# Linear Regression with Multiple Variables

## Multivariate Linear Regression

- Multiple features (variables)
  - $n$ = number of features
  - $x^{(i)}$ = input (features) of $i^{th}$ training example.
  - $x_j^{(i)}$ = value of feature $j$ in $i^{th}$ training example.
- Hypotesis
  - Previously: $h_\theta(x) = \theta_0 + \theta_1 x$
  - $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$
    - For convenience of notation, define $x_0 = 1$
    - $x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}, h_\theta(x) = \theta^T x$

## Gradient Descent for Multiple Variables

- Hypothesis: $h_\theta(x) = \theta^T x = \theta_0 x_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$
- Parameters: $\theta_0, \theta_1, \ldots, \theta_n$
- Cost function: $J(\theta_0, \theta_1, \ldots, \theta_n) = \frac{1}{2m} \sum_{i=1}^m \left( h_\theta(x^{(i)}) - y^{(i)} \right)^2$
  - or $J(\theta) = \frac{1}{2m} \sum_{i=1}^m (\theta^T x^{(i)} - y^{(i)})^2$
- Gradient descent

  > repeat {
  >
  > $\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \ldots, \theta_n)$
  >
  > (simultaneously update for every $j = 0, \ldots, n$)
  >
  > }

  or

  > repeat {
  >
  > $\theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^m \left( h_\theta(x^{(i)}) - y^{(i)} \right) x_j^{(i)}$
  >
  > (simultaneously update for every $j = 0, \ldots, n$)
  >
  > }

# Gradient Descent in Practice

## Feature Normalization

Idea: Make sure featueres are on a similar scale.

- We can speed up gradient descent by having each of our input values in roughly the same range. This is because $\theta$ will descend quickly on small ranges and slowly on large ranges, and so will oscillate inefficiently down to the optimum when the variables are very uneven.
- The way to prevent this is to modify the ranges of our input variables so that they are all roughly the same.

### Example

- $x_1$ = size (0-2000 $feet^2$)
- $x_2$ = number of bedrooms (1-5)

$x_1$ has a much larger range of values than $x_2$. So the $J(\theta_1, \theta_2)$ can be a very very skewed elliptical shape. And if you run gradient descents on this cost function, your gradients may end up taking a long time and can oscillate back and forth and take a long time before it can finally find its way to the global minimum.

### Feature scaling

Get every feature into approcimately $-1 \leq x_i \leq 1$ range.

- Feature scaling involves dividing the input values by the range (i.e. the maximum value minus the minimum value) of the input variable, resulting in a new range of just 1.
- These aren't exact requirements; we are only trying to speed things up.
- $-3 \leq x_i \leq 3$ or $-\frac{1}{3} \leq x_i \leq \frac{1}{3}$ just is fine.

### Mean normalization

Replace $x_i$ with $x_i - \mu_i$ to make features have approximately zero mean (Do not apply to $x_0 = 1$)

- E.g. $x_1 = \frac{size-1000}{2000}, x_2 = \frac{bedrooms-2}{5}$
- $x_i = \frac{x_i - \mu_i}{s_i}$
    - $\mu_i$ is the average value of $x_i$ in training set.
    - $s_i$ is the range ($x_{imax} - x_{imin}$) or standard deviation ($\sigma$)

## Learning Rate

- "Debugging": **How to make sure gradient descent is working correctly**
    - Make a plot with *number of iterations* on the x-axis. Now plot the cost function, $J(\theta)$ over the number of iterations of gradient descent.
        - For sufficient small $\alpha$, $J(\theta)$ should decreases on every iteration.
        - But if $\alpha$ is too small, gradient descent can be slow to converge.

    - If $J(\theta)$ ever increases, then you probably need to use smaller $\alpha$.
  - Example automatic convergence test
    - Declare convergence if $J(\theta)$ decreases by less than $\epsilon$ (e.g., $10^{-3}$) in one iteration.
- **How to choose learing rate $\alpha$**
  - So just try running gradient descent with a range of values for $\alpha$, like 0.001 and 0.01. And for these different values of $\alpha$ are just plot $J(\theta)$ as a function of number of iterations, and then pick the value of $\alpha$ that seems to be causing $J(\theta)$to decrease rapidly.
  - Andrew Ng recommends decreasing $\alpha$ by multiples of 3. And then try to pick the largest possible value, or just something slightly smaller than the largest reasonable value.
  - E.g. $\dots, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1, \dots$

## Features and Polynomial Regression

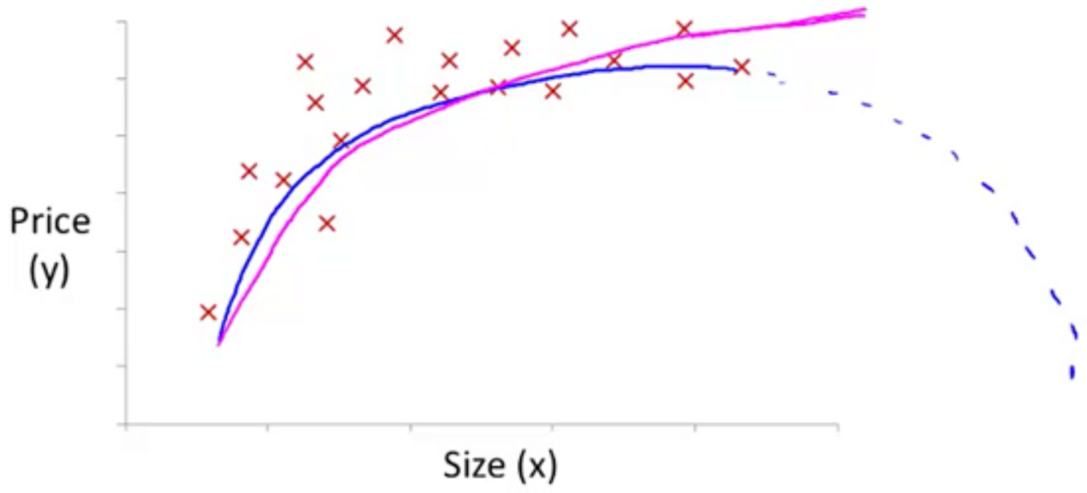
### Choice of features

- We can improve our features and the form of our hypothesis function in a couple different ways.
- We can **combine** multiple features into one. For example, we can combine $x_1$ and $x_2$ into a new feature $x_3$ by taking $x_1 \cdot x_2$. (E.g. $HouseArea = Frontage \times Depth$)

### Polynomial Regression

Our hypothesis function need not be linear (a straight line) if that does not fit the data well.

- We can **change the behavior or curve** of our hypothesis function by making it a quadratic, cubic or square root function (or any other form).
  - For example, if our hypothesis function is $h_\theta(x) = \theta_0 + \theta_1 x_1$ then we can create additional features based on $x_1$, to get the quadratic function $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2$ or the cubic function $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3$.
  - In the cubic version, we have created new features $x_2$ and $x_3$ where $x_2 = x_1^2$ and $x_3 = x_1^3$.
  - To make it a square root function, we could do: $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 \sqrt{x_1}$
  - Note that at 2:52 and through 6:22 in the "Features and Polynomial Regression" video, the curve that Prof Ng discusses about "doesn't ever come back down" is in reference to the hypothesis function that uses the `sqrt()` function (shown by the solid purple line), not the one that uses $size^2$ (shown with the dotted blue line). The quadratic form of the hypothesis function would have the shape shown with the blue dotted line if $\theta_2$ was negative.



- One important thing to keep in mind is, if you choose your features this way then **feature scaling becomes very important**.
    - E.g. if $x_1$ has range $1 - 1000$ then range of $x_1^2$ becomes $1 - 1000000$ and that of $x_1^3$ becomes $1 - 1000000000$
    - So you should scale $x_1$ before using polynomial regression.

## Computing Parameters Analytically

### Normal Equation

The "Normal Equation" (正规方程) is a method of finding the optimum $\theta$ **without iteration.**

> There is **no need** to do feature scaling with the normal equation.

#### Intuition

- $\theta \in R^{n+1}, J(\theta_0, \theta_1, \ldots, \theta_m) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$
- Set $\frac{\partial}{\partial \theta_j} J(\theta) = \cdots = 0$ (for every $j$), solve for $\theta_0, \theta_1, \ldots, \theta_m$

#### Method

We have $m$ examples $(x^{(1)}, y^{(1)}), \ldots, (x^{(m)}, y^{(m)})$ and $n$ features. (Note that $x_0^{(i)} = 0$)

$$x^{(i)} = \begin{bmatrix} x_0^{(i)} \\ x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_n^{(i)} \end{bmatrix}$$

And construct the $m \times (n+1)$ matrix $X$

$$X = \begin{bmatrix} (x^{(1)})^T \\ (x^{(2)})^T \\ \vdots \\ (x^{(m)})^T \end{bmatrix}$$

And the $m$-dimension vector $y$

$$y = \begin{bmatrix} y^{(i)} \\ y^{(i)} \\ y^{(i)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

Finally, we can get

$$\theta = (X^T X)^{-1} X^T y$$

## Example

Suppose you have the training in the table below:

| age ($x_1$) | height in cm ($x_2$) | weight in kg ($y$) |
|---|---|---|
| 4 | 89 | 16 |
| 9 | 124 | 28 |
| 5 | 103 | 20 |

You would like to predict a child's weight as a function of his age and height with the model

$$weight = \theta_0 + \theta_1 age + \theta_2 height$$

Then you can construct $X$ and $y$

$$X = \begin{bmatrix} 1 & 4 & 89 \\ 1 & 9 & 124 \\ 1 & 5 & 103 \end{bmatrix}$$

$$Y = \begin{bmatrix} 16 \\ 28 \\ 20 \end{bmatrix}$$

## Usage in Octave

```
pinv (X'*X)*X'*y
```

### Comparison of gradient descent and the normal equation

$m$ training examples and $n$ features.

| Gradient Descent | Normal Equation |
|---|---|
| Need to choose $\alpha$ | No need to choose $\alpha$ |
| Needs many iterations | No need to iterate |
| $O(kn^2)$ | $O(n^3)$, need to calculate $(X^TX)^{-1}$ |
| Works well when $n$ is large | Slow if $n$ is very large |

With the normal equation, computing the inversion has complexity $O(n^3)$. So if we have a very large number of features, the normal equation will be slow. In practice, **when $n$ exceeds 10,000 it might be a good time to go from a normal solution to an iterative process.**

## Normal Equation Noninvertibility

$$\theta = (X^TX)^{-1}X^Ty$$

- What if $X^TX$ is non-invertible (不可逆的) ? (singular/ degenerate)
- Octave: `pinv(X'*X)*X"*y`
    - There's two functions in Octave for inverting matrices, `pinv` (pseudo-inverse, 伪逆) and `inv` (inverse).
    - As long as you use the `pinv` function then this will actually compute the value of data that you want even if X transpose X is non-invertible.
    - So when implementing the normal equation in octave we want to use the `pinv` function rather than `inv`.
- $X^TX$ may be **noninvertible**. The common causes are:
    - Redundant features, where two features are very closely related (i.e. they are linearly dependent)
        - E.g. $x_1$ = size in $feet^2$, and $x_2$ = size in $m^2$. So you'll always have $x_1 = (3.28)^2x_2$
    - Too many features (e.g. $m \leq n$).
        - In this case, delete some features or use "regularization".

# Octave/Matlab Tutorial

## Basic Operations

- Print specific decimals: `disp(sprintf('6 decimals: %0.6f', a)) // 6 decimals:`

`3.141593`

- `v = 1:0.2:2 // [1.0 1.2 1.4 1.6 1.8 2.0]`
- `ones`, `zeros`, `rand`, `randn` (生成正态分布的随机数矩阵), `eye` (生成单位矩阵)
- `hist` (直方图，第二个参数课自定义条数)
- `size` (返回矩阵的行数与列数 [m n] )
- `length` (返回向量的维数)

## Moving Data Around

- Use `load` to load data set.

  ```
  load featureX.dat
  load('priceY.dat')
  ```

- Use `who` to show all variables in Octave workspace
  - `whos` for detail information
  - `clear` to delete a variable

    ```
    clear featureX
    ```

- Get first ten elements of a matrix

  ```
  v = priceY(1:10)
  ```

- Use `save` to save your variable

  ```
  save hello.mat v
  ```

  - By default the data is saved in binary. You can save it to ASCII by

    ```
    save hello.txt v -ascii
    ```

- Use `A(3, 2)` to get $A_{32}$, or `A(2, :)` to get every element along the second row
  - `A([1, 3], :)` to get everything in the first and third rows
  - `A(:, 2) = [10; 11; 12]` to change the value of elements in second column.
  - `A = [A, [100; 101; 102]]` to append another column vector to right
  - `A(:)` to put all elements of $A$ into a single vector

## Computing on Data

- Use `max` to get the largest element in a vector

```
a = [1 15 2 0.5];
[val, ind] = max(a); // val = 15, ind = 2
```

  - If you do `max(A)`, where $A$ is a matrix, what this does is this actually does the column wise maximum.

    ```
    A = [1 2; 3 4; 5 6];
    max(A) // [5 6]

    A = [8 1 6; 3 5 7; 4 9 2];
    max(A, [], 1) // [8 9 7] (get the column wise maximum)
    max(A, [], 2) // [8 7 9] (get the row wise maximum)
    max(max(A)) // 9
    max(A(:)) // 9
    ```

- `a < 3` does the element wise operation, you'll get `[1 0 1 1]`
  - `find(a<3)` gets `[1 3 4]`
- `magic(3)` gets a 3x3 magic matrix
- `sum`, `prod`, `floor`, `ceil`, `flipud`

## Plotting Data

- `plot`
- `hold on`, `figure`, `subplot`
- `xlabel`, `ylabel`, `legend`, `title`, `axis`
- `print -dpng 'myPlot.png'`
- `imagesc(A)` to visualize a matrix
  - `imagesc(A), colorer, colormap gray` to be in gray scale.

## Control Statements: for, while, if statement

## Vectorization

Vectorization is the process of taking code that relies on **loops** and converting it into **matrix operations**. It is more efficient, more elegant, and more concise.

As an example, let's compute our prediction from a hypothesis. Theta is the vector of fields for the hypothesis and x is a vector of variables.

With loops ($h_\theta(x) = \theta_0 x_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$):

```
prediction = 0.0;
for j = 1:n+1,
  prediction += theta(j) * x(j);
end;
```

With vectorization ($h_\theta(x) = \theta^T x$):

```
prediction = theta' * x;
```