Quantum bits

Quantum computing

G. Chênevert

January 6, 2023

Motivation

The basic unit of information in classical computing is a **bit**

that can take two symbolic values: 0 and 1

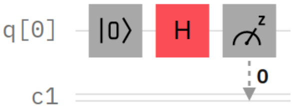
Quantum computing works with **quantum bits** (or **qubits**)

that can be simultaneously 0 and 1 — in various proportions!

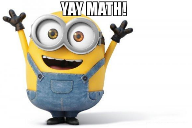
By the end of today we'll be storing information on qubits and running programs on quantum computers

Spoiler warning

This is what our first quantum computer program will look like:



To make sense of what it does, we first need to review some concepts and formalism from Quantum Mechanics



Quantum bits

Quantum systems

Dirac formalism

Quantum bits

Recall: Wave function

A quantum system can be described by a (complex-valued) **wave function** $\Psi(\mathbf{x}, t)$

satisfying **Schrödinger's equation**:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi$$

where

- $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator,
- $V(\mathbf{x}, t)$ the potential function representing the environment.

Stationary states

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\,\Psi$$

Let's assume that the potential $V = V(\mathbf{x})$ is independent of t and look for *separable solutions* of the form

$$\Psi(\mathbf{x}, t) = \chi(t)\,\phi(\mathbf{x}).$$

The equation becomes:
$$i\hbar \frac{\partial \chi}{\partial t}\,\phi = \chi\left(-\frac{\hbar^2}{2m}\Delta\phi + V\phi\right)$$

or

$$\frac{i\hbar}{\chi}\frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\Delta\phi}{\phi} + V.$$

Separable solutions

$$\frac{i\hbar}{\chi}\frac{\partial \chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\Delta\phi}{\phi} + V = \text{constant} =: E$$

reduces to

$$\begin{cases} \dfrac{\partial \chi}{\partial t} = \dfrac{\chi E}{i\hbar} = -\dfrac{iE}{\hbar}\chi \\ -\dfrac{\hbar^2}{2m}\Delta\phi + V\phi = E\,\phi \end{cases}$$

$$\Longrightarrow \begin{cases} \chi(t) = A\, e^{-\frac{iE}{\hbar}t} \quad \text{and} \\ \widehat{H}\,\phi = E\,\phi \qquad \text{where} \qquad \widehat{H} = -\dfrac{\hbar^2}{2m}\Delta + V \end{cases}$$

Quantization

Given boundary conditions on $\phi(\mathbf{x})$, the reduced Hamiltonian operator $\widehat{H}$ only has countably many (real) eigenvalues:

$$E_1 \le E_2 \le \cdots \le E_n \le \cdots,$$

corresponding to countably many eigenfunctions:

$$\phi_1, \quad \phi_2, \quad \ldots \quad \phi_n, \quad \ldots$$

hence we get countably many separable solutions

$$\Psi_n(\mathbf{x}, t) = A_n \, e^{-\frac{iE_n}{\hbar} t} \, \phi_n(\mathbf{x}).$$

Quantum states

In general, the state of a quantum system can be written as a linear combination

$$\Psi(\mathbf{x}, t) = \sum_n A_n \, e^{-i \frac{E_n}{\hbar} t} \, \phi_n(\mathbf{x})$$

where the ϕ_n are eigenfunctions for the reduced Hamiltonian operator:

$$\widehat{H} \, \phi_n = E_n \, \phi_n.$$

These eigenstates are orthogonal with respect to the Hermitian product

$$\langle \phi \,|\, \psi \rangle = \int \phi(\mathbf{x})^* \, \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

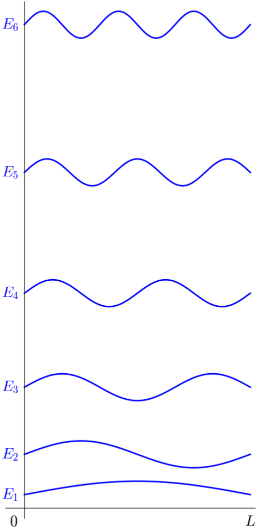
Remember the 1-D infinite potential well

$$V(x) = \begin{cases} 0 & 0 \leq x \leq L \\ +\infty & \text{elsewhere} \end{cases}$$

$$\Longrightarrow \phi_n(x) = \sin \frac{n\pi x}{L}$$

with energy

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}$$



Quantum bits

Quantum systems

Dirac formalism

Quantum bits

Braket notation

The instantaneous states $\phi(\mathbf{x}) = \Psi(\mathbf{x}, t_0)$ form a vector space $\mathcal{V}$ spanned by the ϕ_n:

$$\phi(\mathbf{x}) = \sum_n \alpha_n \, \phi_n(\mathbf{x}) \qquad \text{with} \qquad \alpha_n \in \mathbb{C}.$$

Hermitian product: if the ϕ_n are **normalized** ($\|\phi_n\| = \sqrt{\langle \phi_n | \phi_n \rangle} = 1$) then for

$$\phi = \sum_n \alpha_n \, \phi_n, \qquad \psi = \sum_n \beta_n \, \phi_n,$$

we have

$$\langle \phi \,|\, \psi \rangle = \sum_n \alpha_n^* \, \beta_n = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \end{bmatrix}^* \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix} = |\phi\rangle^\dagger \, |\psi\rangle$$

Measurement

When we measure a **mixed state**

$$|\phi\rangle = \sum_n \alpha_n |\phi_n\rangle \in \mathcal{V} \setminus \{\mathbf{0}\} :$$

it gets projected on the **pure state** $|\phi_n\rangle$ with energy E_n with probability

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |\phi_n\rangle\,\big] = \frac{|\langle \phi \mid \phi_n \rangle|^2}{\|\phi\|^2} = \frac{|\alpha_n|^2}{\|\phi\|^2}.$$

If $|\phi\rangle$ is normalized, this is just

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |\phi_n\rangle\,\big] = |\langle \phi \mid \phi_n \rangle|^2 = |\alpha_n|^2.$$

Exercise

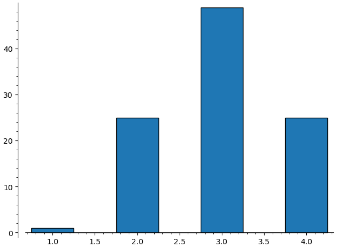
We measure the mixed quantum state

$$|\phi\rangle = |\phi_1\rangle + (3+4i)|\phi_2\rangle + 7|\phi_3\rangle + 5i|\phi_4\rangle.$$

What do we expect to see ?

Answer:

$$\mathbb{P}\left[\,\mathcal{M}|\phi\rangle = |\phi_n\rangle\,\right] = \begin{cases} 1\% & n=1 \\ 25\% & n=2 \\ 49\% & n=3 \\ 25\% & n=4 \end{cases}$$



Quantum bits

Quantum systems

Dirac formalism

Quantum bits

Computational quantum systems

N-**level** quantum system: when $\dim_{\mathbb{C}} \mathcal{V} = N$.

Basis of pure (eigen) states $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_N\rangle$.

Computational basis : to simplify notation let us write

$$|n\rangle := |\phi_{n+1}\rangle \qquad (0 \leq n < N)$$

and $\mathcal{V}_N$ for the standard N-level state space with pure states

$$|0\rangle, |1\rangle, \ldots, |N-1\rangle.$$

$N = 2$: Quantum bits (or qubits)

The state of a qubit can be thought of as a nonzero linear combination

$$|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad \alpha, \beta \in \mathbb{C}.$$

When we measure it:

$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |0\rangle\,\big] = \frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2}, \qquad \mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |1\rangle\,\big] = \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}.$$

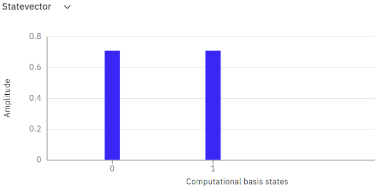
For a normalized state, $|\alpha|^2 + |\beta|^2 = 1$ so this is just

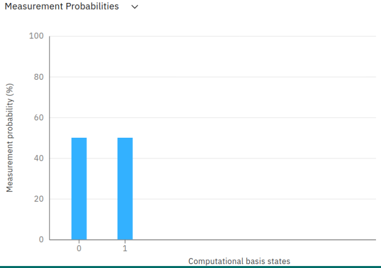
$$\mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |0\rangle\,\big] = |\alpha|^2, \qquad \mathbb{P}\big[\,\mathcal{M}|\phi\rangle = |1\rangle\,\big] = |\beta|^2.$$

Example

$$|\phi\rangle = |0\rangle + |1\rangle, \qquad |\tilde{\phi}\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \text{ normalized}$$

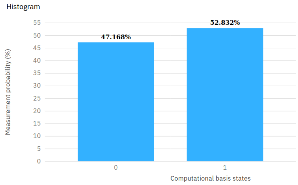
$$\mathbb{P}\left[\,\mathcal{M}|\phi\rangle = |0\rangle\,\right] = \mathbb{P}\left[\,\mathcal{M}|\phi\rangle = |1\rangle\,\right] = \frac{1}{2}$$



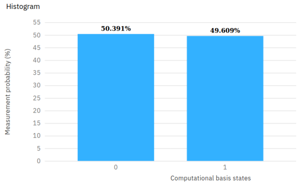


IBM Q Experience results

Result of 1024 *simulations*:



Result of 1024 *executions* on `ibmq_armonk` (a real physical qubit):



For next time

Start messing around with qubits yourself by creating an account on

IBM Q Experience

https://quantum-computing.ibm.com/

Suggestion:

yields $|\phi\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$