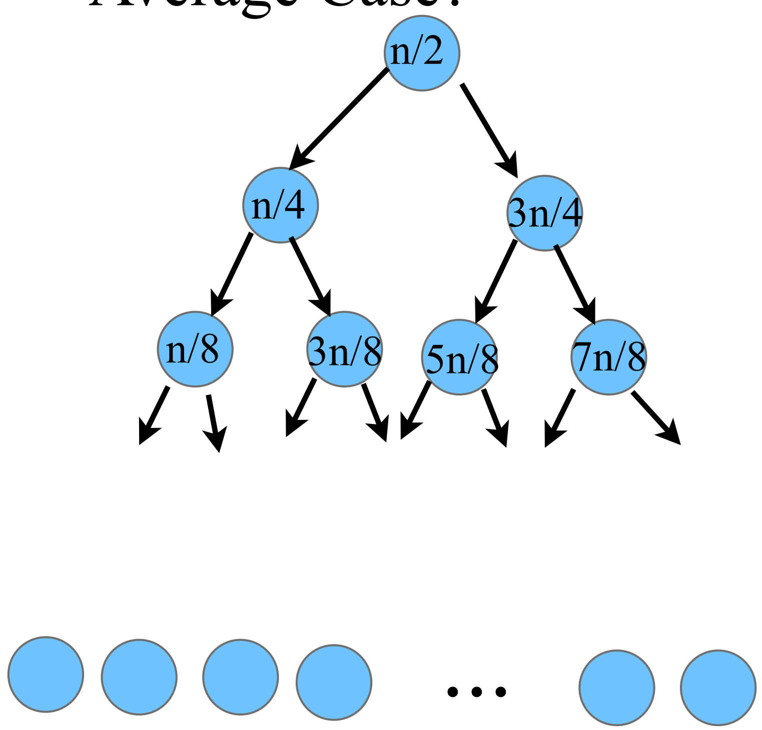
Motivation

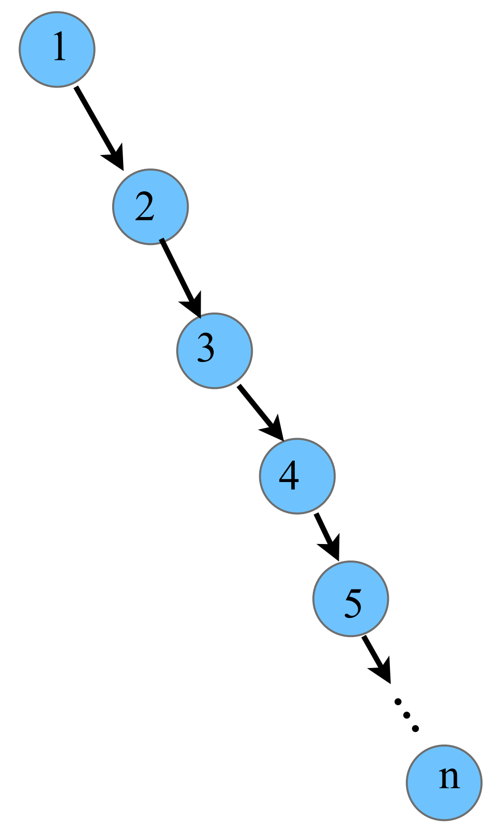
Finding an item in a binary search tree:

How long does find take in the worst case?

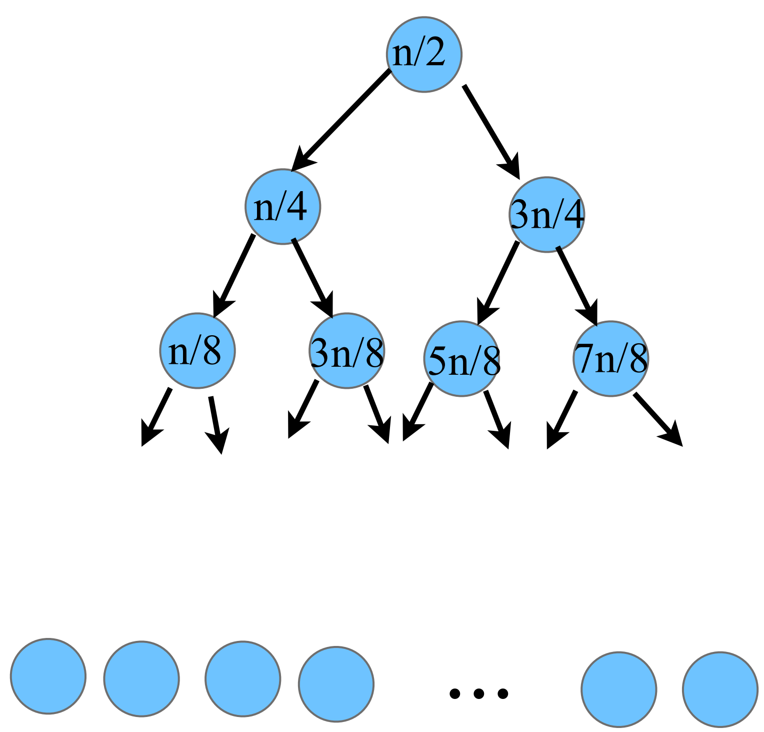
Best case?

Average Case?

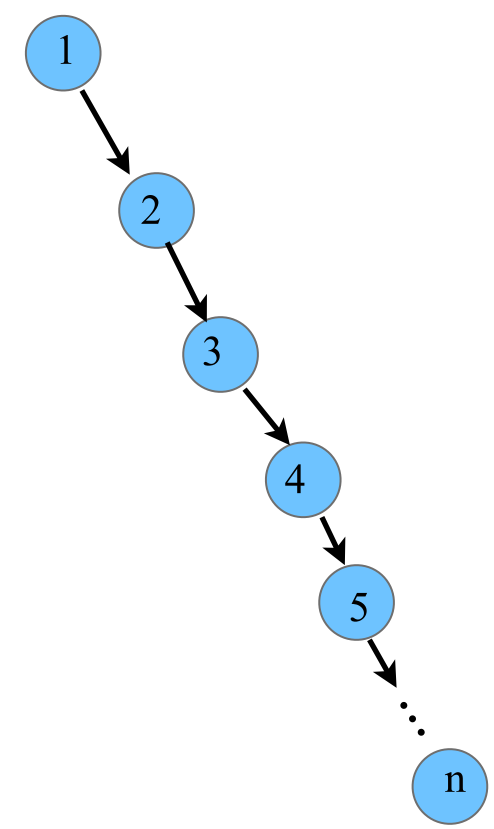




Why do we care about the height of a binary search tree?



In a balanced tree it takes O(log(n)) time to search for an item



In an unbalanced tree it takes O(n) time to search for an item

Lecture 16 cont

Red-Black Trees
(Building a Balanced Binary Search Tree)

So popular - it has its own youtube video!

http://www.youtube.com/watch?v=vDHFF4wjWYU

Red-Black trees are a type of binary search tree guaranteed to have logarithmic depth

Red-Black Tree

A Red-Black tree is a binary search tree that obeys 4 properties:

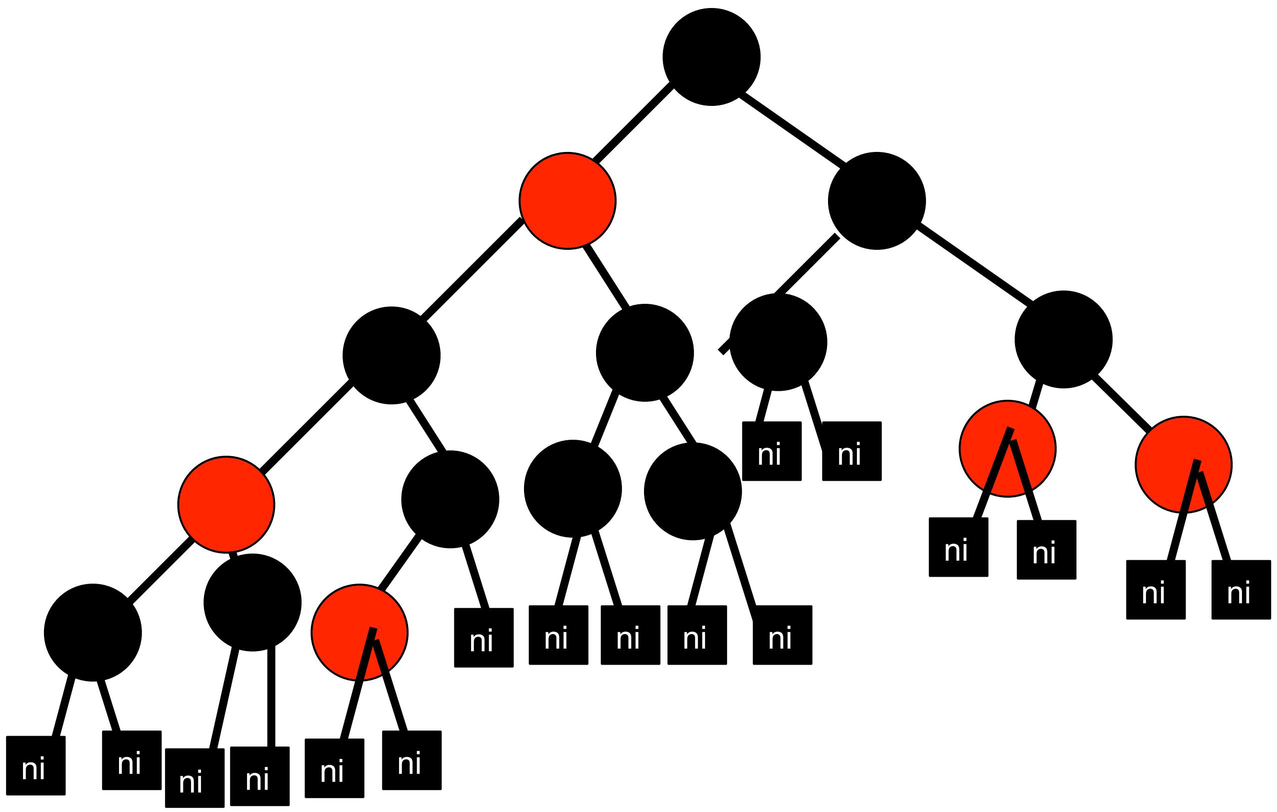
1) Every node is colored red or black
2) All children of a red node are black
3) [black property] For every node in the tree, all paths from that node down to a nullptr have the same number of black nodes along the path
4) The root is black (This property is non - essential to maintain a balanced tree)

red-black balancing rules

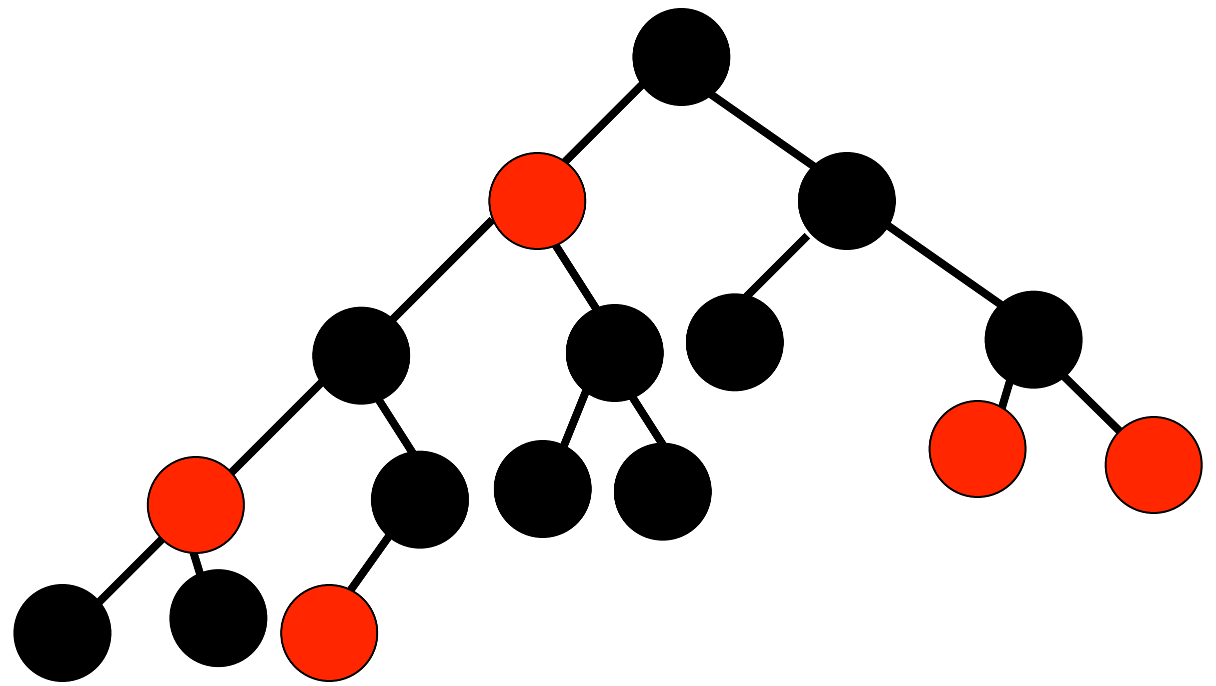
NOTE: We adopt the convention that nullptrs are viewed as pseudo node and are black. i.e. we will call them nil nodes. They do not contain any data.

creates a cleaner case analysis

Red-Black Tree

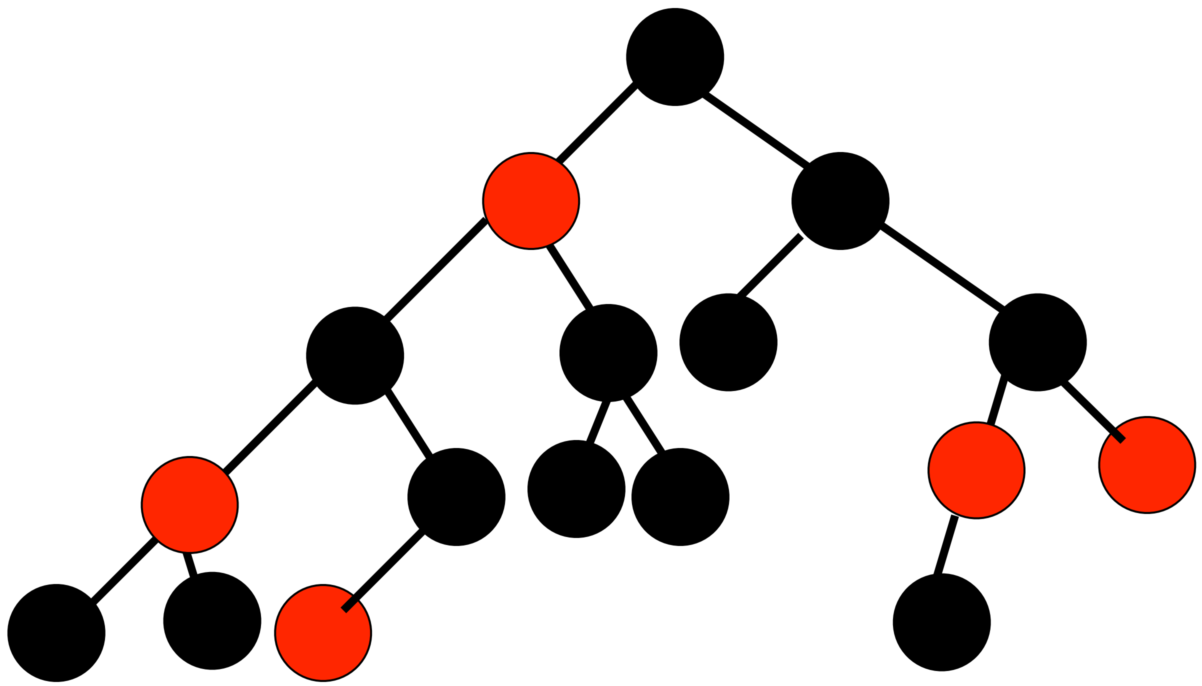


Red-Black Tree

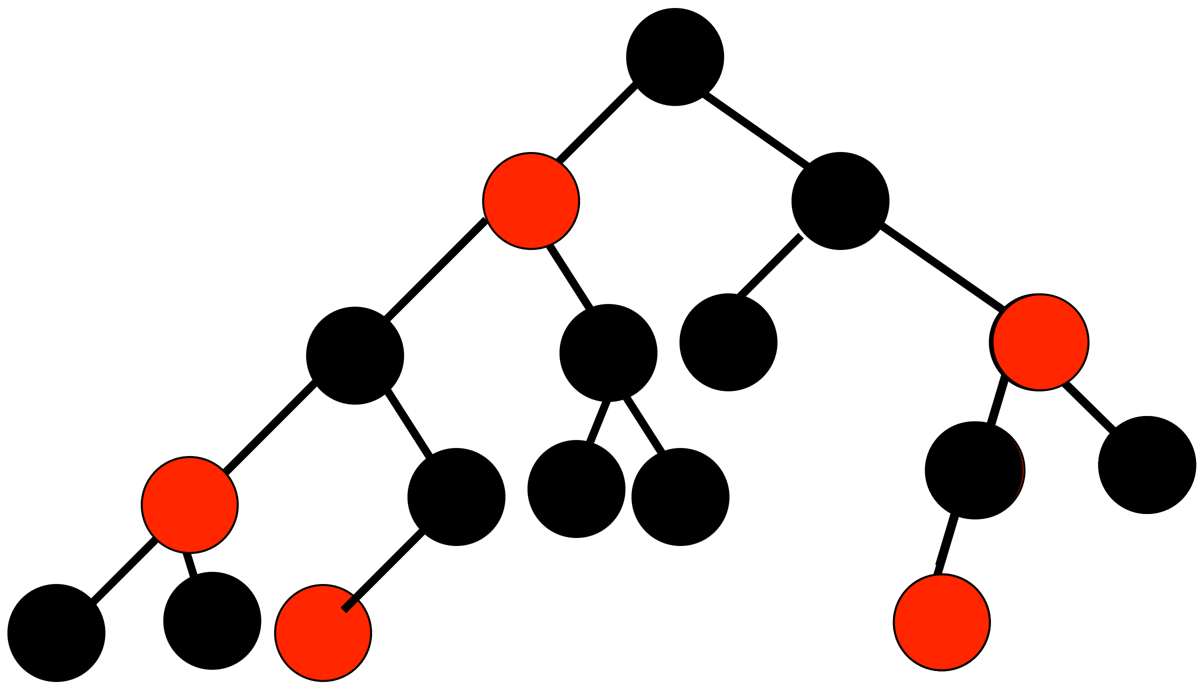


When drawing a tree, we will omit the nil nodes

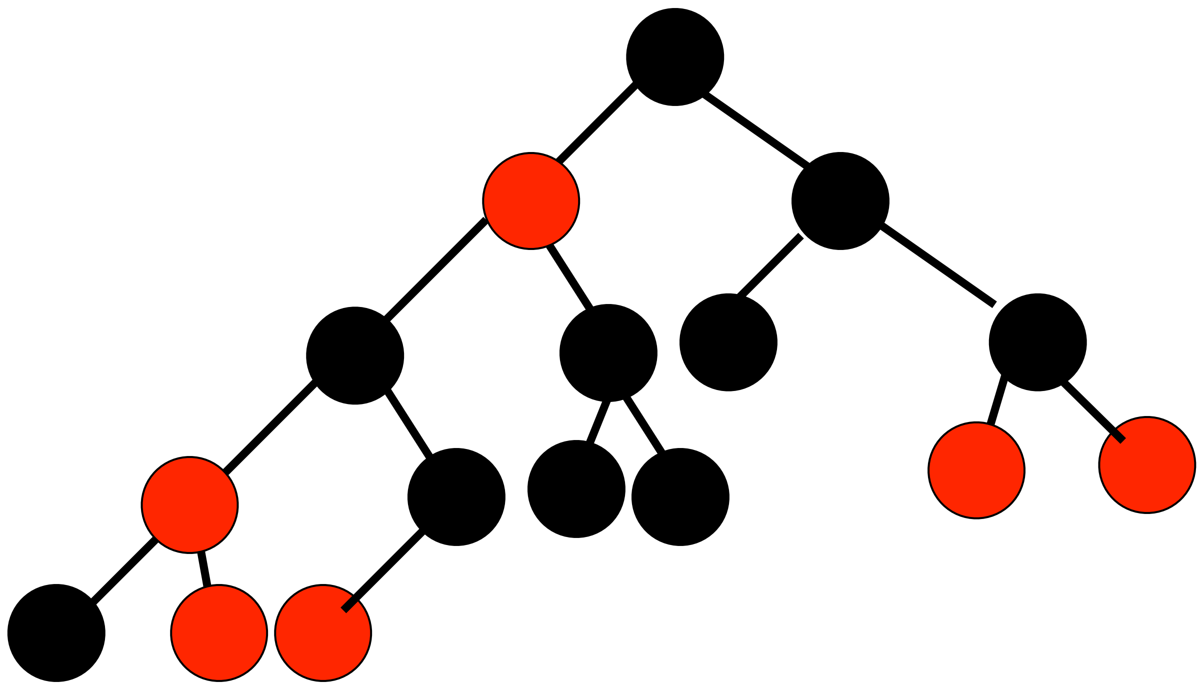
Not a Red-Black Tree



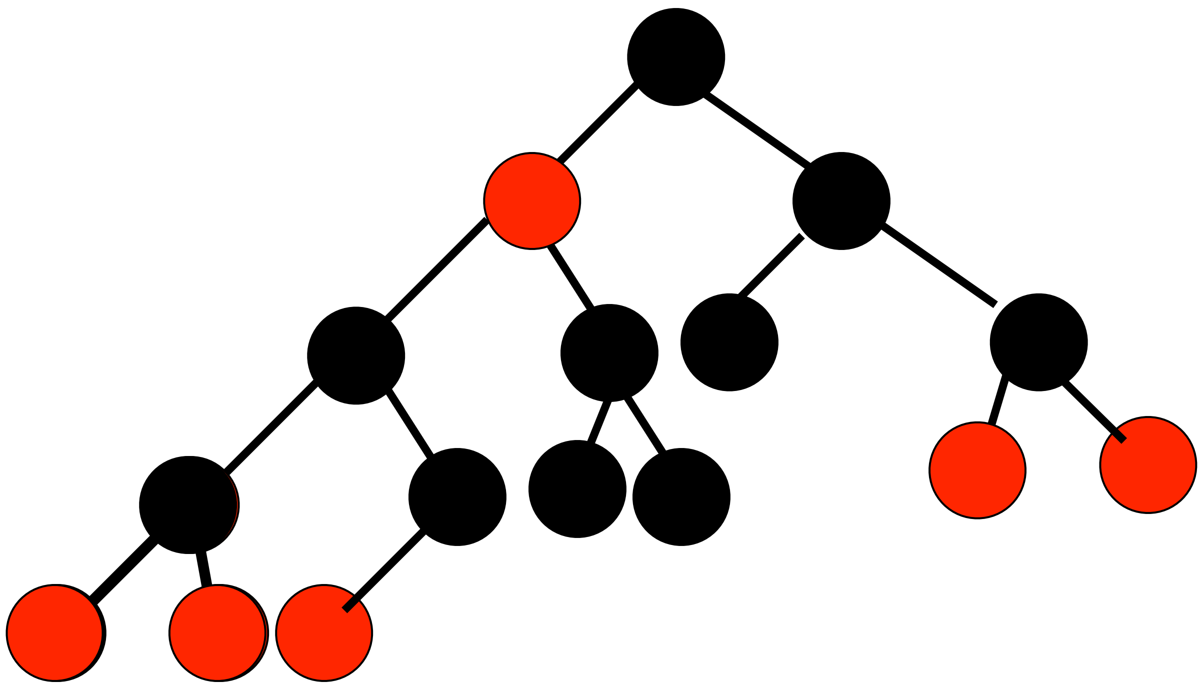
A Red-Black Tree



Not a Red-Black Tree



A Red-Black Tree



Black nodes in a Red-Black Tree

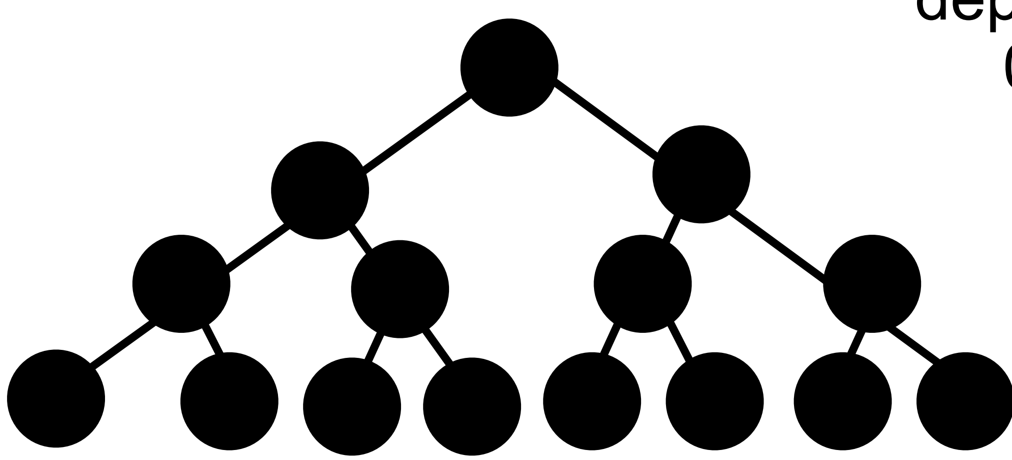
Let bh of a node be the number of black nodes on a path from the node to nil (including nil) but not including the node itself

We will prove on the next slide that any RedBlack Tree has at least = $2^{bh}-1$ internal nodes.

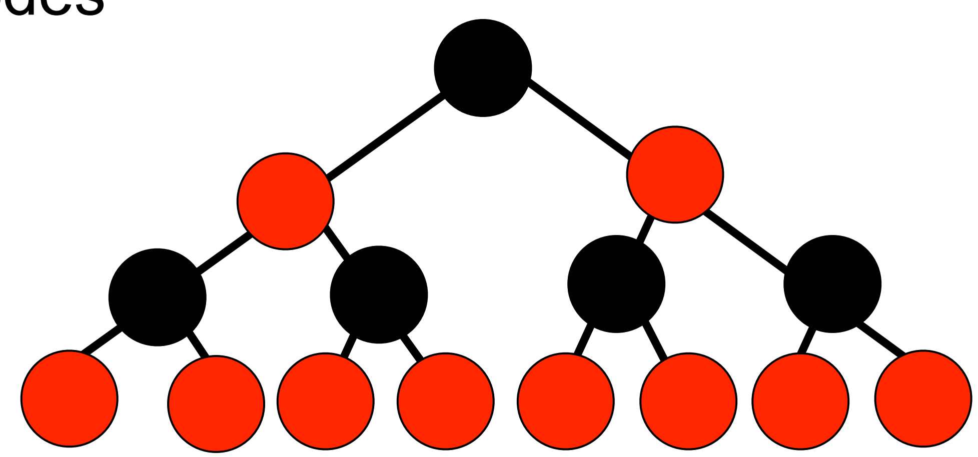
Any RedBlack Tree has at most bh red nodes on a path from the root to a leaf.

Every red node has a black parent.

Any RedBlack Tree with Black height bh has at height at most 2bh.



depth	#nodes
0	1
1	2
2	4
3	8



Max #nodes at depth i = 2^i

#black nodes $1 + 2 + 4 + 8 = 2^4-1$

#black nodes $1 + 2 + 4 + 8 + \cdots + 2^{bh-1} = 2^{bh}-1$

#black nodes $1 + 4 > 2^2 - 1$

#Black nodes in a Red-Black Subtree

Any Red-Black tree with black height bh, has at least 2^{bh}-1 internal (i.e. non nil) nodes.

The proof of this fact is based on induction.

Induction Hypothesis: Any subtree rooted at x of a Red-Black tree has 2^{c}-1 internal (i.e. non nil) nodes where c is the black height of the node x.

Base Case: If the height, is zero. Then x is a leaf and the black height is zero $2^{c}-1 = 2^{0}-1 = 0$

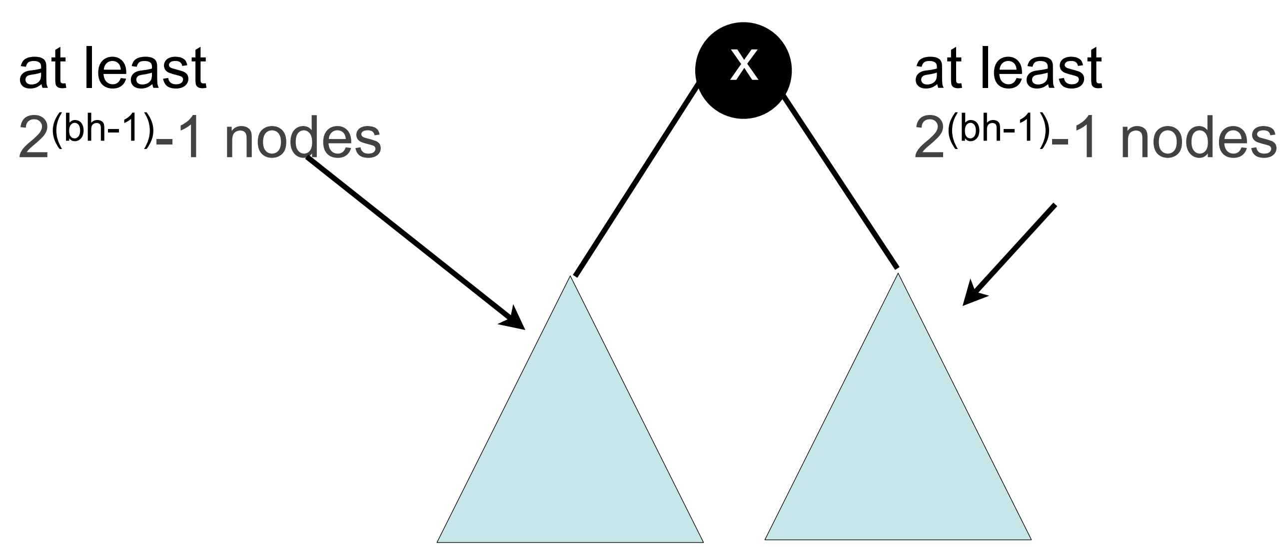
Induction Step:

We assume the result is true for any subtree of a Red-Black tree, whose height is less than the height of x.

Let bh be the black height of x.

Thus x's left and right subtree have black height at least (bh-1), and consequently each subtree has at least $2^{(bh-1)}$-1 internal nodes

Therefore the subtree rooted at x has at least $2^{(bh-1)}-1 + 2^{(B-1)}-1 + 1 = 2^{bh} - 1$ internal nodes



Height and Size of RB Trees

- If every path from the root down to nil has bh black nodes, then tree has at least $2^{bh}-1$ internal nodes (we showed this fact by induction)

  $\Rightarrow$ N >= $2^{bh}-1$ (since number nodes >= number black nodes)

  $\Rightarrow$ log(N + 1) >= bh

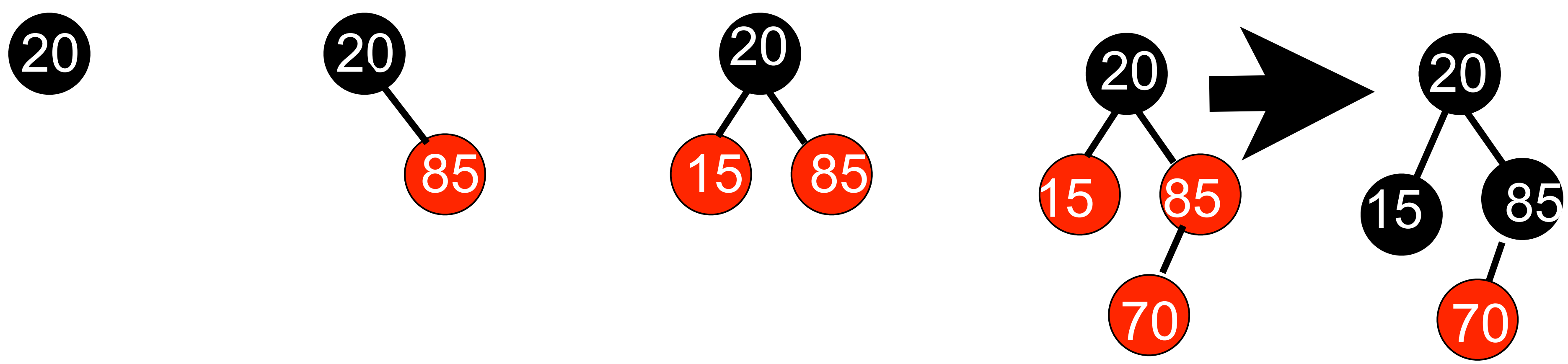
  $\Rightarrow$ log(N + 1) >= ½ height of tree (since bh >= ½ height)

  $\Rightarrow$ Height of tree <= 2 log (N+1) = O(log N)

  RED-BLACK TREES HAVE O(log N) height

Insertion into a Red-Black Tree

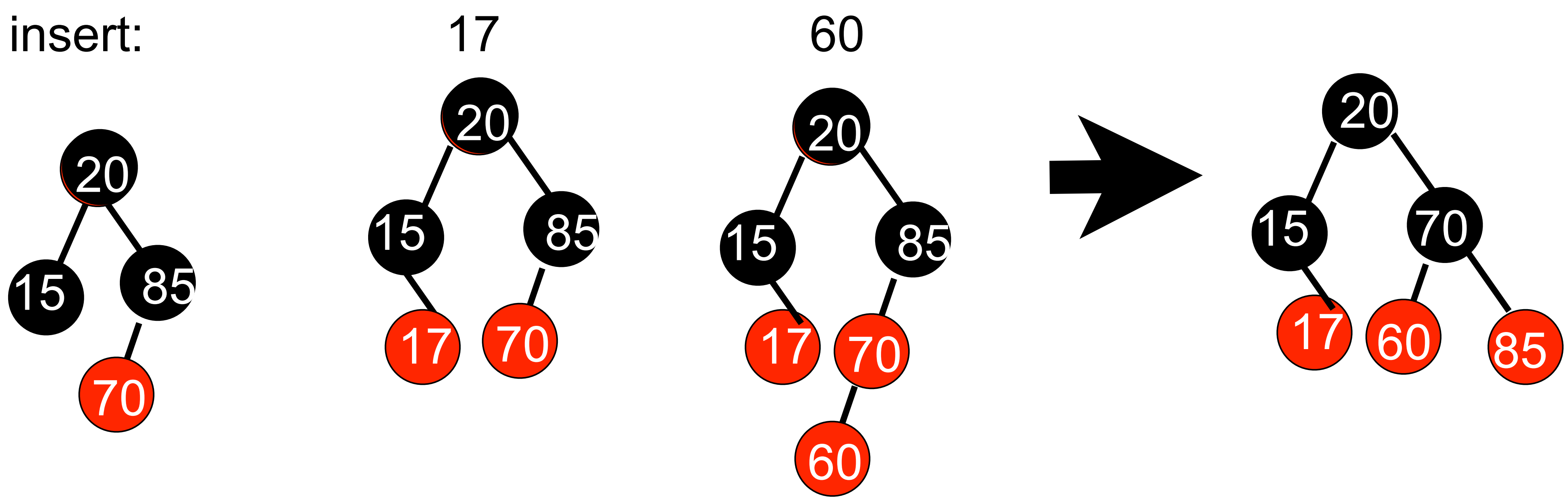
insert: 20 85 15 70



We will cxatorgize the different types of red-red violations. Case I will be distinguished by the parent node of the red-red violation having a red sibling

Double Red Problem
case 1 sibling of parent is red

Insertion into a Red-Black Tree



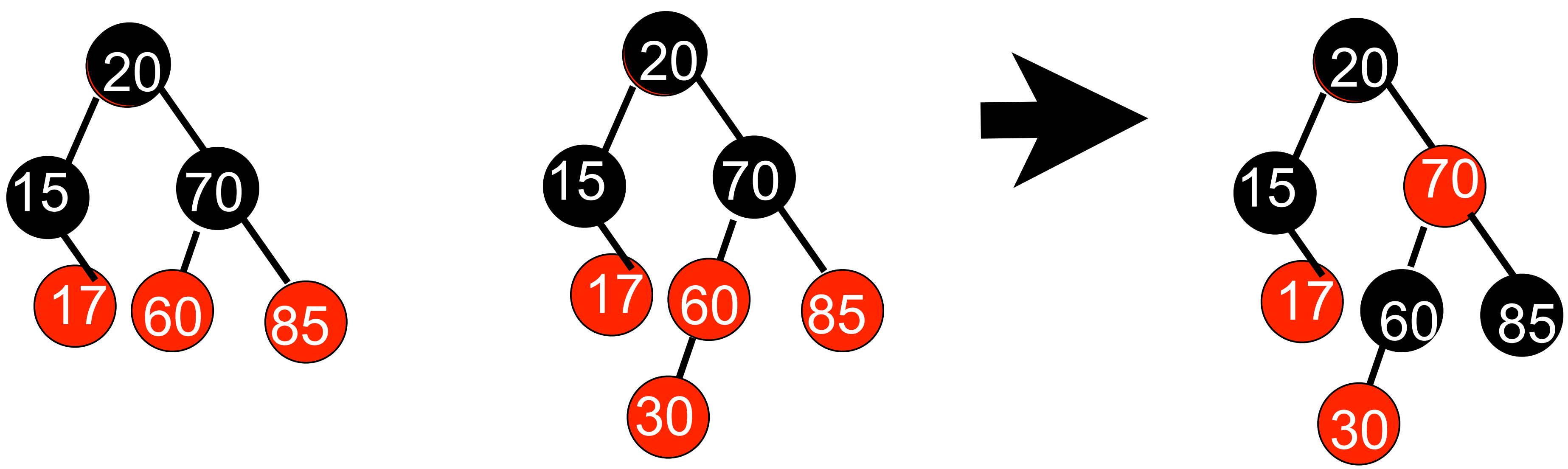
Double Red Problem

<u>case 3</u> sibling of parent is BLACK (or nil)

Right Rotate

Insertion into a Red-Black Tree

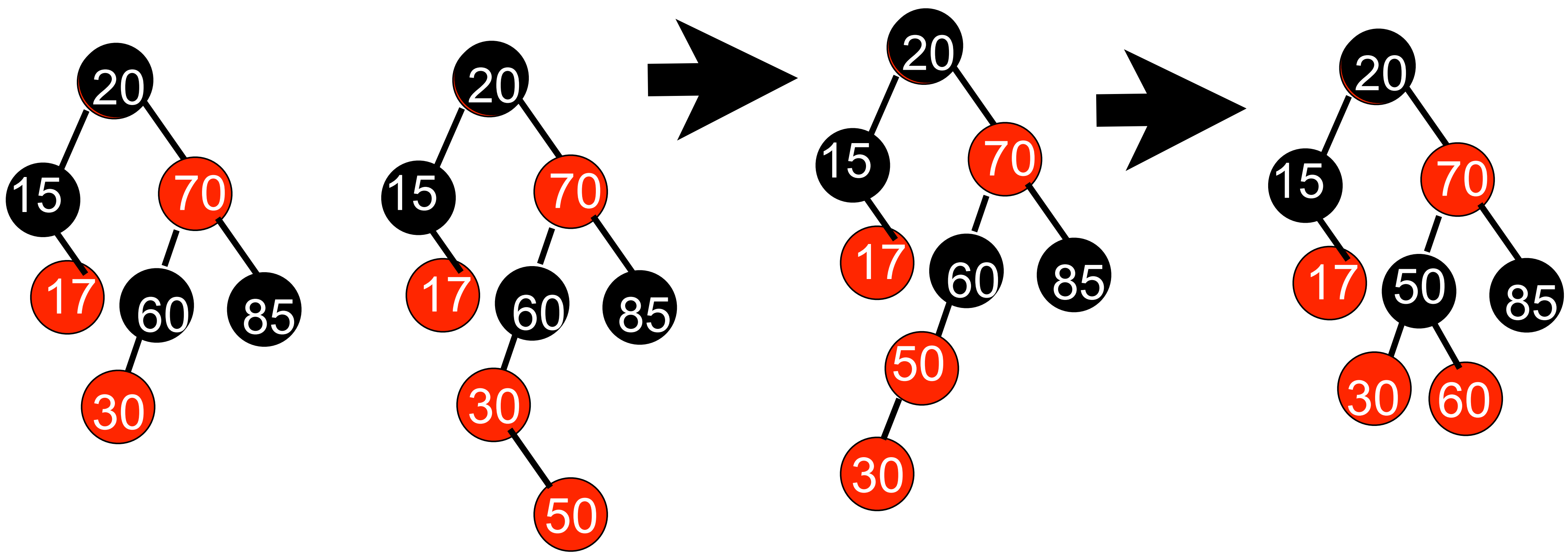
insert: 30



Double Red Problem
<u>case 1</u> sibling of parent
is RED

Insertion into a Red-Black Tree

insert: 50



If a red-red violation occurs, we we call it a case 2 or case 3 if the sibling of the parent is a black or nil node.
Look for later slides to help you distinguish a case 2 from a case 3

Double Red Problem
<u>case 2</u> sibling of parent is BLACK (or nil)

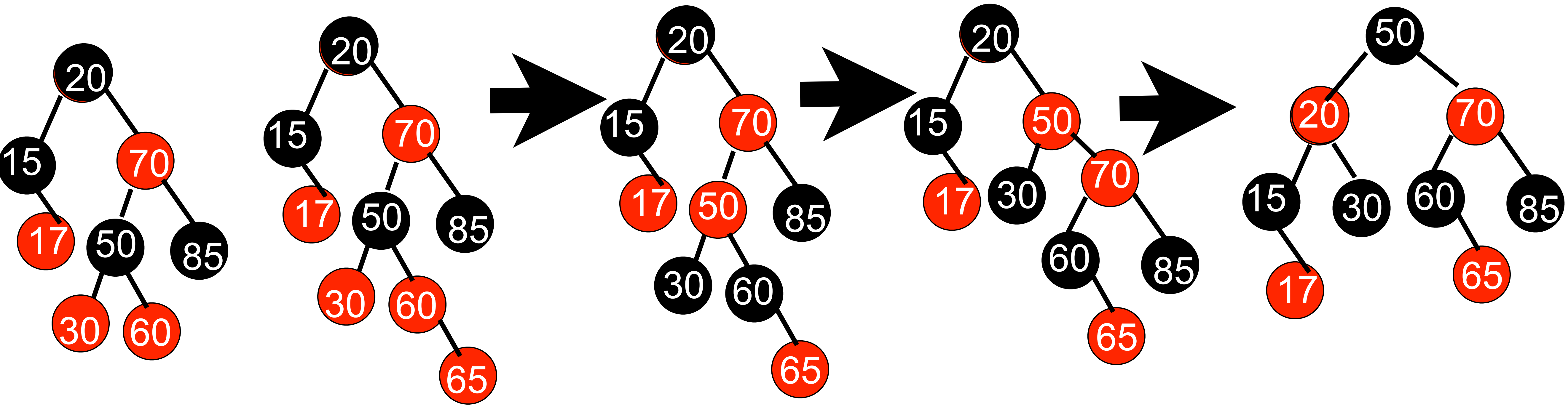
Left Rotate

Double Red Problem
<u>case 3</u> sibling of parent is BLACK (or nil)

Right Rotate

Insertion into a Red-Black Tree

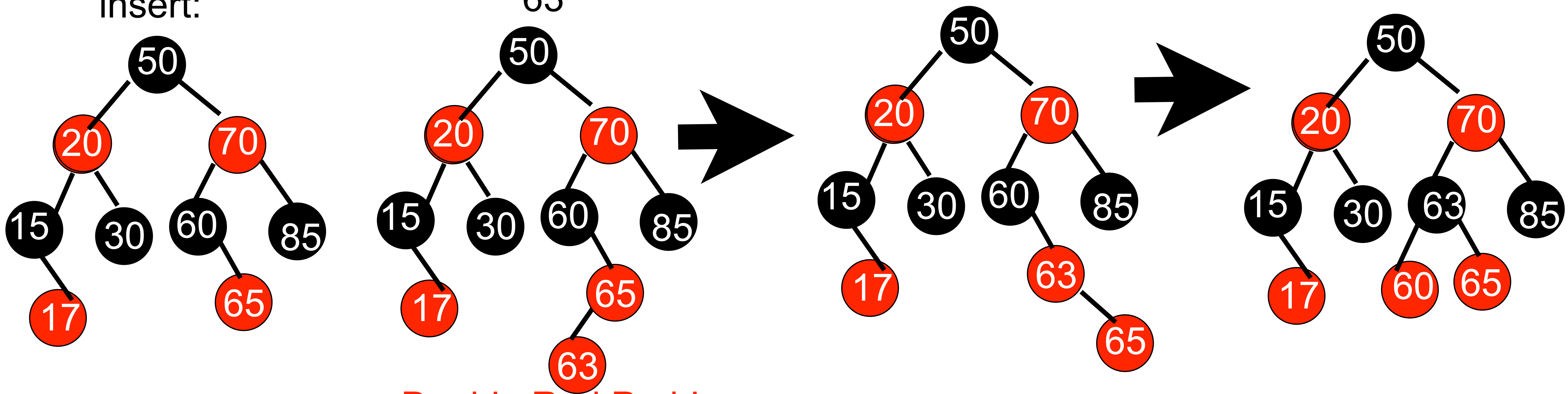
insert: 65



Double Red Problem case 1 sibling of parent is RED

Double Red Problem case 2 sibling of parent is BLACK or NULL

Double Red Problem case 3 sibling of parent is BLACK or NULL insert: 63



Double Red Problem
case 2 sibling of parent is BLACK or NULL

Double Red Problem
case 3 sibling of parent is BLACK or NULL

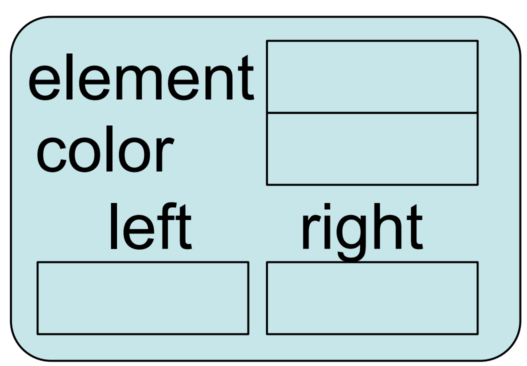
Bottom-Up Insertion into RB tree

- First insert as you would in standard binary search tree
- Color new node red
- If parent is black, done
- If parent is red, have red-red violation.
  - Fix by recoloring and (maybe) rotation.
  - May just move red-red violation further up the tree.
  - If so, repeat. Continue until
    - no red-red violation
    - red-red violation is root and one of its children. Color root black.
- Bottom-up insertion is only partially covered in our textbook

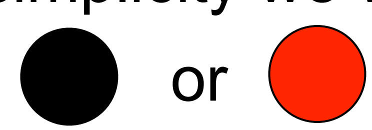
A Node of the Red-Black Tree

```
template <class Comparable>
class RedBlackNode
{
    Comparable    element;
    RedBlackNode *left;
    RedBlackNode *right;
    int           color;

    RedBlackNode( const Comparable & theElement = Comparable( ),
          RedBlackNode *lt = nullptr, RedBlackNode *rt = nullptr,
          int c = RedBlackTree<Comparable>::RED )
  : element( theElement ), left( lt ), right( rt ), color( c ) { }
    friend class RedBlackTree<Comparable>;
};
```

```
template< class
Comparable>
class RedBlackTree
{

    enum { RED, BLACK };

};
```



For simplicity we will represent a node by:



Fixing a red-red violation

- Call inserted node X
- May need to manipulate the following nodes:

  P: Parent of X

  G: Grandparent of X

  S: Sibling of parent of X (uncle)
- 3 Cases –

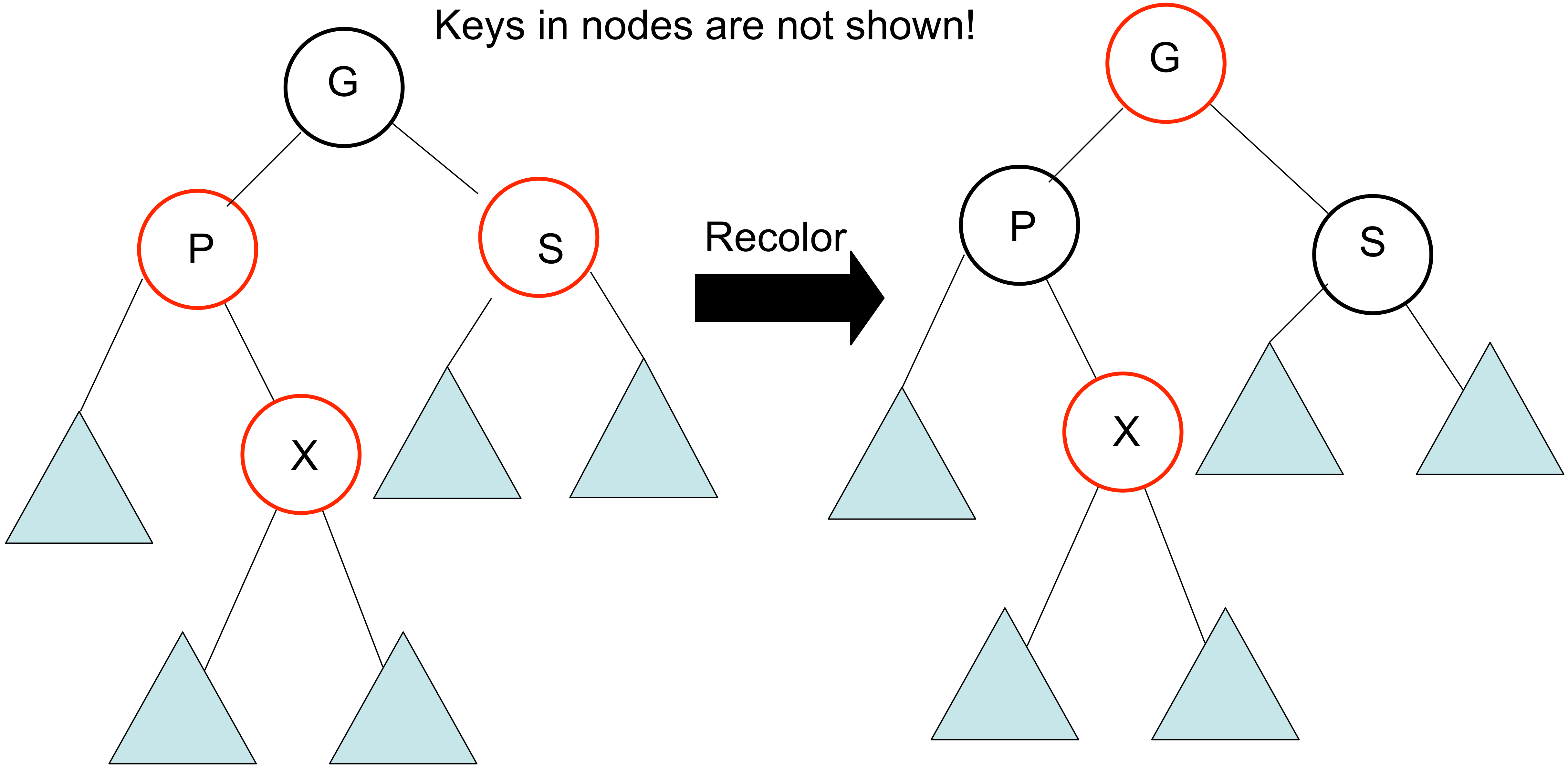
  Case 1 is easy (but may move problem up tree),

  Case 3 requires one rotation

  Case 2 you do one rotation and get to Case 3 (for a total of 2 rotations)
- When applying the following, if S is nil, treat it as a black node

Case 1: Sibling of Parent is Red

X is node causing a problem, P is parent, G is grandparent, S is sibling of parent
Keys in nodes are not shown!

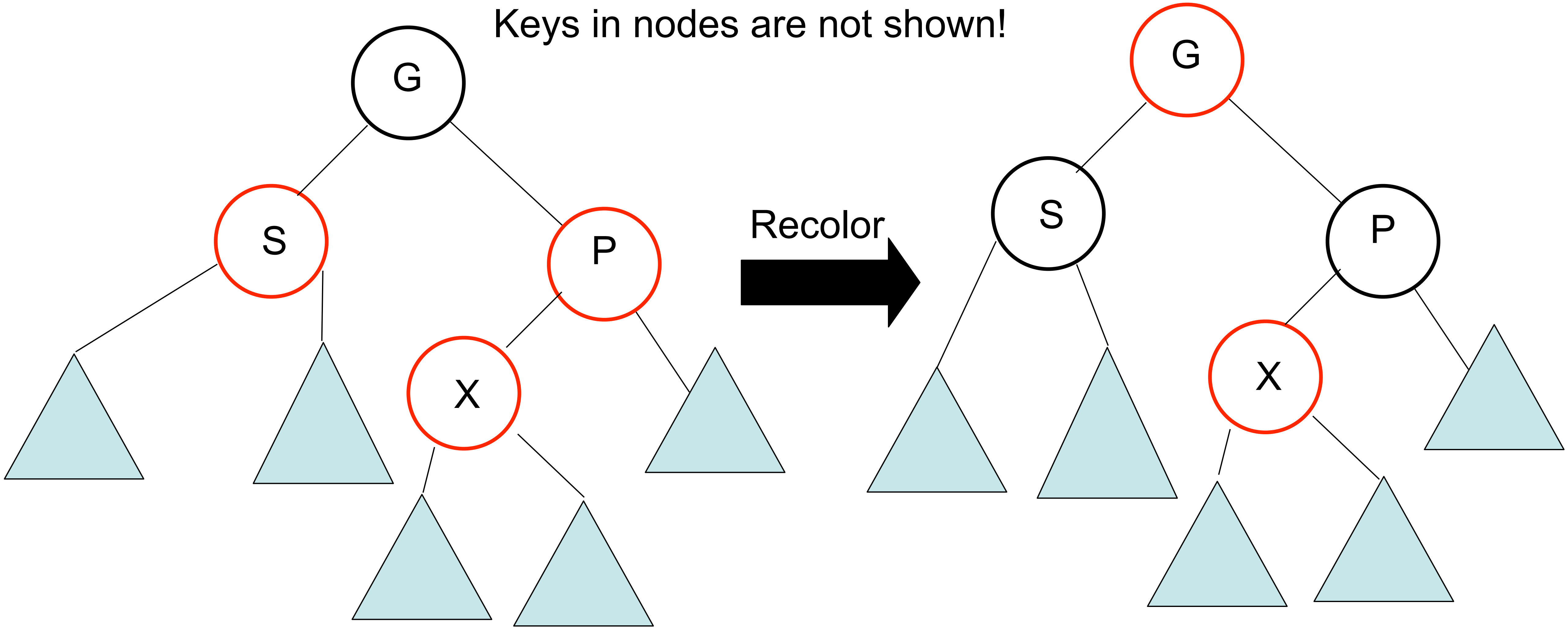


After recolor: If G is root, color it black.
If parent of G is red, continue repair with G being the new X.

Another Example of Case 1: Sibling of Parent is Red

X is node causing a problem, P is parent, G is grandparent, S is sibling of parent

Keys in nodes are not shown!



After recolor: If G is root, color it black.
If parent of G is red, continue repair with G being the new X.

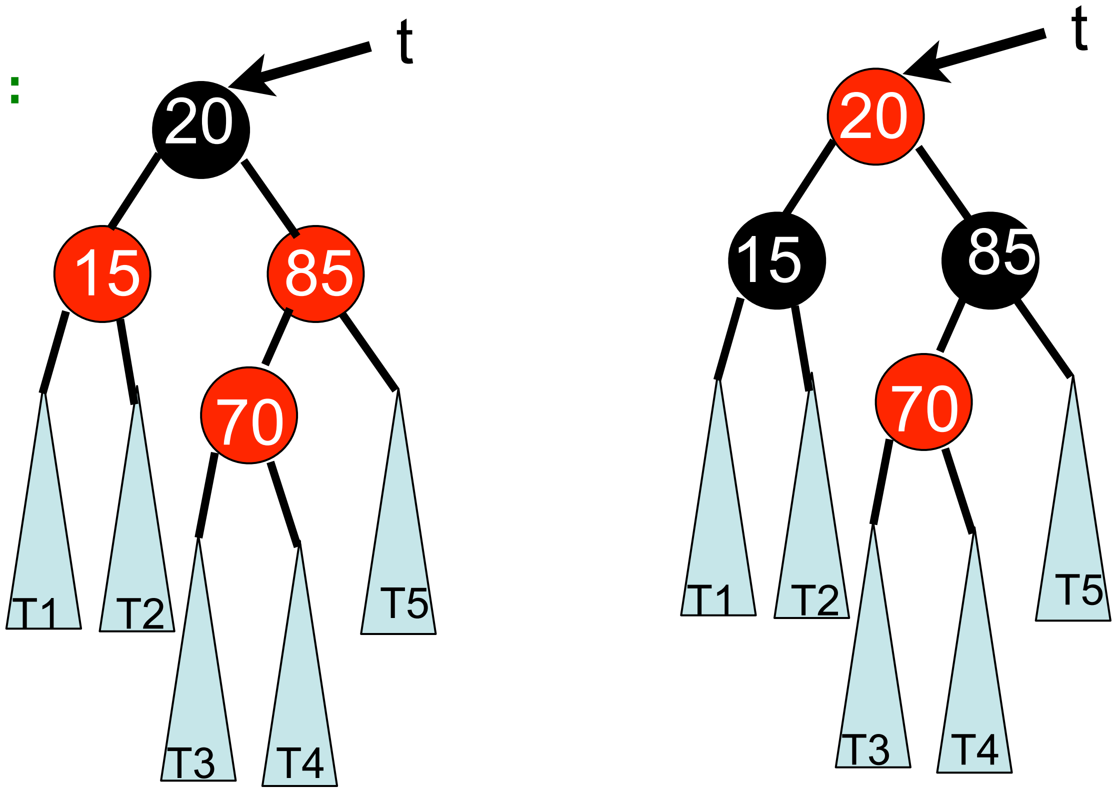
Sibling of parent is red

```
// Flips node and children's color
template <class Comparable>
void RedBlackTree<Comparable>::colorFlip( Node * t )
{
        t->color = RED;
        t->left->color = BLACK;
        t->right->color = BLACK;
}
```

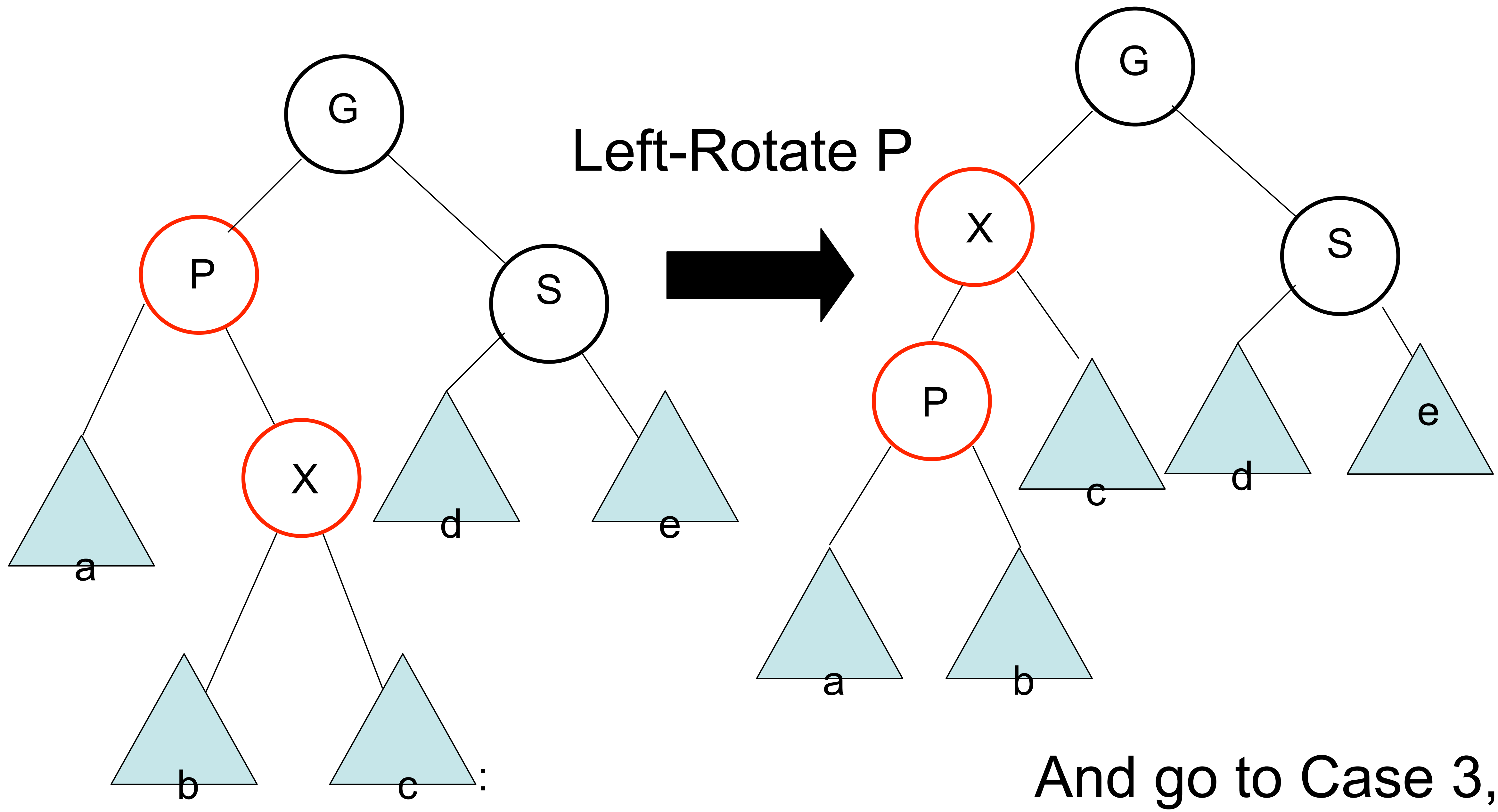
```
template< class
Comparable>
class RedBlackTree
{

    enum { RED, BLACK };

};
```

```
template <class Comparable>
class RedBlackNode
{
    Comparable    element;
    RedBlackNode *left;
    RedBlackNode *right;
    int           color;

    // constructors removed
};
```

Case 1:



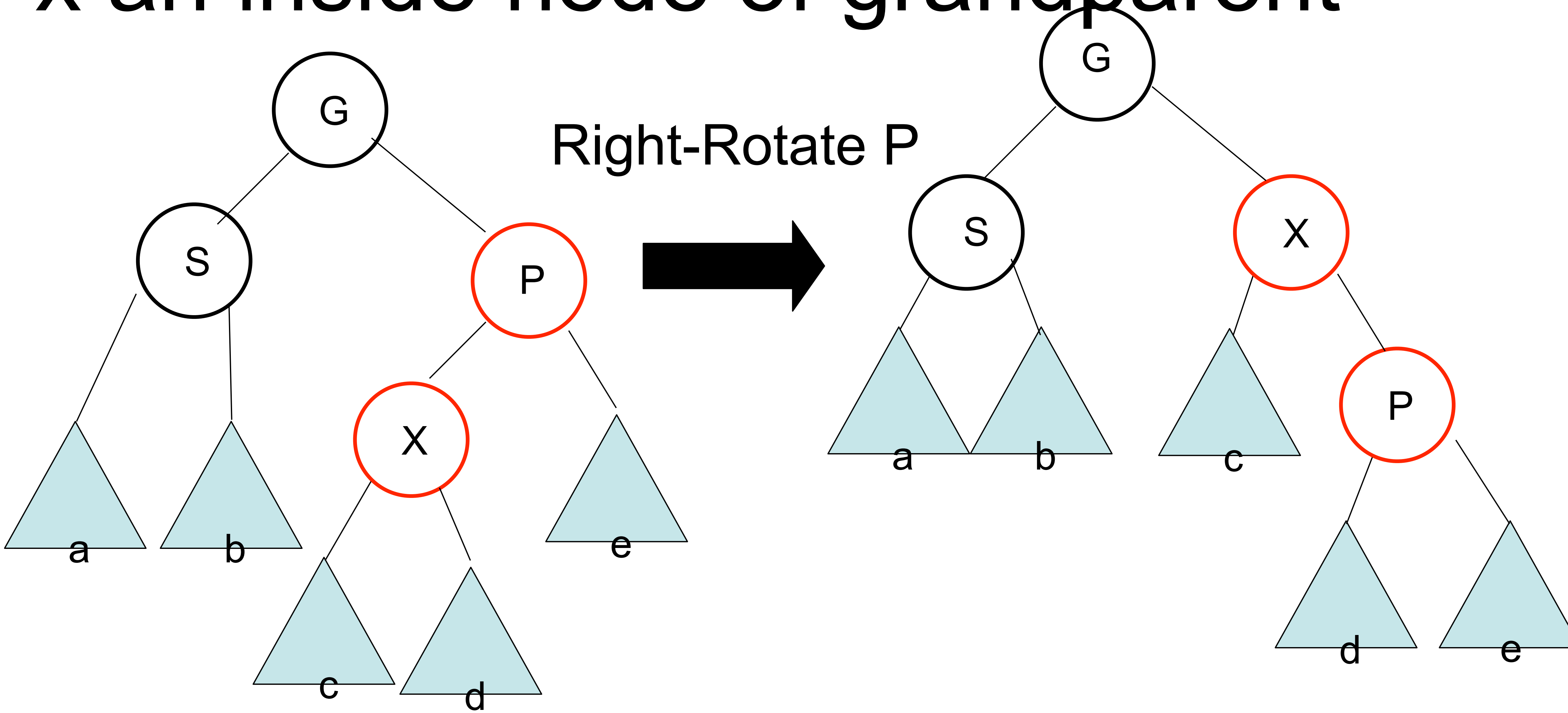
Case 2: Sibling of Parent is Black, x an inside node of grandparent



And go to Case 3, with P as new X

Here we left-rotate P. Right rotate in Case 2 if P, X are in right subtree of G.

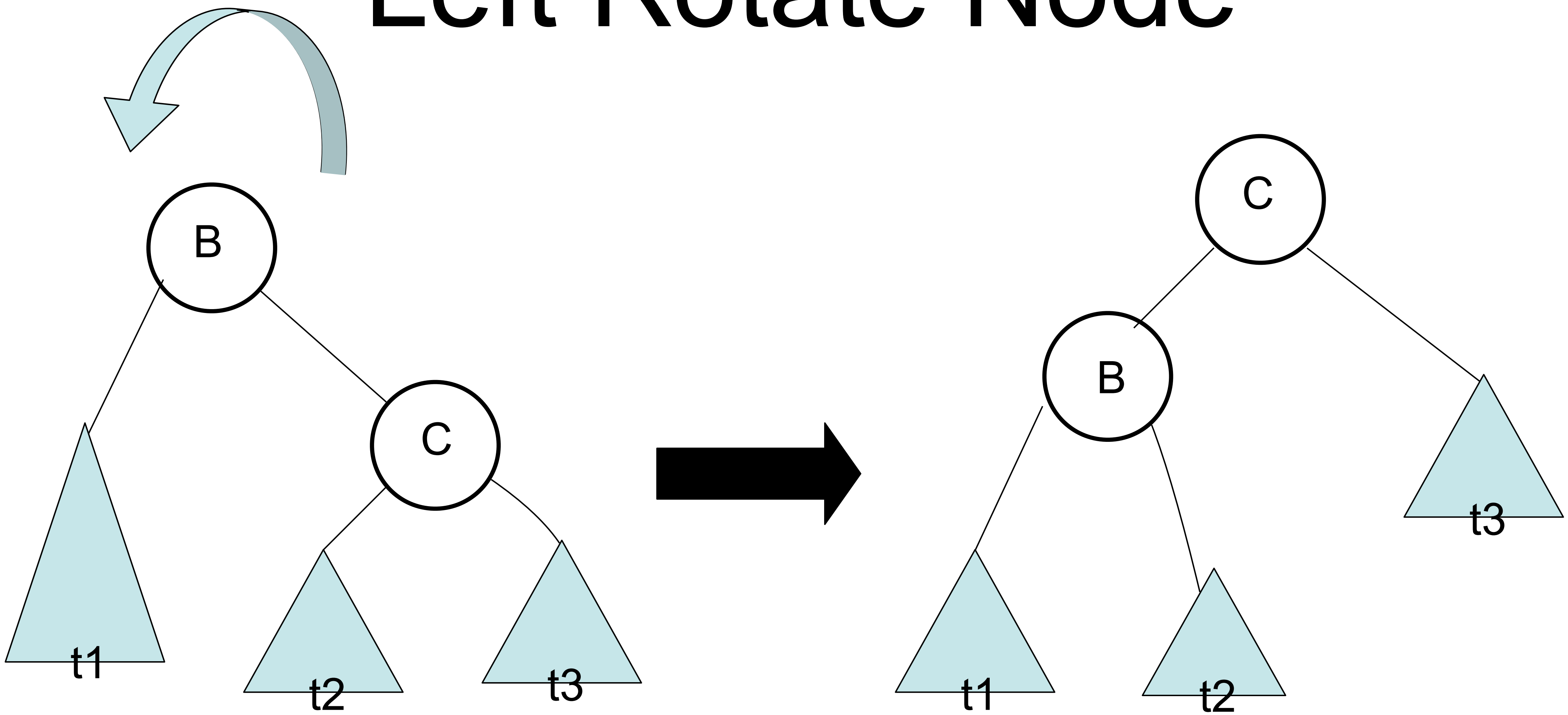
The Other Case 2: Sibling of Parent is Black, x an inside node of grandparent



And go to Case 3, with P as new X

Here we right-rotate P.

Left Rotate Node



In text book, this is called “rotate node with right child”

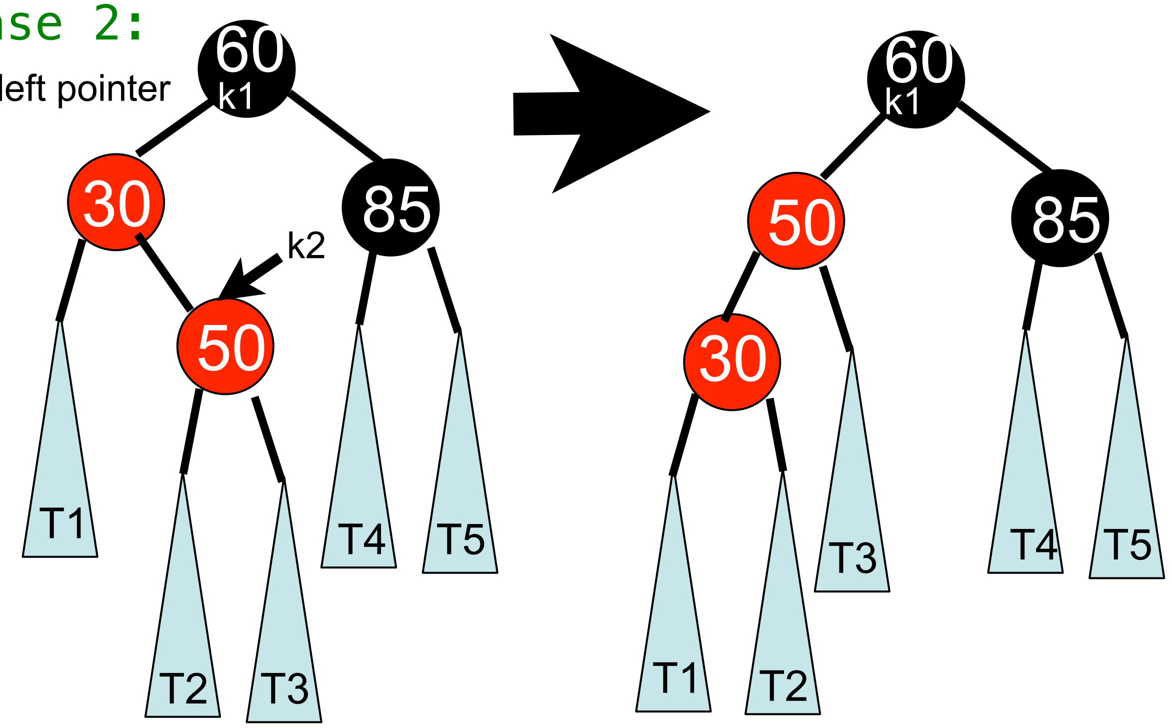
```
// Rotate binary tree node with right child
template <class Comparable>
void RedBlackTree<Comparable>::leftRotate( Node * & k1 ) const
{
    Node *k2 = k1->right;
    k1->right = k2->left;
    k2->left = k1;
    k1 = k2;
}
```

```
template< class Comparable>
class RedBlackTree
{
    enum { RED, BLACK };
};
```

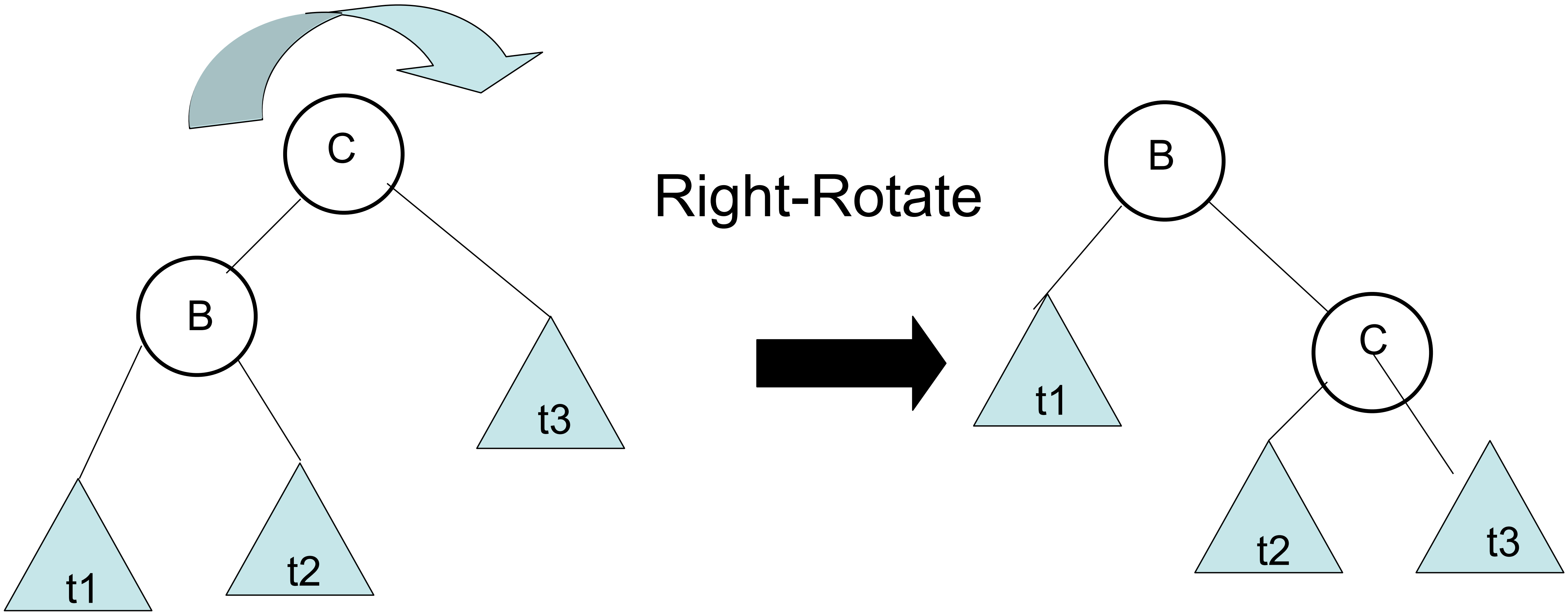
```
template <class Comparable>
class RedBlackNode
{
    Comparable    element;
    RedBlackNode *left;
    RedBlackNode *right;
    int           color;

    // constructors removed
};
```

Case 2:

k1 is the left pointer



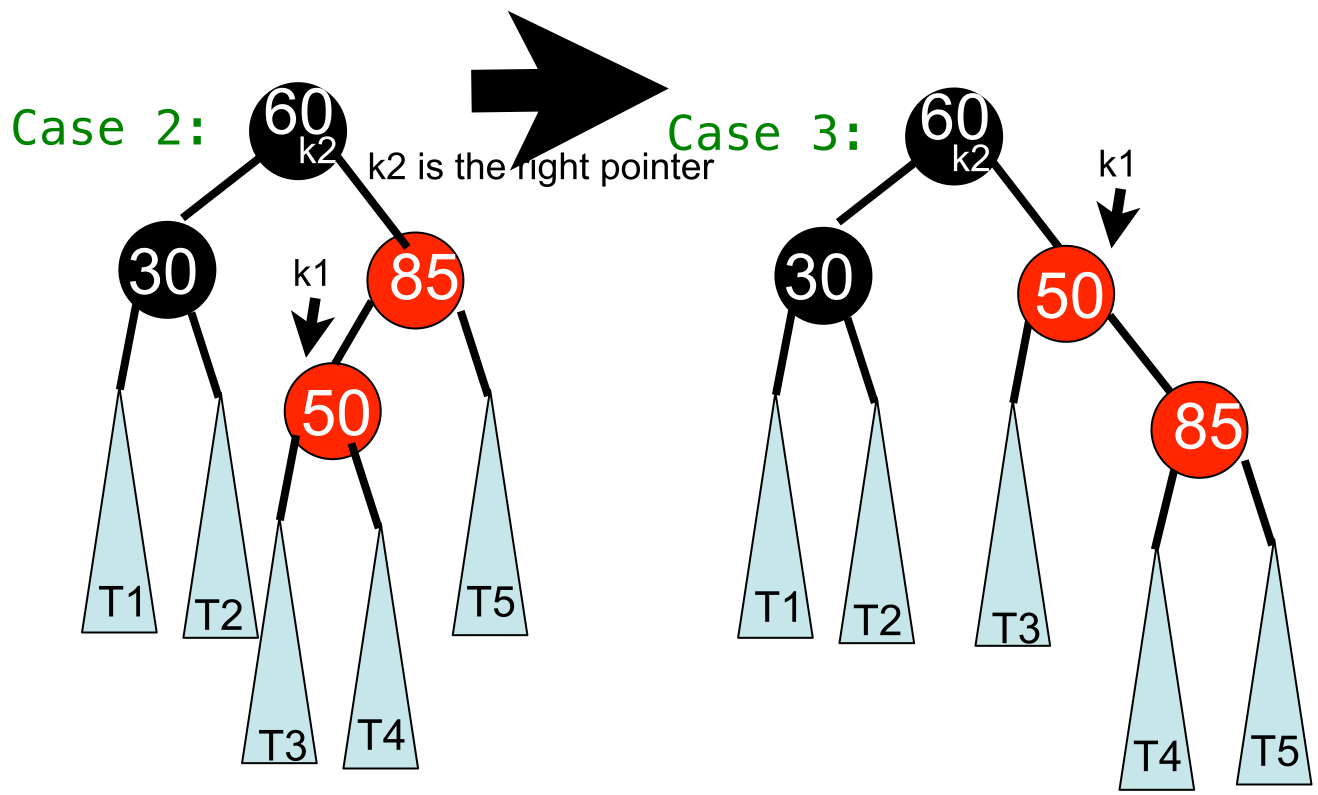
Right Rotate Node



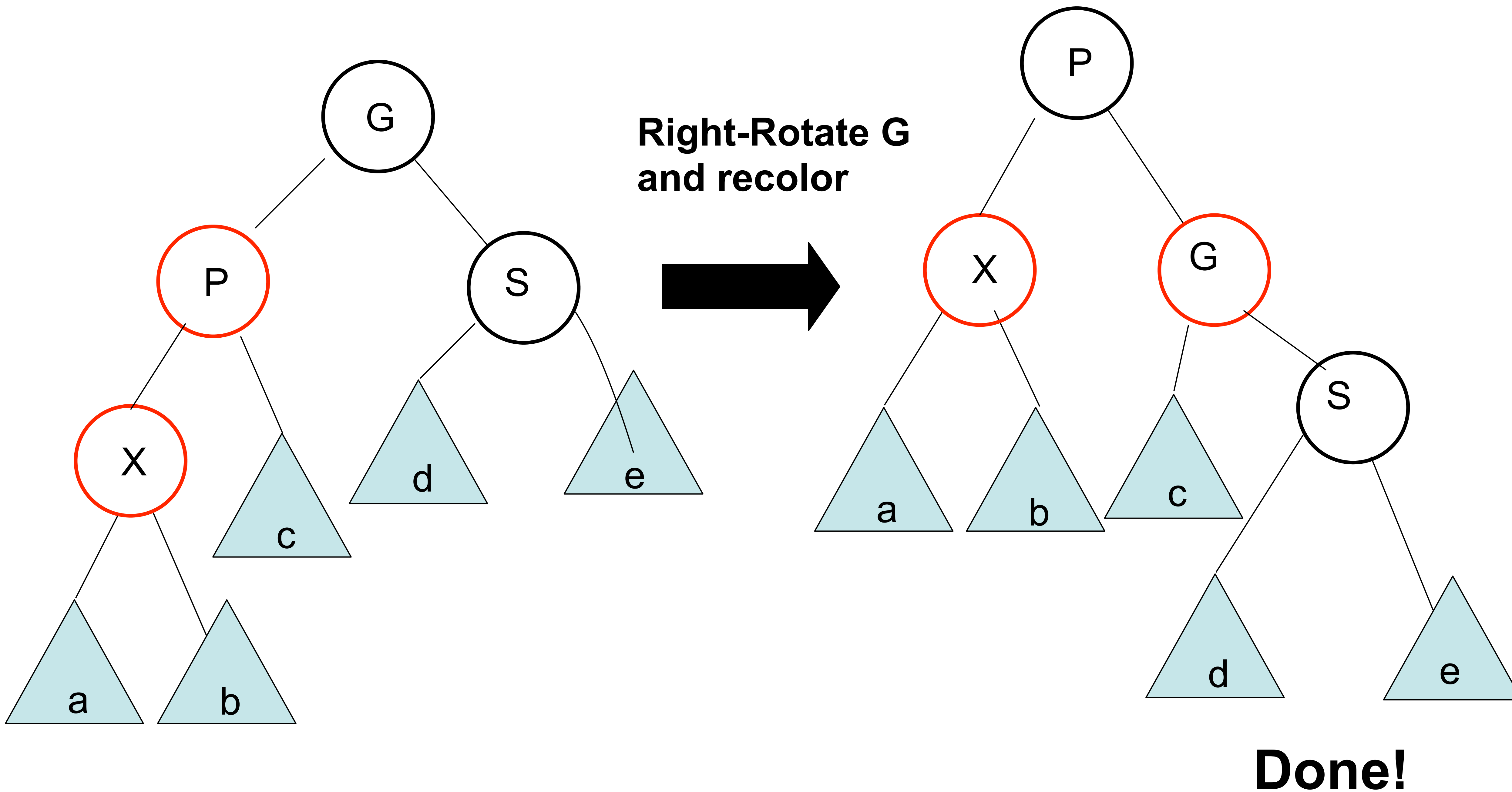
In text book, this is called “rotate node with left child”

Rotate with Left Child

```
// Rotate binary tree node with left child.
template <class Comparable>
void RedBlackTree<Comparable>::rightRotate( Node * & k2 ) const
{

            Homework!

}
```

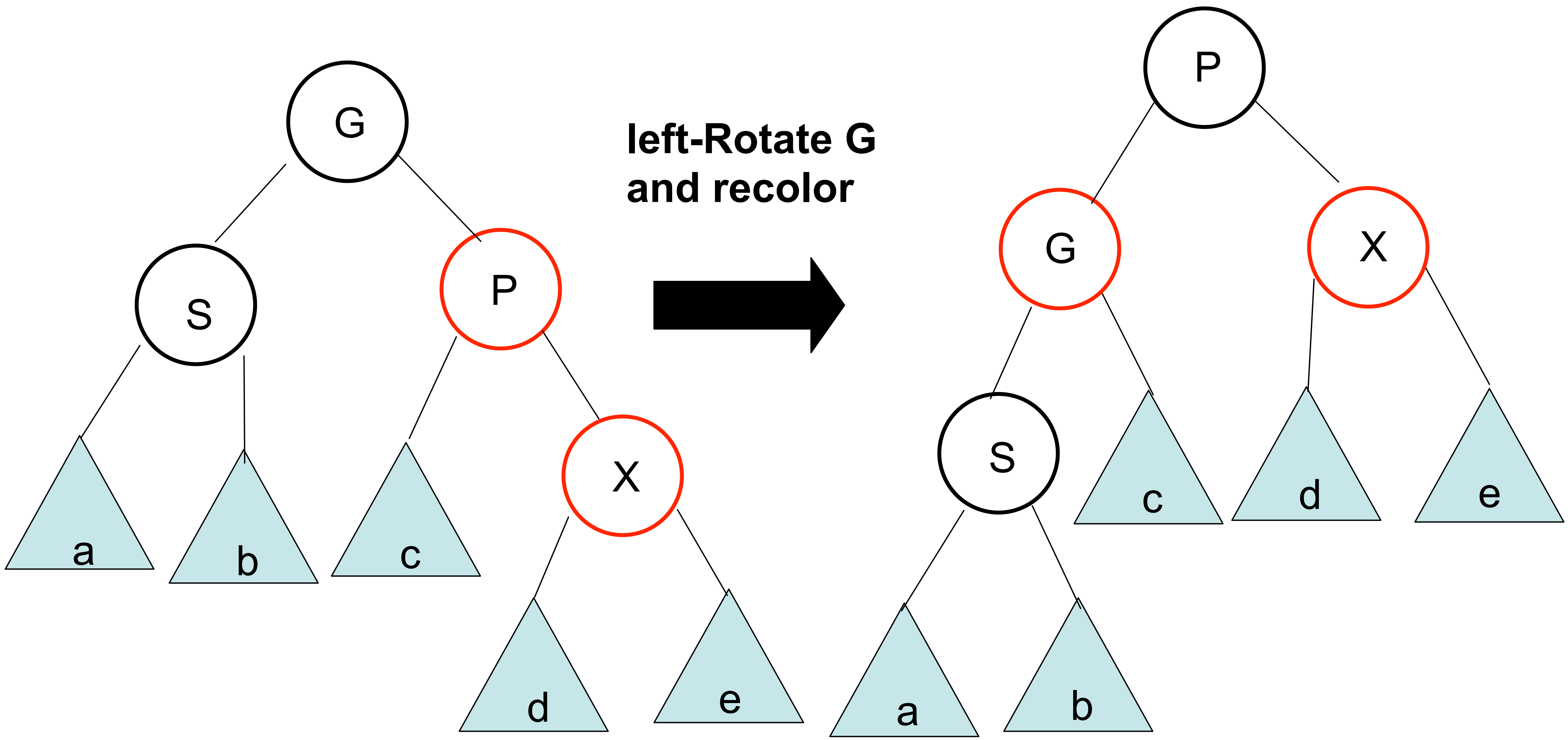


Case 3: Sibling of Parent is Black, x an outside node of grandparent



Here we right-rotate G. Left-rotate in Case 3 if P,X are in right subtree of G.

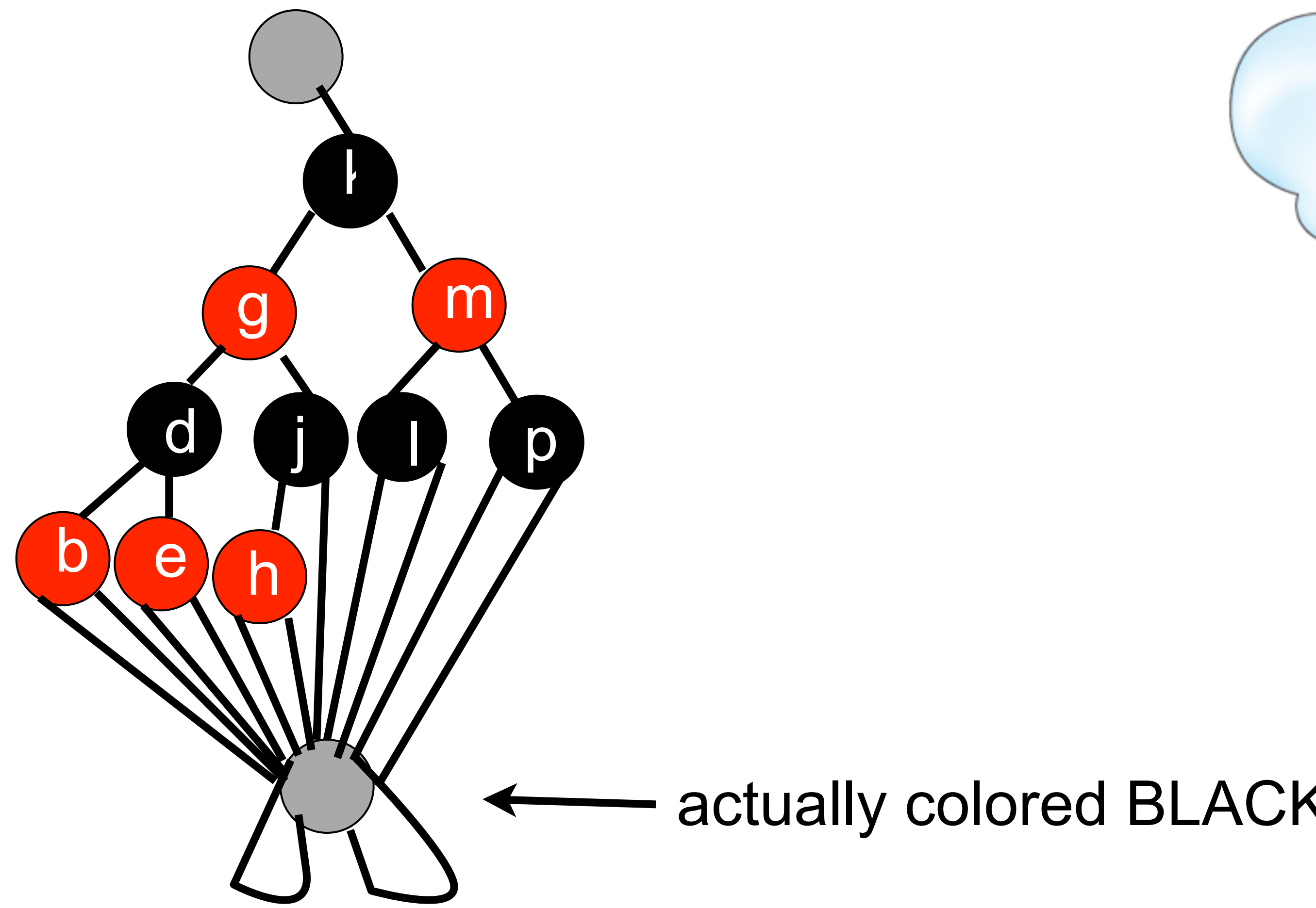
The OtherCase 3: Sibling of Parent is Black, x an outside node of grandparent



Done!

Here we left-rotate G.

Trick to Simplify the Coding of the Red-Black Tree



Coding trick: nullNode points to itself

Insert (very basic)

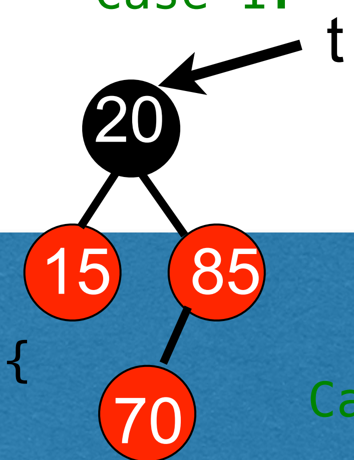
```
template <class Comparable>
void RedBlackTree<Comparable>::insert( const Comparable & x, Node * & t )
{
   if( t == nullNode )
   {
      t = new Node( x, nullNode, nullNode, RED );
      return;
   }

   if( x < t->element)
     insert( x, t->left);
   else if ( x > t->element )
     insert(x, t->right);
   else if ( t->element == x )
     throw DuplicateItemException( );

   // Case 1: t is grand parent
   if( t->left->color == RED && t->right->color == RED ){
     colorFlip(t); // might do unnecessary change....
        return;
   }

   // Case 2: t is grand parent
   if ( t->left->color == RED && t->left->right->color == RED )
     leftRotate( t->left );
   if ( t->right->color == RED && t->right->left->color == RED )
     rightRotate( t->right );

   // Case 3: t is grand parent
   if ( t->left->color == RED && t->left->left->color == RED )
     rightRotateRecolor( t );
   if ( t->right->color == RED && t->right->right->color == RED )
    leftRotateRecolor( t );
}
```

The driver for the insert method ensures the root node is black

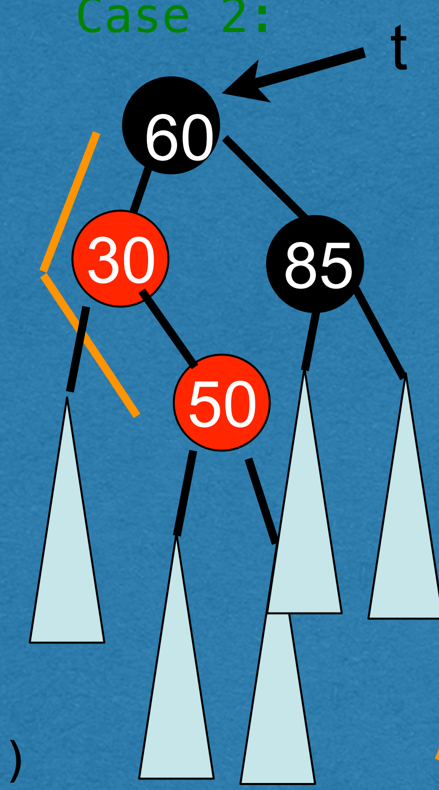
Check out these other implementations of red-black trees:
http://www.cs.princeton.edu/~rs/talks/LLRB/LLRB.pdf

http://www.eternallyconfuzzled.com/tuts/datastructures/jsw_tut_rbtree.aspx

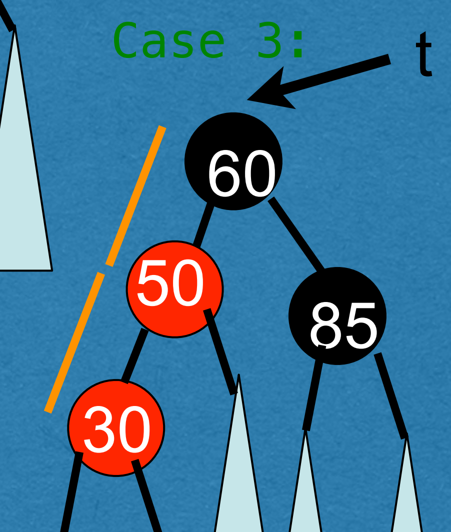
Case 1:



Case 2:



Case 3:



Maps

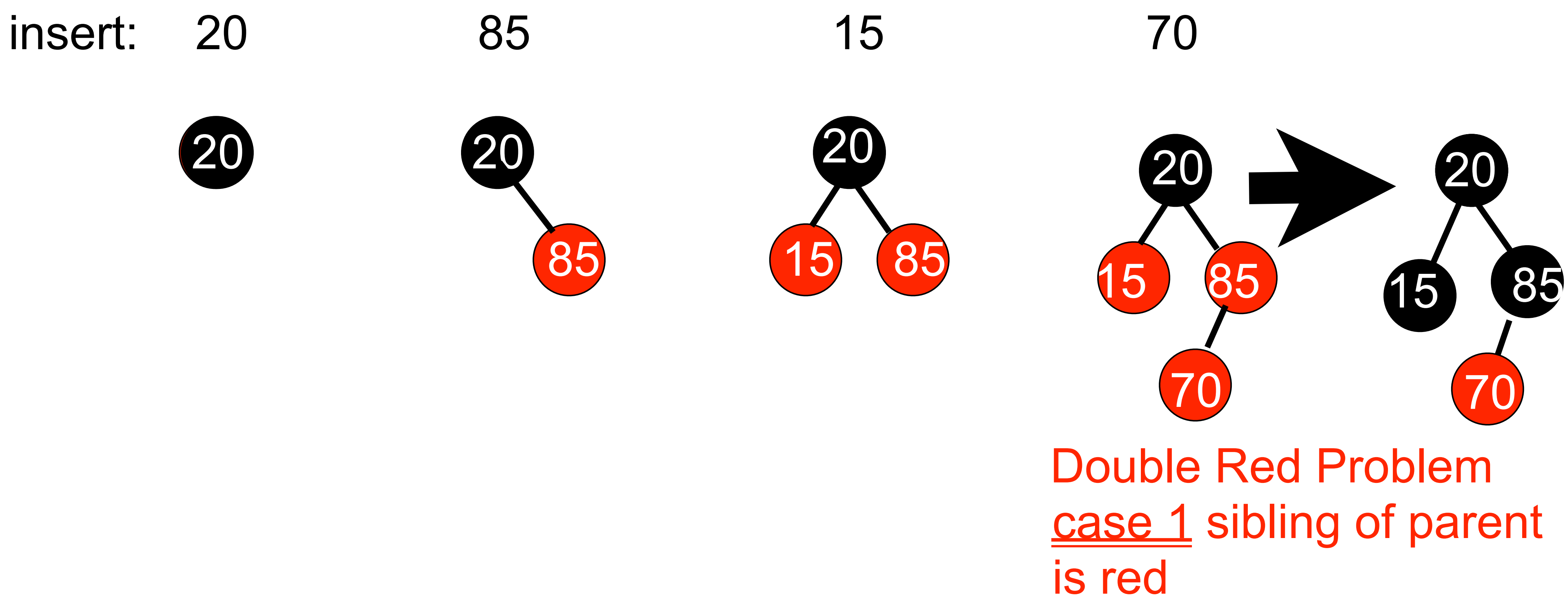
- Objects in a set can be inserted or deleted, but keys cannot be modified, since modification may destroy sorted order
- Maps associate unique key with data
- Can insert/delete (key,data) pairs
- Can modify data in an object that’s in the map
- Cannot modify key of object in the map

Map - Bidirectional Iterator

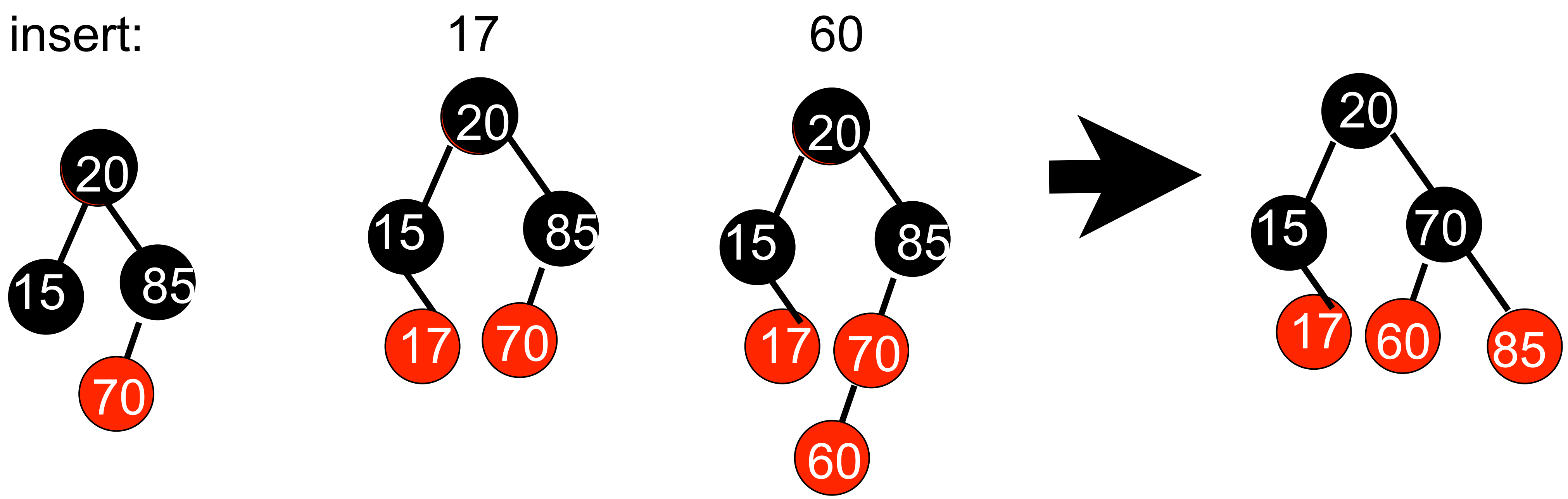
- m.insert(pair) O(log(n))
- m.find(key) O(log(n))
- m.size() O(1)
- m.begin() O(1)
- m.end() O(1)
- m.lower_bound(key) O(log(n))
- m.upper_bound(key) O(log(n))
- m[key] O(log(n))
- m.clear() O(n)
- m.erase(key) & m.erase(iterator) O(log(n))
  O(1) amortized

data structure	build	insert	find
vector	O(n)	O(1)	O(n)
sorted vector	O(n log n)	O(n)	O(log n)
set or map	O(n log n)	O(log n)	O(log n)
list	O(n)	O(1)	O(n)
sorted list	O(n log n)	O(n)	O(n)

Insertion into a Red-Black Tree



Insertion into a Red-Black Tree



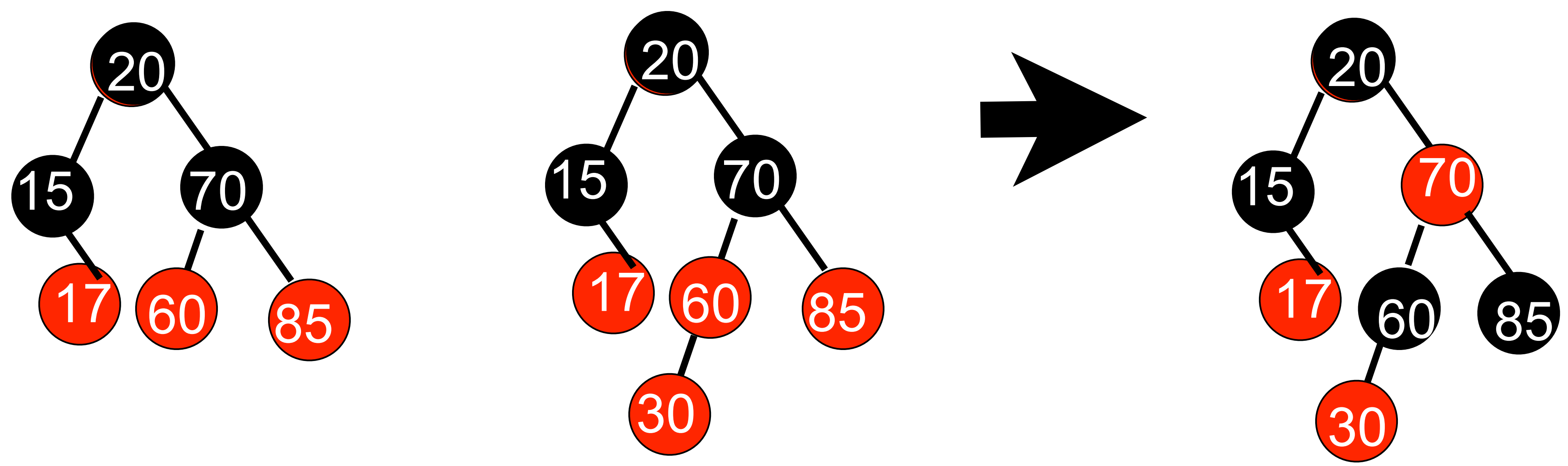
Double Red Problem

case 3 sibling of parent is BLACK or nil

Right Rotate

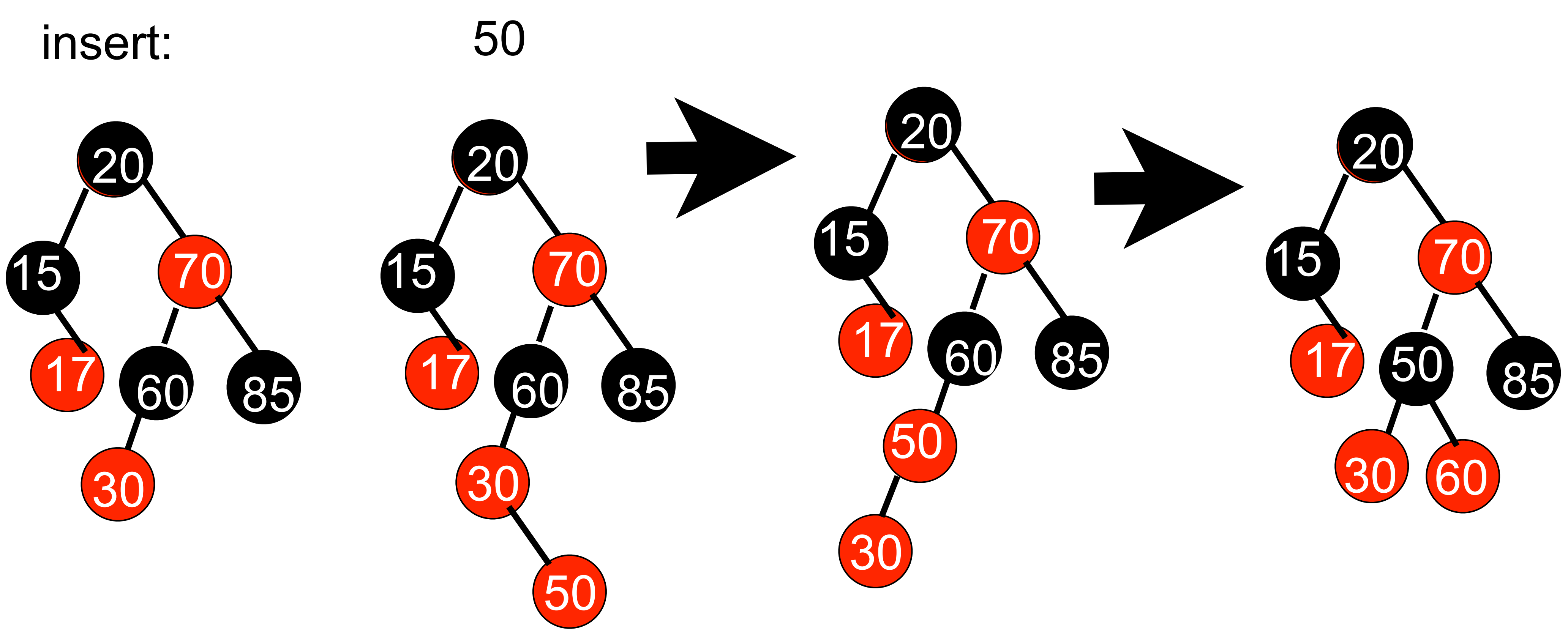
Insertion into a Red-Black Tree

insert: 30



Double Red Problem
case 1 sibling of parent
is RED

Insertion into a Red-Black Tree



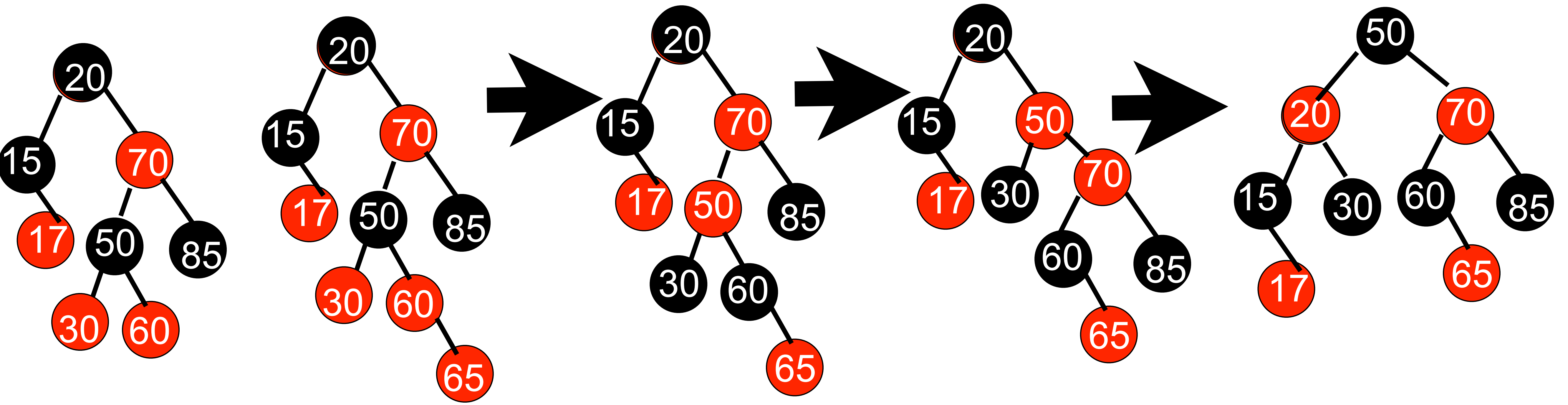
Double Red Problem
case 2 sibling of parent
is BLACK or nil

Double Red Problem
case 3 sibling of parent
is BLACK or nil

Right Rotate

Insertion into a Red-Black Tree

insert: 65

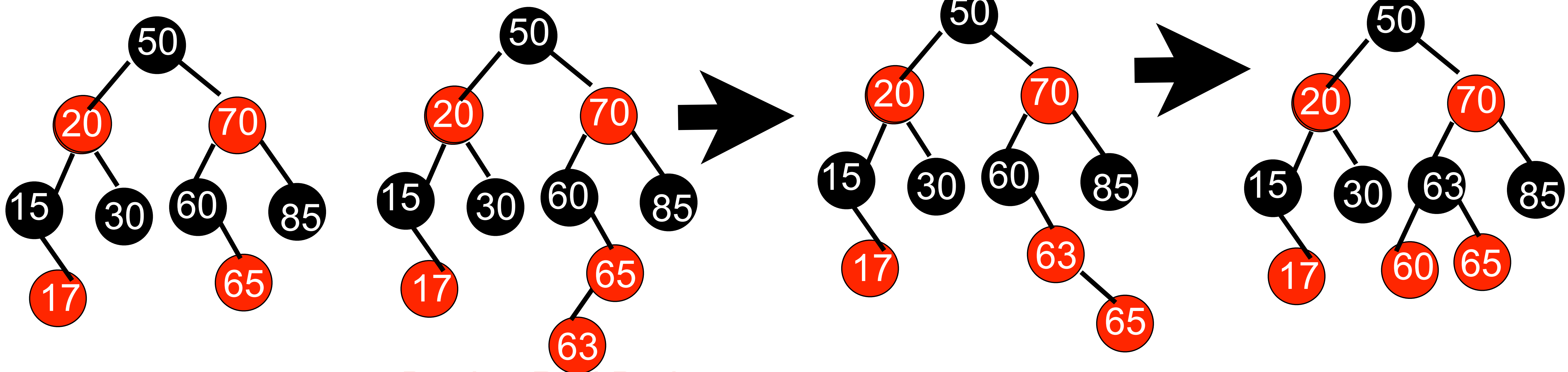


Double Red Problem
case 1 sibling of parent is RED

Double Red Problem
case 2 sibling of parent is BLACK or nil

Double Red Problem
case 3 sibling of parent is BLACK or nil

insert: 63

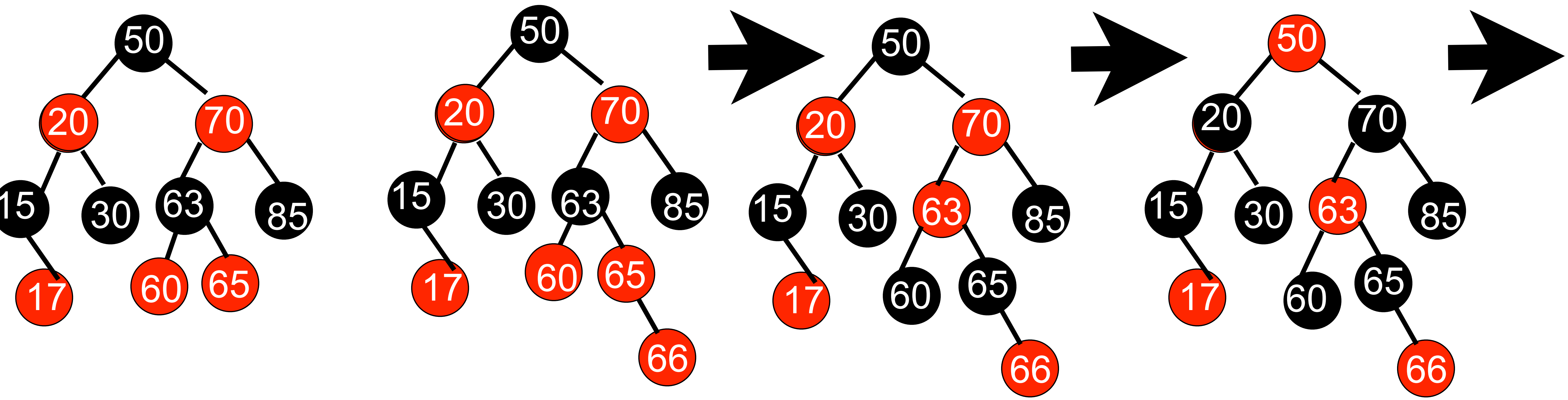


Double Red Problem
case 2 sibling of parent is BLACK or nil

Double Red Problem
case 3 sibling of parent is BLACK or nil

insert:

66



Double Red Problem

case 1 sibling of parent is RED

Double Red Problem

case 1 sibling of parent is RED

Root is not red