Binary Values and Number Systems

CPSC 1050 – Chapter 2

Objectives

- Distinguish among categories of numbers
- Describe positional notation
- Convert numbers in other bases to base 10
- Convert decimal numbers to numbers in other bases
- Describe the relationship between bases 2, 8, and 16
- Explain the importance of bases that are powers of 2

Introduction

"There are only 10 kinds of people in the world –

Those who understand *binary* numbers

and those who don't!"

Numbers

- Natural Numbers
  - Zero and any number obtained by repeatedly adding one to it
  - 0, 100, 485858
- Negative Numbers
  - A value less than 0, with a – sign
  - -38, -1, -85775, …
- Integers
  - A natural number or a negative number
- Rational Numbers
  - An integer or quotient of two integers
  - 33, -443, 1, 0, -1, 3/5, -22/56, ...
- Real Numbers
  - 1, 2.45, -0.546, ...

Number Systems

We are going to learn how to write this table!

decimal	binary	octal	hexadecimal
0	0000	0	0
1	0001	1	1
2	0010	2	2
3	0011	3	3
4	0100	4	4
5	0101	5	5
6	0110	6	6
7	0111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Base of a Number System

- The base of a number specifies the number of digits used in the system and the value of digit positions
- The digits in any base always begin with 0 and continue through one less than the base
    - Base 2 has 2 digits: 0 and 1
    - Base 8 has 8 digits: 0 to 7
    - Base 10 has 10 digits: 0 to 9
- For example
    - 567 can be a number in base 8 or higher but not in a base lower than 8
    - 123 can be a number in base 4 and higher but not in a base lower than 4
- What bases can these numbers be in?
    - 122, 198, 178, 0110

Bases Higher than 10

- For bases higher than 10, we need symbols to represent digits that correspond to the decimal values 10 and beyond
  - We use letters as digits
- Base 16 (Hexadecimal) has 16 digits
  - 0, 1 ,2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F
- What are Base 12 digits? Base 14?

Positional Notation

- Numbers are written using positional notation
- The value is represented as a polynomial in the base of the number system

A polynomial is a sum of two or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative integral power

- The formula is:

$$d_n * R^{n-1} + d_{n-1} * R^{n-2} + \ldots + d_2 * R + d_1$$

  - d_i is the digit at the i^{th} position in the number
  - n is the number of digits in the number
  - R is the base of the number

Positional Notation

- For example, 642 in base 10 positional notation is:

$$\begin{aligned} & 6 \times 10^2 = 6 \times 100 = 600 \\ + & 4 \times 10^1 = 4 \times 10 = 40 \\ + & 2 \times 10^0 = 2 \times 1 = 2 \\ & = 642_{10} \end{aligned}$$

  - The power indicates the position of the digit

Positional notation formula converts a number in any base to decimal (base 10)

Octal / Hex to Decimal

- What if 642_8 is in base 8 (Octal)?

$$
\begin{aligned}
& 6 \times 8^2 = 6 \times 64 = 384 \\
+\ & 4 \times 8^1 = 4 \times \ \ 8 = \ \ 32 \\
+\ & 2 \times 8^0 = 2 \times \ \ 1 = \ \ \ \ 2 \\
& = 418_{10}
\end{aligned}
$$

- What if 642_{16} is in base 16 (Hexadecimal)?

$$
\begin{aligned}
& 6 \times 16^2 = 6 \times 256 = 1536 \\
+\ & 4 \times 16^1 = 4 \times \ \ 16 = \ \ \ \ 64 \\
+\ & 2 \times 16^0 = 2 \times \ \ \ \ 1 = \ \ \ \ \ \ 2 \\
& = 1602_{10}
\end{aligned}
$$

Binary to Decimal

- The decimal equivalent of the binary number 1101010_2

$$
\begin{aligned}
& 1 \times 2^6 = 1 \times 64 = 64 \\
+ & 1 \times 2^5 = 1 \times 32 = 32 \\
+ & 0 \times 2^4 = 0 \times 16 = 0 \\
+ & 1 \times 2^3 = 1 \times 8 = 8 \\
+ & 0 \times 2^2 = 0 \times 4 = 0 \\
+ & 1 \times 2^1 = 1 \times 2 = 2 \\
+ & 0 \times 2^0 = 0 \times 1 = 0 \\
& = 106_{10}
\end{aligned}
$$

- So we use the positional notation formula to convert from any base to decimal (base 10)

Checkpoint

- $1010011_2 = (?)_{10}$
- $BAD_{16} = (?)_{10}$
- $265_8 = (?)_{10}$
- $847_{12} = (?)_{10}$
- $A49_{16} = (?)_{10}$

Decimal to other Bases

- Algorithm for converting number in base 10 to other bases
  - While (the quotient is not zero)
    - Divide the decimal number by the new base
    - Make the remainder the next digit to the left in the answer
    - Replace the original decimal number with the quotient

Decimal to Binary

- The binary equivalent of 23_{10}

11	5	2	1	0	Quotients
$2\overline{)23}$	$2\overline{)11}$	$2\overline{)5}$	$2\overline{)2}$	$2\overline{)1}$	
$\underline{22}$	$\underline{10}$	$\underline{4}$	$\underline{2}$	$\underline{0}$	
1	1	1	0	1 (first)	Remainders

Write in reverse order

Answer is : 10111_2

- Stop dividing when quotient is zero!

Decimal to Octal

- The octal equivalent of 1988_{10}

$$
\begin{array}{llll}
\;\;248 & \;\;31 & \;\;3 & \;\;0 \\
8\overline{)1988} & 8\overline{)248} & 8\overline{)31} & 8\overline{)3} \\
\;\;\underline{16} & \;\;\underline{24} & \;\;\underline{24} & \;\;\underline{0} \\
\;\;\;38 & \;\;\;08 & \;\;\;7 & \;\;3\text{ (first)} \\
\;\;\;\underline{32} & \;\;\;\;\underline{8} & & \\
\;\;\;\;68 & \;\;\;\;0 & & \\
\;\;\;\;\underline{64} & & & \\
\;\;\;\;\;4\text{ (last position)} & & &
\end{array}
$$

Answer is : 3704_8

Decimal to Hexadecimal

- The Hexadecimal equivalent of 3567_{10}

$16\overline{)3567}$ = 222	$16\overline{)222}$ = 13	$16\overline{)13}$ = 0
32	16	0
36	62	13
32	48	D
47	14	
32	E	
15		
F		

Answer is : DEF_{16}

Checkpoint

- $100_{10} = (?)_2$
- $100_{10} = (?)_8$
- $100_{10} = (?)_{16}$

Binary to Decimal Conversion Shortcut!

- Binary to decimal
  - $101101_2 = 32 + 0 + 8 + 4 + 0 + 1 = 45_{10}$

Powers of 2	32	16	8	4	2	1
Binary	1	0	1	1	0	1
Multiplication	32	0	8	4	0	1

  - $101111101_2 = 256 + 64 + 32 + 16 + 8 + 4 + 1 = 381_{10}$

Powers of 2	256	128	64	32	16	8	4	2	1
Binary	1	0	1	1	1	1	1	0	1
Multiplication	256	0	64	32	16	8	4	0	1

Decimal to Binary Conversion Shortcut!

- Decimal to binary
  - $79_{10} = 1001111_2$

Powers of 2	64	32	16	8	4	2	1
Quotient	1	0	0	1	1	1	1
Remainder	79-64=15	-	-	15-8=7	7-4=3	3-2=1	2-1=1

  - $893_{10} = 1101111101_2$

Powers of 2	512	256	128	64	32	16	8	4	2	1
Quotient	1	1	0	1	1	1	1	1	0	1
Remainder	381	125	-	61	29	13	5	1	-	0

Binary to Octal Shortcut!

- There is an easier way to convert among powers of 2
- Mark groups of three digits from right
- Convert each group to octal

  $10101011_2 \longrightarrow \underline{10}\ \ \underline{101}\ \ \underline{011}$

  2 5 3

  10101011_2 is 253_8

- The reverse works the same (octal to binary)
  - Example: $742_8 \longrightarrow$ 7 4 2

    $\underline{111}\ \underline{100}\ \underline{010}$

    742_8 is 111100010_2

Octal	0	1	2	3	4	5	6	7
Binary	000	001	010	011	100	101	110	111

Binary to Hexadecimal Shortcut!

- Mark groups of four digits from right
- Convert each group to hexadecimal

  $10101011_2 \longrightarrow \underset{A}{\underline{1010}}\ \ \underset{B}{\underline{1011}}$

  10101011_2 is AB_{16}

- The reverse works the same (hexadecimal to binary)
  - Example: $75C_{16} \longrightarrow \underset{\underline{0111}}{7}\ \underset{\underline{0101}}{5}\ \underset{\underline{1100}}{C}$

  $75C_{16}$ is 011101011100_2

binary	hexadecimal
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	8
1001	9
1010	A
1011	B
1100	C
1101	D
1110	E
1111	F

Checkpoint

- $111010011_2 = (?)_8$
- $1010111011110111_2 = (?)_{16}$
- $BAD_{16} = (?)_2$
- $265_8 = (?)_{16}$
- $847_{10} = (?)_{16}$
- $783_{10} = (?)_2$

Binary Addition

- Remember that there are only 2 digits in binary, 0 and 1
- 1 + 1 is 0 with a carry
- Addition:

$$\begin{array}{r} \color{red}{1\,0\,1\,1\,1\,1\,1} \\ 1\,0\,1\,0\,1\,1\,1_2 \\ \underline{+\,1\,0\,0\,1\,0\,1\,1_2} \\ 1\,0\,1\,0\,0\,0\,1\,0_2 \end{array}$$

Carry Values

Binary Subtraction

- Apply the concept of borrowing to base 2
- Subtraction:

$$
\begin{array}{r}
\color{red}{1\ 2\ \ \ \ \ \ \ \ \ \ \ \ \ \ } \\
\color{red}{2\ 0\ 2\ \ \ \ \ \ \ \ \ \ } \\
1\ 0\ 1\ 0\ 1\ 1\ 1_2 \\
\underline{-\ \ \ \ 1\ 1\ 1\ 0\ 1\ 1_2} \\
0\ 0\ 1\ 1\ 1\ 0\ 0_2
\end{array}
$$

Octal Addition

- Remember that there are only 8 digits in Octal
- Addition:

$$\begin{array}{r} \color{red}{1\,1\,1} \\ 5\,6\,2\,3_8 \\ \underline{+\,4\,1\,7\,2_8} \\ 1\,2\,0\,1\,5_8 \end{array}$$

Carry Values

Octal Subtraction

- Apply the concept of borrowing to base 8
- The column borrowed from is reduced by 1
- The amount of the borrow (8) is added to the next column
- Subtraction:

$$
\begin{array}{r}
\color{red}{10\ 11} \\
\color{red}{6\ \ 2\ \ 3\ 14} \\
7\ 3\ 4\ 6_8 \\
\underline{-\ 2\ 4\ 5\ 7_8} \\
4\ 6\ 6\ 7_8
\end{array}
$$

Hexadecimal Addition

- There are only 16 digits in hexadecimal
- Addition:

$$
\begin{array}{r}
{\color{red}1\ 1\ 1\ \ } \\
5\ A\ B\ 3_{16} \\
\underline{+\ F\ C\ 7\ 9}_{16} \\
1\ 5\ 7\ 2\ C_{16}
\end{array}
$$

Carry Values

  - ○ $7 + B = 7 + 11 = 18 = 16 + 2$
    - 16 is carried to the next bit as 1
  - ○ $1 + A + C = 1 + 10 + 12 = 23 = 16 + 7$

Hexadecimal Subtraction

- Apply the concept of borrowing to base 16
- The column borrowed from is reduced by 1
- The amount of the borrow (16) is added to the next column
- Subtraction:

$$
\begin{array}{r}
\color{red}{21\ 20} \\
\color{red}{2\ \ 5\ \ 4\ 18} \\
3\ 6\ 5\ 2_{16} \\
\underline{-\ 1\ C\ 9\ 8_{16}} \\
1\ 9\ B\ A_{16}
\end{array}
$$

Checkpoint

- $11001010_2 + 111100_2 = ?$
- $11000111_2 - 11111_2 = ?$
- $346_8 + 127_8 = ?$
- $6543_8 - 354_8 = ?$
- $764A_{16} + BCD_{16} = ?$
- $876D_{16} - 89C_{16} = ?$

Summary

- Numbers are written using positional notation
  - The number is equal to the sum of the products of each digit by its place value
    - The place values are powers of the base of the number system
- Arithmetic can be performed on numbers in any base represented in positional notation
- Base 2, base 8, and base 16 are all powers of 2
  - Provides a quick way to convert between numbers in these bases
- Computer hardware is designed using numbers in base 2

Additional Resources

- To Read!
  - Number Systems: Naturals, Integers, Rationals, Irrationals, Reals, and Beyond – Varsity Tutors
  - Computer - Number Systems – tutorialspoint
  - Number Systems: An Introduction to Binary, Hexadecimal, and More – tutsplus

This is a good one!

- To Watch!
  - Introduction to number systems and binary – Khan Academy
- To Practice!
  - The Number Base Calculator – Cleave Books