Isomorphism is an Equivalence Relation

Proof Walkthrough

Adam Grant

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Isomorphic

The nicest definition of isomorphism I've seen is by Roland Backhouse:

Definition

Object x and y in the same category are said to be *isomorphic* if there are two arrows $f \in x \leftarrow y$ and $g \in y \leftarrow x$ such that $f \cdot g = id_x$ and $g \cdot f = id_y$, i.e. they are each other's inverses. If this is the case we write $x \cong y$ or, if we want to be explicit about the arrows, we write $f \in x \cong y \ni g$.

In this presentation, we will be explicit about the arrows. We'll also make the existential quantification explicit and write:

$$x \cong y \equiv \exists(f, g :: f \in x \cong y \ni g)$$

which will help when writing proofs.

Our task

Show that $\cong$ is an *equivalence relation*.

Equivalence Relation

Definition (equivalence relation)

A relation is an *equivalence relation* if it is *reflexive*, *symmetric*, and *transitive*.

Equivalence Relation

Definition (reflexive)

A relation R is *reflexive* when $\forall(a :: aRa)$.

Definition (symmetric)

A relation R is *symmetric* when $\forall(b, a : bRa : aRb)$.

Definition (transitive)

A relation R is *transitive* when $\forall(c, b, a : cRb \wedge bRa : cRa)$.

Proof obligations

Show that $\cong$ is an Equivalence Relation

- Reflexivity: $\forall(a :: a \cong a)$
- Symmetry: $\forall(b, a : b \cong a : a \cong b)$
- Transitivity: $\forall(c, b, a : c \cong b \land b \cong a : c \cong a)$

Reflexivity: $\forall(a :: a \cong a)$

- $a \cong a$

$\equiv$ Definition of $\cong$

$\exists(f, g :: f \in a \cong a \ni g)$

$\equiv$ Expand abbreviation

$\exists(f, g :: f \cdot g = id_a \wedge g \cdot f = id_a)$

$\Leftarrow$ $\exists$-introduction, with $f, g := id_a, id_a$

$id_a \cdot id_a = id_a \wedge id_a \cdot id_a = id_a$

$\equiv$ $\wedge$-idempotent

$id_a \cdot id_a = id_a$

$\square$

Proof obligations

Show that $\cong$ is an Equivalence Relation

- ✓ Reflexivity: $\forall(a :: a \cong a)$
- Symmetry: $\forall(b, a : b \cong a : a \cong b)$
- Transitivity: $\forall(c, b, a : c \cong b \wedge b \cong a : c \cong a)$

Symmetry: $\forall(b, a : b \cong a : a \cong b)$

- $b \cong a$

$\equiv$ Definition of $\cong$

$\exists(f, g :: f \in b \cong a \ni g)$

$\equiv$ Expand abbreviation

$\exists(f, g :: f \cdot g = id_b \wedge g \cdot f = id_a)$

$\equiv$ $\wedge$-symmetric; Rearrange dummies

$\exists(g, f :: g \cdot f = id_a \wedge f \cdot g = id_b)$

$\equiv$ Abbreviate

$\exists(g, f :: g \in a \cong b \ni f)$

$\equiv$ Definition of $\cong$

$a \cong b$

$\square$

Proof obligations

Show that $\cong$ is an Equivalence Relation

- ✓ Reflexivity: $\forall(a :: a \cong a)$
- ✓ Symmetry: $\forall(b, a : b \cong a : a \cong b)$
- Transitivity: $\forall(c, b, a : c \cong b \wedge b \cong a : c \cong a)$

Transitivity: $\forall(c, b, a : c \cong b \land b \cong a : c \cong a)$

- $b \cong a$

Context

- $c \cong b$
- $b \cong a$

Building Context

Context

- $c \cong b$
- $b \cong a$

+ $\exists(f, g :: f \in c \cong b \ni g)$
+ $\exists(f, g :: f \in b \cong a \ni g)$
+ Define f,g
  $f \in c \cong b \ni g$
  $f \cdot g = id_c \wedge g \cdot f = id_b$
+ Define h,j
  $h \in b \cong a \ni j$
  $h \cdot j = id_b \wedge j \cdot h = id_a$

Building Context

Express id_c in terms of f, g, h, k

- id_c

$=$ Definition of f, g

$f \cdot g$

$=$ Identity

$f \cdot id_b \cdot g$

$=$ Definition of h, j

$f \cdot h \cdot j \cdot g$

$\square$

Building Context

Context

...

- Define f,g

  $f \in c \cong b \ni g$

  $f \cdot g = id_c \wedge g \cdot f = id_b$
- Define h,j

  $h \in b \cong a \ni j$

  $h \cdot j = id_b \wedge j \cdot h = id_a$
- $id_c = f.h.j.k$

Building Context

Express id_a in terms of f, g, h, k

- id_a

$=$ Definition of h, j

$j \cdot h$

$=$ Identity

$j \cdot id_b \cdot h$

$=$ Definition of h, j

$j \cdot g \cdot f \cdot h$

$\square$

Building Context

Context

$\ldots$

- $+ \; id_c = f \cdot h \cdot j \cdot g$
- $+ \; id_a = j \cdot g \cdot f \cdot h$

Transitivity: $\forall(c, b, a : c \cong b \wedge b \cong a : c \cong a)$

- $b \cong a$

$\equiv$ Definition of $\cong$

$\exists(f, g :: f \in c \cong a \ni g)$

$\equiv$ Expand abbreviation

$\exists(f, g :: f \cdot g = id_c \wedge g \cdot f = id_a)$

$\Leftarrow$ $\exists$-introduction

$(f \cdot g = id_c \wedge g \cdot f = id_a)[f, g := f \cdot h, j \cdot g]$

$\equiv$ Textual substitution

$(f \cdot h) \cdot (j \cdot g) = id_c \wedge (j \cdot g) \cdot (f \cdot h) = id_a$

$\equiv$ Context

true

$\square$

Proof obligations

Show that $\cong$ is an Equivalence Relation

- ✓ Reflexivity: $\forall(a :: a \cong a)$
- ✓ Symmetry: $\forall(b, a : b \cong a : a \cong b)$
- ✓ Transitivity: $\forall(c, b, a : c \cong b \wedge b \cong a : c \cong a)$