Statistical Inference
- Naïve Bayes -

Artificial Intelligence

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Essential Probability Concepts

- Marginalization: $P(B) = \sum_{v \in \mathrm{values}(A)} P(B \wedge A = v)$

- Conditional Probability: $P(A \mid B) = \dfrac{P(A \wedge B)}{P(B)}$

- Bayes' Rule: $P(A \mid B) = \dfrac{P(B \mid A) \times P(A)}{P(B)}$

- Independence:

$$
\begin{aligned}
A \perp\!\!\!\perp B \quad &\leftrightarrow \quad P(A \wedge B) = P(A) \times P(B) \\
&\leftrightarrow \quad P(A \mid B) = P(A) \\
A \perp\!\!\!\perp B \mid C \quad &\leftrightarrow \quad P(A \wedge B \mid C) = P(A \mid C) \times P(B \mid C)
\end{aligned}
$$

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Joint Distribution

- Experimental Settings
  - Suppose each of two urns contains twice as many red balls as blue balls
  - Suppose one ball is randomly selected from each urn
  - Red ball: 2/3, blue ball: 1/3

- Joint probability distribution

	A=Red	A=Blue	P(B)
B=Red	(2/3)(2/3)=4/9	(1/3)(2/3)=2/9	4/9+2/9=2/3
B=Blue	(2/3)(1/3)=2/9	(1/3)(1/3)=1/9	2/9+1/9=1/3
P(A)	4/9+2/9=2/3	2/9+1/9=1/3	

From Wikipedia: https://en.wikipedia.org/wiki/Joint_probability_distribution

Joint Distribution

- Learning a Joint Distribution

Step 1:
Build a JD table for your attributes in which the probabilities are unspecified

A	B	C	Prob
0	0	0	?
0	0	1	?
0	1	0	?
0	1	1	?
1	0	0	?
1	0	1	?
1	1	0	?
1	1	1	?

Step 2:
Then, fill in each row with:

$$\hat{P}(\text{row}) = \frac{\text{records matching row}}{\text{total number of records}}$$

A	B	C	Prob
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	$\boxed{0.25}$
1	1	1	0.10

Fraction of all records in which A and B are true but C is false

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Joint Distribution

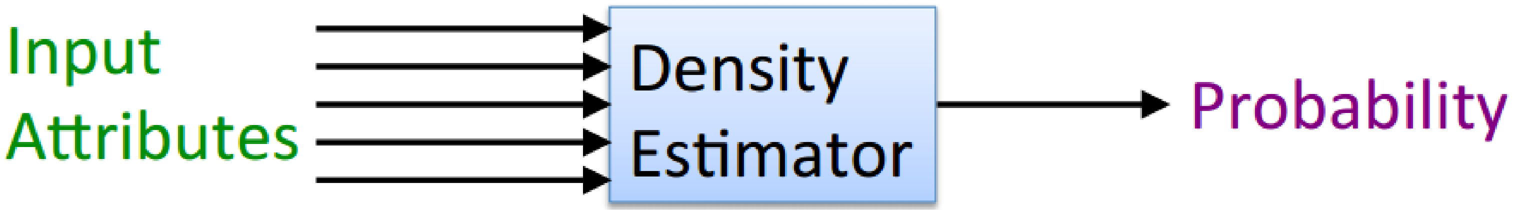
- Example of Learning a Joint PD

This Joint PD was obtained by learning from three attributes in the UCI "Adult" Census Database [Kohavi 1995]

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

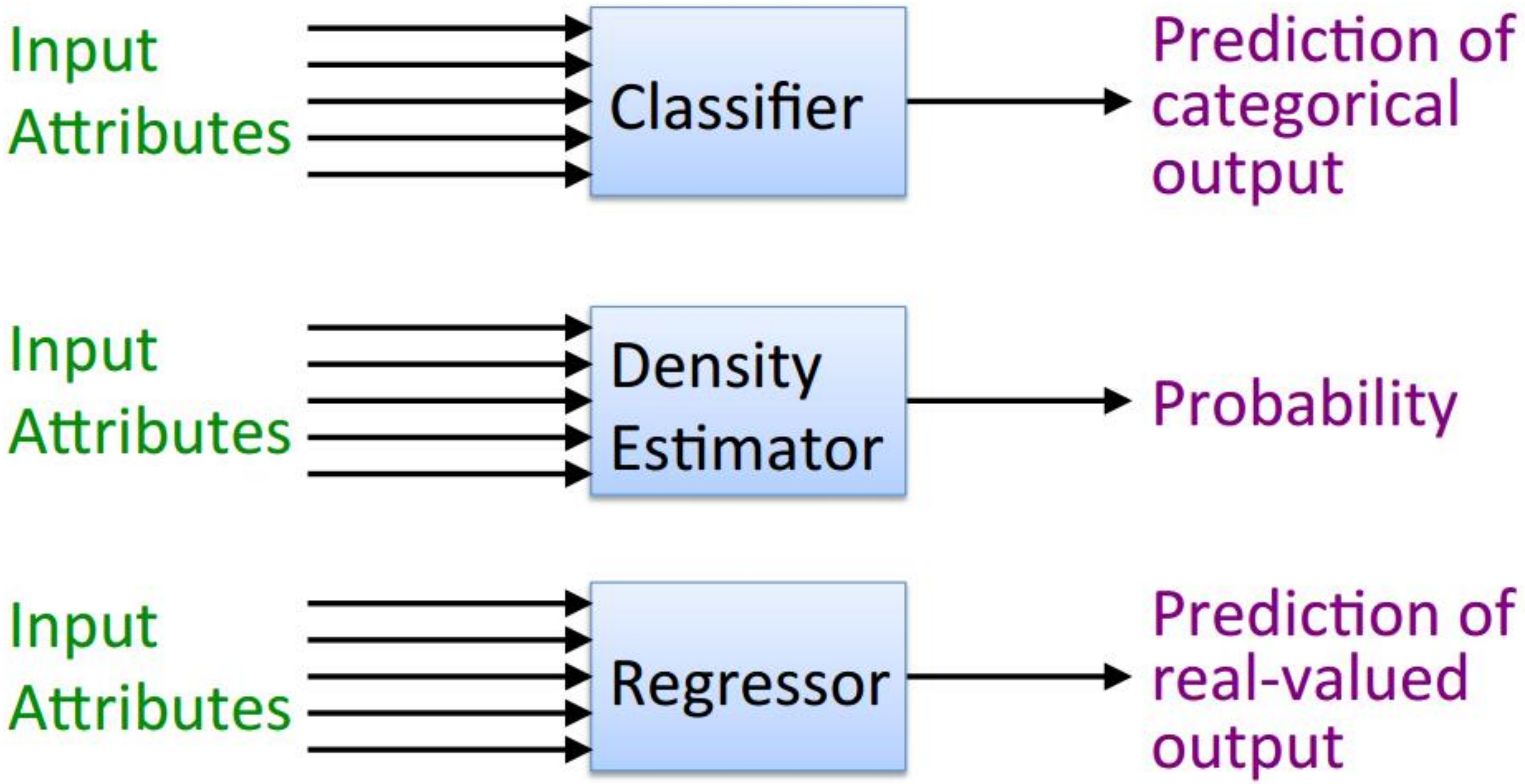
Density Estimation

- Our joint distribution learner is an example of something called **Density Estimation**
- A Density Estimator learns a mapping from a set of attributes to a probability



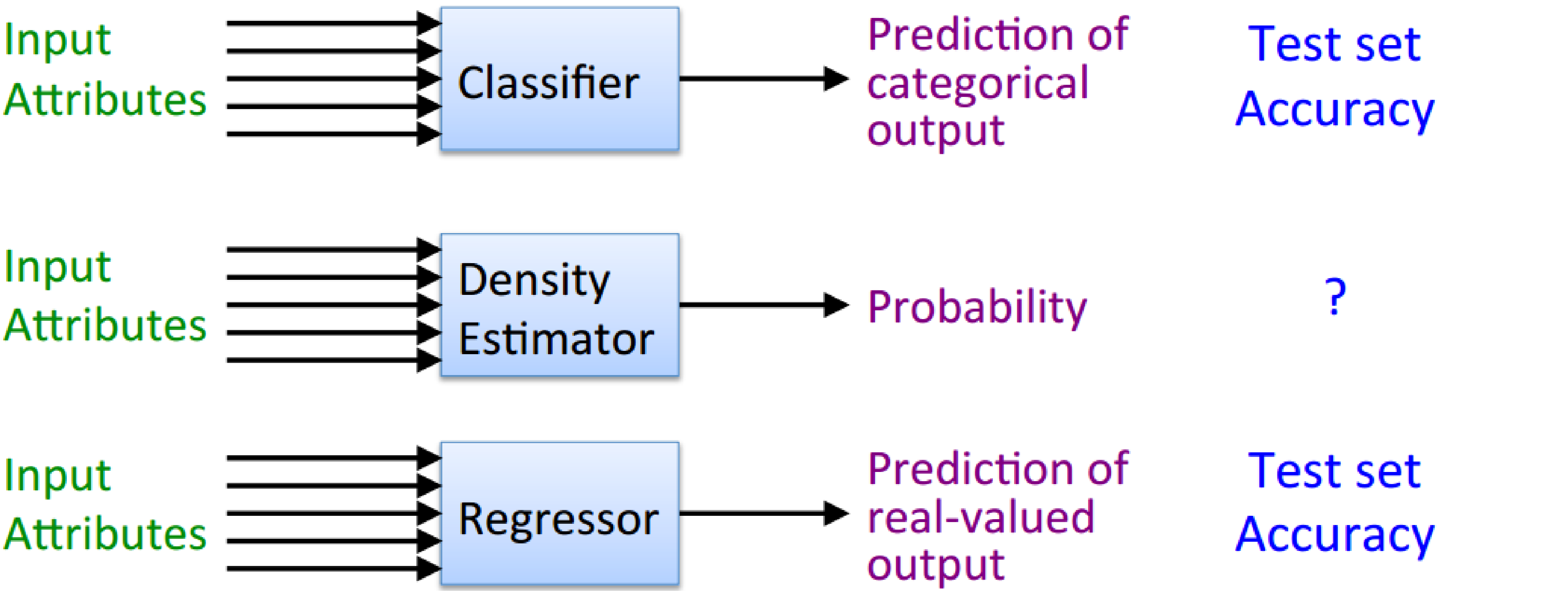
Density Estimation

Compare it against the two other major kinds of models:



Density Estimation

Test-set criterion for estimating performance on future data



Density Estimation

❖ Evaluating a Density Estimator

- Given a record $\mathbf{x}$, a density estimator *M* can tell you how likely the record is:

$$\hat{P}(\mathbf{x} \mid M)$$

- The density estimator can also tell you how likely the dataset is:
  - Under the assumption that all records were independently generated from the Density Estimator's JD (that is, i.i.d.)

$$\hat{P}(\underbrace{\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_n}_{\text{dataset}} \mid M) = \prod_{i=1}^{n} \hat{P}(\mathbf{x}_i \mid M)$$

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Density Estimation

- Example Small Dataset: Miles Per Gallon

From the UCI repository (thanks to Ross Quinlan)

- 192 records in the training set

mpg	modelyear	maker
good	75to78	asia
bad	70to74	america
bad	75to78	europe
bad	70to74	america
bad	70to74	america
bad	70to74	asia
bad	70to74	asia
bad	75to78	america
:	:	:
:	:	:
:	:	:
bad	70to74	america
good	79to83	america
bad	75to78	america
good	79to83	america
bad	75to78	america
good	79to83	america
good	79to83	america
bad	70to74	america
good	75to78	europe
bad	75to78	europe

mpg	modelyear	maker	
bad	70to74	america	0.27551
		asia	0.0255102
		europe	0.0153061
	75to77	america	0.153061
		asia	0.0255102
		europe	0.0357143
	78to83	america	0.0561224
		asia	Never
		europe	Never
good	70to74	america	0.0102041
		asia	0.0306122
		europe	0.0459184
	75to77	america	0.0306122
		asia	0.0408163
		europe	0.0357143
	78to83	america	0.112245
		asia	0.0714286
		europe	0.0357143

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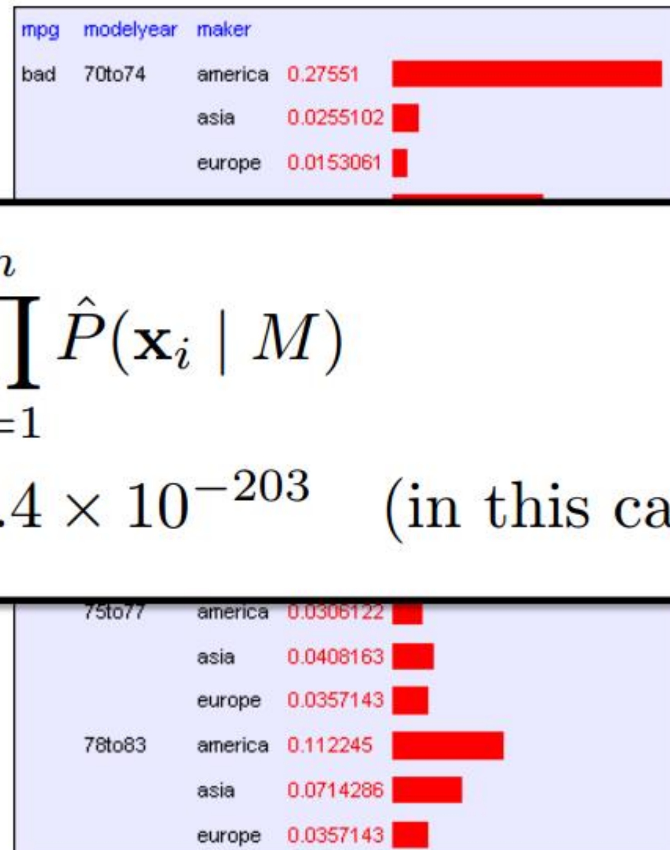
Density Estimation

- Example Small Dataset: Miles Per Gallon

  From the UCI repository (thanks to Ross Quinlan)

  - 192 records in the training set

mpg	modelyear	maker
good	75to78	asia
bad	70to74	america
bad	75to78	america
good	79to83	america
good	79to83	america
bad	70to74	america
good	75to78	europe
bad	75to78	europe



$$\hat{P}(\text{dataset} \mid M) = \prod_{i=1}^{n} \hat{P}(\mathbf{x}_i \mid M)$$

$$= 3.4 \times 10^{-203} \quad \text{(in this case)}$$

Density Estimation

- Log Probabilities
  - For decent sized data sets, this product will underflow

$$\hat{P}(\text{dataset} \mid M) = \boxed{\prod_{i=1}^{n}} \hat{P}(\mathbf{x}_i \mid M)$$

  - Therefore, since probabilities of datasets get so small, we usually use log probabilities

$$\log \hat{P}(\text{dataset} \mid M) = \log \prod_{i=1}^{n} \hat{P}(\mathbf{x}_i \mid M) = \sum_{i=1}^{n} \log \hat{P}(\mathbf{x}_i \mid M)$$

Slide Credit: Eric Eaton

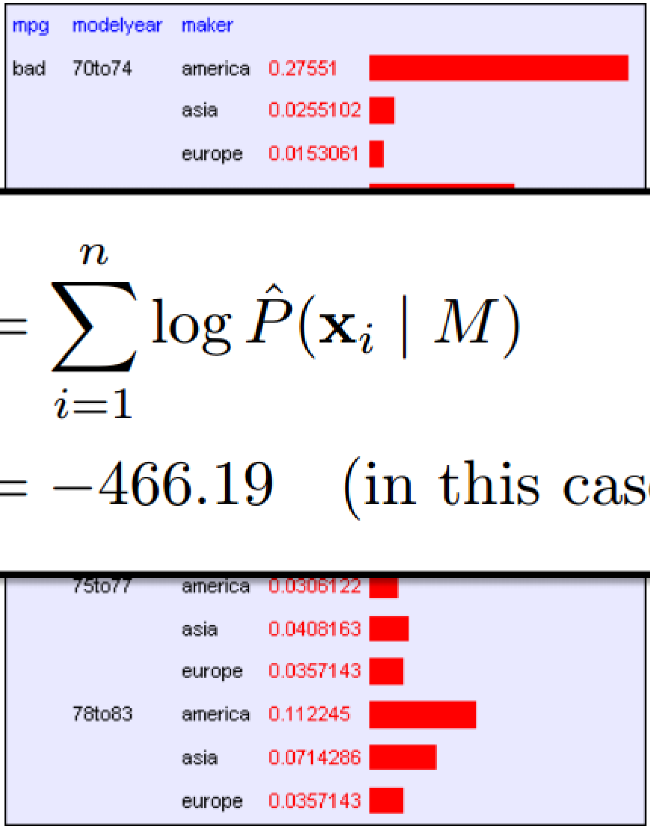
Density Estimation

- Example Small Dataset: Miles Per Gallon

From the UCI repository (thanks to Ross Quinlan)

- 192 records in the training set

mpg	modelyear	maker
good	75to78	asia
bad	70to74	america
bad	75to78	america
good	79to83	america
good	79to83	america
bad	70to74	america
good	75to78	europe
bad	75to78	europe



$$\log \hat{P}(\text{dataset} \mid M) = \sum_{i=1}^{n} \log \hat{P}(\mathbf{x}_i \mid M)$$

$$= -466.19 \quad (\text{in this case})$$

Slide by Andrew Moore

The Naïve Bayes Classifier

❖ Bayes' Rule

- Recall Baye's Rule:

$$P(\text{hypothesis} \mid \text{evidence}) = \frac{P(\text{evidence} \mid \text{hypothesis}) \times P(\text{hypothesis})}{P(\text{evidence})}$$

- Equivalently, we can write:

$$P(Y = y_k \mid X = \mathbf{x}_i) = \frac{P(Y = y_k)P(X = \mathbf{x}_i \mid Y = y_k)}{P(X = \mathbf{x}_i)}$$

where X is a random variable representing the evidence and Y is a random variable for the label

- This is actually short for:

$$P(Y = y_k \mid X = \mathbf{x}_i) = \frac{P(Y = y_k)P(X_1 = x_{i,1} \wedge \ldots \wedge X_d = x_{i,d} \mid Y = y_k)}{P(X_1 = x_{i,1} \wedge \ldots \wedge X_d = x_{i,d})}$$

where X_j denotes the random variable for the j^{th} feature

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The Naïve Bayes Classifier

Idea: Use the training data to estimate

$$P(X \mid Y) \quad \text{and} \quad P(Y) \; .$$

Then, use Bayes rule to infer $P(Y|X_{\text{new}})$ for new data

Easy to estimate from data

Impractical, but necessary

$$P(Y = y_k \mid X = \mathbf{x}_i) = \frac{\boxed{P(Y = y_k)}\boxed{P(X_1 = x_{i,1} \wedge \ldots \wedge X_d = x_{i,d} \mid Y = y_k)}}{\boxed{P(X_1 = x_{i,1} \wedge \ldots \wedge X_d = x_{i,d})}}$$

Unnecessary, as it turns out

- Recall that estimating the joint probability distribution $P(X_1, X_2, \ldots, X_d \mid Y)$ is not practical

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The Naïve Bayes Classifier

Problem: estimating the joint PD or CPD isn't practical

– Severely overfits, as we saw before

However, if we make the assumption that the attributes are independent given the class label, estimation is easy!

$$P(X_1, X_2, \ldots, X_d \mid Y) = \prod_{j=1}^{d} P(X_j \mid Y)$$

- In other words, we assume all attributes are *conditionally independent* given Y
- Often this assumption is violated in practice, but more on that later…

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The Naïve Bayes Classifier

- Training Naïve Bayes

Estimate $P(X_j \mid Y)$ and $P(Y)$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
rainy	cold	high	strong	warm	change	*no*
sunny	warm	high	strong	cool	change	*yes*

$$\mathrm{P}(\mathrm{play}) = ?$$
$$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \mathrm{play}) = ?$$
$$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \mathrm{play}) = ?$$
$$\ldots$$

$$\mathrm{P}(\neg\mathrm{play}) = ?$$
$$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \neg\mathrm{play}) = ?$$
$$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \neg\mathrm{play}) = ?$$
$$\ldots$$

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

- Training Naïve Bayes

Estimate $P(X_j \mid Y)$ and $P(Y)$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
rainy	cold	high	strong	warm	change	*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P(play)} = 3/4$

$\mathrm{P(Sky = sunny \mid play)} = ?$

$\mathrm{P(Humid = high \mid play)} = ?$

$\ldots$

$\mathrm{P(\neg play)} = 1/4$

$\mathrm{P(Sky = sunny \mid \neg play)} = ?$

$\mathrm{P(Humid = high \mid \neg play)} = ?$

$\ldots$

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

- Training Naïve Bayes

Estimate $P(X_j \mid Y)$ and $P(Y)$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny						*yes*
sunny						*yes*
rainy	cold	high	strong	warm	change	*no*
sunny						*yes*

$\mathrm{P}(\mathrm{play}) = 3/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \mathrm{play}) = 1$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \mathrm{play}) = ?$

$\ldots$

$\mathrm{P}(\neg\mathrm{play}) = 1/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \neg\mathrm{play}) = ?$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \neg\mathrm{play}) = ?$

$\ldots$

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

- Training Naïve Bayes

Estimate $P(X_j \mid Y)$ and $P(Y)$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
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sunny	warm	high	strong	warm	same	*yes*
rainy						*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P(play)} = 3/4$

$\mathrm{P(Sky = sunny \mid play)} = 1$

$\mathrm{P(Humid = high \mid play)} = ?$

$\ldots$

$\mathrm{P(\neg play)} = 1/4$

$\mathrm{P(Sky = sunny \mid \neg play)} = 0$

$\mathrm{P(Humid = high \mid \neg play)} = ?$

$\ldots$

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The Naïve Bayes Classifier

❖ Training Naïve Bayes

Estimate $P(X_j \mid Y)$ and $P(Y)$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
		normal				*yes*
		high				*yes*
rainy	cold	high	strong	warm	change	*no*
		high				*yes*

$\mathrm{P(play)} = 3/4$ $\qquad$ $\mathrm{P}(\neg\mathrm{play}) = 1/4$

$\mathrm{P(Sky = sunny} \mid \mathrm{play}) = 1$ $\qquad$ $\mathrm{P(Sky = sunny} \mid \neg\mathrm{play}) = 0$

$\mathrm{P(Humid = high} \mid \mathrm{play}) = 2/3$ $\qquad$ $\mathrm{P(Humid = high} \mid \neg\mathrm{play}) = ?$

$\ldots$ $\qquad$ $\ldots$

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

- Training Naïve Bayes

Estimate $P(X_j \mid Y)$ and $P(Y)$ directly from the training data by counting!

Sky	Temp	Humid	Wind	Water	Forecast	Play?
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
		high				*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P}(\mathrm{play}) = 3/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \mathrm{play}) = 1$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \mathrm{play}) = 2/3$

$\ldots$

$\mathrm{P}(\neg\mathrm{play}) = 1/4$

$\mathrm{P}(\mathrm{Sky} = \mathrm{sunny} \mid \neg\mathrm{play}) = 0$

$\mathrm{P}(\mathrm{Humid} = \mathrm{high} \mid \neg\mathrm{play}) = 1$

$\ldots$

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The Naïve Bayes Classifier

❖ Laplace Smoothing

- Notice that some probabilities estimated by counting might be zero
  - Possible overfitting!

- Fix by using Laplace smoothing:
  - Adds 1 to each count

$$P(X_j = v \mid Y = y_k) = \frac{c_v + 1}{\sum_{v' \in \text{values}(X_j)} c_{v'} + |\text{values}(X_j)|}$$

where

- c_v is the count of training instances with a value of v for attribute j and class label y_k
- $|\text{values}(X_j)|$ is the number of values X_j can take on

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

❖ Laplace Smoothing

- Notice that some probabilities estimated by counting might be zero
  - Possible overfitting!

- Fix by using Laplace smoothing:
  - Adds 1 to each count

$$P(X_j = v \mid Y = y_k) = \frac{c_v + 1}{\sum_{v' \in \text{values}(X_j)} c_{v'} + |\text{values}(X_j)|}$$

where

- c_v is the count of training instances with a value of v for attribute j and class label y_k
- $|\text{values}(X_j)|$ is the number of values X_j can take on

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

- Training Naïve Bayes with Laplace Smoothing

Estimate $P(X_j \mid Y)$ and $P(Y)$ directly from the training data by counting with Laplace smoothing:

Sky	**Temp**	**Humid**	**Wind**	**Water**	**Forecast**	**Play?**
sunny	warm	normal	strong	warm	same	*yes*
sunny	warm	high	strong	warm	same	*yes*
		high				*no*
sunny	warm	high	strong	cool	change	*yes*

$\mathrm{P(play)} = 3/4$

$\mathrm{P(Sky = sunny \mid play)} = 4/5$

$\mathrm{P(Humid = high \mid play)} = 3/5$

$\ldots$

$\mathrm{P(\neg play)} = 1/4$

$\mathrm{P(Sky = sunny \mid \neg play)} = 1/3$

$\mathrm{P(Humid = high \mid \neg play)} = 2/3$

$\ldots$

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The Naïve Bayes Classifier

Using the Naïve Bayes Classifier

- Now, we have

$$P(Y = y_k \mid X = \mathbf{x}_i) = \frac{P(Y = y_k) \prod_{j=1}^{d} P(X_j = x_{i,j} \mid Y = y_k)}{\boxed{P(X = \mathbf{x}_i)}}$$

This is constant for a given instance, and so irrelevant to our prediction

– In practice, we use log-probabilities to prevent underflow

- To classify a new point $\mathbf{x}$,

$$h(\mathbf{x}) = \arg\max_{y_k} \; P(Y = y_k) \prod_{j=1}^{d} P(X_j = \underbrace{x_j}_{j^{\text{th}} \text{ attribute value of } \mathbf{x}} \mid Y = y_k)$$

$$= \arg\max_{y_k} \; \log P(Y = y_k) + \sum_{j=1}^{d} \log P(X_j = x_j \mid Y = y_k)$$

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

❖ The Naïve Bayes Classifier Algorithm

- For each class label y_k
  - Estimate $\mathrm{P}(\mathrm{Y} = y_k)$ from the data
  - For each value $x_{i,j}$ of each attribute X_i
    - Estimate $\mathrm{P}(\mathrm{X}_i = x_{i,j} \mid \mathrm{Y} = y_k)$

- Classify a new point via:

$$h(\mathbf{x}) = \arg\max_{y_k} \log P(Y = y_k) + \sum_{j=1}^{d} \log P(X_j = x_j \mid Y = y_k)$$

- In practice, the independence assumption doesn't often hold true, but Naïve Bayes performs very well despite it

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The Naïve Bayes Classifier

❖ Computing Probabilities (Not Just Predicting Labels)

- NB classifier gives predictions, not probabilities, because we ignore $\mathrm{P}(\mathrm{X})$ (the denominator in Bayes rule)
- Can produce probabilities by:
  - For each possible class label y_k, compute

$$\underbrace{\tilde{P}(Y = y_k \mid X = \mathbf{x})}_{\substack{\text{This is the numerator of Bayes rule, and is} \\ \text{therefore off the true probability by a factor} \\ \text{of } \alpha \text{ that makes probabilities sum to 1}}} = P(Y = y_k) \prod_{j=1}^{d} P(X_j = x_j \mid Y = y_k)$$

  - α is given by $\alpha = \dfrac{1}{\sum_{k=1}^{\#classes} \tilde{P}(Y = y_k \mid X = \mathbf{x})}$

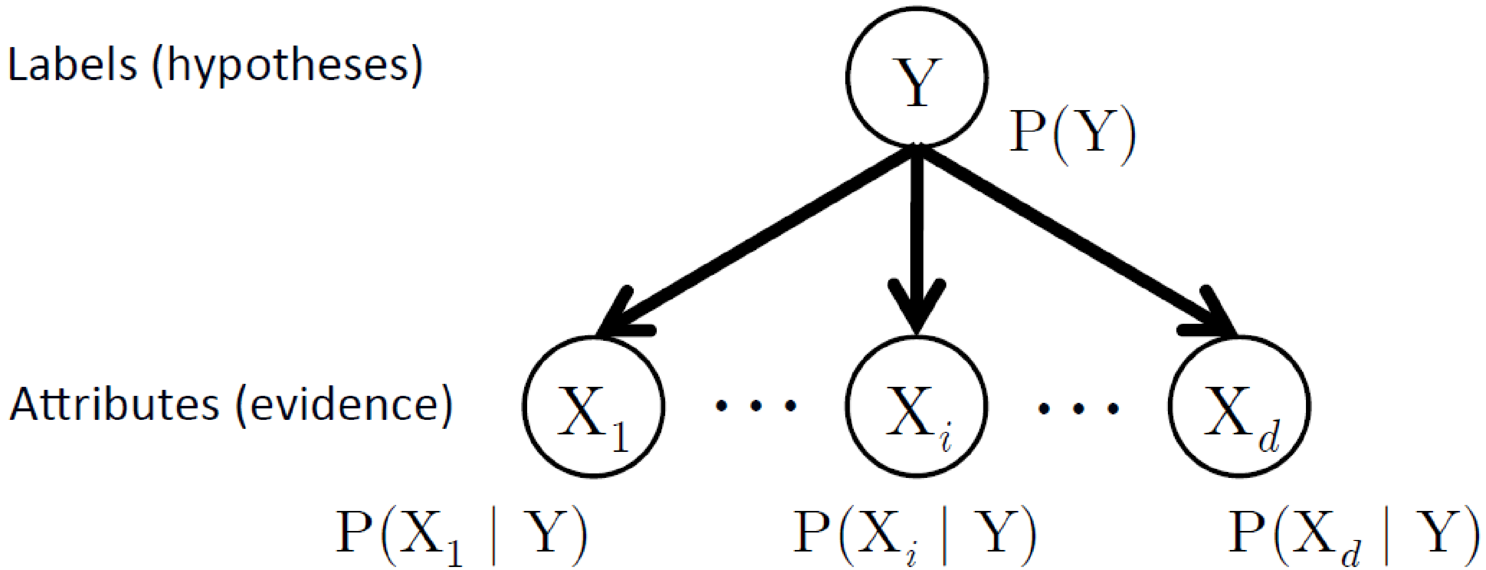
  - Class probability is given by

$$P(Y = y_k \mid X = \mathbf{x}) = \alpha \tilde{P}(Y = y_k \mid X = \mathbf{x})$$

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

❖ The Naïve Bayes Graphical Model



- Nodes denote random variables
- Edges denote dependency
- Each node has an associated conditional probability table (CPT), conditioned upon its parents

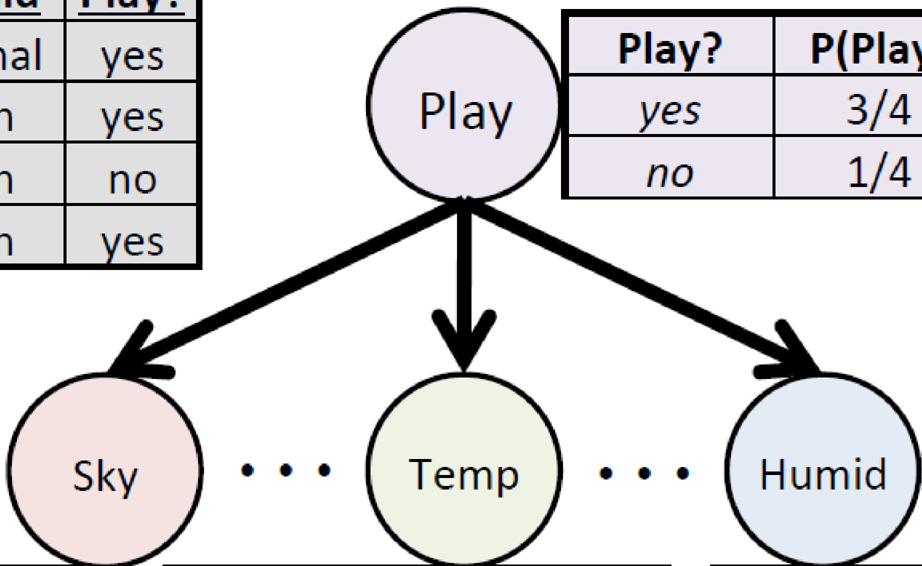
Slide Credit: Eric Eaton

The Naïve Bayes Classifier

- Example NB Graphical Model

Data:

Sky	Temp	Humid	Play?
sunny	warm	normal	yes
sunny	warm	high	yes
rainy	cold	high	no
sunny	warm	high	yes



Play?	P(Play)
yes	3/4
no	1/4

Sky	Play?	P(Sky | Play)
sunny	*yes*	4/5
rainy	*yes*	1/5
sunny	*no*	1/3
rainy	*no*	2/3

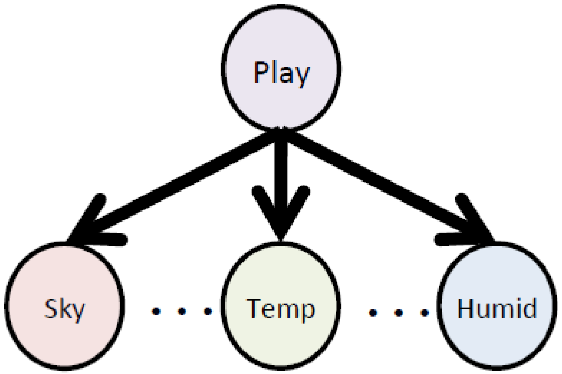
Temp	Play?	P(Temp | Play)
warm	*yes*	4/5
cold	*yes*	1/5
warm	*no*	1/3
cold	*no*	2/3

Humid	Play?	P(Humid | Play)
high	*yes*	3/5
norm	*yes*	2/5
high	*no*	2/3
norm	*no*	1/3

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

❖ Example Using NB for Classification



Play?	P(Play)
yes	3/4
no	1/4

Temp	Play?	P(Temp | Play)
warm	*yes*	4/5
cold	*yes*	1/5
warm	*no*	1/3
cold	*no*	2/3

Sky	Play?	P(Sky | Play)
sunny	*yes*	4/5
rainy	*yes*	1/5
sunny	*no*	1/3
rainy	*no*	2/3

Humid	Play?	P(Humid | Play)
high	*yes*	3/5
norm	*yes*	2/5
high	*no*	2/3
norm	*no*	1/3

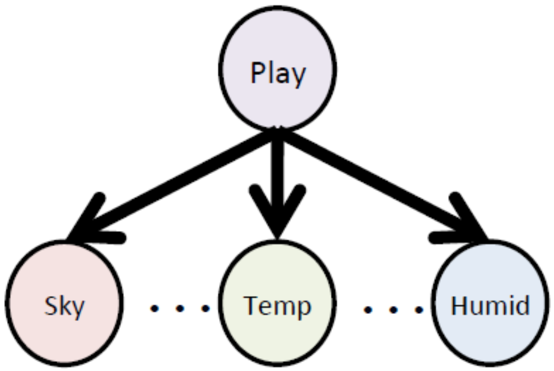
$$h(\mathbf{x}) = \arg\max_{y_k} \ \log P(Y = y_k) + \sum_{j=1}^{d} \log P(X_j = x_j \mid Y = y_k)$$

Goal: Predict label for $\mathbf{x}$ = (rainy, warm, normal)

Slide Credit: Eric Eaton

The Naïve Bayes Classifier

❖ Example Using NB for Classification



Play?	P(Play)
yes	3/4
no	1/4

Temp	Play?	P(Temp | Play)
warm	*yes*	4/5
cold	*yes*	1/5
warm	*no*	1/3
cold	*no*	2/3

Sky	Play?	P(Sky | Play)
sunny	*yes*	4/5
rainy	*yes*	1/5
sunny	*no*	1/3
rainy	*no*	2/3

Humid	Play?	P(Humid | Play)
high	*yes*	3/5
norm	*yes*	2/5
high	*no*	2/3
norm	*no*	1/3

Predict label for:

$\mathbf{x}$ = (rainy, warm, normal)

$$P(\text{play} \mid \mathbf{x}) \propto \log P(\text{play}) + \log P(\text{rainy} \mid \text{play}) + \log P(\text{warm} \mid \text{play}) + \log P(\text{normal} \mid \text{play})$$

$$\propto \log 3/4 + \log 1/5 + \log 4/5 + \log 2/5 = \boxed{-1.319}$$

predict PLAY

$$P(\neg\text{play} \mid \mathbf{x}) \propto \log P(\neg\text{play}) + \log P(\text{rainy} \mid \neg\text{play}) + \log P(\text{warm} \mid \neg\text{play}) + \log P(\text{normal} \mid \neg\text{play})$$

$$\propto \log 1/4 + \log 2/3 + \log 1/3 + \log 1/3 = -1.732$$

Slide Credit: Eric Eaton