Session 1
Constraint Programming: Problems and Models

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with contributions by Philippe Codognet

Course topics and structure

Problems and Models

Constraint Modeling and Programming

Global Constraints

Tools for Constraint Programming

Incomplete methods and Metaheuristics

Session pattern

Mixed: lectures, theory, exercises

Mixed: tutorials on tools and applications to problem-solving

Slides & other stuff (GDrive):

bit.ly/3GvB4sp

Some Context

Declarative programming

Express relations, not actions

Kowalski's equation:

Algorithms = Logic + Control

Argues that the logic describes **what** must be true and the control *may* be used to add information as to **how** to reach that.

Declarative programming

First-order logic, resolution, unification (Herbrand terms), some control hacks

Prolog

Issues with unification, free (unbound) variables and negation (as failure). Ways of dealing with it:

- make sure all variables are ground (not variable anymore)
- delay goals until variables are sufficiently specified

One step further would be to let variables “carry the disjunction”

CLP(*)

Declarative programming

By keeping just:

- The variables,
- Their domains and
- The relations that bind them (the constraints)

We get the topic of this course: **constraint programming**.

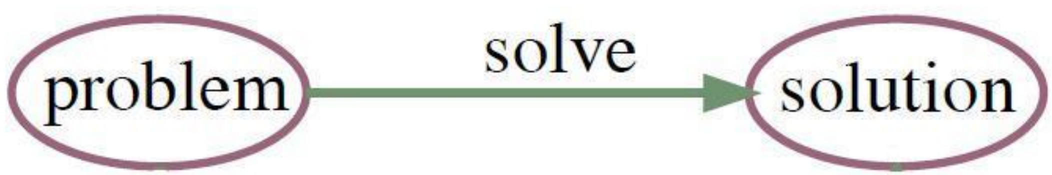
Problems and Models

Problems in the real world

A problem situation may be described by:

- A set of **general rules** (world axioms)
- Entities that represent the **state** of the system
- Specific **limits** on those entities
- Things that **must hold** true, involving the same entities and possibly some external ones

Ideally:

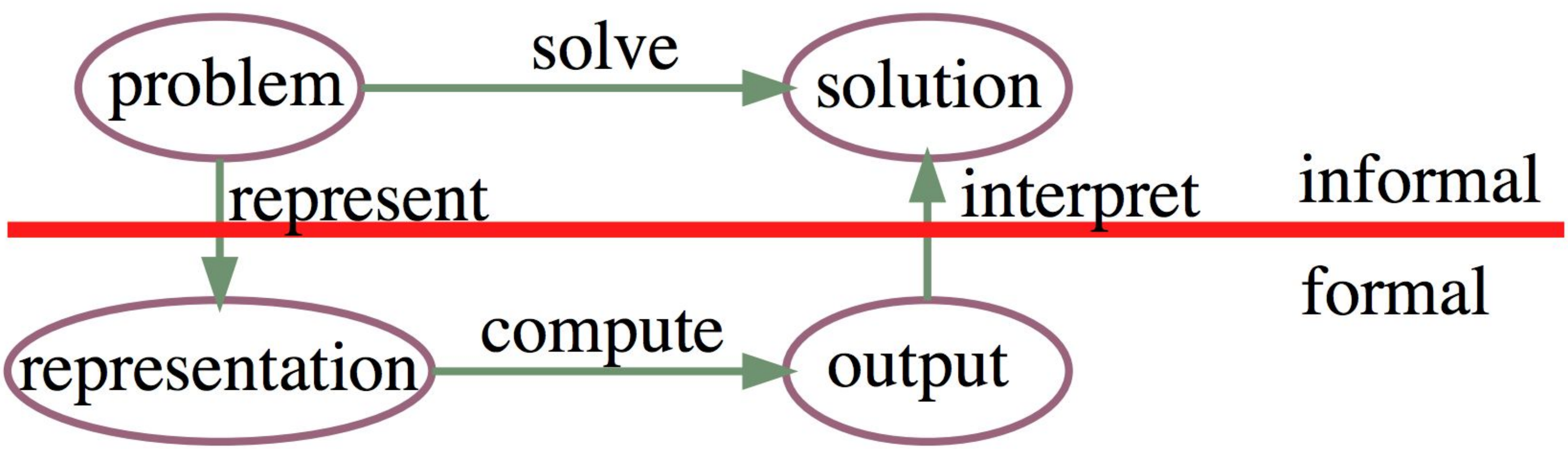


Models

Emerge after consideration of **multiple** concrete problem cases

Allow for solving problems in an **abstract** fashion

In practice



[from Poole & Mackworth 2010]

Take a real, concrete, problem situation

Let's pack a bunch of **boxes...** into a container, like this van:

Consider another -apparently similar- problem

A different set of boxes...

...into another van

After a few cases (instances)

We see **patterns** emerge (insights)

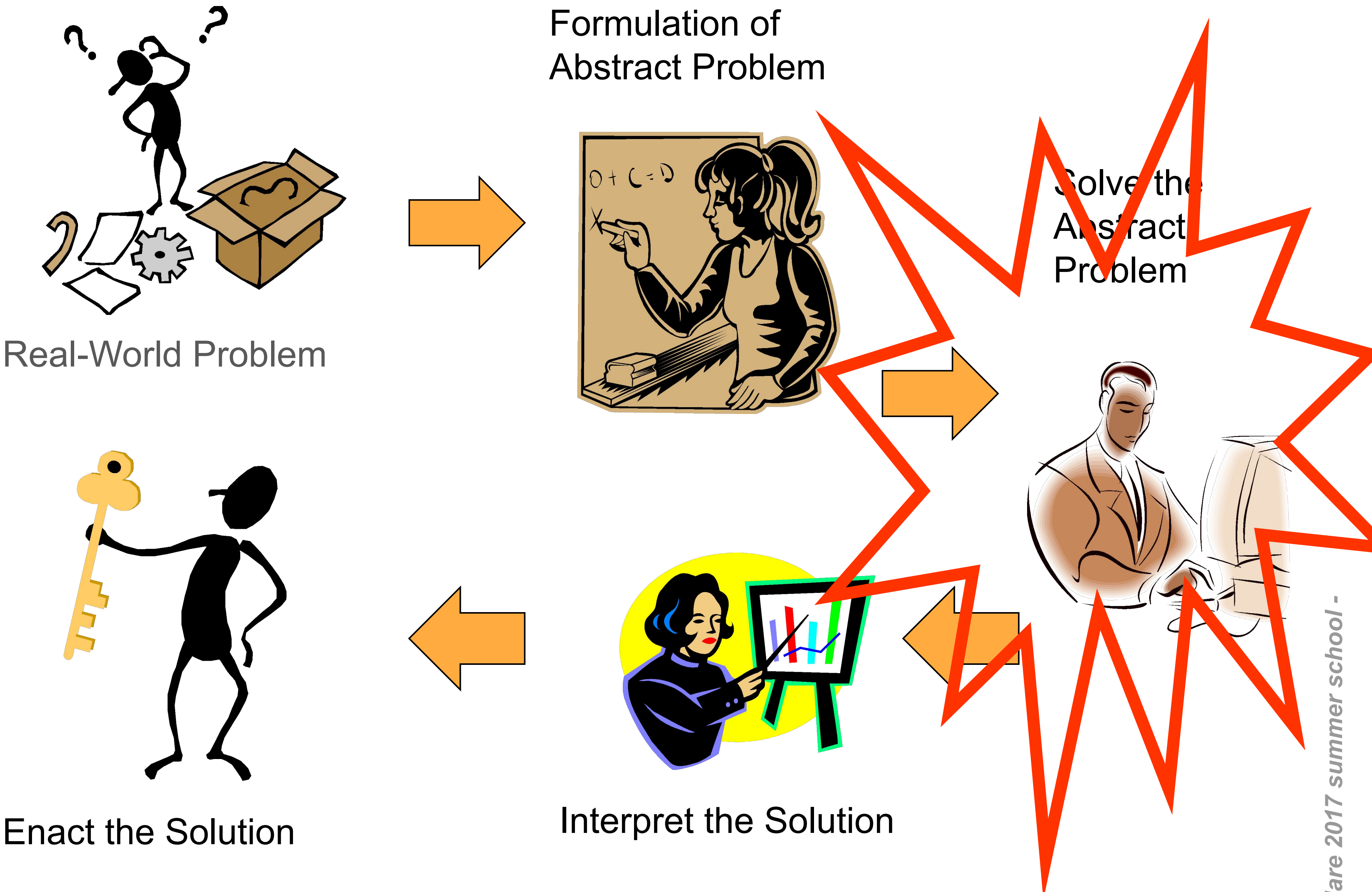
They will guide the abstract problem-solving process, i.e. the design of a **model**

For example,

- No two distinct boxes may occupy the same location
- All boxes must be placed inside the van
- ...

This leads us to a model for the problem, e.g. as a ***set of variables*** and a ***system of relations*** among the said variables.

Conceptually close to *systems of equations*, only it's more general.



[From A. Løkketangen]

Problem Representation

We want a representation to be:

- **Rich** enough to express the knowledge needed to solve the problem.
- As **close** as possible **to the problem**: compact, natural and maintainable.
- Amenable to **efficient** computation:
  - Able to express features of the problem that can be exploited for computational gain
  - Able to trade off accuracy and computation time or space
- Able to be acquired from people, data and past **experiences**.

[from Poole & Mackworth 2010]

Modeling

- We have to represent (model) the problem
  … in a modeling language
- and to have a notion of “solution”
  by reduction / simplification of the problem
- may use the whole mathematics toolbox:
  Logic, polynomial equations, differential equations, …
- key: model must be (efficiently) executable by a computer
- Modeling Language or Modeling Paradigm with associated computation algorithm(s)

Modeling

We may be aware that some forms are better than others.

- There is no unique formulation
- Some models may incorporate expert knowledge
- Some models may be more appropriate for describing (and solving) particular problems, but not others
- Sometimes it may be useful to combine different, otherwise sufficient, models (hybrid models)

What is a Solution?

- Formula to be satisfied or set of conditions to be achieved
- Issue: Unique solution? Several solutions?
- Issue: Some solutions are better than others?
    - Do we care?
    - Optimal solution?
- Issue: Sometimes too hard to find
    - Seek approximate solution
- Issue: Solution quality vs. time spent looking for it:
    - Anytime algorithms

Crypto-arithmetic:

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline = & M & O & N & E & Y \end{array}$$

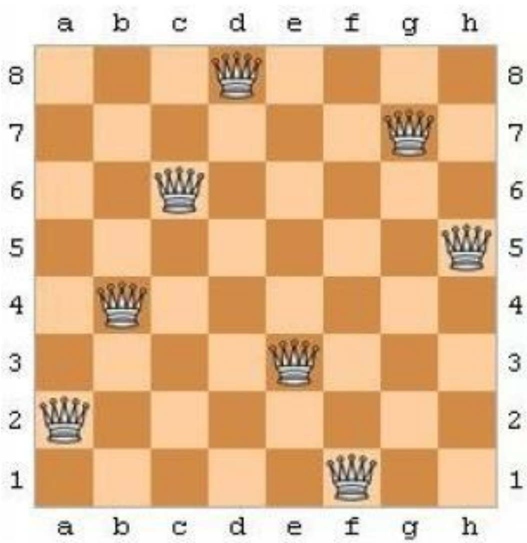
Mathematical and Logic Puzzles

boolean formulas (SAT)

Magic squares:

"Melencolia", Albrecht Dürer, 1514

N-Queens



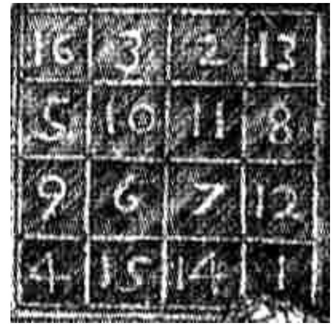
Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

1μs/config: 10^{-6}s; 10^{13}s

Take Dürer's magic square:

16^{16} combinations $>10^{19}$…



Benjamin Franklin's Magic Square

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

52	73	7	64	21	15	35	98	99	44
91	58	25	6	66	19	41	79	84	43
31	60	62	11	5	26	29	68	36	74
100	40	2	3	20	61	65	86	24	88
4	38	14	76	87	71	16	80	53	97
34	22	85	89	82	18	77	69	47	56
8	9	57	67	50	78	42	10	96	70
90	1	13	39	46	33	81	49	27	59
83	30	48	12	51	45	55	92	28	23
95	93	63	32	72	17	94	75	37	54

Consider 10x10 magic square

- naïve search space = $100^{100} = 10^{200}$
- "much better" with permutations:

  100! ≈ *only* 10^{158}

For a 400x400 square:

- We'll have 400x400! = 160000! ≈ 10^{763175}

There are methods which can solve 400x400 in less than one hour CPU-time

Example: Simple Scheduling

What is the **minimal time** to build the house?
How to schedule the tasks to achieve the goal in minimal time (satisfying the precedence constraints)?

Task	Description	Duration	Predecessors
A	Masonry	7	none
B	Carpentry	3	A
C	Plumbing	8	A
D	Ceiling	3	A
E	Roofing	1	B
F	Painting	2	D
G	Windows	1	E
H	Facade	2	E, C
I	Garden	1	E, C
J	Moving in	1	G, H, I, F

Modeling

Variables: **start date of each task**

Constraints:

$B \geq A + 7$ $C \geq A + 7$ $D \geq A + 7$
$E \geq B + 3$ $F \geq D + 3$ $G \geq E + 1$
$H \geq E + 1$ $H \geq C + 8$ $I \geq E + 1$
$I \geq C + 8$ $J \geq G + 1$ $J \geq H + 2$
$J \geq I + 1$ $J \geq F + 2$

Minimise J (optimization problem)

Solution: 17 days:

A=0 B=C=D=7 E=F=10
G=11 H=I=15 J=**17**

Task	Description	Duration	Predecessors
A	Masonry	7	none
B	Carpentry	3	A
C	Plumbing	8	A
D	Ceiling	3	A
E	Roofing	1	B
F	Painting	2	D
G	Windows	1	E
H	Facade	2	E, C
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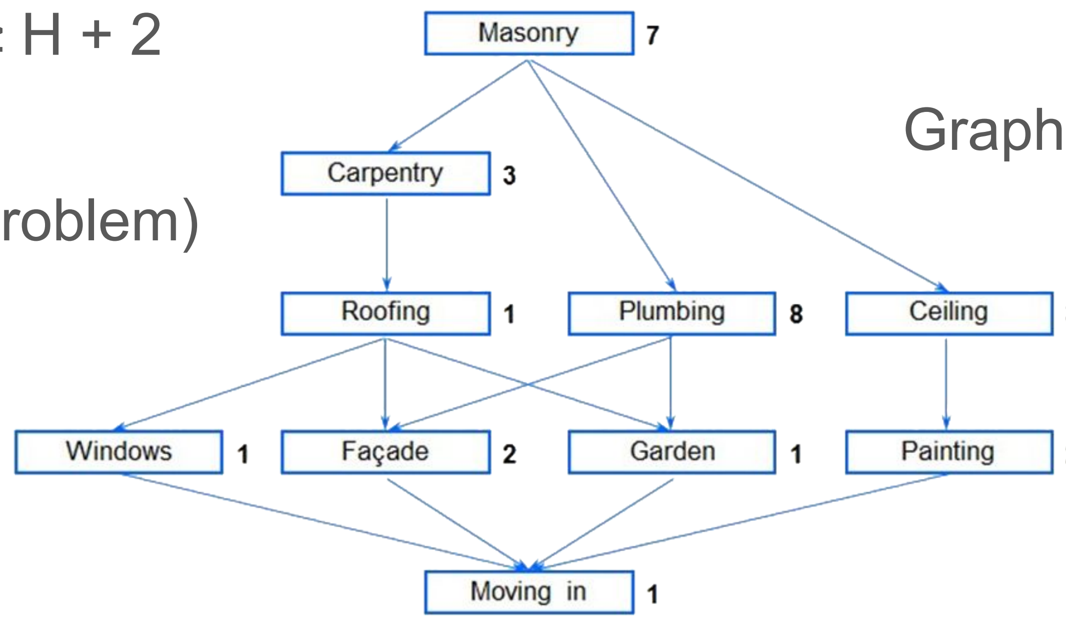


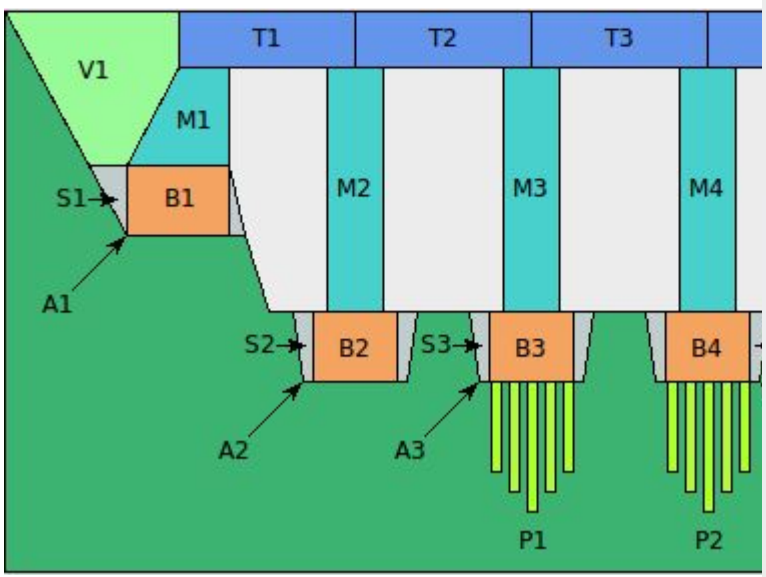
Graph

Example: Disjunctive S

Build a bridge in minimal tim

and resource requirements (

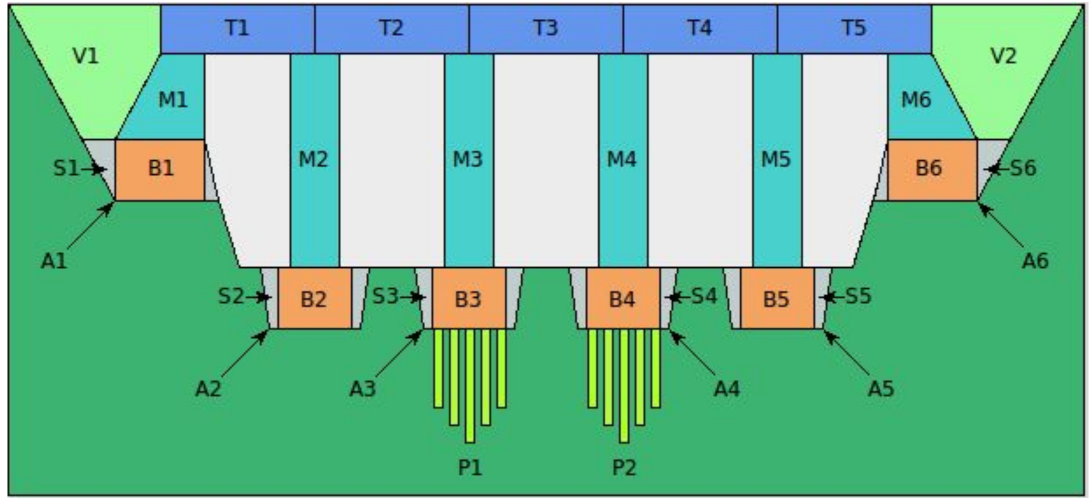
cannot overlap).

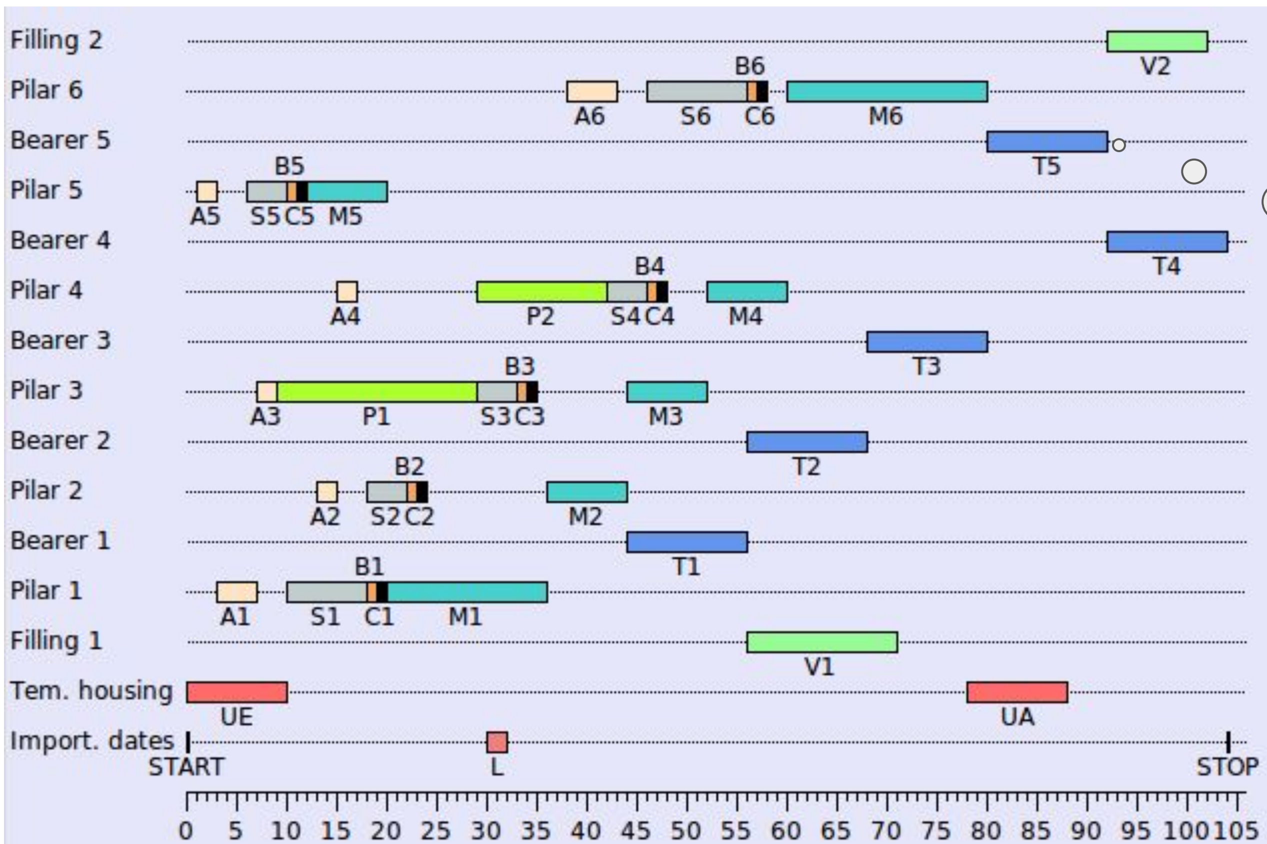


Task	Description	Dur.	Precedences	Resource
START	Beginning of the project	0		
A1	Excavation pillar 1 (abutment)	4	START	Excavator
A2	Excavation pillar 2	2	START	Excavator
A3	Excavation pillar 3	2	START	Excavator
A4	Excavation pillar 4	2	START	Excavator
A5	Excavation pillar 5	2	START	Excavator
A6	Excavation pillar 6 (abutment)	5	START	Excavator
P1	Extra foundation pillar 3	20	A3	Pile driver
P2	Extra foundation pillar 4	13	A4	Pile driver
UE	Erection of temporary housing	10	START	
S1	Formwork pillar 1	8	A1	Carpentry
S2	Formwork pillar 2	4	A2	Carpentry
S3	Formwork pillar 3	4	P1	Carpentry
S4	Formwork pillar 4	4	P2	Carpentry
S5	Formwork pillar 5	4	A5	Carpentry
S6	Formwork pillar 6	10	A6	Carpentry
B1	Concrete foundation pillar 1	1	S1	Concrete mixer
B2	Concrete foundation pillar 2	1	S2	Concrete mixer
B3	Concrete foundation pillar 3	1	S3	Concrete mixer
B4	Concrete foundation pillar 4	1	S4	Concrete mixer
B5	Concrete foundation pillar 5	1	S5	Concrete mixer
B6	Concrete foundation pillar 6	1	S6	Concrete mixer
C1	Concrete setting time pillar 1	1	B1	
C2	Concrete setting time pillar 1	1	B2	
C3	Concrete setting time pillar 2	1	B3	
C4	Concrete setting time pillar 3	1	B4	
C5	Concrete setting time pillar 4	1	B5	
C6	Concrete setting time pillar 6	1	B6	
M1	Masonry work pillar 1	16	C1	Bricklaying
M2	Masonry work pillar 2	8	C2	Bricklaying
M3	Masonry work pillar 3	8	C3	Bricklaying
M4	Masonry work pillar 4	8	C4	Bricklaying
M5	Masonry work pillar 5	8	C5	Bricklaying
M6	Masonry work pillar 6	20	C6	Bricklaying
L	Delivery of preformed bearers	2		Crane
T1	Positioning preformed bearer 1	12	M1 M2 L	Crane
T2	Positioning preformed bearer 2	12	M2 M3 L	Crane
T3	Positioning preformed bearer 3	12	M3 M4 L	Crane
T4	Positioning preformed bearer 4	12	M4 M5 L	Crane
T5	Positioning preformed bearer 5	12	M5 M6 L	Crane
UA	Removal of temporary housing	10		
V1	Filling 1	15	T1	Bulldozer
V2	Filling 2	10	T5	Bulldozer
STOP	End of the project	0	T2 T3 T4 V1 V2 UA	

A solution

Optimal end time = **103**





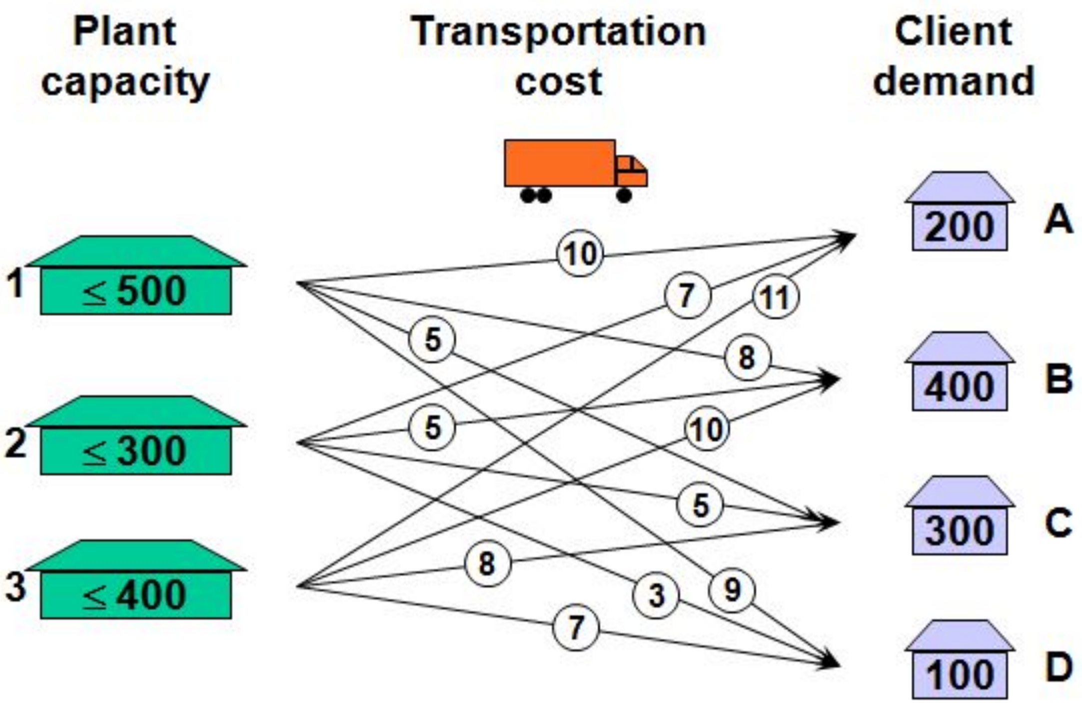
Color designates tasks (and also resources)

Example: Resource Allocation

Constraints:

A1 + A2 + A3 = 200
B1 + B2 + B3 = 400
C1 + C2 + C3 = 300
D1 + D2 + D3 = 100

A1 + B1 + C1 + D1 ≤ 500
A2 + B2 + C2 + D2 ≤ 300
A3 + B3 + C3 + D3 ≤ 400



Minimise the total cost:

10*A1 + 7*A2 + 11*A3
+ 8*B1 + 5*B2 + 10*B3
+ 5*C1 + 5*C2 + 8*C3
+ 9*D1 + 3*D2 + 7*D3

Solution: minimal cost = **6600**

A3=200, B1=200, B2=200, C1=300, D2=100
(all others vars = 0)

What do all these problems have in common?

- Variables take values over a **finite subset of integers**
- Clear constraints (easy to state in natural language)
- Large search space
- Well-identified “goal”
  Notion of solution is easy to define (declaratively)
- But we don’t know how to compute it!
- No algorithm to build a solution (incrementally or otherwise)
- Hence:
  - Need to explore (search) the possible solution space (all possibilities)
  - Either exhaustively or in an “intelligent”, “guided” manner

Methods covered in these lectures

- Constraint Programming
  - Represent problem by variables and constraints
  - Use specific solving algorithms to speed search up
  - Use “complete” solvers
- Modeling with constraints
  - Constraints over finite domains
  - Useful global constraints
- Constraint Optimisation vs. Constraint Satisfaction

Methods lightly approached in these lectures

- *Local Search and Metaheuristics*
    - *Evaluation function to check if state is “good” or not*
    - *Optimization of the evaluation function*
    - *Sacrifice completeness to (hopefully) quickly find a solution*
- *SAT (solvers for boolean-variable formulations)*
- *Modeling Languages*

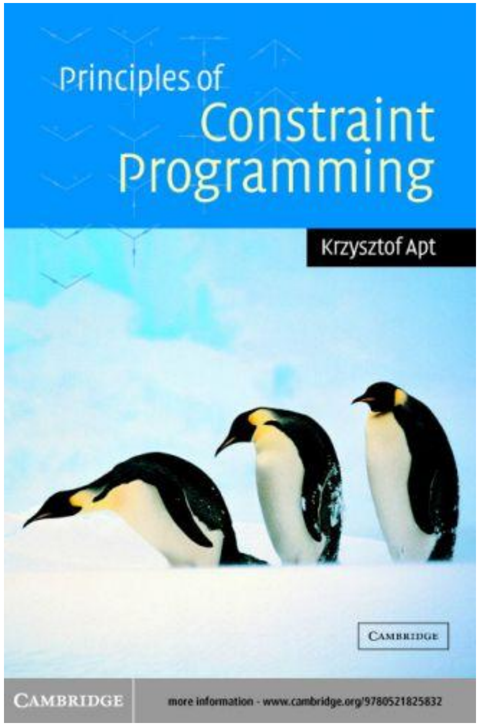
Methods **NOT** covered in this lecture series

- Graph Search
  - Representation of states and transitions/actions between states → graph
  - Explored explicitly or implicitly
- Numerical Optimization Methods
  - For continuous domains & twice differentiable functions
- Linear Optimization methods
  - For linear constraints & rational domains
  - Simplex algorithm, Interior Point Methods
  - (Mixed-)Integer Programming, cutting plane methods
- Dynamic Programming
  - Decomposable problem, recursive relation
- … :)

Useful Resources

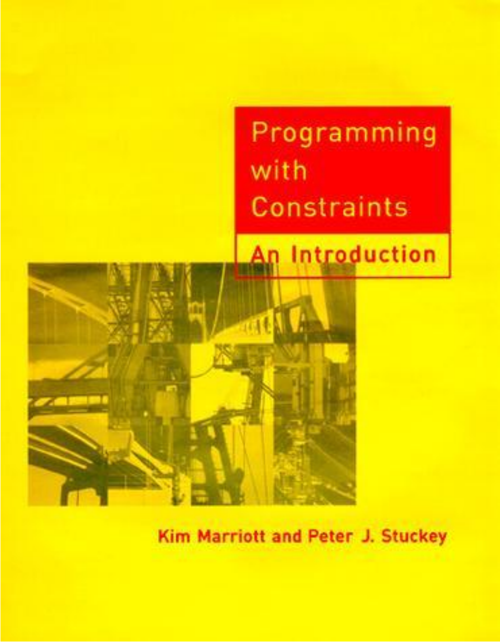
K. Apt

Principles of Constraint Programming,
Cambridge University Press 2003



K. Marriott and P. J. Stuckey

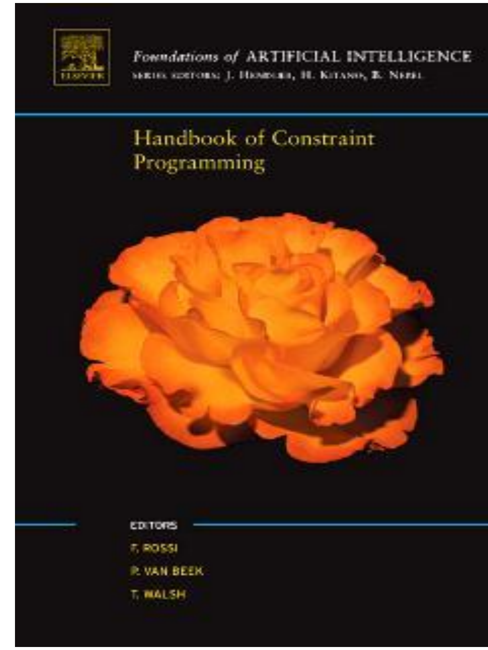
Programming with Constraints:
An Introduction
MIT Press, 1998



F. Rossi, P. Van Beek and T. Walsh

Handbook of Constraint Programming,

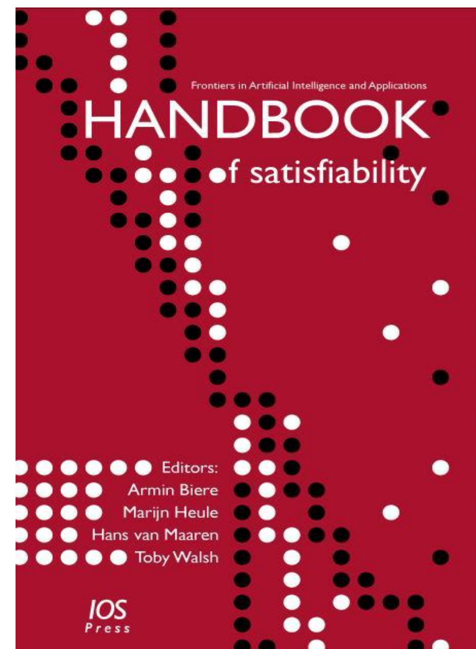
Elsevier 2006



A. Biere, M. Heule, H. van Maaren & T. Walsh

Handbook of Satisfiability,

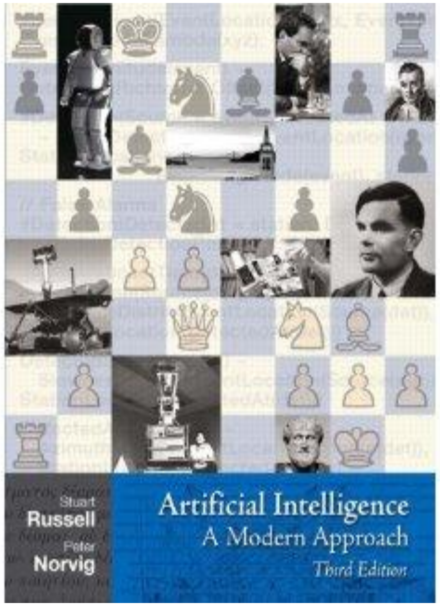
IOS Press 2009



S. Russell & P. Norvig

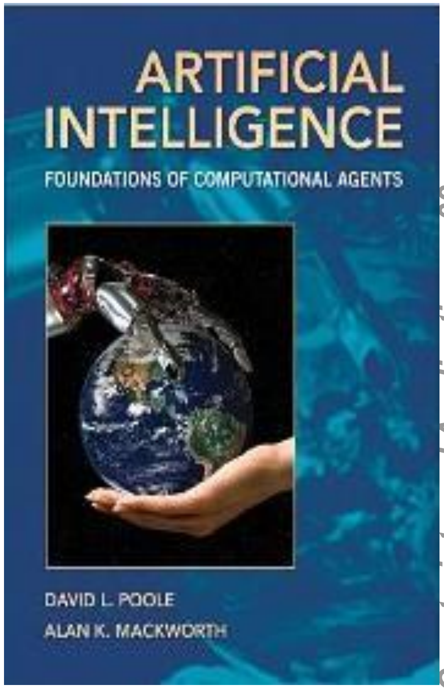
Artificial Intelligence: A Modern Approach, 3rd edition, Pearson 2010

http://aima.cs.berkeley.edu/



D. Poole & A. Mackworth, Artificial Intelligence: Foundation of Computational Agents, Cambridge University Press 2010

http://artint.info/



F. Rothlauf

Design of Modern Heuristics,

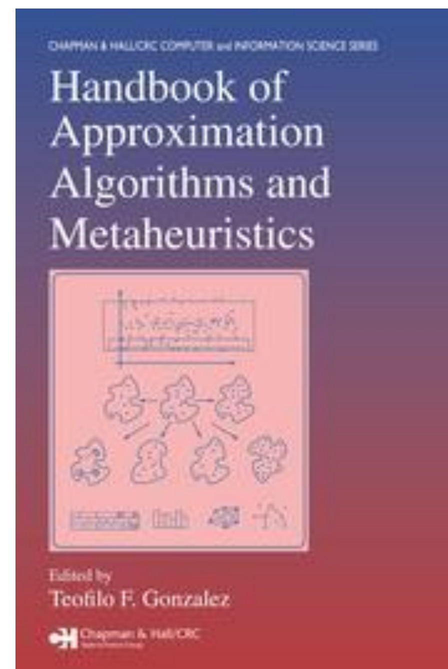
Springer Verlag 2011

T. Gonzalez

Handbook of Approximation

Algorithms and Metaheuristics,

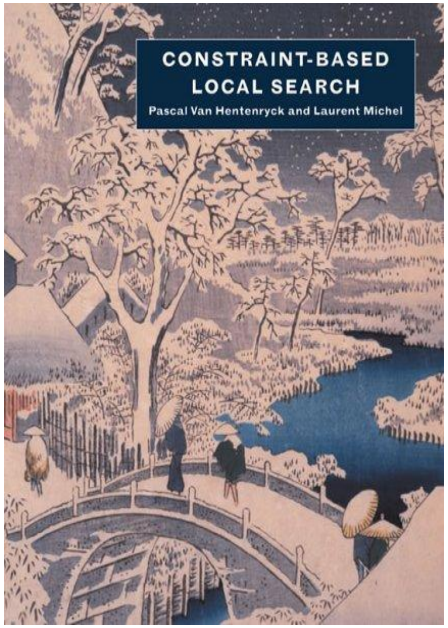
Chapman & Hall/CRC 2010



P. Van Hentenryck and L. Michel

Constraint-based Local Search

MIT Press 2005



Exercise: **ad-hoc** problem solving

Implement in **Java** (or Python) a solution for these problems:

1) TWO + TWO = FOUR (*)
2) N-queens (**), Try N=8, 10, 11, 12,...
3) Magic squares (**). Try 3x3, can you solve 4x4? Larger? Why?
4) ~~Square packing (***)~~
5) ~~Stable marriage (****)~~

Questions:

1) Is it ***easy*** to program?
2) Can it be ***generalized*** (for arbitrary size problems)?
3) Is it ***efficient***? Does it ***scale well*** (in terms of execution time)?
4) Measure execution time for different size problems