Blackjack Strategy Analysis

An In-Depth Statistical Approach

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1 Intro

Blackjack is one of the fairest games in the casino—if you apply a good strategy. There are many options for the player: Hit, Stand, Split, Double Down, and Insurance. In this work, we use tools from Markov Chain theory and conditional expectations to find the basic strategies and expected value of the game. For simplicity, we assume the probability of drawing a particular card is independent of previous cards drawn, or we simply assume an infinite deck. Although there is a minor effect due to cards that have been seen influencing future draws, this effect is negligible. Casinos typically operate the game with 6 to 8 decks, and the mathematics are not drastically different between 8 decks and an infinite deck. However, the basic strategy may differ in a few borderline cases, which we will point out later.

1.1 Game Procedures

In Blackjack, the player and dealer are each initially dealt two cards, with one of the dealer's cards face down. Next, the player adds up their card values: Cards 2 through 9 have values 2 through 9, ten and picture cards have a value of 10, and aces are valued at either 1 or 11.

The player may take additional cards (hit) as long as their total value is under 21. If the total exceeds 21 (bust), the player loses their bet. If the player stops taking cards (stand/stay), the dealer then takes cards until the dealer's total is 17 or higher. If neither busts, the one with a total closer to 21 wins the bet. If the player and dealer have the same total value, it is a tie (push).

If a hand contains an ace and the total can be valued in two ways, it is called a soft hand. For example, an ace and a 6 is a soft 17 because it can also be valued at 7. However, an ace, a 6, and a ten will total 17, as there is only one way to value the hand.

If the player's initial two cards add up to 21 (an ace and a ten-value card), it is called a blackjack. A player with blackjack wins the bet immediately, with the casino paying out at three to two (1.5 to 1). However, if the dealer also has blackjack, the result is a push.

1.2 Actions

We have discussed the hit and stand options. Sometimes, the player has additional options. After receiving their initial two cards, the player can double their bet and receive exactly one more card (double). If the player's two cards have the same value, they can split the cards and play two separate hands for an additional bet. If the player splits two aces, the dealer will give exactly one more card for each ace. If the player's total becomes 21 after a split, it does not count as a blackjack.

Another option is surrender. If the dealer does not have a blackjack, the player can surrender by forfeiting half of their wager, keeping the other half, and ending their hand. This option is only available on the initial two cards. The last option is “insurance.” If the dealer shows an ace, they will offer a side bet called “insurance.” This bet pays 2 to 1 if the dealer has a blackjack; otherwise, it loses.

2 Dealer’s Probability

2.1 Markov Chain Matrix Method

Markov chains are a key type of random process characterized by transitions between states. To define a Markov chain, we need a state space and a transition matrix. In this paper, we develop a Markov chain model to represent the play of a single hand in Blackjack, using absorbing states to calculate the probability that a dealer starting in state i will reach an absorbing state j. [1]

We define each possible total value of the dealer’s hand as a state. Additionally, for hands with a "soft" value (where an ace is counted as 11 without busting), we label these states with an “s” prefix.

The set of states is:

$$U = \{1, 2, 3, ... ,15, 16, s12, s13, s14, s15, s16, 17, 18, 19, 20, 21, bust\}$$

We will construct a transition matrix $P_{27\times 27}$ where each element P_{ij} represents the probability of transitioning from state i to state j.

Specifically,

$$P\_ij = Prob(X\_(n+1) = j|X\_n = i), \forall i, j \in U$$

According to dealer’s procedure, the states 17, 18, 19, 20, 21 and bust are absorbing states. This means $P_{mn} = 0, if\ m \neq n$ and $P_{mn} = 1, if\ m = n$. $\forall m \in \{17, 18, \ldots, 21, bust\}$. Since our goal is to find the optimal play, we assume the dealer does not start with a blackjack. This implies $P_{1,21} = 0$ meaning the probability that a dealer starting with an ace reaches a total of 21 in a single transition is zero.

The transition matrix P is

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & s12 & s13 & s14 & s15 & s16 & 17 & 18 & 19 & 20 & 21 & bust \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ s12 \\ s13 \\ s14 \\ s15 \\ s16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ bust \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{9}&\frac{1}{9}&\frac{1}{9}&\frac{1}{9}&\frac{1}{9}&\frac{1}{9}&\frac{1}{9}&\frac{1}{9}&\frac{1}{9}&0&0 \\
0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{4}{13}&\frac{1}{13}&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{4}{13}&\frac{1}{13}&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&\frac{1}{13}&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&\frac{1}{13}&0 \\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13} \\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{5}{13} \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{6}{13} \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{7}{13} \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{8}{13} \\
0&0&0&0&0&0&0&0&0&0&0&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0 \\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0 \\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0 \\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0 \\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{4}{13}&0&0&0&0&0&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&\frac{1}{13}&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\end{array}
$$

Define $Q = \lim_{n\to\infty} P^n$, Q is a 27×27 matrix with rank seven, which each row represents a stationary distribution π. Additionally, $Q_{ij} = 0$ if $j \notin \{17, 18, 19, 20, 21, bust\}$.

$$
Q = \begin{pmatrix}
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.188917 & 0.188917 & 0.188917 & 0.188917 & 0.0778062 & 0.166525 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.139809 & 0.134907 & 0.129655 & 0.124026 & 0.117993 & 0.353608 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.135034 & 0.130482 & 0.125581 & 0.120329 & 0.1147 & 0.373875 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.13049 & 0.125938 & 0.121386 & 0.116485 & 0.111233 & 0.394468 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.122251 & 0.122251 & 0.1177 & 0.113148 & 0.108246 & 0.416404 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.165438 & 0.106267 & 0.106267 & 0.101715 & 0.0971633 & 0.42315 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.368566 & 0.137797 & 0.0786254 & 0.0786254 & 0.0740737 & 0.262312 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.128567 & 0.359336 & 0.128567 & 0.0693949 & 0.0693949 & 0.244741 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.119995 & 0.119995 & 0.350765 & 0.119995 & 0.0608238 & 0.228425 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.111424 & 0.111424 & 0.111424 & 0.342194 & 0.111424 & 0.212109 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.111424 & 0.111424 & 0.111424 & 0.111424 & 0.342194 & 0.212109 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.103465 & 0.103465 & 0.103465 & 0.103465 & 0.103465 & 0.482673 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0960751 & 0.0960751 & 0.0960751 & 0.0960751 & 0.0960751 & 0.519625 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0892126 & 0.0892126 & 0.0892126 & 0.0892126 & 0.0892126 & 0.553937 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0828402 & 0.0828402 & 0.0828402 & 0.0828402 & 0.0828402 & 0.585799 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0769231 & 0.0769231 & 0.0769231 & 0.0769231 & 0.0769231 & 0.615385 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.151008 & 0.151008 & 0.151008 & 0.151008 & 0.151008 & 0.244958 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.145501 & 0.145501 & 0.145501 & 0.145501 & 0.145501 & 0.272495 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.14001 & 0.14001 & 0.14001 & 0.14001 & 0.14001 & 0.299951 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.134561 & 0.134561 & 0.134561 & 0.134561 & 0.134561 & 0.327196 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.129176 & 0.129176 & 0.129176 & 0.129176 & 0.129176 & 0.354121 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1.
\end{pmatrix}
$$

2.2 Analyzing Dealer's hand

Dealer will start with one of the following states $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Where 1 stands for Ace. We define a function for the dealer, starting with a face-up card $i \in V$, that eventually ends in an absorbing state j.

Define:

$$D(i, j) = Prob(\lim_{n \to \infty} X_n = j | X_0 = i)$$

It is clear that $D(i, j) = Q_{ij}, \forall i \neq 10$. We need to calculate $D(10, j)$ separately because $Q_{10,j}$ includes the probability that dealer gets a blackjack.

$$D(10, j) \; = \; \sum_k D(k, j) \times Prob(X_1 = k | \; X_0 = 10)$$

For example:

$$\begin{aligned} D\ 10, 17 \; &= D\ 12, 17 \; \times \frac{1}{12} + D(13, 17) \times \frac{1}{12} \; + \cdots + D(20, 17) \times \frac{4}{12} \\ &= \; 0.103465 \times \frac{1}{12} + 0.096075 \times \frac{1}{12} + \cdots + \; 0 \times \frac{4}{12} \; = \; 0.12071 \end{aligned}$$

Notice that the likelihood of moving from state 10 to state 12 in one transition is 1/12 instead of 1/13. This adjustment is because Ace is not an option for the next card (Otherwise, it would result in a blackjack).

By repeating this process for $D(10,18), \cdots, D(10,21)$. We can complete a table for $D(i,j)$. The table below shows the values of $D(i,j)$, which represent the probability that the dealer, starting in state i, will eventually reach absorbing state j:

	17	18	19	20	21	Bust
1	0.188917	0.188917	0.188917	0.188917	0.077806	0.166525
2	0.139809	0.134907	0.129655	0.124026	0.117993	0.353608
3	0.135034	0.130482	0.125581	0.120329	0.1147	0.373875
4	0.13049	0.125938	0.121386	0.116485	0.111233	0.394468
5	0.122251	0.122251	0.1177	0.113148	0.108246	0.416404
6	0.165438	0.106267	0.106267	0.101715	0.097163	0.42315
7	0.368566	0.137797	0.078625	0.078625	0.074074	0.262312
8	0.128567	0.359336	0.128567	0.069395	0.069395	0.244741
9	0.119995	0.119995	0.350765	0.119995	0.060824	0.228425
10	0.12071	0.12071	0.12071	0.37071	0.037376	0.229785

Example: to find $D(2,21)$. That is the probability that dealer starts with 2 and ends at 21. We simply look up the value in the 2nd row and 5^{th} column of the table, which is 0.117993.

3 Optimal Play

In this section, we will find the optimal play for the player. Let

$$C = \{4, 5, 6, \ldots, 21, s12, s13, \ldots, s21\}$$

Represent all possible states the player can achieve. Unlike dealers, players do not follow a fixed procedure, so we need to evaluate every state $c \in C$ that player can reach. This set includes both soft and hard totals, accounting for combinations of at least two cards.

3.1 Expectation of Stand a Hand

When the player chooses to stand, their total sum is fixed. The possible outcomes for the player are $\{4,5,6,\ldots,21, bust\}$. There is no need to separately calculate soft totals in this case.

Define:

$$S(i,k) = \mathrm{E}[\text{Player stands at } i \mid \text{dealer has initial card } k] \tag{1}$$

By the law of total expectation:

$$S(i,k) = \sum_{\mathrm{j}} \mathrm{E}[\text{Player stands at } i, \text{Dealer ends at total } j | \text{ dealer has initial value } k] \times \mathrm{Prob}(\text{ Dealer ends at total } j | \text{ dealer has initial value } k)$$

For example:

$$\begin{aligned} S(18,10) &= 1 \times D(10,17) + 0 \times D(10,18) + (-1) \times D(10,19) + (-1) \times D(10,20) \\ &\quad + (-1) \times D(10,21) + 1 \times D(10, bust) \\ &= 0.12071 - 0.12071 - 0.37071 - 0.037376 + 0.22985 = -0.1783 \end{aligned}$$

This result means that if a player stands on 18 while the dealer shows a 10, the player would, on average, lose 17.83% of their total wager.
By repeating the process for all i, k. We can construct a table for $S(i,k)$.
Note that $S(i,k) = S(16,k), \forall i < 16$.

	Dealer's up card									
	Ace	2	3	4	5	6	7	8	9	10
16	-0.6670	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5404
17	-0.4780	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4197
18	-0.1002	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.1783
19	0.2776	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631
20	0.6555	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545
21	0.9222	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626

Example for using the chart:
To find expected value of standing on a total of 18 vs dealer's initial value 9, look up the value on the 3rd row and 9th column, which gives -0.1832. If For $S(14,9)$, since $14 < 16$, we use $S(16,9) = -0.5431$.

3.2 Expectation of Hit and Stand

We define a recurrence function $HS(i,k)$ be the expectation of player is in state i when dealer has a card k shows.

$$HS(i,k) = max[\sum_{j} HS(j,k) \times Prob(X_1 = j | X_0 = i) \ , S(i,k)] \tag{2}$$

e:

$X \in \{4, 5, 6, \ldots, 21, s12, s13, \ldots, s21, bust\}$

$k \in \{Ace, 2, 3, \ldots, 10\}$

y that $HS\ \ bust, k\ \ = \ -1, \forall k$

This definition captures that if the player busts, the expected outcome is a loss, represented by -1.

Prob$(X_1 = j | X_0 = i)$ refers the probability that the player is current in state i, will move to j after drawing one more card.

We can compute $HS(i, k)$ by working backwards, starting from the state where the player has a total of 21. For example, let's calculate HS(21,10):

Since the player is already at a total of 21, they cannot move to any other state except "bust" if they draw another card. Therefore:

$$Prob\ \ X_1 = j | X_0 = 21\ \ = 0\ \forall j \neq bust, Prob\ \ X_1 = bust | X_0 = 21\ \ = 1$$

Thus, the summation simplifies to:

$$\sum_j HS(j, 10) \times Prob(X_1 = j | X_0 = 21) = HS(bust, j) \times 1 = -1$$

Therefore,

$$HS(21, 10) = max(-1, S(21{,}10)) = max(-1, 0.9626) = 0.9626$$

Since HS(21,10) = S(21,10) = 0.9626, the optimal strategy in this case (21 vs. dealer's 10) is to stand.

Then we can calculate $HS(20, 10)$.

$$HS(20, 10)\ = max\{HS\ \ 21, 10\ \ \times \left(\frac{1}{13}\right) + HS\ \ bust, 10\ \ \times \left(\frac{12}{13}\right), S\ \ 20{,}10\ \ \}$$
$$= \max\{0.9626 \times\ \ 1/13\ \ +\ \ -1\ \ ) \times\ \ 12/13\ \ , S(20{,}10)\}$$
$$= max\{\ -0.8490, S\ \ 20, 10\ \ \} = \max\{-0.8490{,}0.5545\} = 0.5545$$

Since HS(20,10) $=$ S(20,10) $=$ 0.5545, the optimal strategy for a total of 20 vs. dealer's 10 is also to stand.

By repeating this process for all values of i and k, we can construct a complete table for $HS(i,k)$.

Dealer's Up card

	2	3	4	5	6	7	8	9	10	Ace
4	-0.1149	-0.0826	-0.0494	-0.0124	0.0111	-0.0883	-0.1593	-0.2407	-0.2892	-0.2531
5	-0.1282	-0.0953	-0.0615	-0.0240	-0.0012	-0.1194	-0.1881	-0.2666	-0.3134	-0.2786
6	-0.1408	-0.1073	-0.0729	-0.0349	-0.0130	-0.1519	-0.2172	-0.2926	-0.3377	-0.3041
7	-0.1092	-0.0766	-0.0430	-0.0073	0.0292	-0.0688	-0.2106	-0.2854	-0.3191	-0.3101
8	-0.0218	0.0080	0.0388	0.0708	0.1150	0.0822	-0.0599	-0.2102	-0.2494	-0.1970
9	0.0744	0.1013	0.1290	0.1580	0.1960	0.1719	0.0984	-0.0522	-0.1530	-0.0657
10	0.1825	0.2061	0.2305	0.2563	0.2878	0.2569	0.1980	0.1165	0.0253	0.0814
11	0.2384	0.2603	0.2830	0.3073	0.3337	0.2921	0.2300	0.1583	0.1195	0.1430
12	-0.2534	-0.2337	-0.2111	-0.1672	-0.1537	-0.2128	-0.2716	-0.3400	-0.3810	-0.3505
13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.2691	-0.3236	-0.3872	-0.4253	-0.3969
14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3213	-0.3719	-0.4309	-0.4663	-0.4400
15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3698	-0.4168	-0.4716	-0.5044	-0.4800
16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4148	-0.4584	-0.5093	-0.5398	-0.5171
17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4197	-0.4780
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.1783	-0.1002
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

Dealer's Up card

soft	2	3	4	5	6	7	8	9	10	Ace
12	0.0818	0.1035	0.1266	0.1565	0.1860	0.1655	0.0951	0.0001	-0.0700	-0.0205
13	0.0466	0.0741	0.1025	0.1334	0.1617	0.1224	0.0541	-0.0377	-0.1049	-0.0573
14	0.0224	0.0508	0.0801	0.1119	0.1392	0.0795	0.0133	-0.0752	-0.1395	-0.0939
15	-0.0001	0.0292	0.0593	0.0920	0.1182	0.0370	-0.0271	-0.1122	-0.1737	-0.1300
16	-0.0210	0.0091	0.0400	0.0734	0.0988	-0.0049	-0.0668	-0.1486	-0.2074	-0.1656
17	-0.0005	0.0290	0.0593	0.0912	0.1281	0.0538	-0.0729	-0.1498	-0.1969	-0.1796
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1007	-0.1438	-0.0929
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

3.3 Expectation of Hit, Stand and Double

Define function $DB(i,k)$ refers to expectation of double when dealer shows an k.

$$DB(i,k) = 2\sum_j \mathrm{E}(\mathit{Player\ stands\ at\ j}|\ \mathit{Dealer\ shows\ k}) \times \mathit{Prob}(X_1 = j|X_0 = i) \tag{3}$$

$$= 2\sum_j S(j,k) \times \mathit{Prob}(X_1 = j|X_0 = i)$$

Then we define function $HSD(i,k)$ which represents the expected value of the current state when the options of hit, stand, and double are all allowed:

$$HSD(i,k) = \max\ \{DB(i,k), HS(i,k)\}$$

If $HSD(i,k) = DB(i,k)$, the optimal play is to double. If $HSD(i,k) = S(i,k)$, the optimal play is to stand. Otherwise, the optimal play is to hit.

Example: To find the optimal play when the player has a total of 16 and the dealer shows a 10, with hit, stand, and double options available:

$$DB\ 16{,}10\ = 2\big(S\ 17{,}10\ \times\ 1/13\ + S\ 18{,}10\ \times\ 1/13\ + \cdots + S\ 21{,}10\ \times\ 1/13\ + S\ \mathit{bust}, 10\ \times\ 8/13\ \big)$$
$$= 2(-0.4197 \times 1/13 + (-0.1783) \times 1/13 + \cdots + 0.9626 \times 1/13 + (-1) \times 8/13) = -1.07965$$

$$HSD\ 16, 10\ = \mathit{max}\{-1.07965, HS\ 16, 10\ \} = \max\{-1.0797, -0.5398\} = -0.5398$$

Since -0.5398 is neither$S$(16,10) nor $DB(16,10)$, the optimal play is to hit.
We repeat the process and construct a complete table for $HSD(i,k)$

Dealer's Up card

	2	3	4	5	6	7	8	9	10	Ace
4	-0.1149	-0.0826	-0.0494	-0.0124	0.0111	-0.0883	-0.1593	-0.2407	-0.2892	-0.2531
5	-0.1282	-0.0953	-0.0615	-0.0240	-0.0012	-0.1194	-0.1881	-0.2666	-0.3134	-0.2786
6	-0.1408	-0.1073	-0.0729	-0.0349	-0.0130	-0.1519	-0.2172	-0.2926	-0.3377	-0.3041

7	-0.1092	-0.0766	-0.0430	-0.0073	0.0292	-0.0688	-0.2106	-0.2854	-0.3191	-0.3101
8	-0.0218	0.0080	0.0388	0.0708	0.1150	0.0822	-0.0599	-0.2102	-0.2494	-0.1970
9	0.0744	0.1208	0.1819	0.2431	0.3171	0.1719	0.0984	-0.0522	-0.1530	-0.0657
10	0.3589	0.4093	0.4609	0.5125	0.5756	0.3924	0.2866	0.1443	0.0253	0.0814
11	0.4706	0.5178	0.5660	0.6147	0.6674	0.4629	0.3507	0.2278	0.1797	0.1430
12	-0.2534	-0.2337	-0.2111	-0.1672	-0.1537	-0.2128	-0.2716	-0.3400	-0.3810	-0.3505
13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.2691	-0.3236	-0.3872	-0.4253	-0.3969
14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3213	-0.3719	-0.4309	-0.4663	-0.4400
15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3698	-0.4168	-0.4716	-0.5044	-0.4800
16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4148	-0.4584	-0.5093	-0.5398	-0.5171
17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4197	-0.4780
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.1783	-0.1002
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

	Dealer's Up card									
	2	3	4	5	6	7	8	9	10	Ace
12	0.0818	0.1035	0.1266	0.1565	0.1860	0.1655	0.0951	0.0001	-0.0700	-0.0205
13	0.0466	0.0741	0.1025	0.1334	0.1797	0.1224	0.0541	-0.0377	-0.1049	-0.0573
14	0.0224	0.0508	0.0801	0.1260	0.1797	0.0795	0.0133	-0.0752	-0.1395	-0.0939
15	-0.0001	0.0292	0.0593	0.1260	0.1797	0.0370	-0.0271	-0.1122	-0.1737	-0.1300
16	-0.0210	0.0091	0.0584	0.1260	0.1797	-0.0049	-0.0668	-0.1486	-0.2074	-0.1656
17	-0.0005	0.0551	0.1187	0.1824	0.2561	0.0538	-0.0729	-0.1498	-0.1969	-0.1796
18	0.1217	0.1776	0.2370	0.2952	0.3815	0.3996	0.1060	-0.1007	-0.1438	-0.0929
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

3.4 Expectation of Hit, Stand, Double and Surrender

This step is straightforward. According to the rules of Blackjack, the expected value of surrendering is -0.5. since the player forfeits half of their wager. We define the function $HSDR(i, k)$, which represents the expected value when the player has a current state iii and the dealer shows an up card k, with surrender as an option.

$$HSDR(i, k) \;=\; max\{HSD(i, k), -0.5\}$$

If $HSDR(i,k) = -0.5$, the optimal play is to surrender. Otherwise, the optimal play remains the same as when surrender is not allowed.

To obtain the $HSDR(i,k)$ table, we can simply modify the $HSD(i,k)$ table by replacing any value below -0.5 with -0.5.

By observation, there are only four situations where this occurs:

15 vs. 10
16 vs. 9
16 vs. 10
16 vs. Ace

Therefore, if the player is initially dealt one of these situations, the optimal play is to surrender.

3.5 Expectation of Split

When the player is initially dealt a pair, the option to split is available. If the player chooses to split, they must place a second bet, and their original hand is split into two separate hands. The player is then dealt one additional card for each hand, playing each hand independently, except when splitting aces. When aces are split, the player receives only one additional card per ace, with no further actions allowed.

In most casinos, players are allowed to split up to 3 hands. For simplicity, in this model, we assume players can split up to 2 hands to make the calculations more manageable.

Let $X = \{ace, 2, 3, \ldots, 21, s12, \ldots, s21, bust\}$, and define the function $Sp\{i,k\}$ as the expected value when the player splits a pair of i while the dealer shows k.

$$Sp(i,k) \;=\; 2\; \times \sum_j HSD(j,k) \times Prob(X_1 = j | X_0 = i), \forall i \neq ace \tag{4}$$

$$Sp(i,k) \;=\; 2\; \times \sum_j S(j,k) \times Prob(X_1 = j | X_0 = ace)$$

Next, we define the decision function $NP(i,k)$, which determines whether or not to split. The possible outcomes are $\{Y, N\}$, where:

$NP(i,k) = Y$ if $Sp(i,k) > HSDR(2i,k)$, meaning the optimal play is to split.

$NP(i,k) = N$ if $Sp(i,k) \leq HSDR(2i,k)$, meaning the optimal play is **not** to split.

Example: When player has a pair of 8s and dealer shows a 10.

$$Sp\ 8{,}10 = 2 \times (HSD(10,10) \times (1/13) + HSD(11,10) \times (1/13) + \ldots + HSD(18,10) \times (4/13) + HSD(s19,10) \times (1/13)) = 2 \times (-0.24474) = -0.4895$$

For $HSDR$(16,10)

$$HSDR(16,10) = -0.5 \text{ (optimal play is to surrender)}$$

Since $Sp(16,10) > HSDR$(16,10), we have $NP(16,10) = Y$, and the optimal play is to split.
Using this process, we can construct the table for $Sp(i,k)$ and also create the table for $NP(i,k)$, summarizing when to split based on expected outcomes.

Dealer's Up card

	2	3	4	5	6	7	8	9	10	Ace
2	-0.0889	-0.0256	0.0429	0.1272	0.1948	-0.0074	-0.1741	-0.3651	-0.4747	-0.4067
3	-0.1382	-0.0639	0.0146	0.1023	0.1694	-0.0678	-0.2297	-0.4152	-0.5214	-0.4559
4	-0.1669	-0.0913	-0.0116	0.0803	0.1460	-0.1294	-0.2865	-0.4664	-0.5691	-0.5062
5	-0.1935	-0.1167	-0.0330	0.0599	0.1243	-0.1918	-0.3440	-0.5183	-0.6176	-0.5571
6	-0.2186	-0.1367	-0.0496	0.0440	0.1079	-0.2568	-0.4023	-0.5703	-0.6662	-0.6083
7	-0.1555	-0.0748	0.0105	0.1000	0.1877	-0.0905	-0.3890	-0.5558	-0.6288	-0.6201
8	0.0193	0.0869	0.1566	0.2283	0.3255	0.2115	-0.0876	-0.4054	-0.4895	-0.3941
9	0.1846	0.2421	0.3015	0.3633	0.4434	0.3700	0.2153	-0.0937	-0.2966	-0.1314
10	0.3650	0.4122	0.4609	0.5125	0.5756	0.5138	0.3959	0.2331	0.0506	0.1629
Aces	0.4706	0.5178	0.5660	0.6147	0.6674	0.4629	0.3507	0.2278	0.1797	0.1091

$Sp(i,k)$ table

Dealer's Up card

Pair	2	3	4	5	6	7	8	9	10	Ace
2	Y	Y	Y	Y	Y	Y	N	N	N	N
3	Y	Y	Y	Y	Y	Y	N	N	N	N
4	N	N	N	Y	Y	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N
6	Y	Y	Y	Y	Y	N	N	N	N	N
7	Y	Y	Y	Y	Y	Y	N	N	N	N
8	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
9	Y	Y	Y	Y	Y	N	Y	Y	N	N
10	N	N	N	N	N	N	N	N	N	N
Aces	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

3.6 Insurance

If the dealer's up card is an Ace, they offer the player a side bet, called "insurance," on whether the dealer's two cards will form a blackjack. Insurance pays 2 to 1 if the dealer has blackjack, but the player loses the insurance bet if the dealer does not.

The probability that the dealer has blackjack when showing an Ace is 4/13. Therefore, the expectation for the insurance bet is

$$\text{Expectation} = 2(4/13) + (-1)(9/13) = -0.0769.$$

Since this expectation is bad, it indicates that insurance is a poor bet. The optimal play is not to take insurance.

3.7 Blackjack Basic Strategy

By combining the results from sections 3.1 to 3.6, we can determine the optimal strategy for all possible situations by selecting the action with the highest expected value. The following abbreviations are used:

- **S**: Stand
- **H**: Hit
- **D**: Double if allowed; otherwise, Stand
- **Dh**: Double if allowed; otherwise, Hit
- **R**: Surrender

(No insurance)

Differences for 6-8 Decks

For Blackjack with 6-8 decks, the strategy differs slightly in some marginal cases:

- **Soft 15 vs 4**: "Dh" instead of "H"
- **Soft 13 vs 5**: "Dh" instead of "H"

All other strategies remain the same.

Dealer's Up card

Hard	2	3	4	5	6	7	8	9	10	Ace
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	Dh	Dh	Dh	Dh	H	H	H	H	H
10	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	H	H
11	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	H
12	H	H	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H
14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	R	H
16	S	S	S	S	S	H	H	R	R	R
17	S	S	S	S	S	S	S	S	S	S
18	S	S	S	S	S	S	S	S	S	S
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

Soft	2	3	4	5	6	7	8	9	10	Ace
12	H	H	H	H	H	H	H	H	H	H
13	H	H	H	H	Dh	H	H	H	H	H
14	H	H	H	Dh	Dh	H	H	H	H	H
15	H	H	H	Dh	Dh	H	H	H	H	H
16	H	H	Dh	Dh	Dh	H	H	H	H	H
17	H	Dh	Dh	Dh	Dh	H	H	H	H	H
18	S	Ds	Ds	Ds	Ds	S	S	H	H	H
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

Pair	2	3	4	5	6	7	8	9	10	Ace
2	Y	Y	Y	Y	Y	Y	N	N	N	N
3	Y	Y	Y	Y	Y	Y	N	N	N	N
4	N	N	N	Y	Y	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N
6	Y	Y	Y	Y	Y	N	N	N	N	N
7	Y	Y	Y	Y	Y	Y	N	N	N	N

8	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
9	Y	Y	Y	Y	Y	N	Y	Y	N	N
10	N	N	N	N	N	N	N	N	N	N
Aces	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

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4 Expectation of Blackjack

If players follow optimal strategy, Blackjack is the fairest game in the casino. In this section, we will compute the expected value of the game by calculating the probability of each possible starting state for the player.

The probability of drawing a specific initial pair of two cards is $1/13^2$ if neither card is a 10-value card, and $4/13^2$ if one card is a 10-value card. We will calculate pairs separately, as splitting may be an option.

Example Calculation:

The probability of starting with a hard total of 5 is:

$Prob(first\ is\ card\ 2\ and\ second\ card\ is\ 3) + Prob(first\ card\ is\ 3\ and\ second\ card\ is\ 2) = 1/13^2 + 1/13^2 = 0.011834$

By repeating this process, we can find the probabilities of all possible starting hands, excluding blackjack, pairs, and soft hands.

Hard	Probability
5	0.01183
6	0.01183
7	0.02367
8	0.02367
9	0.03550
10	0.03550
11	0.04734
12	0.08284
13	0.08284
14	0.07101
15	0.07101
16	0.05917

17	0.05917
18	0.04734
19	0.04734

We define function $I(i,k)$ be the probability that the player's initial state is iii and the dealer's up card is k.

$$I(i,k) \;=\; Prob(player\ starts\ at\ i\ and\ dealer\ has\ an\ up\ card\ k) \qquad (5)$$

Example Calculation:

$$I(5,9) = Prob(player\ starts\ at\ 5\ and\ dealer\ has\ an\ up\ card\ 9)$$

$$= Prob(player\ starts\ at\ 5) \times Prob(dealer\ has\ an\ up\ card\ 9) = 0.01183 \times (1/13)$$
$$= 0.00091$$

Since this is the simplified model, the likelihood of drawing each card is considered an independent event.

Now we construct the table for $I(i,k)$. We need to be careful when dealer starts with a 10 or ace.

$I(5{,}10) \;=\; 0.01183 \times (4/13) \times (12/13) \;=\; 0.00336$. Here, we multiply by 12/13 at the end because this is the probability that the dealer's hole card is not an Ace, preventing the dealer from having blackjack.

Below is the table of probabilities $I(i,k)$, representing the likelihood of the player starting with a hard hand total iii while the dealer shows an up card k:

	Dealer's up card									
Hard	2	3	4	5	6	7	8	9	10	Ace
5	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
6	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
7	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00672	0.00126
8	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00672	0.00126
9	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.01008	0.00189
10	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.01008	0.00189

11	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252
12	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.02353	0.00441
13	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.02353	0.00441
14	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.02017	0.00378
15	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.02017	0.00378
16	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.01681	0.00315
17	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.01681	0.00315
18	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252
19	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252

Note that the sum of all values in this table is 0.676447, representing the probability that the player is initially dealt a hard hand.

Next, we will calculate the probabilities for soft hands and pairs using the same method. By considering the probability of drawing each card (with 1/13 for non-10-value cards and 4/13 for 10-value cards) and adjusting for the odds of the dealer not getting blackjack (multiplying by 12/13 if the dealer's up card is 10 and 9/13 if the dealer's up card is Ace), we can complete the table for $I(i,k)$, where i includes:

- Hard totals $\{5,6,\ldots,19\}$
- Soft totals $\{s13, s14, \ldots, s21\}$
- Pairs $\{pair\ aces,\ pair\ 2,\ \ldots,\ pair\ 10\}$

This covers all possible starting hands for the player.

	Dealer's up card									
soft	2	3	4	5	6	7	8	9	10	Ace
13	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
14	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
15	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
16	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
17	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
18	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
19	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
20	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
21	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252

Pair										
2	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
3	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
4	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
5	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
6	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
7	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
8	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
9	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
10	0.00728	0.00728	0.00728	0.00728	0.00728	0.00728	0.00728	0.00728	0.02689	0.00504
Aces	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032

Now, by summing all values in the probability charts for the player's initial hands (hard hands, soft hands, and pairs), the total is 0.952660. The remaining probability corresponds to the dealer getting a blackjack, which we calculate as follows:

$$\text{Prob}(\textit{Dealer gets blackjack}) = 2 \times (1/13 \times 4/13) = 0.04734$$

Adding $0.95266 + 0.04734 = 1$, confirming that we have accounted for all possible starting scenarios, completing the probability chart.

Next, we calculate the expected values of each starting hand for the player against the dealer's up card. Define function $EV(i,k)$ as the expected value of each starting hand i vs. dealer's up card.

Definitions of $EV(i,k)$:

1. If $i \neq s21$ (blackjack) or pairs, then:
$$EV\ \ i,k\ \ = I(j,k) \times HSDR(i,k)$$
2. If the player has a pair of m and ($NP\ \ m,k\ \ = N$ (do not split), then:
$$EV\ \ i,k\ \ = I(i,k) \times HSDR(2m,k)$$
3. If the player has a pair of m and ($NP\ \ m,k\ \ = Y$ (split), then:
$$EV\ \ i,k\ \ = I(i,k) \times Sp(j,k)$$
4. If $i = s21$ (Blackjack), then:
$$EV\ \ i,k\ \ = 1.5$$

With these definitions, we can fill in the table for $EV(i,k)$, calculating the expected values for each possible starting hand.

Dealer's up card

Hard	2	3	4	5	6	7	8	9	10	Ace
5	-1.17E-04	-8.68E-05	-5.60E-05	-2.18E-05	-1.08E-06	-1.09E-04	-1.71E-04	-2.43E-04	-1.05E-03	-1.76E-04
6	-1.28E-04	-9.77E-05	-6.64E-05	-3.18E-05	-1.18E-05	-1.38E-04	-1.98E-04	-2.66E-04	-1.14E-03	-1.92E-04
7	-1.99E-04	-1.39E-04	-7.83E-05	-1.32E-05	5.31E-05	-1.25E-04	-3.83E-04	-5.20E-04	-2.14E-03	-3.91E-04
8	-3.97E-05	1.46E-05	7.06E-05	1.29E-04	2.09E-04	1.50E-04	-1.09E-04	-3.83E-04	-1.68E-03	-2.48E-04
9	2.03E-04	3.30E-04	4.97E-04	6.64E-04	8.66E-04	4.69E-04	2.69E-04	-1.42E-04	-1.54E-03	-1.24E-04
10	9.80E-04	1.12E-03	1.26E-03	1.40E-03	1.57E-03	1.07E-03	7.83E-04	3.94E-04	2.55E-04	1.54E-04
11	1.71E-03	1.89E-03	2.06E-03	2.24E-03	2.43E-03	1.69E-03	1.28E-03	8.29E-04	2.42E-03	3.60E-04
12	-1.61E-03	-1.49E-03	-1.34E-03	-1.07E-03	-9.79E-04	-1.36E-03	-1.73E-03	-2.17E-03	-8.97E-03	-1.55E-03
13	-1.87E-03	-1.61E-03	-1.34E-03	-1.07E-03	-9.79E-04	-1.71E-03	-2.06E-03	-2.47E-03	-1.00E-02	-1.75E-03
14	-1.60E-03	-1.38E-03	-1.15E-03	-9.13E-04	-8.40E-04	-1.75E-03	-2.03E-03	-2.35E-03	-9.40E-03	-1.66E-03
15	-1.60E-03	-1.38E-03	-1.15E-03	-9.13E-04	-8.40E-04	-2.02E-03	-2.28E-03	-2.58E-03	-1.01E-02	-1.82E-03
16	-1.33E-03	-1.15E-03	-9.61E-04	-7.61E-04	-7.00E-04	-1.89E-03	-2.09E-03	-2.28E-03	-8.40E-03	-1.58E-03
17	-6.96E-04	-5.34E-04	-3.67E-04	-2.05E-04	5.34E-05	-4.86E-04	-1.74E-03	-1.93E-03	-7.05E-03	-1.51E-03
18	4.43E-04	5.40E-04	6.40E-04	7.27E-04	1.03E-03	1.45E-03	3.86E-04	-6.67E-04	-2.40E-03	-2.53E-04
19	1.41E-03	1.47E-03	1.54E-03	1.60E-03	1.81E-03	2.24E-03	2.16E-03	1.05E-03	8.49E-04	7.00E-04

soft

13	4.25E-05	6.75E-05	9.33E-05	1.21E-04	1.64E-04	1.11E-04	4.92E-05	-3.43E-05	-3.52E-04	-3.61E-05
14	2.04E-05	4.63E-05	7.29E-05	1.15E-04	1.64E-04	7.24E-05	1.21E-05	-6.84E-05	-4.69E-04	-5.92E-05
15	-1.10E-07	2.65E-05	5.40E-05	1.15E-04	1.64E-04	3.37E-05	-2.46E-05	-1.02E-04	-5.84E-04	-8.19E-05
16	-1.91E-05	8.25E-06	5.32E-05	1.15E-04	1.64E-04	-4.45E-06	-6.08E-05	-1.35E-04	-6.97E-04	-1.04E-04
17	-4.47E-07	5.02E-05	1.08E-04	1.66E-04	2.33E-04	4.90E-05	-6.64E-05	-1.36E-04	-6.62E-04	-1.13E-04
18	1.11E-04	1.62E-04	2.16E-04	2.69E-04	3.47E-04	3.64E-04	9.65E-05	-9.17E-05	-4.83E-04	-5.86E-05
19	3.52E-04	3.68E-04	3.85E-04	4.00E-04	4.52E-04	5.61E-04	5.41E-04	2.62E-04	2.12E-04	1.75E-04
20	5.83E-04	5.92E-04	6.02E-04	6.10E-04	6.41E-04	7.04E-04	7.21E-04	6.90E-04	1.86E-03	4.13E-04
21	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	2.02E-02	3.78E-03

Pair

2	-4.05E-05	-1.17E-05	1.95E-05	5.79E-05	8.87E-05	-3.37E-06	-7.25E-05	-1.10E-04	-4.86E-04	-7.97E-05
3	-6.29E-05	-2.91E-05	6.66E-06	4.66E-05	7.71E-05	-3.08E-05	-9.89E-05	-1.33E-04	-5.68E-04	-9.58E-05

4	-9.92E-06	3.64E-06	1.77E-05	3.65E-05	6.64E-05	3.74E-05	-2.73E-05	-9.57E-05	-4.19E-04	-6.21E-05
5	1.63E-04	1.86E-04	2.10E-04	2.33E-04	2.62E-04	1.79E-04	1.30E-04	6.57E-05	4.25E-05	2.57E-05
6	-9.95E-05	-6.22E-05	-2.26E-05	2.00E-05	4.91E-05	-9.69E-05	-1.24E-04	-1.55E-04	-6.40E-04	-1.10E-04
7	-7.08E-05	-3.40E-05	4.78E-06	4.55E-05	8.54E-05	-4.12E-05	-1.69E-04	-1.96E-04	-7.84E-04	-1.39E-04
8	8.78E-06	3.95E-05	7.13E-05	1.04E-04	1.48E-04	9.63E-05	-3.99E-05	-1.85E-04	-8.23E-04	-1.24E-04
9	8.40E-05	1.10E-04	1.37E-04	1.65E-04	2.02E-04	1.82E-04	9.80E-05	-4.26E-05	-3.00E-04	-3.16E-05
10	4.66E-03	4.74E-03	4.81E-03	4.88E-03	5.13E-03	5.63E-03	5.77E-03	5.52E-03	1.49E-02	3.30E-03
Aces	2.14E-04	2.36E-04	2.58E-04	2.80E-04	3.04E-04	2.11E-04	1.60E-04	1.04E-04	3.02E-04	3.44E-05

Summing all values in this table gives a total expected value of 0.0402480.0402480.040248, which does not yet account for the dealer's blackjack probability.

$$Prob(Dealer\ gets\ Blackjack\ and\ Player\ does\ not)$$
$$= (2 \times 1/13 \times 4/13) \times (1 - 2 \times 1/13 \times 4/13)$$
$$= 0.0451$$

The expected value when the dealer has Blackjack, and the player does not is:

$$\mathrm{E}[Dealer\ gets\ a\ Blackjack\ and\ Player\ does\ not] = -0.0451$$

The total expectation for an infinite-deck Blackjack game, after accounting for the dealer's blackjack, is

$$0.040248 - 0.045096 = -0.004849$$

This result of -0.004849 reflects a slight house edge, confirming that Blackjack is close to fair when optimal strategy is used but still favors the house by a small margin.

5 Conclusion

This project presents an in-depth analysis of Blackjack strategy using statistical modeling, probability calculations, and Markov chains. We calculated expected values for various player hands against different dealer up cards, identifying strategies that help players minimize losses and, at times, maximize their winnings. Our results show that Blackjack offers one of the lowest house edges in the casino, especially when players follow mathematically optimal strategies.

To keep things simple, we used an "infinite deck" model, which is slightly different from the 6- to 8-deck games typically found in casinos. In a real 6-deck game, the true expected value for the player is about -0.0036. This small negative expected value means that, on average, a player loses just 36 cents for every $100 wagered—very low compared to other casino games. For instance, in roulette, the expected value is -0.0526, or about $5.26 lost per $100 wagered. This low house edge makes Blackjack one of the best bets in the casino.

Additionally, the small average loss in Blackjack is often offset by casino “comps,” or free perks, such as complimentary drinks, meals, or even hotel stays. With a low expected loss rate, players in Blackjack often find that the experience and comps more than make up for the slight house advantage, making it one of the most appealing games for casino players.

This analysis shows not only the strategic depth and fairness of Blackjack but also the value of mathematical models in understanding casino games. Blackjack's favorable odds make it unique among casino games, offering a more balanced experience for those who play with strategy.