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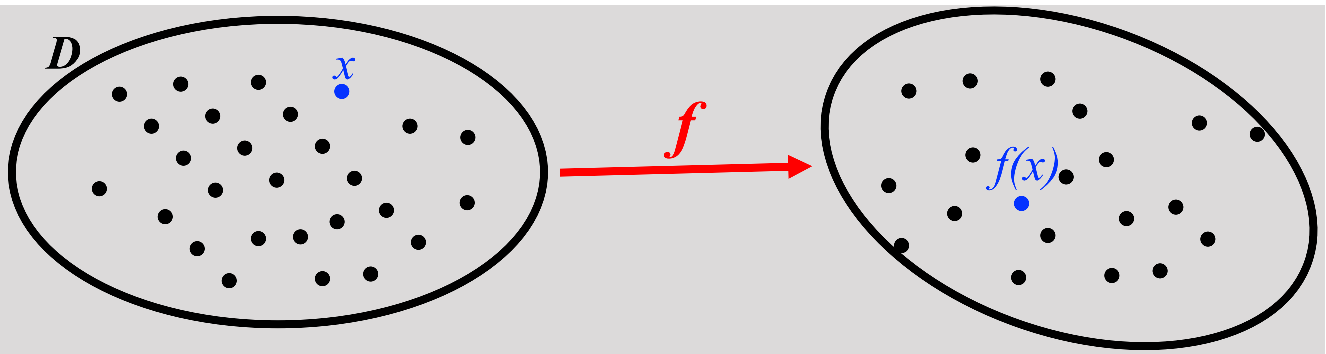
Single variable function *f(x)*

- Definition
  - It is a rule that assigns to each element *x* in a set *D* a unique element.
- Composite function:

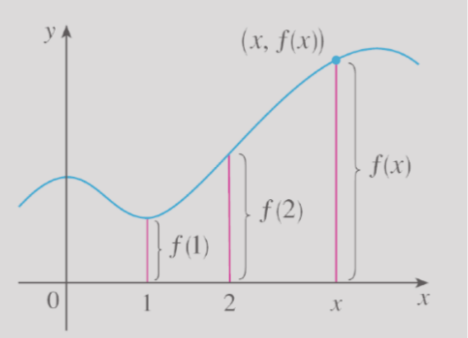
$$(g \circ f)(x) = g(f(x))$$

- Inverse function
  - Reverse process done by *f* : $(g \circ f)(x) = x$

Illustration of *f(x)*



Graph



Limit of a function

- Definition
  - The limit of *f(x)* at *a* is *L* if the value of *f(x)* approaches the real number L as *x* approaches as close as possible (but NEVER equal) to *a*.

$$\lim_{x \to a} f(x) = L$$

Derivative

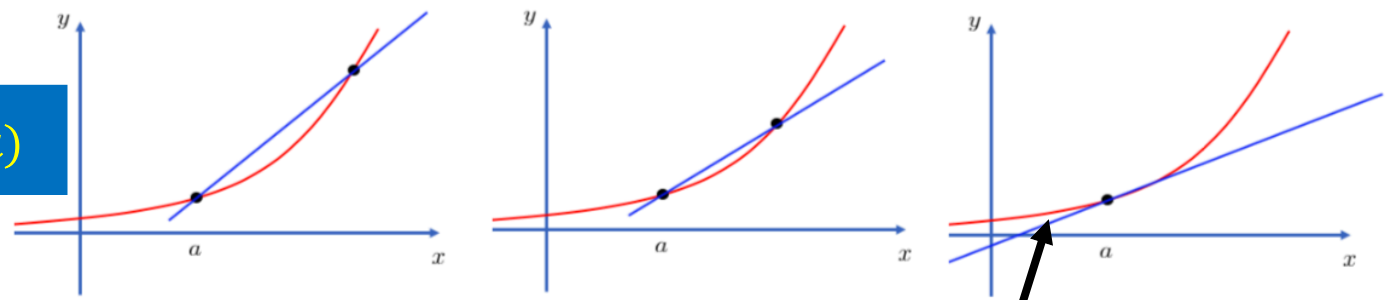
$f'(x)$ or $\frac{dy}{dx}$ or $\frac{d}{dx}f(x)$

Let $f(x)$ be a function and a be a real number. The number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if the above limit exists, is called the **derivative** of $f(x)$ at $x = a$ or the slope of the tangent line of $y = f(x)$ at $x = a$.

Illustration of $f'(a)$



Tangent line at $x = a$:

$$y = f'(a)(x - a) + f(a)$$

Four rules

Scalar coefficient: $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x)$, $\quad k$ is a scalar.

Sum rule: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

Product rule: $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

Quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

Chain rules

If $u = g(x)$, $y = f(u) = f(g(x))$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Applications of Derivative

Problem of finding rate of change::

- Given one rate of change $\dfrac{dy}{dt}$, we want to find another rate of change $\dfrac{dz}{dt}$
- The procedure is to find an equation that relates the two quantities y and z and then use the Chain Rule to differentiate both sides with respect to t.

The Closed Interval Method: To find the maximum and minimum values of a continuous function $f(x)$ on a closed interval $a \leq x \leq b$.

(1) Find the values of f at stationary point(s).

(2) Find the values of f at the endpoints of the interval: $f(a)$, $f(b)$.

(3) The largest of the values from Step (1) and (2) is the maximum; the smallest of the values from Step (1) and (2) is the minimum.

L'Hospital's Rule:
Suppose f and g are differentiable, and by direct substitution we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}.$$

Here a can be a real number or $\pm\infty$.

Indefinite and Definite Integrals

A function $F(x)$ is an **antiderivative** of $f(x)$ on an interval (a, b) if

$$F'(x) = f(x) \quad \text{for all } x \in (a, b).$$

Indefinite Integral:

All antiderivatives of f differ by a constant. Thus, the most general antiderivative of f on (a, b) is called the **indefinite integral** of f, and is denoted by

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is an antiderivative of $f(x)$ and C is an arbitrary constant.

Definite Integral:

We obtain the **definite integral** of f over the interval $[a, b]$, denoted by $\int_a^b f(x)\,dx$, by subtracting the value of an antiderivative $F(x)$ at a from that at b:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

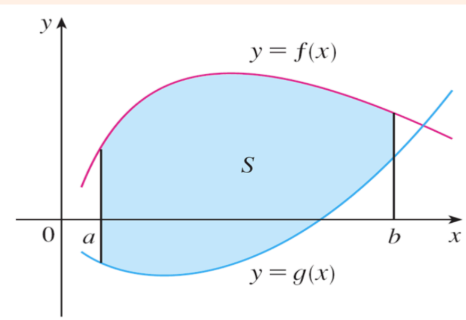
where $F(x)$ is an antiderivative of $f(x)$.

Fundamental Theorem of Calculus.

Area between two curves:

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$ is

$$A = \int_a^b \Big(\underbrace{f(x)}_{\text{top curve}} - \underbrace{g(x)}_{\text{bottom curve}} \Big)\,dx.$$



$$A = \int_a^b \Big(\underbrace{f(x)}_{\text{top curve}} - \underbrace{g(x)}_{\text{bottom curve}} \Big)\,dx.$$

Techniques of Integration

Substitution Rule:
Steps when applying the Substitution Rule to integrate

$$\int_a^b f(x)\,dx$$

- Think of a function $u = g(x)$.
- Compute $\frac{du}{dx} = g'(x) \implies dx = \frac{1}{g'(x)}\,du$.
- Convert $f(x)\,dx$ into an expression in terms of u and du.
- Replace the lower limit a by $g(a)$, and the upper limit b by $g(b)$.
- Integrate with respect to u.

Integration by Parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

In the case of a definite integral, we have

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx.$$

Partial Fraction Decomposition:

- Start with a rational function $\frac{P(x)}{Q(x)}$, where the degree of P is **strictly less** than the degree of Q.

  This is important. Otherwise, we will first do a **long division** to expand the function.

- We factor the denominator $Q(x)$ as completely as possible into irreducible factors. For our purposes, $Q(x)$ only contains **linear** or **quadratic** factors.

Each factor in the denominator (and their multiplicity $r = 1, 2, 3, \cdots$) will determine the term(s) that occur in the partial fraction decomposition.

Factor in $Q(x)$	**Term in partial fraction decomposition**
$(ax+b)^r$	$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}$
$(ax^2+bx+c)^r$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

Ordinary Differential Equation

Type of ODE	Method	
1st order Separable	Separating variables	Method 1
1st order Homogeneous	Substituting $y = vx$	Method 2
1st order Linear	Integrating factor	Method 3
1st order Bernoulli's equation	Divide by y^n, reduce to linear	Method 4
2nd order Homogeneous	Auxiliary equation	
2nd order Inhomogenous	Homogeneous solution $+$ particular solution	

Method 1: Separating variables

- We can 'separate' the y-factors and the x-factors into opposite sites:
$$g(y)\,dy = f(x)\,dx.$$
- Then solve the ODE by integrating both sides:
$$\int g(y)\,dy = \int f(x)\,dx.$$

$$\frac{dy}{dx} + Py = Q$$

Method 3: Integrating Factor

- Multiply both sides of the equation by $e^{\int P\,dx}$, called the **integrating factor**. (When evaluating $\int P\,dx$, we will ignore any constant.)
- This converts the left-hand side of the ODE into the derivative of the product $ye^{\int P\,dx}$.
- Integrate both sides to obtain a solution.

Method 2: Substituting y=vx
(For 1st Order Homogeneous ODE)

- Let $y = vx$.
- Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ by Product Rule.
- Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ into the equation. This converts 1st order homogeneous ODE into a separable ODE in terms of v and x for which we can solve by separating variables.
- Convert back into terms of y and x.

Method 4: Bernoulli's Equation

- To solve $\frac{dy}{dx} + Py = Qy^n$, first divide both sides by y^n.
- Put $z = y^{1-n}$ and convert the ODE into a linear 1st order ODE:
$$\frac{dz}{dx} + P^*z = Q^*,$$
where $P^* = (1-n)P$, $Q^* = (1-n)Q$.
- Solve the linear ODE using the method of integrating factor.
- Convert back into terms of y and x.

Ordinary Differential Equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=r(x)$$

Type of ODE	Method	
1st order Separable	Separating variables	
1st order Homogeneous	Substituting $y = vx$	
1st order Linear	Integrating factor	
1st order Bernoulli's equation	Divide by y^n, reduce to linear	
2nd order Homogeneous	Auxiliary equation	Method 5
2nd order Inhomogenous	Homogeneous solution + particular solution	Method 6

Method 5: Homogeneous second order ODE

- Let w be a variable, and m_1, m_2 be the roots of the auxiliary equation $aw^2 + bw + c = 0$.
- Recall that the roots of the auxiliary (quadratic) equation is given by

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

- Let m_1, m_2 be the roots of the auxiliary equation. There are three cases:
  - (i) Real and distinct roots i.e. $m_1 \neq m_2$. The general solution is

  $$y = Ae^{m_1x} + Be^{m_2x}.$$

  - (ii) Real and equal roots i.e $m_1 = m_2 = m$. The general solution is

  $$y = (Ax + B)e^{mx}.$$

  - (iii) Complex roots i.e. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$. The general solution is

  $$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Here, A and B are arbitrary constants.

Method 6: Inhomogeneous second order ODE

- Find the general solution $y_h(x)$ of the homogeneous ODE $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$. It is called the homogeneous solution.
- Find ANY solution $y_p(x)$ of the non-homogeneous ODE $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$. It is called a particular solution.
- The general solution to (2) is

$$y = y_h(x) + y_p(x).$$

- First, we guess the form of $y_p(x)$ based on the form of $r(x)$. This is given in the table below.

$r(x)$	$y_p(x)$
Kx^n	$C_nx^n + C_{n-1}x^{n-1} + \cdots + C_1x + C_0$
$Ke^{\alpha x}$	$Ce^{\alpha x}$
$K\cos(\beta x)$	$C\cos(\beta x) + D\sin(\beta x)$
$K\sin(\beta x)$	$C\cos(\beta x) + D\sin(\beta x)$
$Ke^{\alpha x}\cos(\beta x)$	$e^{\alpha x}(C\cos(\beta x) + D\sin(\beta x))$
$Ke^{\alpha x}\sin(\beta x)$	$e^{\alpha x}(C\cos(\beta x) + D\sin(\beta x))$

- The various constants appearing in $y_p(x)$ can be determined by assuming that $y_p(x)$ satisfies the ODE.

Sequences

A **sequence** can be regarded as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots.$$

Limit of a sequence:

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that terms of the sequence $\{a_n\}$ approach L as n gets larger and larger.

- If L is a real number, then we say that $\{a_n\}$ **converges** to L (or is **convergent**).
- Otherwise, we say that the sequence **diverges** (is **divergent**).

Series

If we add the terms of a sequence $\{a_n\}_{n=1}^{\infty}$, we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

which is called a **series**

Limit of Series

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \underbrace{\sum_{n=1}^{k} a_n}_{\text{partial sum}} .$$

If the above limit exists and is equal to S, then the series $\sum a_n$ is called **convergent**, and the number S is called the **sum** of the series. Otherwise, the series is said to be **divergent**.

Geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

- converges to $\dfrac{a}{1-r}$, if $|r| < 1$.
- diverges is $|r| \geq 1$.

Harmonic series (divergent)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- convergent if $p > 1$;
- divergent if $p \leq 1$.

Convergence Tests for Series

(i) Divergence Test

(ii) Integral Test

(iii) Absolute Convergence Test

(iv) Ratio Test

(v) Root Test

Convergence Tests for Series

Divergence Test:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$,

then the series $\sum_{n=1}^{\infty} a_n$ is **divergent**.

The Integral Test:

Suppose f is a **continuous**, **positive**, **decreasing** function on $[c, \infty)$, and let $a_n = f(n)$.

(i) If $\int_c^{\infty} f(x)\,dx$ is **convergent (i.e. equals a real number)**, then the series $\sum_{n=c}^{\infty} a_n$ is **convergent**.

(ii) If $\int_c^{\infty} f(x)\,dx$ is **divergent**, then $\sum_{n=c}^{\infty} a_n$ is **divergent**.

Absolute Convergence Test:

If $\sum |a_n|$ converges, then the series $\sum a_n$ is convergent.

Ratio Test:

Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(i) If $\rho < 1$, then $\sum a_n$ **converges absolutely** (so it converges by the Absolute Convergence Test).

(ii) If $\rho > 1$ or $\rho = \infty$, then $\sum a_n$ **diverges**.

(iii) If $\rho = 1$, then Ratio Test is inconclusive (the series may converge or diverge).

Root Test:

Let $\{a_n\}$ be a sequence and assume that the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

(i) If $L < 1$, then $\sum a_n$ **converges absolutely**.

(ii) If $L > 1$ or $L = \infty$, then $\sum a_n$ **diverges**.

(iii) If $L = 1$, the Root Test is inconclusive. The series may converge or diverge.