

CSCI 3022

intro to data science with probability & statistics

Lecture 17
March 14, 2018

Introduction to p -values and hypothesis testing



Department of Computer Science
UNIVERSITY OF COLORADO BOULDER



Get Paid to Code

Build on SafeTrek's life-saving API for some extra cash and professional experience.

SafeTrek is a tech company on a mission to help users feel safe and protected so that they can live life freely. SafeTrek recently released their connected safety API and are looking for students who can use it to build integrations that could make a big impact in people's lives.

But wait, there's more! SafeTrek is offering the opportunity to make over \$300, complete freedom to build whatever you want, and reference letters to boost your portfolio.

Sound good?

Contact [**Benjamin at SafeTrek**](#) today to sign up and get started.

Space is limited, so act fast!

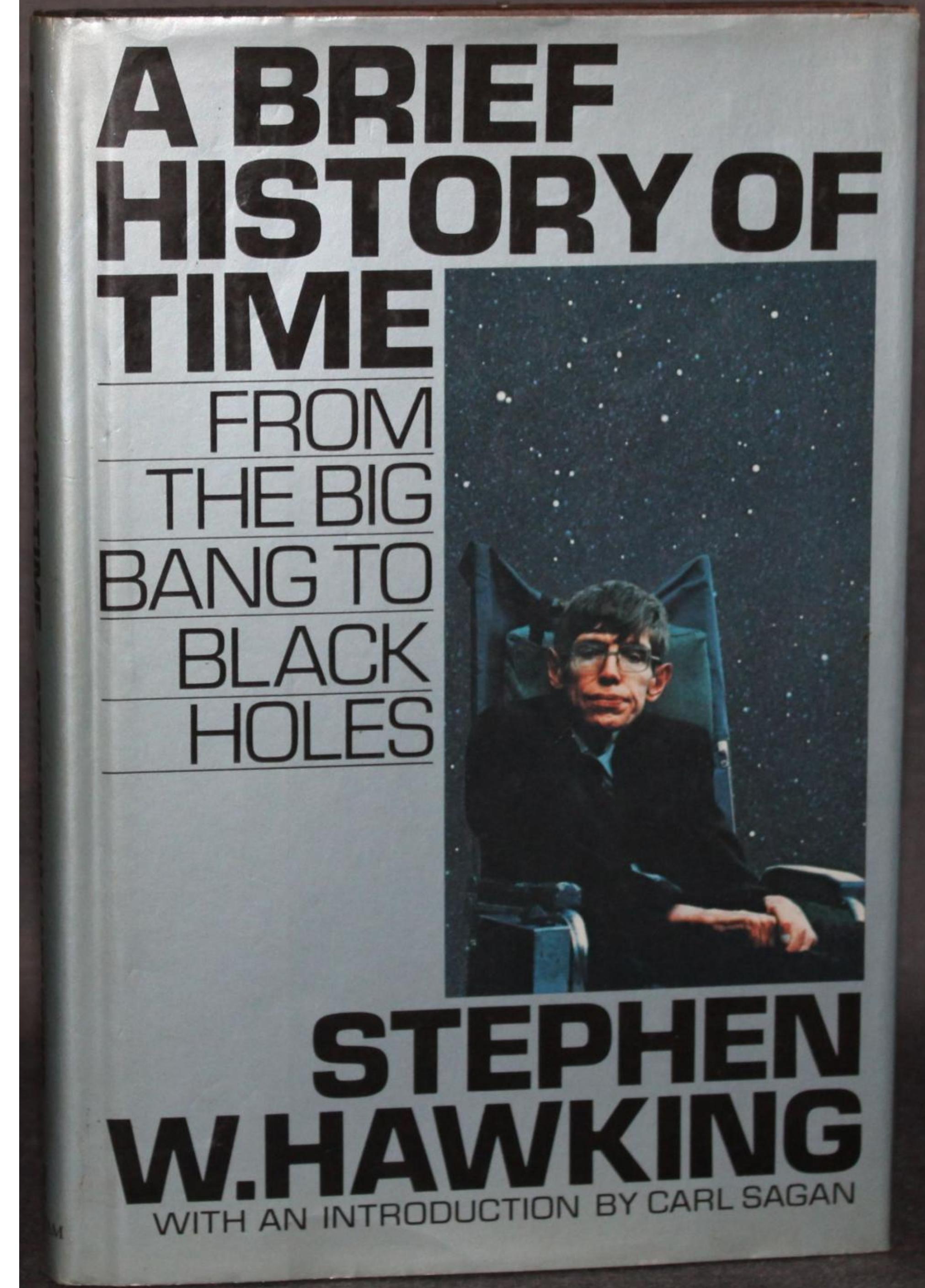


"We recently opened our connected safety API to CU students and are offering a Junior Developer Program to pay those who complete integration projects with it.

These will be paid and students will gain experience with REST APIs and OAuth2 flows, and will own their own code."

- [Link 1](#)
- [Link2](#)

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Switching advertising strategies

$$H_0: \mu = 200$$

$$H_1: \mu > 200$$

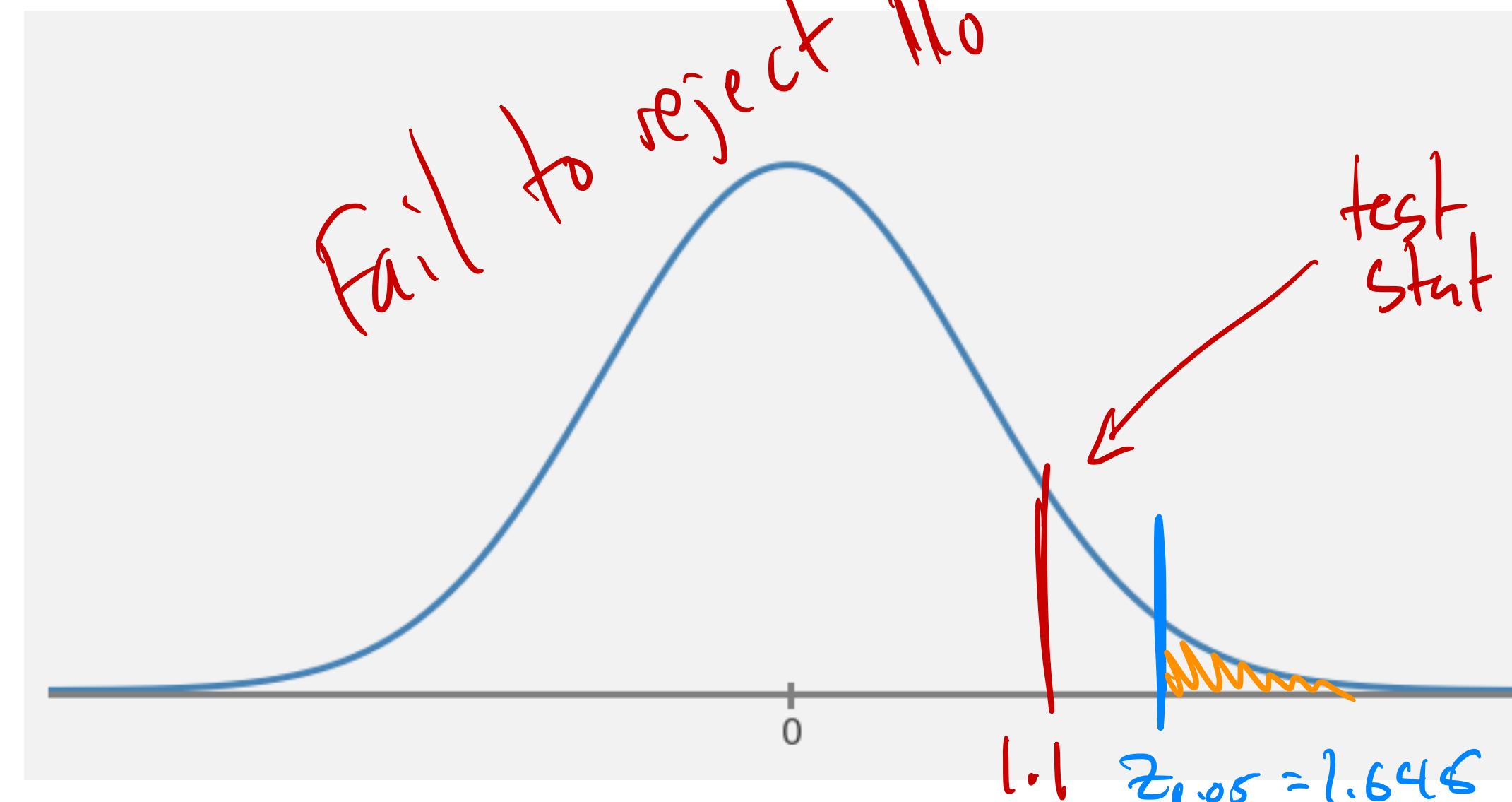
- **Example:** Suppose a company is considering hiring a new outside advertising company to help generate traffic to their website. Under their current advertising they get, on average, 200 thousand hits per day with a standard deviation of 50 thousand hits per day. You decide to hire the new ad company for a 30 day trial. During those 30 days, your website gets 210 thousand hits per day. Perform a hypothesis test to determine if the new ad campaign outperforms the old one at the .05 significance level.

$$\text{CLT } N\left(\mu, \frac{\sigma^2}{n}\right)$$

If null H_0 were true

$$\bar{X} \sim N\left(200, \frac{50^2}{30}\right)$$

$$Z = \frac{\bar{X} - \mu}{\sigma} \sim N(0, 1)$$



$$z_{\alpha} = z_{0.05} = 1.645$$
$$\frac{210 - 200}{50/\sqrt{30}} = 1.1$$

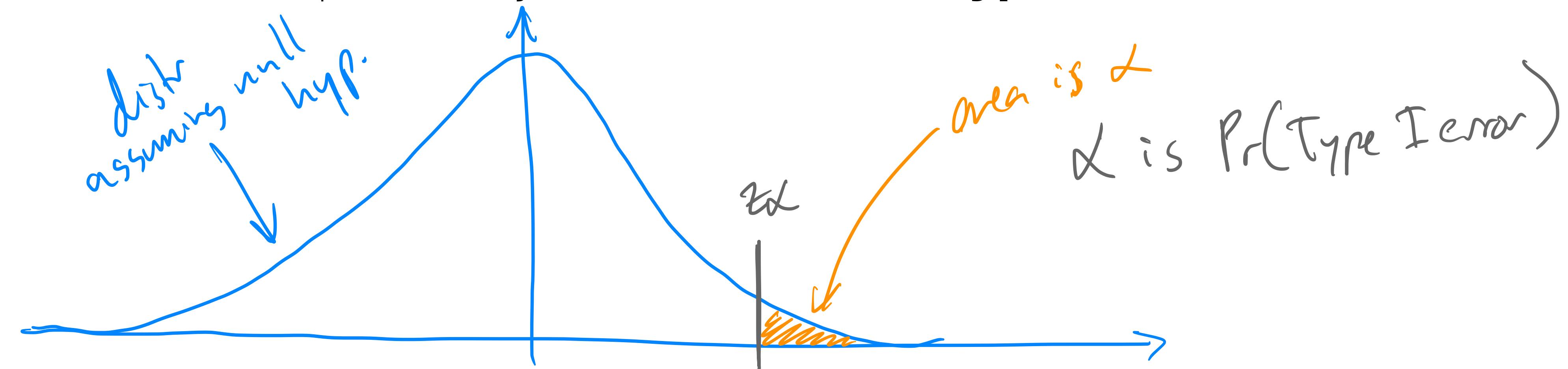
Important assumptions

- **Question:** What assumptions did we make in the previous example?

- ① Assumed that CLT would hold. $n=30$ samples (days)
- ② Assumed that we can represent the involved distributions as Normal.

Errors in hypothesis testing

- **Definitions:**
- A **Type I Error** occurs when the Null hypothesis is rejected, but the Null hypothesis is in fact true (**False Positive**)
*Ads are same.
we conclude: different.*
- A **Type II Error** occurs when the Null hypothesis is not rejected, but the Null hypothesis is in fact false (**False Negative**)
*Ads are diff.
we conclude: same.*
- **Question:** What is the probability that we commit a **Type I Error**?



Errors in hypothesis testing

- **Definitions:**
- A **Type I Error** occurs when the Null hypothesis is rejected, but the Null hypothesis is in fact true (**False Positive**)
- A **Type II Error** occurs when the Null hypothesis is not rejected, but the Null hypothesis is in fact false (**False Negative**)
- **Question:** What is the probability that we commit a **Type I Error?**

- **Answer:** this is exactly the significance level α

- **Consequence:** choose α by considering willingness to risk a Type I error.

Rejection region refresher

- **Example:** The 1999 Volkswagen Jetta was one of the first VW models produced both in Germany and Mexico. The life expectancy of Jettas produced in Germany was found to follow a normal distribution with mean 300K miles and standard deviation 150K miles. Life expectancy of models made in Mexico were recorded for a sample of size 100. The sample mean of these Jettas was found to be 250K miles.
- **Question 1:** What are the Null hypothesis and alternative hypothesis to test the claim that there is statistical evidence that 1999 Jettas made in Mexico have a smaller life expectancy than those made in Germany?

$$H_0: \mu_{\text{Mx}} = 300,000$$

$$H_1: \mu_{\text{Mx}} < 300,000$$

Rejection region refresher

$$H_0: \mu = 300$$
$$H_1: \mu < 300$$

1-tailed test

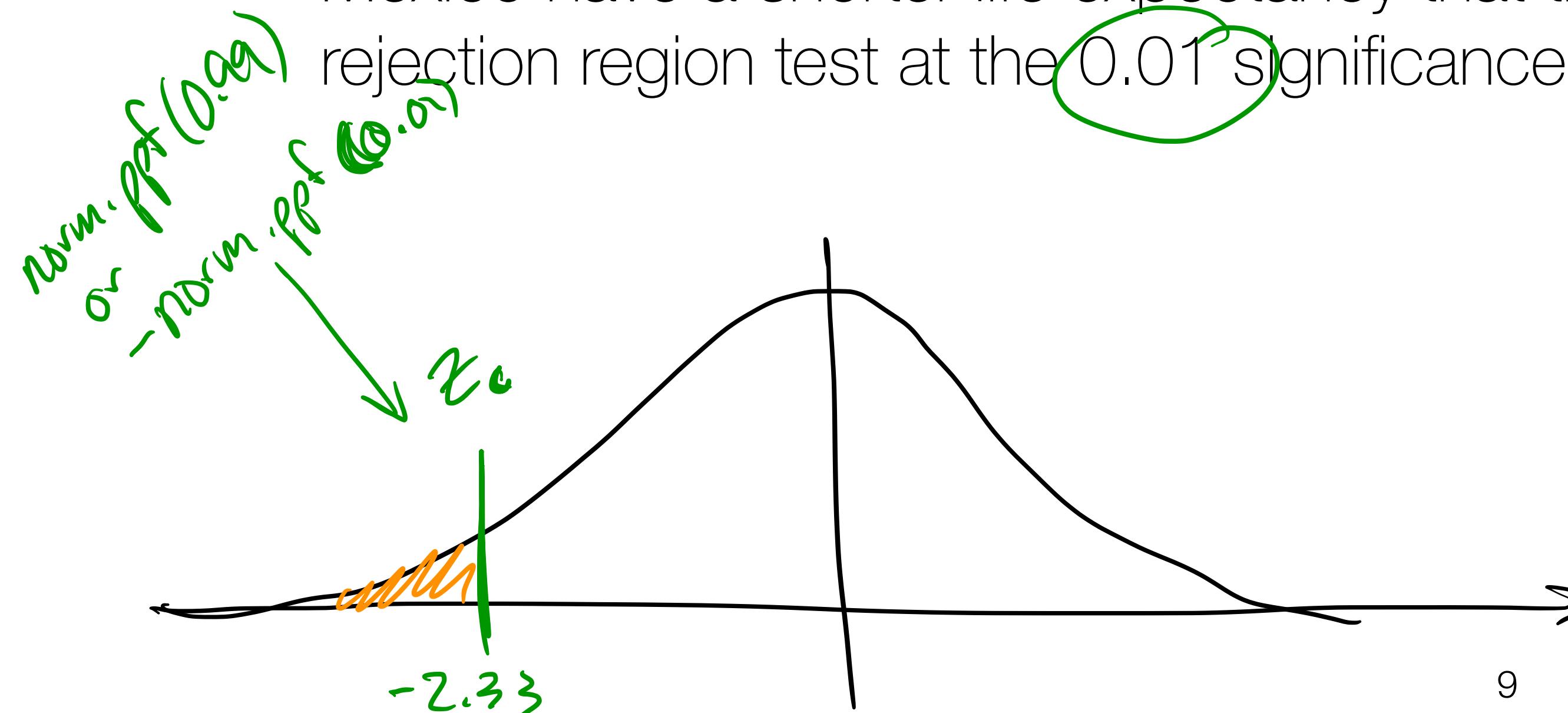
- **Example:** The 1999 Volkswagen Jetta was one of the first VW models produced both in Germany and Mexico. The life expectancy of Jettas produced in Germany was found to follow a normal distribution with mean 300K miles and standard deviation 150K miles. Life expectancy of models made in Mexico were recorded for a sample of size 100. The sample mean of these Jettas was found to be 250K miles.
- **Question 2:** Is there sufficient evidence to conclude that, in fact, 1999 Jettas made in Mexico have a shorter life expectancy than those made in Germany? Carry out a rejection region test at the 0.01 significance level.

$$\alpha = 0.01 \quad n = 100$$

$$\mu = 300$$

$$\sigma = 150$$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$



Rejection region refresher

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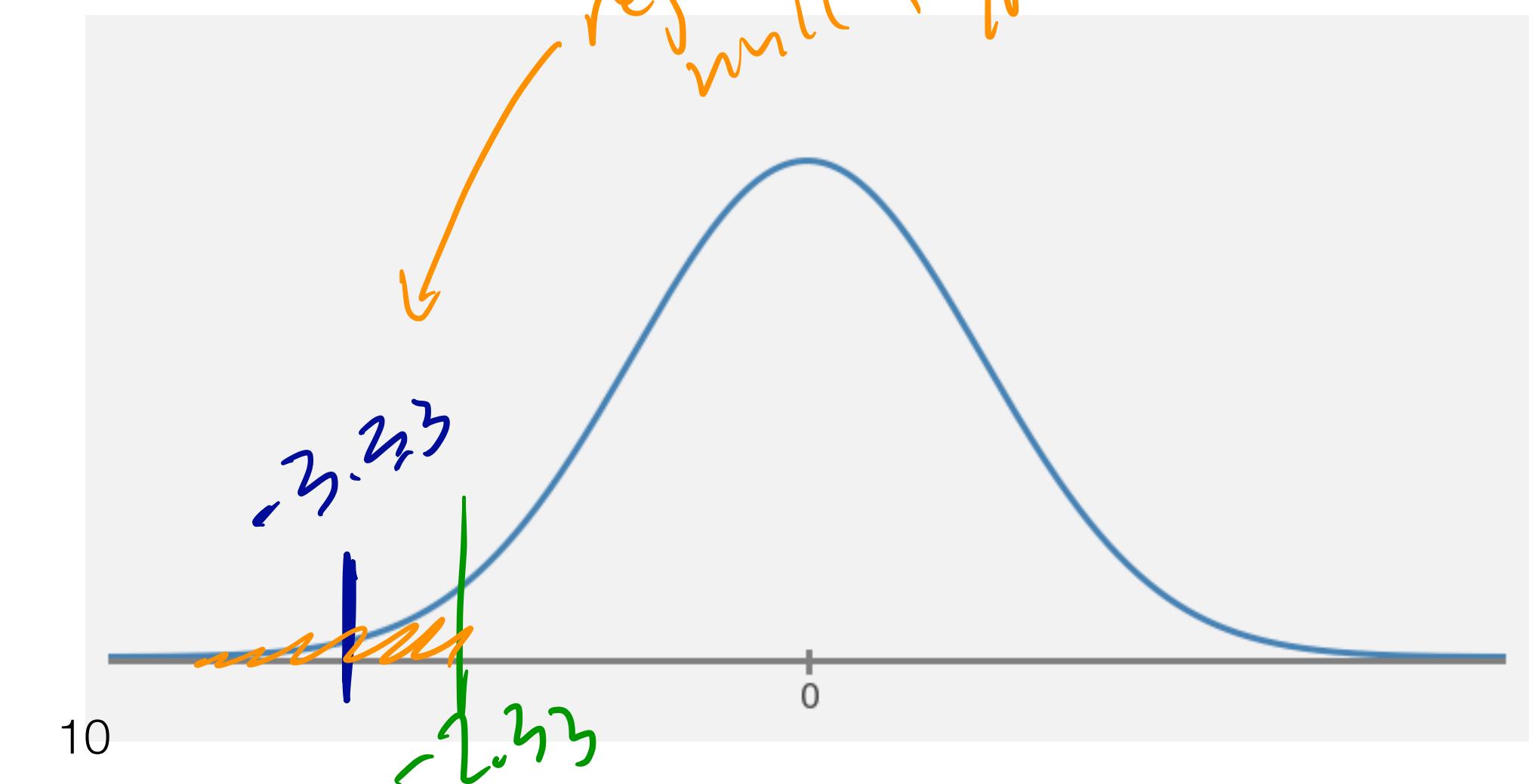
data ↓

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

CLT
sample size
null hypothesis

$$\frac{250 - 300}{150/\sqrt{100}} = -3.33$$

$$Z_c > -2.33$$



Rejection region & critical value summary

Alternative Hypothesis

$$H_1 : \theta > \theta_0$$

$$H_1 : \theta < \theta_0$$

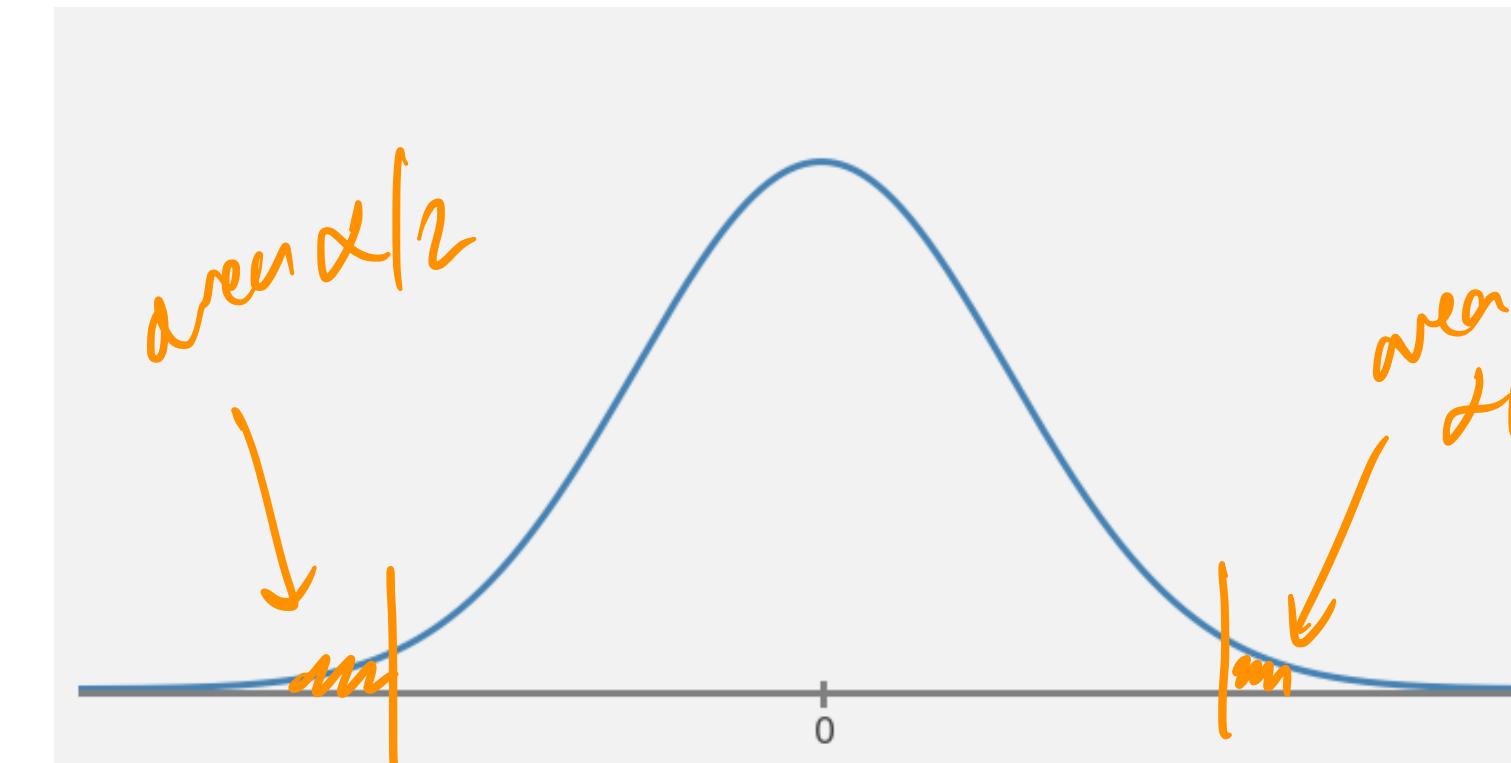
$$H_1 : \theta \neq \theta_0$$

Rejection Region

$$z \geq z_\alpha$$

$$z \leq -z_\alpha$$

$$z \leq -z_{\alpha/2} \text{ or } z \geq z_{\alpha/2}$$



$$-z_\alpha$$

$$-z_{\alpha/2}$$

$$z_{\alpha/2}$$

$$z_\alpha$$

Critical region HT summary

- **Critical Region** is region where test statistic has low probability under Null Hypothesis.
- Requires normally distributed data, or large enough sample for Central Limit Theorem.
- Under these assumptions we call this a Z-Test
- Rejecting the Null when the Null is true is called a Type I Error
- The probability of committing a Type I Error is α , the significance level of the test.
- Failing to reject the Null when the Null is false is called a Type II Error

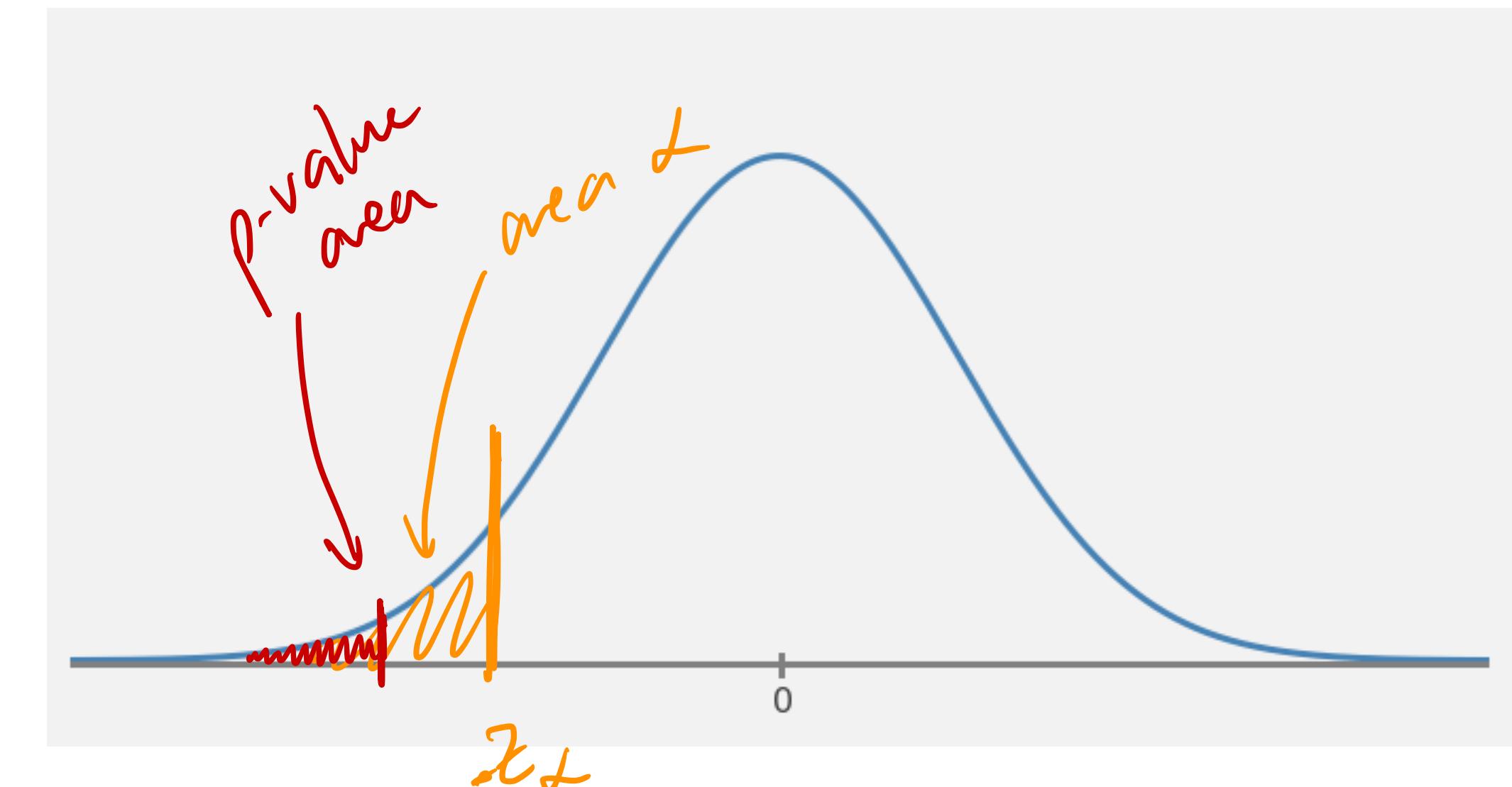
Introduction to p-values

- Another way to view the critical region hypothesis test is through a so-called p-value
- This framework for HT is very popular in scientific study and reporting
- **Example:** Consider a lower-tail critical region test with the following hypotheses.

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu < \mu_0$$

- The critical region test is:



p-values for various hypothesis tests

- **Def:** A p-value is the probability, under the Null hypothesis, that we would get a test statistic at least as extreme as the one we calculated.
- **Def:** For a lower-tailed test with test statistic x , the p-value is equal to $P(X \leq x | H_0)$
- Intuition: The p-value assesses the extremeness of the test statistic. The smaller the p-value, the more evidence we have against the Null hypothesis
- **Important Notes:**
 - The p-value is calculated under the assumption that the Null hypothesis is true
 - The p-value is always a value between 0 and 1
 - The p-value is NOT the probability that the Null is true!!

conditioned
on H_0

The p-value decision rule

- As before, select a significance level α before performing the hypothesis test
- Then the decision rule is:
 - If p-value $\leq \alpha$ then reject the Null hypothesis
 - If p-value $> \alpha$ then fail to reject the Null hypothesis
- Thus if the p-value exceeds the selected significance level then we cannot reject the Null hypothesis.

e.g. if $p = 0.1$ and $\alpha = 0.05$, we cannot reject.

- Note: The p-value can be thought of as the smallest significance level at which the Null hypothesis can be rejected.

Jetta life expectancy with p-values

- **Example:** The 1999 Volkswagen Jetta was one of the first VW models produced both in Germany and Mexico. The life expectancy of Jettas produced in Germany was found to follow a normal distribution with mean 300K miles and standard deviation 150K miles. Life expectancy of models made in Mexico were recorded for a sample of size 100. The sample mean of these Jettas was found to be 250K miles.
- Is there sufficient evidence to conclude that 1999 Jettas made in Mexico have a shorter life expectancy than those made in Germany? Carry out the p-value test at the 0.01 SL.

test statistic

$$\frac{\bar{X} - \mu_0}{\sigma_0 / \sqrt{n}} = \frac{250 - 300}{150 / \sqrt{100}} = -3.33$$



$P(Z \leq -3.33) \rightarrow \text{CDF!}$ $p\text{-value} = \Phi(-3.33) = 0.00043 \leq 0.01$

Reject Null Hypothesis

p-values for different z-tests

Alternative Hypothesis

$$H_1 : \theta > \theta_0$$

$$H_1 : \theta < \theta_0$$

$$H_1 : \theta \neq \theta_0$$

Critical Region Level α Test

$$z \geq z_\alpha$$

$$z \leq -z_\alpha$$

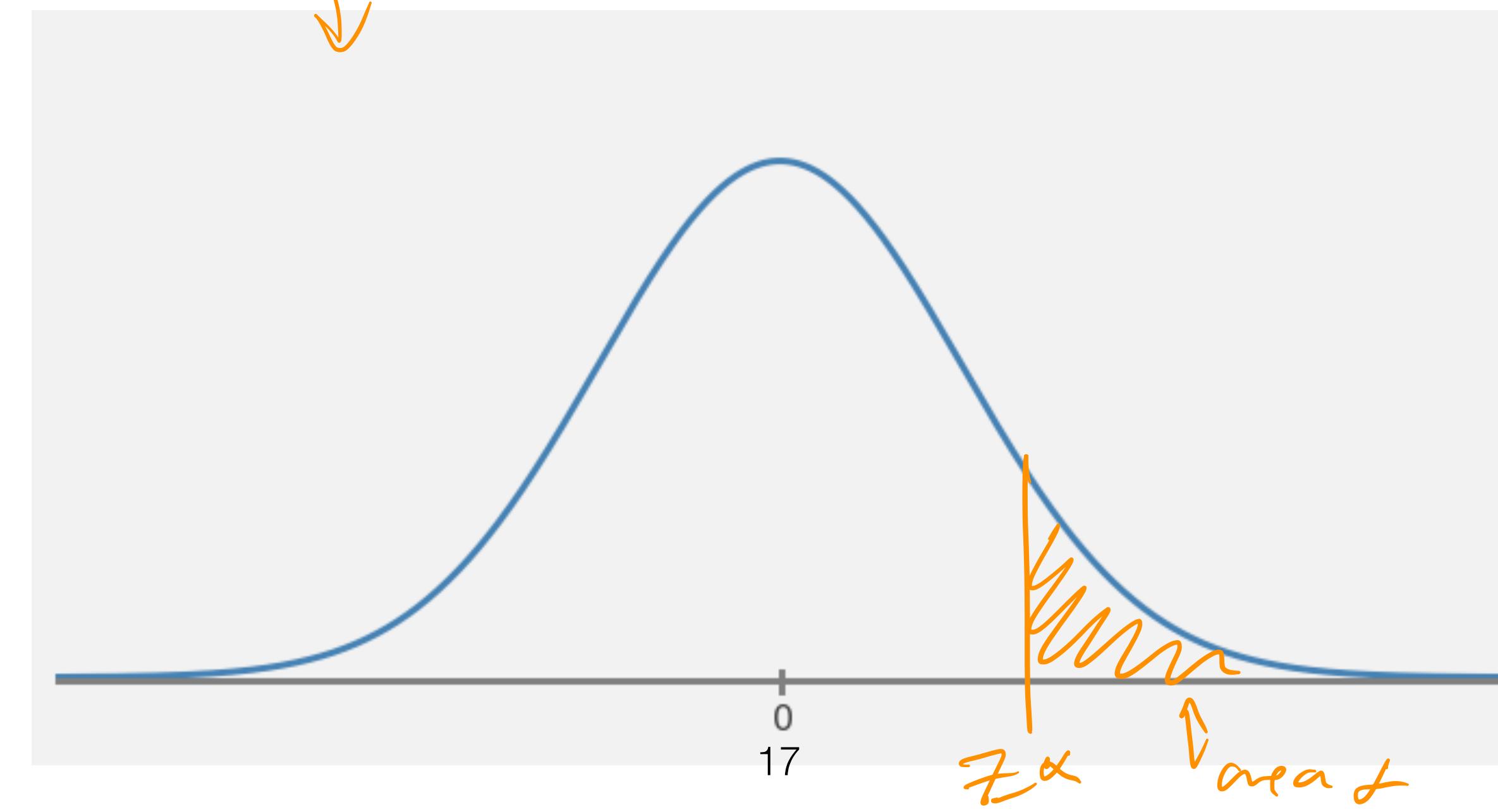
$$z \leq -z_{\alpha/2} \text{ or } z \geq z_{\alpha/2}$$

p-value Level α Test

If $1 - \Phi(z) \leq \alpha$

If $\Phi(z) \leq \alpha$

Next slide.



p-values for different z-tests

Alternative Hypothesis

$$H_1 : \theta > \theta_0$$

$$H_1 : \theta < \theta_0$$

$$H_1 : \theta \neq \theta_0$$

Critical Region Level α Test

$$z \geq z_\alpha$$

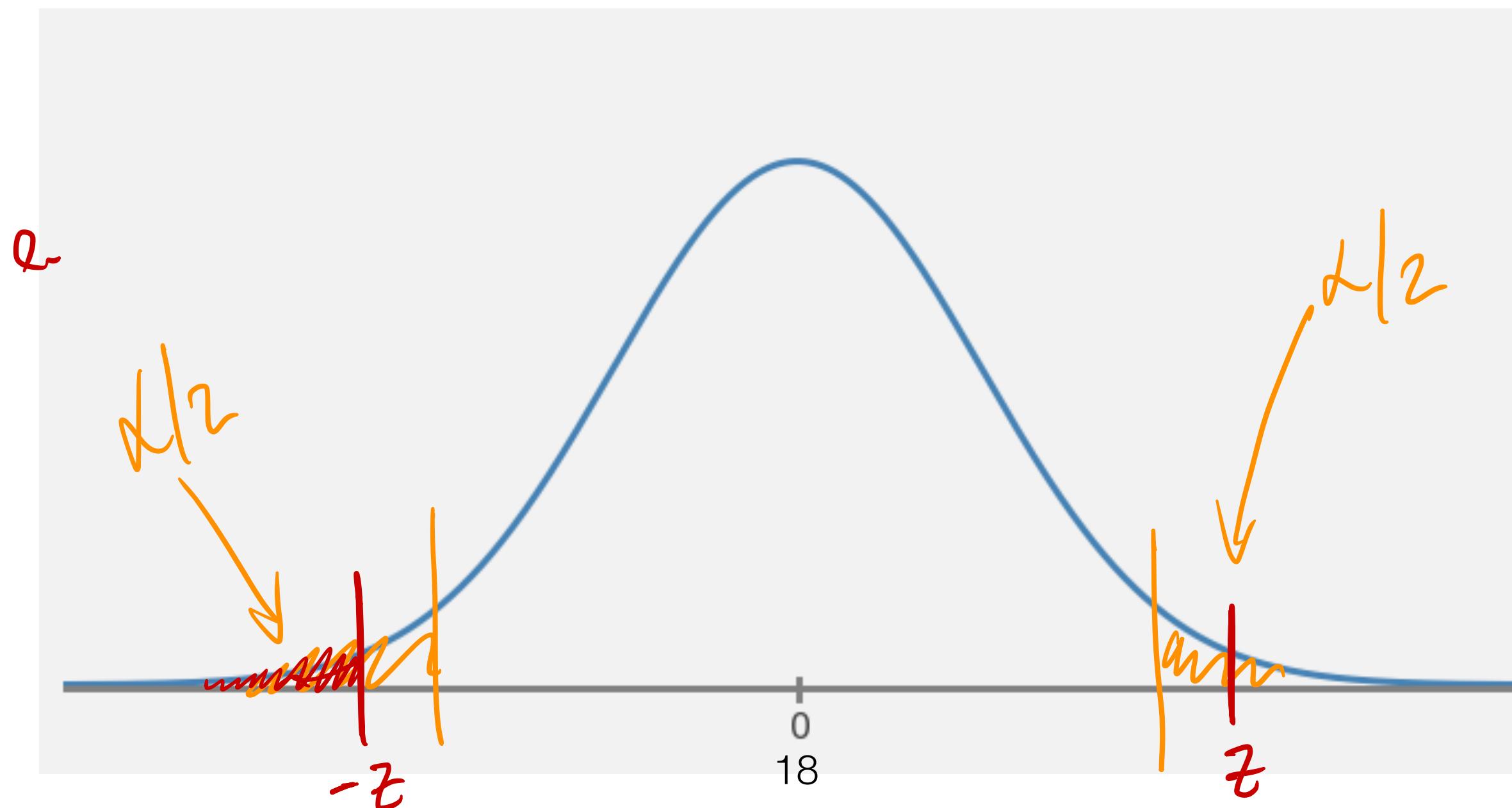
$$z \leq -z_\alpha$$

$$z \leq -z_\alpha \text{ or } z \geq z_\alpha$$

p-value Level α Test

$$\text{p-value: } 2 \times \Phi(-|z|) \leq \alpha$$

z is test statistic
 z comes from data
 α is our Type I error tolerance



p-values for different z-tests

Alternative Hypothesis

$$H_1 : \theta > \theta_0$$

$$H_1 : \theta < \theta_0$$

$$H_1 : \theta \neq \theta_0$$

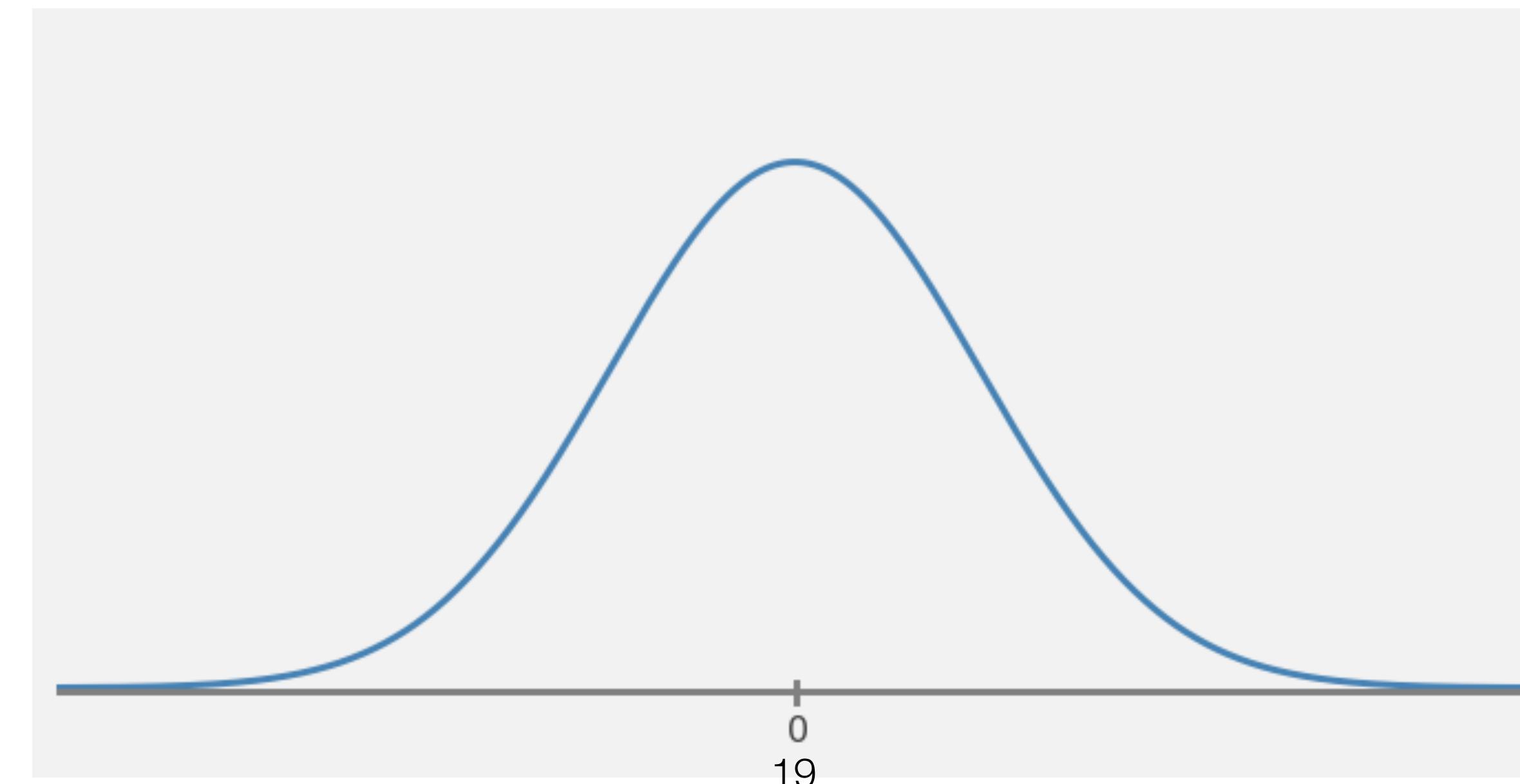
Critical Region Level α Test

$$z \geq z_\alpha$$

$$z \leq -z_\alpha$$

$$z \leq -z_\alpha \text{ or } z \geq z_\alpha$$

p-value Level α Test



Is the Belgian 1 Euro biased?

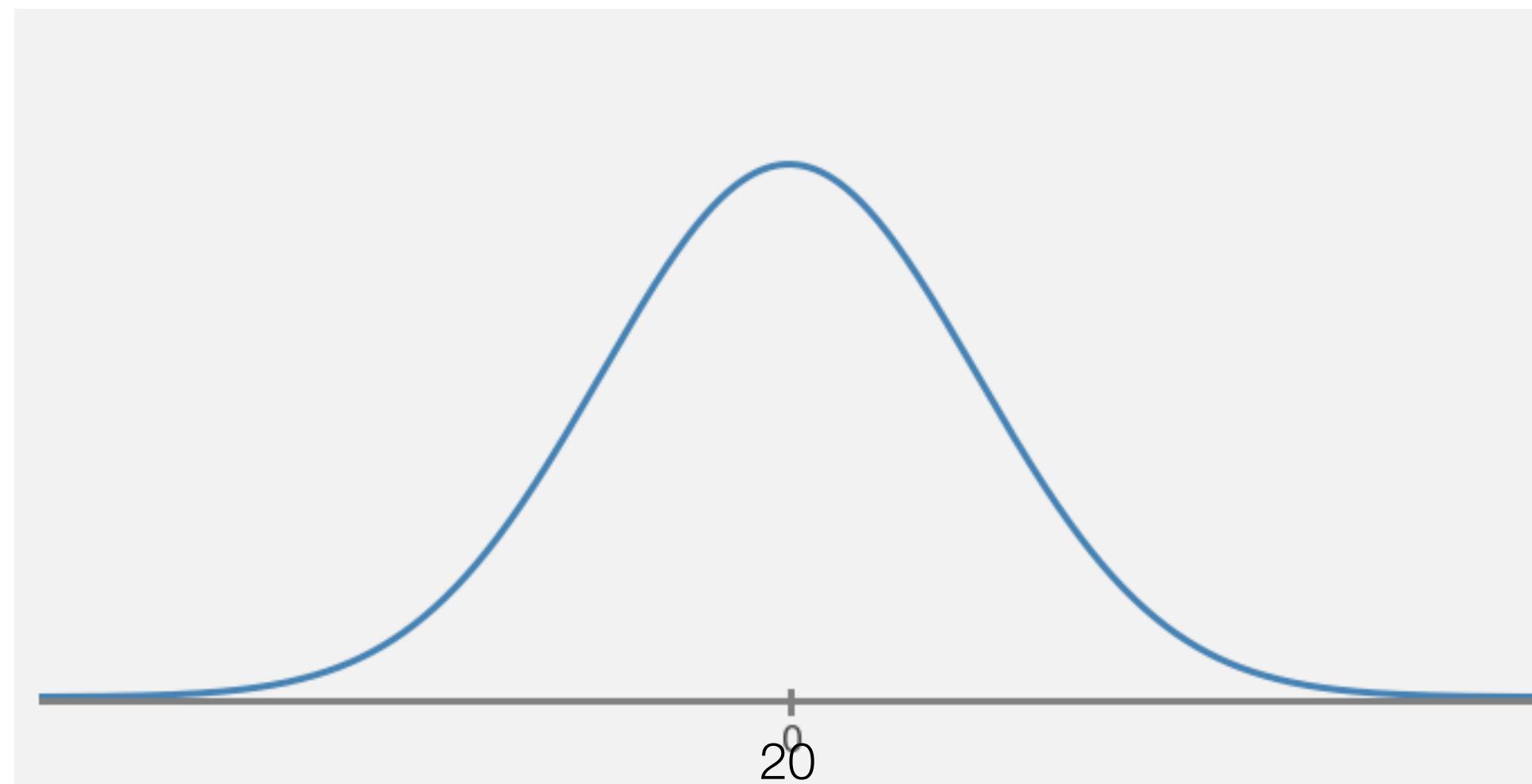
- Example: To test if the Belgian 1 Euro coin is fair you flip it 100 times and observe 38 Heads. Perform a p-value Z-test at the .05 significance level.

$$H_0: p = 0.5 \text{ fair}$$

$$H_1: p \neq 0.5 \text{ biased}$$

$$\hat{p} = 0.38 \text{ (data)}$$

$$z = \frac{0.38 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{100}}} = -2.4$$



$$\alpha = 0.05 \text{ two sided}$$

$$2 \times \Phi(-|-2.4|)$$

$$= 0.0164 \text{ p-value}$$

$$0.0164 \leq 0.05$$

Reject H_0 !

Coin is not fair.



Two-Sample Testing for Difference of Means

- Suppose we want to test whether or not the difference of sample means from two populations is nonzero, or equal to a particular value.
- **Question:** What kinds of Null and alternative hypotheses might we want to test?

$$H_0: \mu_1 - \mu_2 = C \quad \text{Some constant}$$
$$H_1: \mu_1 - \mu_2 \neq C$$
$$H_1: \mu_1 - \mu_2 < C$$
$$H_1: \mu_1 - \mu_2 > C$$
$$\frac{(\mu_1 - \mu_2) - C}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim N(0, 1)$$

CLT

Two-Sample Testing for Difference of Means

- Suppose we want to test whether or not the difference of sample means from two populations is nonzero, or equal to a particular value. C
- Assuming that our sample sizes are large enough, we can standardize our test statistics as:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - C}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}$$

- We can then compute an appropriate p-value in the usual way!

Yay!

Two-Sample Testing for Difference of Means

$$z = \frac{(\mu_1 - \mu_2) - c}{\sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{m}}}$$

- **Example:** Data on calorie intake both for a sample of teens that reported that they do not typically eat fast food and another sample of teens who said they did usually eat fast food is as follows:

Fast Food	Sample Size	Sample Mean	Sample SD
pop 2 → No	663 n	2258 μ_2	1519 s_2
pop 1 → Yes	413 m	2637 μ_1	1138 s_1

- Does this data provide statistical evidence at the 0.05 significance level that true average calorie intake for teens who typically eat fast food exceeds that of teens who do not typically eat fast food by more than 200 cals per day?

$$H_0: \mu_1 - \mu_2 = 200$$

$$H_1: \mu_1 - \mu_2 > 200$$

$$z = \frac{(2637 - 2258) - 200}{\sqrt{\frac{1138^2}{413} + \frac{1519^2}{663}}} = 2.20$$

p-value = ??