Null Space**
  - The vectors attached to free variables in parametric form of solution set of $A\vec{x}=\emptyset$ form a basis of Nul(A).

ex. $\to$ $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix} x_4$

$Nul(A) = \left\{ \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

- **Rank** > The rank of A is the dimension of column space A.
- **Nullity** > The nullity of A is the dimension of the null space of A.

IV.I - IV.II

THE DETERMINANTS.

Definitions

- The **determinant** of a function is real # describing a square matrix.
- We have 4 ways to find it/define it...
- **Row Opperations** (easiest)
  - There are 4 ways row opperations effect determinants
    - row replacement $\to$ doesn't change det(A)
    - scaling by c $\to$ multiplies $c \cdot \det(A)$
    - swapping rows $\to$ multiplies $-1 \cdot \det(A)$
    - identity matrix $\to$ $\det(I) = 1$
  - so we work backwards from RREF...

ex $\to$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 6 & 8 \\ -2 & 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/4 \\ 0 & 0 & 1 \end{bmatrix} = B$

$\det(A) \cdot \frac{1}{6} \cdot \frac{1}{4} = \det(B) = 1 \Rightarrow \det(A) = 24$

- **Big Formula**
  - $\det(A) = \sum (\text{sgn}\, P)(\text{prod}\, P)$
    - P: patterns — pick n #s from an n×n matrix so there's one in each row and column. n! total.
    - signature — for each pattern check to see how many #s in the pattern are above and to the right of each other
    - product — the product of all #s in the pattern
    - sum — sum all possible patterns

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $P_i = d, b, i$ — 1 is above + to right

ex $\to$ A again

patterns $\to$	1,6,0	2,8,-2	3,1,-4	1,8,-4	2,1,0	3,6,-2	
sgn $\to$	-1^0	-1^2	-1^2	-1^1	-1^1	-1^3	
products	0	-32	-12	-32	0	-36	sum = 24

- **QR Decomposition**
  - find the QR decomposition of a matrix then multiply the diagnol of R.

ex. $\to$ $A = QR = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{30} & 2/\sqrt{5} \\ -1/\sqrt{6} & \sqrt{5/6} & 0 \\ -2/\sqrt{6} & \sqrt{2/15} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 8\sqrt{2/3} & 3/\sqrt{6} \\ 0 & 2\sqrt{10/3} & 37/\sqrt{30} \\ 0 & 0 & 6/\sqrt{5} \end{bmatrix}$

$\det(A) = \sqrt{6} \cdot 2 \cdot \sqrt{10/3} \cdot 6/\sqrt{5} = 24$

- **Cofactor Expansion**

this is a cross product $\to$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ (mini dets)

ex $\to$ $1\begin{vmatrix} 6 & 8 \\ 4 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 8 \\ -2 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 6 \\ -2 & 4 \end{vmatrix}$

$= 32 - 32 + 24 = 24$

Properties

- **Invertible** $\to$ a square matrix is invertible if and only if $\det(A) \neq \emptyset$
- **Multiplicativity** $\to$ if A and B are n×n, then $\det(AB) = \det(A)\det(B)$.
- **Transpose** $\to$ for any square matrix A, $\det(A) = \det(A^T)$

VI.I - VI.V

THE ORTHOGONALS.

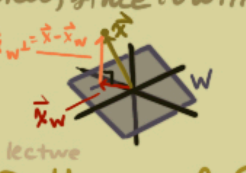
Definitions

- **dot product** > where $\vec{x} \cdot \vec{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n y_n$ produces a scalar.
- **distance** > between points x and y is $dist(x,y) = \|y - x\|$ where $\|y-x\| = \left\| \begin{bmatrix} y_x - x_x \\ y_y - x_y \end{bmatrix} \right\| = \sqrt{(y_x - x_x)^2 + (y_y - x_y)^2}$
- **unit vector** > a vector $\vec{x}$ w/ length $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = 1$
- **Orthogonal** > two vectors $\vec{x}$ and $\vec{y}$ are orthogonal if $\vec{x} \cdot \vec{y} = \emptyset$. aka perpendicular
- **Orthogonal Compliment** > If W is a subspace it's orthogonal compliment is a subspace where all vectors in the compliment are orthogonal to all vectors in W. written $\to$ $W^\perp$

- Computing Orthogonal Compliments
  1. define a matrix A where W = Col(A)
     ex. $\to$ $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$ $A = \begin{bmatrix} 1 & -2 \\ 7 & 3 \\ 2 & 1 \end{bmatrix}$
  2. $\boxed{W^\perp = Nul(A^T)}$
     ex. $\to$ $A^T = \begin{bmatrix} 1 & 7 & 2 \\ -2 & 3 & 1 \end{bmatrix} \to Nul(A^T) = \begin{bmatrix} 1/17 \\ -5/17 \\ 1 \end{bmatrix}$ $W^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ -5 \\ 17 \end{bmatrix} \right\}$

- **Orthogonal Projection** > the closest vector in the subspace W to the vector $\vec{x}$ is called the orthogonal projection, since it will be orthogonal to $\vec{x}_{W^\perp} = \vec{x} - \vec{x}_W$

Written $\to$ $\vec{x}_W$ (in txtbk) or $\vec{b}^{\parallel}$ (in lecture)



$\vec{x}_{W^\perp}$ (in txtbk) aka $\vec{b}^\perp$ (in lecture)

- Compute Orthogonal Compliment
  1. define a matrix A where W = Col(A). let $\vec{x}$ be a vector in $\mathbb{R}^m$. And let $\vec{x}_W = A\vec{c}$ for any solution $\vec{c}$.
  2. set up equation $A^TA\vec{c} = A^T\vec{x}$
  3. solve $\to$ $\boxed{\vec{x}_W = A(A^TA)^{-1}A^T\vec{x}}$

- **Orthogonal Decomposition** > An equation breaking down the relationship between the orthogonal projection and orthogonal compliment...

W $\to$ subspace in $\mathbb{R}^n$
$\vec{x}$ $\to$ vector in $\mathbb{R}^n$

$\boxed{\vec{x} = \vec{x}_W + \vec{x}_{W^\perp}}$

- Computing w/ Orthogonal Decomposition
  - properties of the orthogonal decomposition are...

$$\vec{x}_W = A\vec{c} = A(A^TA)^{-1}A^T\vec{x}$$
$$\vec{x}_{W^\perp} = \vec{x} - \vec{x}_W = \vec{x} - A(A^TA)^{-1}A^T\vec{x}$$

Projection Formula

- Instead of using orthogonal decomposition to find the orthogonal projection we're going to use the...
- **Projection Formula** > W is a subspace. $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is an orthogonal basis for W. So the orthogonal projection of $\vec{x}$ is...

what do these mean?

$$\vec{x}_W = \frac{\vec{x} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{x} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 + \dots + \frac{\vec{x} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m}\vec{u}_m$$

and if $\{u_1, u_2, \dots, u_m\}$ is orthonormal...

$$\vec{x}_W = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2 + \dots + (\vec{x} \cdot \vec{u}_3)\vec{u}_3$$

- **Orthogonal Basis** > a set of nonzero vectors is called orthogonal when every vector is orthogonal to every other vector. AKA if we have vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ they are orthogonal if $\vec{u}_i \cdot \vec{u}_j = \emptyset$ for every i and j where $i \neq j$.
- **Orthonormal Basis** > a set of vectors that are orthogonal and also each vector is a unit vector, aka $\vec{u}_i \cdot \vec{u}_i = 1$, for all $i = 1, 2, \dots, m$.
- We can make an orthogonal basis $\{v_1, v_2, \dots, v_m\}$ orthonormal by doing... $\left\{ \frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{v}_2}{\|\vec{v}_2\|}, \dots \frac{\vec{v}_m}{\|\vec{v}_m\|} \right\}$
- but how can we make any given basis orthogonal?

Gram-Schmidt

- let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be a basis for a subspace W of $\mathbb{R}^n$. To find an **Orthogonal Basis** $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_m\}$ define...
  1. $\vec{u}_1 = \vec{v}_1$
  2. $\vec{u}_2 = (\vec{v}_2) - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1$ — this should remind you of the Projection formula
  3. $\vec{u}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 - \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}u_2$

  ⋮

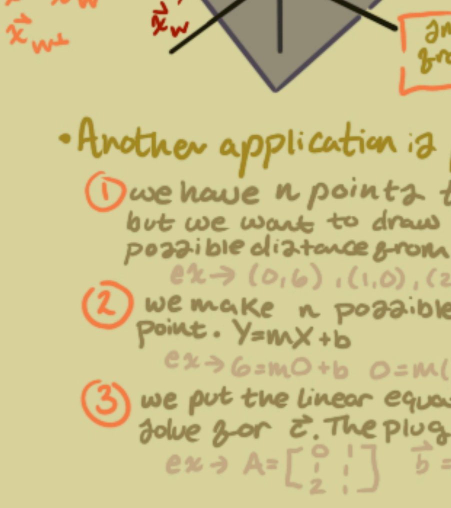
  (m) $\vec{u}_m = \vec{v}_m - \sum_{i=1}^{m-1} \frac{\vec{v}_m \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}\vec{u}_i$
- If we want the **Orthonormal Basis**, we need to divide each vector by it's length...

$$\left\{ \frac{\vec{u}_1}{\|\vec{u}_1\|}, \frac{\vec{u}_2}{\|\vec{u}_2\|}, \dots, \frac{\vec{u}_m}{\|\vec{u}_m\|} \right\}$$

- We can now decide if we want to find $\vec{x}_W$ with the projection formula or the orthogonal decomposition.
  - We might use proj. formula if $\to$
    - we already have an orthogonal basis
    - we have to find mult. projections
  - We might use orthogonal decomp. if $\to$
    - we only have to proj a few vectors.

Applications

- But why are we going through all this trouble?
- One of the main applications of all this is to find the closest distance from a point in $\mathbb{R}^3$ to a plane...



this drawing should look familiar because it's a drawing of the orthogonal projection.

And since we know how to solve all that we know...

$\boxed{\text{smallest distance from W to } x = \vec{x}_{W^\perp} = \vec{x} - \vec{x}_W}$ — and by now we should def know how to find this

- Another application is **Least Squares** aka **Best Fit**...
  1. we have n points that are not linear/not in a line but we want to draw a line/model that is the smallest possible distance from each point.
     ex $\to$ (0,6), (1,0), (2,0)
  2. we make n possible linear equations, one for each point. $Y = mX + b$
     ex $\to$ $6 = m0 + b$ $0 = m(1) + b$ $0 = m(2) + b$
  3. we put the linear equations into matrix form $A\vec{c} = \vec{b}$ and solve for $\vec{c}$. The plug $\vec{c}$ back into $y = mx + b$.
     ex $\to$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} m \\ b \end{bmatrix}$ $\to \left[\begin{array}{cc|c} 0 & 1 & 6 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{solve}}$ $m = -3$, $b = 5$

     $y = -3x + 5$
- we can find the least squares solution of a matrix equation $A\vec{x} = \vec{b}$ that minimizes the distance between $\vec{b}$ and $A\vec{x}$ by solving... $\boxed{A^TA\vec{x} = A^T\vec{b}}$ ...for $\vec{x}$.

Good Luck