Math 615: Homework Due 12/9

Page 304, number 2. Here is the transition matrix “trans”.

```
trans1 = N[ (1/6   0    0    0    0   0)
            (1/6  2/6   0    0    0   0)
            (1/6  1/6  3/6   0    0   0)
            (1/6  1/6  1/6  4/6   0   0)
            (1/6  1/6  1/6  1/6  5/6  0)
            (1/6  1/6  1/6  1/6  1/6  1) ];
```

```
Manipulate[NumberForm[MatrixForm[
   MatrixPower[trans1, exponent] ], {4, 3}],
 {exponent, 1, 60, 1, Appearance → "Labeled"}]
```

exponent 1

```
0.167 0.000 0.000 0.000 0.000 0.000
0.167 0.333 0.000 0.000 0.000 0.000
0.167 0.167 0.500 0.000 0.000 0.000
0.167 0.167 0.167 0.667 0.000 0.000
0.167 0.167 0.167 0.167 0.833 0.000
0.167 0.167 0.167 0.167 0.167 1.000
```

The vector for starting in state 1 after 5 throws is given by

```
MatrixPower[trans1, 5].{{1}, {0}, {0}, {0}, {0}, {0}}
{{0.000128601}, {0.00398663}, {0.0271348}, {0.100437}, {0.27019}, {0.598122}}
```

Side Remark: Is there an easy way to create the unit vector *e_i* = (0, 0, ..., 1, ..., 0)? I don’t know, but here is one possible solution.

```
e[entry_, length_] := Module[ {tempmatrix},
  (tempmatrix = {Table[0, {counter, 1, length}]} );
  tempmatrix[[1, entry]] = 1;
  Transpose[tempmatrix] ]
```

```
MatrixPower[trans1, 5].e[1, 6]
{{0.000128601}, {0.00398663}, {0.0271348}, {0.100437}, {0.27019}, {0.598122}}
```

3. Note that the transition matrix, as written in the textbook, is actually the transpose of what we want.

$$\text{trans2} = \text{Transpose}\left[\begin{pmatrix} .787 & 0 & 0 & .111 & .102 \\ 0 & .966 & .034 & 0 & 0 \\ 0 & .063 & .937 & 0 & 0 \\ 0 & 0 & .074 & .612 & .314 \\ .021 & .009 & .005 & .010 & .954 \end{pmatrix}\right];$$

```
Manipulate[NumberForm[MatrixForm[
   MatrixPower[trans2, exponent] ], {4, 3}],
 {exponent, 1, 60, 1, Appearance → "Labeled"}]
```

exponent 1

$$\begin{pmatrix} 0.787 & 0.000 & 0.000 & 0.000 & 0.021 \\ 0.000 & 0.966 & 0.063 & 0.000 & 0.009 \\ 0.000 & 0.034 & 0.937 & 0.074 & 0.005 \\ 0.111 & 0.000 & 0.000 & 0.612 & 0.010 \\ 0.102 & 0.000 & 0.000 & 0.314 & 0.954 \end{pmatrix}$$

b. Letting v0 be the initial distributions, we obtain the distribution after 1-4 iterations. Note they are listed as row vectors.

```
v0 = Transpose[{{.02, .02, .02, .02, .9}}];

(*MatrixPower[trans,1].v0
 MatrixPower[trans,2].v0
 MatrixPower[trans,3].v0
 MatrixPower[trans,4].v0
*)
TableForm[
 Table[MatrixPower[trans2, k].v0, {k, 1, 4}] ,
 TableHeadings → { {"n=1", "n=2", "n=3", "n=4"}, {"NE", "NC", "S", "W", "Z"}}]
```

	NE	NC	S	W	Z
n=1	0.03464	0.02868	0.0254	0.02346	0.86692
n=2	0.045467	0.0371074	0.0308456	0.0268718	0.837941
n=3	0.0533793	0.0453305	0.0363422	0.0298718	0.812471
n=4	0.0590714	0.053391	0.0418667	0.0323313	0.789922

c. For our new distribution, we get

```
v0 = Transpose[{{0, .6522, .3478, 0, 0}}];

(*MatrixPower[trans,1].v0
 MatrixPower[trans,2].v0
 MatrixPower[trans,3].v0
 MatrixPower[trans,4].v0
*)
TableForm[Table[MatrixPower[trans2, k].v0, {k, 1, 4}],
 TableHeadings → {{"n=1", "n=2", "n=3", "n=4"}, {"NE", "NC", "S", "W", "Z"}}]
```

	NE	NC	S	W	Z
n=1	0.	0.651937	0.348063	0.	0.
n=2	0.	0.651699	0.348301	0.	0.
n=3	0.	0.651484	0.348516	0.	0.
n=4	0.	0.65129	0.34871	0.	0.

d. It is easiest to analyze whether the system has reached an equilibrium by using the Manipulate command. The system has not yet reached an equilibrium by n=50. It does not reach an equilibrium until roughly n=200 or 300. At this point, everything is concentrated in the NC and S regions.

```
Manipulate[NumberForm[MatrixForm[
   MatrixPower[trans2, exponent] ], {4, 3}],
 {exponent, 1, 500, 1, Appearance → "Labeled"}]
```

exponent 50

$$\begin{pmatrix} 0.040 & 0.000 & 0.000 & 0.034 & 0.040 \\ 0.337 & 0.652 & 0.646 & 0.382 & 0.333 \\ 0.203 & 0.348 & 0.354 & 0.225 & 0.200 \\ 0.022 & 0.000 & 0.000 & 0.019 & 0.022 \\ 0.373 & 0.000 & 0.000 & 0.317 & 0.374 \end{pmatrix}$$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$0-0$	1	0	0	0	0	0	0	0
$1-0$	0	0.5	0	0	0	0	0	0
$0-1$	0	0.5	0	0	0	0	0	0
$2-0$	0	0	0.25	0	0	0	0	0
$1-1$	0	0	0.5	0	0	0	0	0
$0-2$	0	0	0.25	0	0	0	0	0
$3-0$	0	0	0	0.125	0	0	0	0
$2-1$	0	0	0	0.375	0	0	0	0
$1-2$	0	0	0	0.375	0	0	0	0
$0-3$	0	0	0	0.125	0	0	0	0
$4-0$	0	0	0	0	0.0625	0.0625	0.0625	0.0625
$3-1$	0	0	0	0	0.25	0	0	0
$2-2$	0	0	0	0	0.375	0	0	0
$1-3$	0	0	0	0	0.25	0	0	0
$0-4$	0	0	0	0	0.0625	0.0625	0.0625	0.0625
$4-1$	0	0	0	0	0	0.125	0.125	0.125
$3-2$	0	0	0	0	0	0.3125	0	0
$2-3$	0	0	0	0	0	0.3125	0	0
$1-4$	0	0	0	0	0	0.125	0.125	0.125
$4-2$	0	0	0	0	0	0	0.15625	0.15625
$3-3$	0	0	0	0	0	0	0.3125	0
$2-4$	0	0	0	0	0	0	0.15625	0.15625
$4-3$	0	0	0	0	0	0	0	0.15625
$3-4$	0	0	0	0	0	0	0	0.15625

Markov chains are a widely used application of matrix operations. They also give us an example of the use of matrices where we do not consider the significance of the maps represented by the matrices. For more on Markov chains, there are many sources such as [Kemeny & Snell] and [Iosifescu].

Exercises

1 These questions refer to the coin-flipping game.

(a) Check the computations in the table at the end of the first paragraph.

(b) Consider the second row of the vector table. Note that this row has alternating 0's. Must $p_1(j)$ be 0 when j is odd? Prove that it must be, or produce a counterexample.

(c) Perform a computational experiment to estimate the chance that the player ends at five dollars, starting with one dollar, two dollars, and four dollars.

2 [Feller] We consider throws of a die, and say the system is in state s_i if the largest number yet appearing on the die was i.

(a) Give the transition matrix.

(b) Start the system in state s_1, and run it for five throws. What is the vector at the end?

3 [Kelton] There has been much interest in whether industries in the United States are moving from the Northeast and North Central regions to the South and West, motivated by the warmer climate, by lower wages, and by less unionization. Here is the transition matrix for large firms in Electric and Electronic Equipment.

	NE	*NC*	*S*	*W*	*Z*
NE	0.787	0	0	0.111	0.102
NC	0	0.966	0.034	0	0
S	0	0.063	0.937	0	0
W	0	0	0.074	0.612	0.314
Z	0.021	0.009	0.005	0.010	0.954

For example, a firm in the Northeast region will be in the West region next year with probability 0.111. (The *Z* entry is a "birth-death" state. For instance, with probability 0.102 a large Electric and Electronic Equipment firm from the Northeast will move out of this system next year: go out of business, move abroad, or move to another category of firm. There is a 0.021 probability that a firm in the *National Census of Manufacturers* will move into Electronics, or be created, or move in from abroad, into the Northeast. Finally, with probability 0.954 a firm out of the categories will stay out, according to this research.)

(a) Does the Markov model assumption of lack of history seem justified?

(b) Assume that the initial distribution is even, except that the value at Z is 0.9. Compute the vectors for $n = 1$ through $n = 4$.

(c) Suppose that the initial distribution is this.

NE	*NC*	*S*	*W*	*Z*
0.0000	0.6522	0.3478	0.0000	0.0000

Calculate the distributions for $n = 1$ through $n = 4$.

(d) Find the distribution for $n = 50$ and $n = 51$. Has the system settled down to an equilibrium?

4 [Wickens] Here is a model of some kinds of learning The learner starts in an undecided state s_U. Eventually the learner has to decide to do either response A (that is, end in state s_A) or response B (ending in s_B). However, the learner doesn't jump right from undecided to sure that A is the correct thing to do (or B). Instead, the learner spends some time in a "tentative-A" state, or a "tentative-B" state, trying the response out (denoted here t_A and t_B). Imagine that once the learner has decided, it is final, so once in s_A or s_B, the learner stays there. For the other state changes, we can posit transitions with probability p in either direction.

(a) Construct the transition matrix.

(b) Take $p = 0.25$ and take the initial vector to be 1 at s_U. Run this for five steps. What is the chance of ending up at s_A?

(c) Do the same for $p = 0.20$.

(d) Graph p versus the chance of ending at s_A. Is there a threshold value for p, above which the learner is almost sure not to take longer than five steps?

5 A certain town is in a certain country (this is a hypothetical problem). Each year ten percent of the town dwellers move to other parts of the country. Each year one percent of the people from elsewhere move to the town. Assume that there are two