# Operations Research for Magic Squares

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# 1 Introduction

A magic square of order n is an $n \times n$ square array of non-negative integers having the same sum along each row, column, and diagonal. The easiest way to construct a magic square is by repeating the same number $n \times n$ times, but it is less interesting. In this study, we will use operations research method to construct several kinds of magic squares.

# 2 The Main Results

## 2.1 General Framework

There are many different methods to construct magic squares. We will focus on using operations research method to construct it [1]. Let $S$ denotes the sum of each row, column, and diagonal. Let $x_{ij}$ denotes the entry at row $i$ and column $j$.
Since each row column adds to the same sum, we get $\forall i, j \leq n$

$$\sum_{k=1}^{n} x_{ik} = S$$

and

$$\sum_{k=1}^{n} x_{kj} = S$$

For the diagonals we have

$$\sum_{k=1}^{n} x_{kk} = S$$

and

$$\sum_{k=1}^{n} x_{k,n-k+1} = S$$

## 2.2 Application 1: Lo-shu Magic Square

**Theorem:** magic square contains numbers $1,2,\dots,n^2$ have the line sum $S = n(n^2+1)/2$.

A magic square of order 3 that containing the consecutive integers 1 to 9 is called Lo-shu Magic Square [1]. Notice that $S = n(n^2+1)/2 = 3(10)/2 = 15$.
Variables:

| | | |
|---|---|---|
| $x_{11}$ | $x_{12}$ | $x_{13}$ |
| $x_{21}$ | $x_{22}$ | $x_{23}$ |
| $x_{31}$ | $x_{32}$ | $x_{33}$ |

Row conditions:

$$x_{11} + x_{12} + x_{13} = S$$
$$x_{21} + x_{22} + x_{23} = S$$
$$x_{31} + x_{32} + x_{33} = S$$

Column conditions:

$$x_{11} + x_{21} + x_{31} = S$$
$$x_{12} + x_{22} + x_{32} = S$$
$$x_{13} + x_{23} + x_{33} = S$$

Diagonal conditions:

$$x_{11} + x_{22} + x_{33} = S$$
$$x_{13} + x_{22} + x_{31} = S$$

Since the magic square is made of all nice integers, so $x_{a} \neq x_b, \forall a, b \in (u, v)$, where $a \neq b$.
Objective function:

$$S = 15$$

Solution:

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

Lo-Shu Magic Square

## 2.3 Application 2: Latin Square and Error Correcting Codes

A Latin square of order n is an $n * n$ matrix in which n different symbols occur exactly once in each row and column [2]. To create a Latin square, we can apply the previous method. Notice that diagonal rule is no longer required.
We are going to construct two distinct $3 * 3$ Latin squares as an example.
We assign number 0,1 and 2 as three symbols. Sum of each row and column equal to 3.

Variables: $x_{ij}$ , $0 \leq i, j \leq 3$

Row conditions: $x_{ij} \neq x_{ik}, \forall i, j, k$ where $j \neq k$.

$$x_{11} + x_{12} + x_{13} = S$$
$$x_{21} + x_{22} + x_{23} = S$$
$$x_{31} + x_{32} + x_{33} = S$$

Column conditions: $x_{ij} \neq x_{kj}, \forall i, j, k$ where $i \neq k$.

$$x_{11} + x_{21} + x_{31} = S$$
$$x_{12} + x_{22} + x_{32} = S$$
$$x_{13} + x_{23} + x_{33} = S$$

Objective function:

$$S = 3$$

Solution A = $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and solution B = $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$ .

**Definition**: Two distinct Lain squares $A = (a_{ij})$ and $B = (b_{ij})$ are orthogonal if $n \times n$ ordered pairs $(a_{ij}, b_{ij})$ are all distinct.

**Definition**: a code *C* of length *n*, size *M* and distance d is referred to an $(n, M, d)$ code [3].

**Theorem**: There exists a *q*-ary $(4, q^2, 3)$ code iff there exists a pair of orthogonal Latin squares of order *q* [3].

Since solution A and solution B are orthogonal Latin squares, we can generate (4, 9, 3) ternary code by pairing orthogonal Latin squares.

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# 3 Operations Research with Matrices

Consider the magic square of order *n* can be represented as a Matrix.

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

**Theorem**: Magic square matrices forms a vector space under addition and scalar-matrix product.

We can construct a magic square of order *n* with line sum $S = s$ by taking linear combinations of magic square matrices $M_1, M_2, \cdots, M_m$ with line sums $a_1, a_2, \cdots, a_m$, such that $s = c_1 a_1 + \cdots + c_m a_m$, Where $c_1, \cdots, c_m$ are arbitrary constants.

## 3.1 Unit Magic Square Matrices

**Definition**: A unit magic square matrix is each row column and diagonal add up to 1 with non-negative entries.

This is an example of a unit magic square matrix of order 5.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Notice: A $3 \times 3$ unit magic square matrix does not exist.

## 3.2 Application 1: $4 \times 4$ Magic Square Sudoku

A $4 \times 4$ magic square sudoku is fascinating not just for mathematician. People are intrigued to see all different numbers in a $4 \times 4$ box where not even every row, column, and diagonal have the same sum. The $2 \times 2$ sub-squares and four corners also add up to the same sum.
We can start by finding all $4 \times 4$ unit magic square matrices. There are total 8 of them. We denote them$\{M_1, M_2, \cdots, M_8\}$.

$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, M_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, M_8 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Notice each matrix $M_i$ has the property rows, columns, diagonals, sub-squares and all corners sums are all equal to 1. Linear combination of these matrices preserves these properties.
In this example we are going to construct a magic square Sudoku of order 4 that uses all integers from 1 to 16. We can calculate the line sum is $\frac{4(4^2+1)}{2} = 34.$

Variables: $c_1, c_2, \cdots, c_8$. Let $\hat{C} = < c_1, c_2, \cdots, c_8 >$
Constrains: There is only one condition because of the nature of sudoku, which is for each entry $a_k$ in Sudoku, $a_i \neq a_j, \forall i \neq j$. This is not a necessary condition for constructing a general $4 \times 4$ magic square.

Objective function:

$$\sum_{i=1}^{8} c_i = 34$$

One of the solutions is:

$$\hat{C} = < 13, 2, 0, 1, 5, 3, 1, 9 >$$

Which means that

$$13M_1 + 2M_2 + M_4 + 5M_5 + 3M_6 + M_7 + 9M_8 = \begin{pmatrix} 8 & 11 & 14 & 1 \\ 13 & 2 & 7 & 12 \\ 3 & 16 & 9 & 6 \\ 10 & 5 & 4 & 15 \end{pmatrix}$$

is a solution.

## 3.3 Application 2: a $4 \times 4$ Magic Square Sudoku with Any Given Sum

For any given sum $S = s$, we can construct by taking linear combination of unit magic squares. Then apply the conditions that no entry repeated, and $\sum c_i = s$.
Notice if $s < 34$, there is no solution.
Take a quick example that s = 57.
We get the solution $\hat{C} = < 16, 7, 5, 1, 5, 8, 6, 9 >$

$$16M_1 + 7M_2 + 5M_3 + M_4 + 5M_5 + 8M_6 + 6M_7 + 9M_8 = \begin{pmatrix} 13 & 16 & 22 & 6 \\ 21 & 7 & 12 & 17 \\ 8 & 24 & 14 & 11 \\ 15 & 10 & 9 & 23 \end{pmatrix}$$

is a magic square Sudoku that line sum equal to 57.

# 4 Conclusions

We present several effective ways to create $3 \times 3$ and $4 \times 4$ magic squares where the total are prescribed by the users. We can also easily construct large matrices by ways.

# 5 References


[1] G. L. Frederick Hillier, Introduction to Operations Research 11th Edition, McGraw-Hill Education, 2020.

[2] F. Swetz, Legacy of the Luoshu: The 4,000 Year Search for the Meaning of the Magic Square of Order Three, A K Peters/CRC Press, 2008.

[3] E. Emanouilidis, »Latin and magic squares,« *International Journal of Mathematical Education in Science and Technology,* årg. 36, nr. 5, pp. 546-549, 2005.

[4] C. X. San Ling, Coding Theory: A First Course, Cambridge University Press, 2004.