Concordia University
Department of Mechanical, Industrial and Aerospace Engineering

Aeroelasticity MECH 6481 - Fall 2024

Handout #1 - Eigenvalue analysis

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EIGENVALUE PROBLEM

When using the p method for a system with a high number of degrees of freedom (for an elastic wing, for example), it is often preferred to obtain the "eigenvalues" of the system, for stability analysis, instead of directly finding the roots of the determinant of the matrix of coefficients.
Assume a linear transformation expressed in the matrix form as $\boldsymbol{Au}$, where $\boldsymbol{A}$ may be considered as the transformation matrix and $\boldsymbol{u}$ as a vector. If $\boldsymbol{u}$ is an "eigenvector," then the linear transformation becomes: $\boldsymbol{Au} = \lambda\boldsymbol{u}$, meaning that $\boldsymbol{u}$ is only scaled/stretched by λ which is called the "eigenvalue."
For a dynamical system described as $\dot{\boldsymbol{X}} = \boldsymbol{AX}$, where $\boldsymbol{A}$ is the coefficient matrix and $\boldsymbol{X}$ is the vector of unknowns, one can assume the solution to be in the form of $\boldsymbol{X} = \overline{\boldsymbol{X}}e^{\lambda t}$ and thus the governing equation becomes

$$\lambda\overline{\boldsymbol{X}} = \boldsymbol{A}\overline{\boldsymbol{X}}, \tag{1}$$

which is an eigenvalue problem with λ being an eigenvalue and $\overline{\boldsymbol{X}}$ the corresponding eigenvector.
In MATLAB, `[V,D]=eig(A)` returns diagonal matrix `D` of eigenvalues and full matrix V whose columns are the corresponding right eigenvectors, so that `A*V=V*D`.
For a dynamical system described in the implicit form as $\boldsymbol{B}\dot{\boldsymbol{X}} = \boldsymbol{AX}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are coefficient matrices and $\boldsymbol{X}$ is the vector of unknowns, eigenvalues and eigenvectors are found using `eig` function as follows:
`[V,D]=eig(A,B)` returns diagonal matrix `D` of generalized eignevalues and full matrix `V` whose columns are the corresponding right eigenvectors, so that `A*V=B*V*D`.

State-space or first-order form

The form $\boldsymbol{B}\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X}$ is called the **state-space** or first-order form. For a second-order system like $\boldsymbol{M}\ddot{\boldsymbol{X}} + \boldsymbol{C}\dot{\boldsymbol{X}} + \boldsymbol{K}\boldsymbol{X} = \boldsymbol{0}$, where $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ are the mass, damping and stiffness matrices, respectively, the state-space form is obtained considering a new variable $\boldsymbol{Y} = \dot{\boldsymbol{X}}$; thus,

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{Y}} \\ \dot{\boldsymbol{X}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{K} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix},$$

where $\boldsymbol{I}$ is the identity matrix (i.e. all the elements are zero except the diagonal elements which are 1).
The vector of unknowns can then be written as $\boldsymbol{Q} = \{\boldsymbol{Y}\ \boldsymbol{X}\}^{\mathrm{T}}$, and thus the above equation may be re-written as

$$\boldsymbol{B}\dot{\boldsymbol{Q}} - \boldsymbol{A}\boldsymbol{Q} = \boldsymbol{0},$$

where

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \boldsymbol{A} = -\begin{bmatrix} \boldsymbol{0} & \boldsymbol{K} \\ -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix}.$$

Consider, for example, the equation of motion for a pitching-plunging typical section model with the steady flow theory assumption:

$$\begin{bmatrix} m & mbx_\theta \\ mbx_\theta & I_P \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_h & 2\pi\rho_\infty b\ell U^2 \\ 0 & k_\theta - 2\pi(\frac{1}{2} + a)\rho_\infty b^2\ell U^2 \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Considering the dimensionless time τ as $\tau = Ut/b$, the above equation can be rendered dimensionless as

$$\begin{bmatrix} 1 & x_\theta \\ x_\theta & r^2 \end{bmatrix} \begin{Bmatrix} \ddot{\hat{h}} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \frac{\sigma^2}{V^2} & \frac{2}{\mu} \\ 0 & \frac{r^2}{V^2} - \frac{2}{\mu}(\frac{1}{2} + a) \end{bmatrix} \begin{Bmatrix} \hat{h} \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

where the first matrix is $\boldsymbol{M}$ and the second one is $\boldsymbol{K}$; also, $\boldsymbol{X} = \{\hat{h}\ \theta\}^{\mathrm{T}}$; the overdot means the time derivative with respect to the dimensionless time τ.
The above equation may be expressed in the state-space form as

$$\begin{bmatrix} 1 & x_\theta & 0 & 0 \\ x_\theta & r^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{Q}_3 \\ \dot{Q}_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & \frac{\sigma^2}{V^2} & \frac{2}{\mu} \\ 0 & 0 & 0 & \frac{r^2}{V^2} - \frac{2}{\mu}(\frac{1}{2} + a) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$

where the first matrix is the $\boldsymbol{B}$ matrix while the second one is the $-\boldsymbol{A}$ matrix; also, $Q_1 = \dot{\hat{h}}$, $Q_2 = \dot{\theta}$, $Q_3 = \hat{h}$, and $Q_4 = \theta$.