Blackjack Strategy Analysis

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1 Introduction

Blackjack is one of the fairest games in the casino- if you apply a good strategy. There are many options for the player: Hit, Stand, split, double down and insurance. In this work we use tools from the theory of Markov Chains and conditional expectations to find the basic strategies an expect value of the game. For the sakes of simplicity, we assume the chance of getting a particular card is independent of how the hand is formed, or just simply declare that we are using infinity decks. There is of course some minor effect would be ignored because the cards that have been seen affect of what will be drawn in the future. Consider casinos often operate the game with 6 to 8 decks, the mathematics of the game are not tremendously different between 8 decks and an infinite deck, but the basic strategy will be different in a couple borderline cases which we will point out later.

1.1 Game Procedures

In Blackjack, player and dealer are initially dealt two cards, and one of the dealer's cards are face down. Next, player add up his card values: Cards 2 through 9 have values 2 through 9, ten and picture cards have value 10, and aces are valued at 1 or 11.

Player may take cards (hit) as long as his total value is under 21. If the total exceeds 21(bust), the player loses his bet. If player stops taking cards (stand/stay), then the dealer takes cards until dealer's total is 17 or higher.

If neither of them busts, then whoever has a total closer to 21 wins the bet. If player and dealer have the same total value, then it is a tie (push).

If a hand contains an ace and the total can be used in two ways, it is called a soft hand. For example, an ace and 6 is a soft 17 because it can also be valued at 7. However, an ace, a 6 and a ten will form a 17, because there is only one way to value the hand,

If player get two cards that initially dealt add up to 21 (an ace and a ten-value card), it is called a blackjack. Whoever gets Blackjack wins the bet right away. the casino pays three to two (1.5 to 1) unless dealer also has a blackjack, in which case it is a push.

1.2 Actions

We have talked about hit, stand options. Sometimes, the player has additional options. After player gets his initial two cards, he can double his bet and get exactly one more card (double). If a player gets two cards have the same value, then he can split the cards and play the two hands for an additional bet.

If he splits two aces, then the dealer will give him exactly one more card. If player gets total of 21, that does not count as a blackjack.

Another option is surrender. If dealer does not have a blackjack, player can surrender his hand by giveaway half of his wager and keeps the other half and does not play out his hand. This option available on the initial two cards. The last option is “insurance”. If dealer has an ace showing. He will offer a side bet called “insurance”. This bet pays 2 to 1 if he has a blackjack, loses otherwise.

2 Dealer’s Probability

2.1 Markov Chain Matrix Method

Markov chains are an important class of random walks. Defining a Markov chain requires a state space and a transition matrix. In this paper we develop a Markov chains to model the play of a single hand, and we use absorbing states to compute the likelihood of a dealer that starts in state i that results in the absorbing state j. [1]

We define all the possible total value of dealer’s hands as the states. Note that we also have soft value hand which we will put an “s” in the front. Consider states

$$U = \{1, 2, 3, \ldots, 15, 16, s12, s13, s14, s15, s16, 17, 18, 19, 20, 21, bust\}$$

.We will build a transition matrix $P_{27\times 27}$ that

$$P_{ij} = Prob(X_{n+1} = j | X_n = i), \forall i, j \in U$$

by dealer’s procedure, 17… 21 and bust are the absorbing states. Which mean $P_{mn} = 0, if\ m \neq n$ and $P_{mn} = 1, if\ m = n$. $\forall m \in \{17, 18, \ldots, 21, bust\}$. Since the current goal is to find the optimal play, we will assume dealer does not have a blackjack, which means $P_{1,21} = 0$ (the probability of dealer starts with an ace get 21 after one transition is 0). The transition matrix P is

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	s12	s13	s14	s15	s16	17	18	19	20	21	bust
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0	0
2	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	0	0	0	0	0	0	0	0	0
3	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	0	0	0	0	0	0	0	0
4	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	0	0	0	0	0	0	0
5	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	0	0	0	0	0	0
6	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	0	0	0	0	0
7	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{4}{13}$	$\frac{1}{13}$	0	0	0	0
8	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{4}{13}$	$\frac{1}{13}$	0	0	0
9	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	$\frac{1}{13}$	0	0
10	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	$\frac{1}{13}$	0
11	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0
12	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$
13	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{5}{13}$
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{6}{13}$
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{7}{13}$
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{8}{13}$
s12	0	0	0	0	0	0	0	0	0	0	0	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0
s13	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0
s14	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0
s15	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0
s16	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0	0	0	0	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
bust	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Define $Q = \lim_{n\to\infty} P^n$, Q is a 27×27 rank-seven matrix that each row is a stationary distribution π. And $Q_{ij} = 0$ if $j \notin \{17, 18, 19, 20, 21, bust\}$.

$$
Q = \begin{pmatrix}
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.188917 & 0.188917 & 0.188917 & 0.188917 & 0.0778062 & 0.166525 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.139809 & 0.134907 & 0.129655 & 0.124026 & 0.117993 & 0.353608 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.135034 & 0.130482 & 0.125581 & 0.120329 & 0.1147 & 0.373875 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.13049 & 0.125938 & 0.121386 & 0.116485 & 0.111233 & 0.394468 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.122251 & 0.122251 & 0.1177 & 0.113148 & 0.108246 & 0.416404 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.165438 & 0.106267 & 0.106267 & 0.101715 & 0.0971633 & 0.42315 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.368566 & 0.137797 & 0.0786254 & 0.0786254 & 0.0740737 & 0.262312 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.128567 & 0.359336 & 0.128567 & 0.0693949 & 0.0693949 & 0.244741 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.119995 & 0.119995 & 0.350765 & 0.119995 & 0.0608238 & 0.228425 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.111424 & 0.111424 & 0.111424 & 0.342194 & 0.111424 & 0.212109 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.111424 & 0.111424 & 0.111424 & 0.111424 & 0.342194 & 0.212109 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.103465 & 0.103465 & 0.103465 & 0.103465 & 0.103465 & 0.482673 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0960751 & 0.0960751 & 0.0960751 & 0.0960751 & 0.0960751 & 0.519625 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0892126 & 0.0892126 & 0.0892126 & 0.0892126 & 0.0892126 & 0.553937 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0828402 & 0.0828402 & 0.0828402 & 0.0828402 & 0.0828402 & 0.585799 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.0769231 & 0.0769231 & 0.0769231 & 0.0769231 & 0.0769231 & 0.615385 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.151008 & 0.151008 & 0.151008 & 0.151008 & 0.151008 & 0.244958 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.145501 & 0.145501 & 0.145501 & 0.145501 & 0.145501 & 0.272495 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.14001 & 0.14001 & 0.14001 & 0.14001 & 0.14001 & 0.299951 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.134561 & 0.134561 & 0.134561 & 0.134561 & 0.134561 & 0.327196 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.129176 & 0.129176 & 0.129176 & 0.129176 & 0.129176 & 0.354121 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1.
\end{pmatrix}
$$

2.2 Analyzing Dealer's hand

Dealer will start with one of the following states V = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. 1 stands for Ace. We define a function for dealer start with face up card $i \in V$,and eventually ends in an absorbing state j.

Define:

$$D(i,j) = Prob(\lim_{n\to\infty} X_n = j \mid X_0 = i)$$

It is clear that $D(i,j) = Q_{ij}, \forall i \neq 10$. We need to calculate $D(10,j)$ separately because $Q_{10,j}$ includes the chance dealer gets blackjack.

$$D(10,j) = \sum_{k} D(k,j) \times Prob(X_1 = k \mid X_0 = 10)$$

For example:

$$\begin{aligned} D(10,17) &= D(12,17) \times \frac{1}{12} + D(13,17) \times \frac{1}{12} + \cdots + D(20,17) \times \frac{4}{12} \\ &= 0.103465 \times \frac{1}{12} + 0.096075 \times \frac{1}{12} + \cdots + 0 \times \frac{14}{12} = 0.12071 \end{aligned}$$

Notice the likelihood that moving from state 10 to state 12 in one transition is 1/12 instead of 1/13. That is because Ace is not an option for the next card (Otherwise, that is a blackjack).

By repeating the process for $D(10,18), \cdots, D(10,21)$. We can complete a chart for $D(i,j)$.

	17	18	19	20	21	Bust
1	0.188917	0.188917	0.188917	0.188917	0.077806	0.166525
2	0.139809	0.134907	0.129655	0.124026	0.117993	0.353608
3	0.135034	0.130482	0.125581	0.120329	0.1147	0.373875
4	0.13049	0.125938	0.121386	0.116485	0.111233	0.394468
5	0.122251	0.122251	0.1177	0.113148	0.108246	0.416404
6	0.165438	0.106267	0.106267	0.101715	0.097163	0.42315
7	0.368566	0.137797	0.078625	0.078625	0.074074	0.262312
8	0.128567	0.359336	0.128567	0.069395	0.069395	0.244741
9	0.119995	0.119995	0.350765	0.119995	0.060824	0.228425
10	0.12071	0.12071	0.12071	0.37071	0.037376	0.229785

Example: in order to find $D(2,21)$. That is the probability that dealer starts with 2 and ends at 21. We just need to look up 2nd row and 5th column of the chart, which is 0.117993.

3 Optimal Play

In this section we will find the optimal play. Let

$$C = \{4, 5, 6, \ldots, 21, s12, s13, \ldots, s21, bust\}$$

Since players do not have a standard procedure like dealers. Consider every state $c \in C$ that player can get. It includes soft and hard total and at least contain two cards.

3.1 Expectation of Stand a Hand

The total sum is fixed when player stands a hand. Consider H = $\{4, 5, 6, \cdots, 21, bust\}$ is the set of all the outcomes that players can get. There is no need to calculate soft total when stand is the only option.

Define:

$$S(i,k) = \text{E[Player stands at i | dealer has initial card k]} \tag{1}$$

By the law of total expectation:

$$S(i,k) = \sum_{j} \text{E[Player stands at i|Dealer ends at total j, dealer has initial value k]}$$
$$\times \text{Prob(Dealer ends at total j| dealer has initial value k)}$$
$$\forall i \in H, \forall j \in V \text{ and } \forall k \in U$$

For example:

$$\begin{aligned} S(18,10) &= 1 \times D(10,17) + 0 \times D(10,18) + (-1) \times D(10,19) \\ &\quad + (-1) \times D(10,20) + (-1) \times D(10,21) + 1 \times D(10, bust) \\ &= 0.12071 - 0.12071 - 0.37071 - 0.037376 + 0.22985 \\ &= -0.1783 \end{aligned}$$

Which means if a player stands on 18 vs dealer 10. He would lose 17.83% of his total wager on average.

By repeat the process $\forall i, k$. We can construct a chart for $S(i,k)$. Notice $S(i,k) = S(16,k), \forall i < 16$.

Dealer's up card

	Ace	2	3	4	5	6	7	8	9	10
16	-0.6670	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4754	-0.5105	-0.5431	-0.5404
17	-0.4780	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4197
18	-0.1002	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.1783
19	0.2776	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631
20	0.6555	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545
21	0.9222	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626

Example for using the chart:
To find expectation that stay a total of 18 vs dealer's initial value 9. We look up 3rd row and 9th column, which is -0.1832. If we want to find $S(14,9)$, that is the same as $S(16,9) = -0.5431$ because 14 < 16.

3.2 Expectation of Hit and Stand

Define a recurrence function. Let $HS(i,k)$ be the expectation of player is in state i when dealer has a card k shows.

$$HS(i,k) = max[\sum HS(j,k) \times Prob(X_1 = j|X_0 = i) , S(i,k)] \qquad (2)$$
$$\forall i, j \in C \text{ and } \forall k \in U.$$

$$HS(bust,k) = -1$$

$(X_1 = j|X_0 = i)$ refers player is current in state i, and land on j after taking one card.

We can compute this easily when we work backward, means we can start with $HS(21, k)$. We will use $HS(21, 10)$ as an example. Since player is already at total of 21, $Prob(X_1 = j|X_0 = 21)$ = 0, if $j \neq\ bust$, and $Prob(X_1 = j|X_0 = 21)$ = 1 otherwise. Therefore, most of the terms in summation will be ignored (or evaluated at 0).

$$\sum HS(j, 10) \times Prob(X_1 = j|X_0 = 21) \ = \ HS(bust, j) \times 1 \ = \ -1$$

$$HS(21, 10) \ = \ max(-1, S(21,10)) \ = \ max(-1, 0.9626) \ = \ 0.9626$$

If $HS(i, k) \ = \ S(i, k)$, the optimal strategy for hit/stand is to stand, hit otherwise. In this case since $HS(21, 10) \ = \ S(21, 10) \ = \ 0.9626$, the optimal play for 21 vs 10 is to stand.

Then we can calculate $HS(20, 10)$.

$$\begin{aligned} HS(20, 10) &= max\{HS(21, 10) \times (1/13) \ + \ HS(bust, 10) \times (12/13) \\ &= 0.9626 \times (1/13) + (-1)) \times (12/13) = \ -0.8490, S(20, 10)\} \\ &= max\{-0.8490, 0.5545\} = 0.5545 \end{aligned}$$

Repeating the process will bring us a table for $HS(i, k)$.

	Dealer's Up card									
Hard	2	3	4	5	6	7	8	9	10	Ace
4	-0.1149	-0.0826	-0.0494	-0.0124	0.0111	-0.0883	-0.1593	-0.2407	-0.2892	-0.2531
5	-0.1282	-0.0953	-0.0615	-0.0240	-0.0012	-0.1194	-0.1881	-0.2666	-0.3134	-0.2786
6	-0.1408	-0.1073	-0.0729	-0.0349	-0.0130	-0.1519	-0.2172	-0.2926	-0.3377	-0.3041
7	-0.1092	-0.0766	-0.0430	-0.0073	0.0292	-0.0688	-0.2106	-0.2854	-0.3191	-0.3101
8	-0.0218	0.0080	0.0388	0.0708	0.1150	0.0822	-0.0599	-0.2102	-0.2494	-0.1970
9	0.0744	0.1013	0.1290	0.1580	0.1960	0.1719	0.0984	-0.0522	-0.1530	-0.0657
10	0.1825	0.2061	0.2305	0.2563	0.2878	0.2569	0.1980	0.1165	0.0253	0.0814
11	0.2384	0.2603	0.2830	0.3073	0.3337	0.2921	0.2300	0.1583	0.1195	0.1430
12	-0.2534	-0.2337	-0.2111	-0.1672	-0.1537	-0.2128	-0.2716	-0.3400	-0.3810	-0.3505
13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.2691	-0.3236	-0.3872	-0.4253	-0.3969
14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3213	-0.3719	-0.4309	-0.4663	-0.4400
15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3698	-0.4168	-0.4716	-0.5044	-0.4800
16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4148	-0.4584	-0.5093	-0.5398	-0.5171
17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4197	-0.4780

18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.1783	-0.1002
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

	Dealer's Up card									
Soft	2	3	4	5	6	7	8	9	10	Ace
12	0.0818	0.1035	0.1266	0.1565	0.1860	0.1655	0.0951	0.0001	-0.0700	-0.0205
13	0.0466	0.0741	0.1025	0.1334	0.1617	0.1224	0.0541	-0.0377	-0.1049	-0.0573
14	0.0224	0.0508	0.0801	0.1119	0.1392	0.0795	0.0133	-0.0752	-0.1395	-0.0939
15	-0.0001	0.0292	0.0593	0.0920	0.1182	0.0370	-0.0271	-0.1122	-0.1737	-0.1300
16	-0.0210	0.0091	0.0400	0.0734	0.0988	-0.0049	-0.0668	-0.1486	-0.2074	-0.1656
17	-0.0005	0.0290	0.0593	0.0912	0.1281	0.0538	-0.0729	-0.1498	-0.1969	-0.1796
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1007	-0.1438	-0.0929
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

3.3 Expectation of Hit, Stand and Double

Define function $DB(i,k)$ refers to expectation of double when dealer shows an k.

$$DB(i,k) = 2\sum_{j} \mathrm{E}(Player\ stands\ at\ j|\ Dealer\ shows\ a\ card\ k) \times Prob(X_1 = j|X_0 = i)) \quad (3)$$

$$= \sum S(j,k) \times Prob(X_1 = j|X_0 = i)$$

$\forall i, j \in C$ and $\forall k \in U.$

Then we define function $HSD(i,k)$ = the expectation of the current hand when hit, stand and double options are allowed.

$$HSD(i,k) = max\{DB(i,k), HS(i,k)\} \quad (4)$$

If $HSD(i,k) = DB(i,k)$, then optimal play is to double, or if $HSD(i,k) = S(i,k)$, the optimal play is stand. Otherwise, it is a hit.

Example: to find the optimal play when player has a 16 when dealer shows a 10 when hit, stand and double are allowed.

$$\begin{aligned} DB(16{,}10) &= 2\big(S(17{,}10) \times (1/13) + S(18{,}10) \times (1/13) + \cdots \\ &\qquad + S(21{,}10) \times (1/13) + S(bust, 10) \times (8/13)\big) \\ &= -1.07965 \end{aligned}$$

$$\begin{aligned} HSD(16, 10) &= max\{-1.07965\ , HS(16, 10)\} \\ &= max\{-1.07965, -0.5398\} = -0.5398 \end{aligned}$$

Since -0.5398 $\neq \{S(16{,}10)\ or\ DB(16, 10)\}$, the optimal play is to hit.
We repeat the process and get the table for $HSD(i,k)$.

	Dealer's Up card									
Hard	2	3	4	5	6	7	8	9	10	Ace
4	-0.1149	-0.0826	-0.0494	-0.0124	0.0111	-0.0883	-0.1593	-0.2407	-0.2892	-0.2531
5	-0.1282	-0.0953	-0.0615	-0.0240	-0.0012	-0.1194	-0.1881	-0.2666	-0.3134	-0.2786
6	-0.1408	-0.1073	-0.0729	-0.0349	-0.0130	-0.1519	-0.2172	-0.2926	-0.3377	-0.3041
7	-0.1092	-0.0766	-0.0430	-0.0073	0.0292	-0.0688	-0.2106	-0.2854	-0.3191	-0.3101
8	-0.0218	0.0080	0.0388	0.0708	0.1150	0.0822	-0.0599	-0.2102	-0.2494	-0.1970
9	0.0744	0.1208	0.1819	0.2431	0.3171	0.1719	0.0984	-0.0522	-0.1530	-0.0657
10	0.3589	0.4093	0.4609	0.5125	0.5756	0.3924	0.2866	0.1443	0.0253	0.0814
11	0.4706	0.5178	0.5660	0.6147	0.6674	0.4629	0.3507	0.2278	0.1797	0.1430
12	-0.2534	-0.2337	-0.2111	-0.1672	-0.1537	-0.2128	-0.2716	-0.3400	-0.3810	-0.3505
13	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.2691	-0.3236	-0.3872	-0.4253	-0.3969
14	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3213	-0.3719	-0.4309	-0.4663	-0.4400
15	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.3698	-0.4168	-0.4716	-0.5044	-0.4800
16	-0.2928	-0.2523	-0.2111	-0.1672	-0.1537	-0.4148	-0.4584	-0.5093	-0.5398	-0.5171
17	-0.1530	-0.1172	-0.0806	-0.0449	0.0117	-0.1068	-0.3820	-0.4232	-0.4197	-0.4780
18	0.1217	0.1483	0.1759	0.1996	0.2834	0.3996	0.1060	-0.1832	-0.1783	-0.1002
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

	Dealer's Up card									
Soft	2	3	4	5	6	7	8	9	10	Ace
12	0.0818	0.1035	0.1266	0.1565	0.1860	0.1655	0.0951	0.0001	-0.0700	-0.0205
13	0.0466	0.0741	0.1025	0.1334	0.1797	0.1224	0.0541	-0.0377	-0.1049	-0.0573

14	0.0224	0.0508	0.0801	0.1260	0.1797	0.0795	0.0133	-0.0752	-0.1395	-0.0939
15	-0.0001	0.0292	0.0593	0.1260	0.1797	0.0370	-0.0271	-0.1122	-0.1737	-0.1300
16	-0.0210	0.0091	0.0584	0.1260	0.1797	-0.0049	-0.0668	-0.1486	-0.2074	-0.1656
17	-0.0005	0.0551	0.1187	0.1824	0.2561	0.0538	-0.0729	-0.1498	-0.1969	-0.1796
18	0.1217	0.1776	0.2370	0.2952	0.3815	0.3996	0.1060	-0.1007	-0.1438	-0.0929
19	0.3863	0.4044	0.4232	0.4395	0.4960	0.6160	0.5939	0.2876	0.0631	0.2776
20	0.6400	0.6503	0.6610	0.6704	0.7040	0.7732	0.7918	0.7584	0.5545	0.6555
21	0.8820	0.8853	0.8888	0.8918	0.9028	0.9259	0.9306	0.9392	0.9626	0.9222

Table: HSD(i, k)

3.4 Expectation of Hit, Stand, Double and Surrender

By the rules of Blackjack, the expectation of surrender is -0.5. We can define the function $HSDR(i,k)$ = the expectation of player has a total i and dealer has an up card k when hit, stand, double and surrender are allowed.

$$HSDR(i,k) = max\{HSD(i,k), -0.5\}$$

If $HSDR(i,k) = -0.5$, the optimal play is surrender. Otherwise, the optimal play is the same when surrender is not allowed. By observation, there are only four spots from the table $HSD(i,k)$ is below -0.5.

They are:

15 vs 10
16 vs 9
16 vs 10
16 vs ace

Therefore, if player initially dealt one of these situations, the optimal play is to surrender.

3.5 Expectation of Split

When player initial gets a pair, split option will be available. If player split a pair, he needs to place a second bet and gets two new hands by split his original hand. Player will be dealt two more cards (one for each hand). He plays each hand normally except for splitting aces. When player splits an ace, he gets dealt one card only with no further actions allowed. Common casino rules are players can split up to 3 hands. In my simplified model, we assume players can split up to 2 hands to make the calculation simpler.

Let $X = \{ace, 2, 3, \ldots, 21, s12, \ldots, s21, bust\}$

and define function $Sp\{i,k\}$ = expectation of *split* a pair of i and dealer shows a k.

$$Sp(i,k) \;=\; 2 \times \sum_{j} HSD(j,k) \times Prob(X_1 = j|X_0 = i), \forall i \neq ace \qquad (5)$$

$$Sp(i,k) \;=\; 2 \times \sum S(j,k) \times Prob(X_1 = j|X_0 = i), if \; (i = ace)$$

And define function $NP(i,k)$ be the decision function that possible out-comes are {Y, N}

$NP(i,k) \;=\; Y \;\; if \; Sp(i,k) \;>\; HSDR(2i,k)$, and the play is to split.
$NP(i,k) \;=\; N \;\; if \; Sp(i,k) \;\leq\; HSDR(2i,k)$, and the play is **not** to split.

For example: When player has a pair of 8 and dealer shows a 10.
$Sp(8,10) = \; 2 \times (HSD(10,10) \times (1/13) + HSD(11,10) \times (1/13) + \ldots + HSD(18,10) \times (4/13) + HSD(s19,10) \times (1/13)) = 2 \times (-0.24474) \;=\; -0.4895$

$$HSDR(16,10) \;=\; -0.5 \; (optimal \; play \; is \; to \; surrender)$$

Since $Sp(16,10) > HSDR(16,10), then \; NP(16,10) \;=\; Y$, and the optimal play is to split.
We then construct the table for $Sp(i,k)$ with the same fashion, and we can also make the table for $NP(i,k)$.

	Dealer's Up card									
Pair	2	3	4	5	6	7	8	9	10	Ace
2	-0.0889	-0.0256	0.0429	0.1272	0.1948	-0.0074	-0.1741	-0.3651	-0.4747	-0.4067
3	-0.1382	-0.0639	0.0146	0.1023	0.1694	-0.0678	-0.2297	-0.4152	-0.5214	-0.4559
4	-0.1669	-0.0913	-0.0116	0.0803	0.1460	-0.1294	-0.2865	-0.4664	-0.5691	-0.5062
5	-0.1935	-0.1167	-0.0330	0.0599	0.1243	-0.1918	-0.3440	-0.5183	-0.6176	-0.5571
6	-0.2186	-0.1367	-0.0496	0.0440	0.1079	-0.2568	-0.4023	-0.5703	-0.6662	-0.6083
7	-0.1555	-0.0748	0.0105	0.1000	0.1877	-0.0905	-0.3890	-0.5558	-0.6288	-0.6201
8	0.0193	0.0869	0.1566	0.2283	0.3255	0.2115	-0.0876	-0.4054	-0.4895	-0.3941
9	0.1846	0.2421	0.3015	0.3633	0.4434	0.3700	0.2153	-0.0937	-0.2966	-0.1314
10	0.3650	0.4122	0.4609	0.5125	0.5756	0.5138	0.3959	0.2331	0.0506	0.1629
Aces	0.4706	0.5178	0.5660	0.6147	0.6674	0.4629	0.3507	0.2278	0.1797	0.1091

$Sp(i,k) \; table$

	Dealer's Up card									
Pair	2	3	4	5	6	7	8	9	10	Ace

2	Y	Y	Y	Y	Y	Y	N	N	N	N
3	Y	Y	Y	Y	Y	Y	N	N	N	N
4	N	N	N	Y	Y	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N
6	Y	Y	Y	Y	Y	N	N	N	N	N
7	Y	Y	Y	Y	Y	Y	N	N	N	N
8	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
9	Y	Y	Y	Y	Y	N	Y	Y	N	N
10	N	N	N	N	N	N	N	N	N	N
Aces	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

$NP(i,k)$ *table*

3.6 Insurance

If dealer's up card is an ace, his next action is to offer player a side bet, called "insurance", on whether his two cards form a blackjack. Insurance pays 2 to 1 if dealer gets the blackjack, loses otherwise.

The change of dealer gets blackjack when he has an ace is 4/13. Therefore, the expectation $= 2(4/13) + (-1)(9/13) = -0.0769$. Which is a terrible bet, the optimal play is **not** to bet insurance.

3.7 Blackjack Basic Strategy

By putting 3.1 to 3.6 together, we can find the optimal strategy for all situations by selecting the highest expectation. We will use "s" for stand, "h" for hit, "d" for double if allowed, otherwise stay. "dh" for double if allowed, otherwise hit. "r" for surrender.

	Dealer's Up card									
Hard	2	3	4	5	6	7	8	9	10	Ace
4	H	H	H	H	H	H	H	H	H	H
5	H	H	H	H	H	H	H	H	H	H
6	H	H	H	H	H	H	H	H	H	H
7	H	H	H	H	H	H	H	H	H	H
8	H	H	H	H	H	H	H	H	H	H
9	H	Dh	Dh	Dh	Dh	H	H	H	H	H
10	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	H	H
11	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	Dh	H
12	H	H	S	S	S	H	H	H	H	H
13	S	S	S	S	S	H	H	H	H	H

14	S	S	S	S	S	H	H	H	H	H
15	S	S	S	S	S	H	H	H	R	H
16	S	S	S	S	S	H	H	R	R	R
17	S	S	S	S	S	S	S	S	S	S
18	S	S	S	S	S	S	S	S	S	S
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

Soft	2	3	4	5	6	7	8	9	10	Ace
12	H	H	H	H	H	H	H	H	H	H
13	H	H	H	H	Dh	H	H	H	H	H
14	H	H	H	Dh	Dh	H	H	H	H	H
15	H	H	H	Dh	Dh	H	H	H	H	H
16	H	H	Dh	Dh	Dh	H	H	H	H	H
17	H	Dh	Dh	Dh	Dh	H	H	H	H	H
18	S	Ds	Ds	Ds	Ds	S	S	H	H	H
19	S	S	S	S	S	S	S	S	S	S
20	S	S	S	S	S	S	S	S	S	S
21	S	S	S	S	S	S	S	S	S	S

Pair	2	3	4	5	6	7	8	9	10	Ace
2	Y	Y	Y	Y	Y	Y	N	N	N	N
3	Y	Y	Y	Y	Y	Y	N	N	N	N
4	N	N	N	Y	Y	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N
6	Y	Y	Y	Y	Y	N	N	N	N	N
7	Y	Y	Y	Y	Y	Y	N	N	N	N
8	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
9	Y	Y	Y	Y	Y	N	Y	Y	N	N
10	N	N	N	N	N	N	N	N	N	N
Aces	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

(No insurance)

We claimed earlier that strategies for infinity decks and 6-8 decks are slightly different at border cases. The differences are in 6-8 decks blackjack, soft 15 vs 4 and soft 13 vs 5 are “Dh”. The rests are identical. [2]

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4 Expectation of Blackjack

If players play their cards right, Blackjack is the fairest game in the casino. In this section we will compute the expectation of this game. First, we find the probability of all the starting states of player. The likelihood of getting a particular initial two cards is $1/13^2$ if no 10 involved, and $4/13^2$ if one card is a 10-value card. We will calculate pairs separately because there might be split strategy involved.

Example: the probability of starting a hard total of 5 is:

$$Prob(first\ is\ card\ 2\ and\ second\ card\ is\ 3) + Prob(first\ card\ is\ 3\ and\ second\ card\ is\ 2) = 1/13^2 + 1/13^2 = 0.011834$$

We repeat the process find the likelihood of all possible starting hand without getting blackjack, pairs and soft hand.

Hard	Probability
5	0.01183
6	0.01183
7	0.02367
8	0.02367
9	0.03550
10	0.03550
11	0.04734
12	0.08284
13	0.08284
14	0.07101
15	0.07101
16	0.05917
17	0.05917
18	0.04734
19	0.04734

Then, we define function $I(i,k)$ be the probability of player's initial state is $i\ and$ dealer has an up card k.

$$I(i,k) = Prob(player\ starts\ at\ i\ and\ dealer\ has\ an\ up\ card\ k) \quad (6)$$

For example:

$$I(5,9) = Prob(player\ starts\ at\ 5\ and\ dealer\ has\ an\ up\ card\ 9)$$

Since this is the simplified model, the likelihood of getting each card is an independent event.

$I(5,9) = Prob(player\ starts\ at\ 5) \times Prob(dealer\ has\ an\ up\ card\ 9) = 0.01183 \times (1/13) = 0.00091$

Now we construct the table for $I(i,k)$. We need to be careful when dealer starts with a ten or ace.

For example: $I(5,10) = 0.01183 \times (4/13) \times (12/13) = 0.00336$. The reason we multiply $12/13$ at the end is because that is the odds dealer does not have an ace in the hole card.

	Dealer's up card									
Hard	2	3	4	5	6	7	8	9	10	Ace
5	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
6	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
7	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00672	0.00126
8	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00182	0.00672	0.00126
9	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.01008	0.00189
10	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.00273	0.01008	0.00189
11	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252
12	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.02353	0.00441
13	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.00637	0.02353	0.00441
14	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.02017	0.00378
15	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.00546	0.02017	0.00378
16	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.01681	0.00315
17	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.00455	0.01681	0.00315
18	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252
19	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252

Note that the sum of all the numbers in the table adds to 0.676447. That is the probability that player is initially dealt a hard hand.

Now we will work with soft hands and pairs with the same method.

After multiplying the odds of getting each card (1/13 for non-ten valued card and 4/13 for ten valued card) and avoid odds of dealer getting a blackjack (simply multiply 12/13 when dealer has a ten and 9/13 if dealer has an ace), we can fill up the rest part for

$$I(i,k), i \in \{5, 6, \ldots 19, s13, s14, \ldots s21, pair\ aces, pair\ 2, \ldots, pair\ 10\}$$

which i are all the possible hands player could start.

	Dealer's up card									
soft	2	3	4	5	6	7	8	9	10	Ace
13	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
14	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063

15	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
16	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
17	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
18	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
19	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
20	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00091	0.00336	0.00063
21	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.00364	0.01344	0.00252

Pair										
2	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
3	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
4	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
5	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
6	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
7	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
8	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
9	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032
10	0.00728	0.00728	0.00728	0.00728	0.00728	0.00728	0.00728	0.00728	0.02689	0.00504
Aces	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00046	0.00168	0.00032

Now if we sum all the values in these charts, the total is 0.95266. The part we are missing is the dealer blackjack which can be calculated easily:

$$Prob(Dealer\ gets\ blackjack) = 2 \times (1/13 \times 4/13) = 0.04734$$

The total added up to 1. Therefore, we completed the probability chart.

Next, we are going to find the expected values of starting hand.

Define function $EV(i,k)$= expected value of each starting hand i vs dealer up card.

If $i \neq s21$ (blackjack) or pairs,

$$\mathrm{EV}(i,k) = \mathrm{I(i,k)} \times \mathrm{HSDR(i,k)} \quad (7)$$

If player has a pair of j and $NP(j,k) = N$.

$$\mathrm{EV}(i,k) = \mathrm{I(i,k)} \times \mathrm{HSDR(2j,k)}$$

If player has a pair of j and $NP(j,k) = Y$,

$$\mathrm{EV}(i,k) = \mathrm{I(i,k)} \times \mathrm{Sp(j,k)}$$

If $i = s21$, $\mathrm{EV}(i,k)$= 1.5

Table for $EV(i,k)$ is listed below.

Dealer's up card

Hard	2	3	4	5	6	7	8	9	10	Ace
5	-1.17E-04	-8.68E-05	-5.60E-05	-2.18E-05	-1.08E-06	-1.09E-04	-1.71E-04	-2.43E-04	-1.05E-03	-1.76E-04
6	-1.28E-04	-9.77E-05	-6.64E-05	-3.18E-05	-1.18E-05	-1.38E-04	-1.98E-04	-2.66E-04	-1.14E-03	-1.92E-04

7	-1.99E-04	-1.39E-04	-7.83E-05	-1.32E-05	5.31E-05	-1.25E-04	-3.83E-04	-5.20E-04	-2.14E-03	-3.91E-04
8	-3.97E-05	1.46E-05	7.06E-05	1.29E-04	2.09E-04	1.50E-04	-1.09E-04	-3.83E-04	-1.68E-03	-2.48E-04
9	2.03E-04	3.30E-04	4.97E-04	6.64E-04	8.66E-04	4.69E-04	2.69E-04	-1.42E-04	-1.54E-03	-1.24E-04
10	9.80E-04	1.12E-03	1.26E-03	1.40E-03	1.57E-03	1.07E-03	7.83E-04	3.94E-04	2.55E-04	1.54E-04
11	1.71E-03	1.89E-03	2.06E-03	2.24E-03	2.43E-03	1.69E-03	1.28E-03	8.29E-04	2.42E-03	3.60E-04
12	-1.61E-03	-1.49E-03	-1.34E-03	-1.07E-03	-9.79E-04	-1.36E-03	-1.73E-03	-2.17E-03	-8.97E-03	-1.55E-03
13	-1.87E-03	-1.61E-03	-1.34E-03	-1.07E-03	-9.79E-04	-1.71E-03	-2.06E-03	-2.47E-03	-1.00E-02	-1.75E-03
14	-1.60E-03	-1.38E-03	-1.15E-03	-9.13E-04	-8.40E-04	-1.75E-03	-2.03E-03	-2.35E-03	-9.40E-03	-1.66E-03
15	-1.60E-03	-1.38E-03	-1.15E-03	-9.13E-04	-8.40E-04	-2.02E-03	-2.28E-03	-2.58E-03	-1.01E-02	-1.82E-03
16	-1.33E-03	-1.15E-03	-9.61E-04	-7.61E-04	-7.00E-04	-1.89E-03	-2.09E-03	-2.28E-03	-8.40E-03	-1.58E-03
17	-6.96E-04	-5.34E-04	-3.67E-04	-2.05E-04	5.34E-05	-4.86E-04	-1.74E-03	-1.93E-03	-7.05E-03	-1.51E-03
18	4.43E-04	5.40E-04	6.40E-04	7.27E-04	1.03E-03	1.45E-03	3.86E-04	-6.67E-04	-2.40E-03	-2.53E-04
19	1.41E-03	1.47E-03	1.54E-03	1.60E-03	1.81E-03	2.24E-03	2.16E-03	1.05E-03	8.49E-04	7.00E-04

soft

13	4.25E-05	6.75E-05	9.33E-05	1.21E-04	1.64E-04	1.11E-04	4.92E-05	-3.43E-05	-3.52E-04	-3.61E-05
14	2.04E-05	4.63E-05	7.29E-05	1.15E-04	1.64E-04	7.24E-05	1.21E-05	-6.84E-05	-4.69E-04	-5.92E-05
15	-1.10E-07	2.65E-05	5.40E-05	1.15E-04	1.64E-04	3.37E-05	-2.46E-05	-1.02E-04	-5.84E-04	-8.19E-05
16	-1.91E-05	8.25E-06	5.32E-05	1.15E-04	1.64E-04	-4.45E-06	-6.08E-05	-1.35E-04	-6.97E-04	-1.04E-04
17	-4.47E-07	5.02E-05	1.08E-04	1.66E-04	2.33E-04	4.90E-05	-6.64E-05	-1.36E-04	-6.62E-04	-1.13E-04
18	1.11E-04	1.62E-04	2.16E-04	2.69E-04	3.47E-04	3.64E-04	9.65E-05	-9.17E-05	-4.83E-04	-5.86E-05
19	3.52E-04	3.68E-04	3.85E-04	4.00E-04	4.52E-04	5.61E-04	5.41E-04	2.62E-04	2.12E-04	1.75E-04
20	5.83E-04	5.92E-04	6.02E-04	6.10E-04	6.41E-04	7.04E-04	7.21E-04	6.90E-04	1.86E-03	4.13E-04
21	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	5.46E-03	2.02E-02	3.78E-03

Pair

2	-4.05E-05	-1.17E-05	1.95E-05	5.79E-05	8.87E-05	-3.37E-06	-7.25E-05	-1.10E-04	-4.86E-04	-7.97E-05
3	-6.29E-05	-2.91E-05	6.66E-06	4.66E-05	7.71E-05	-3.08E-05	-9.89E-05	-1.33E-04	-5.68E-04	-9.58E-05
4	-9.92E-06	3.64E-06	1.77E-05	3.65E-05	6.64E-05	3.74E-05	-2.73E-05	-9.57E-05	-4.19E-04	-6.21E-05
5	1.63E-04	1.86E-04	2.10E-04	2.33E-04	2.62E-04	1.79E-04	1.30E-04	6.57E-05	4.25E-05	2.57E-05
6	-9.95E-05	-6.22E-05	-2.26E-05	2.00E-05	4.91E-05	-9.69E-05	-1.24E-04	-1.55E-04	-6.40E-04	-1.10E-04

7	-7.08E-05	-3.40E-05	4.78E-06	4.55E-05	8.54E-05	-4.12E-05	-1.69E-04	-1.96E-04	-7.84E-04	-1.39E-04
8	8.78E-06	3.95E-05	7.13E-05	1.04E-04	1.48E-04	9.63E-05	-3.99E-05	-1.85E-04	-8.23E-04	-1.24E-04
9	8.40E-05	1.10E-04	1.37E-04	1.65E-04	2.02E-04	1.82E-04	9.80E-05	-4.26E-05	-3.00E-04	-3.16E-05
10	4.66E-03	4.74E-03	4.81E-03	4.88E-03	5.13E-03	5.63E-03	5.77E-03	5.52E-03	1.49E-02	3.30E-03
Aces	2.14E-04	2.36E-04	2.58E-04	2.80E-04	3.04E-04	2.11E-04	1.60E-04	1.04E-04	3.02E-04	3.44E-05

Now if we sum up all the values = 0.040248, it is positive because we have not put dealer's blackjack into equation yet.

$$Prob(Dealer\ gets\ a\ blackjack\ and\ Player\ not\ gets\ a\ blackjack)$$
$$= (2 \times 1/13 \times 4/13) \times (1 - 2 \times 1/13 \times 4/13)$$
$$= 0.0451$$

And

$$\mathrm{E}[Dealer\ gets\ a\ blackjack\ and\ Player\ not\ get\ a\ blackjack] = -0.045096$$

The total expectation of the infinity deck Blackjack game is 0.040248 - 0.045096 = -0.004849.

5 References

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