

Modular Response in Free Quantum Fields: A KMS/FDT Theorem and Conditional Extensions

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(Dated:)

Part I (Theoremic core, free/Gaussian Hadamard QFT). We prove that, for small causal diamonds (CHM) in locally Hadamard states and within a safe window $\epsilon_{UV} \ll \ell \ll \min\{L_{\text{curv}}, \lambda_{\text{mfp}}, m_i^{-1}\}$, the MI/moment-kill projector isolates a finite ℓ^4 modular response with coefficient equal to its flat-space value; the projected KMS/FDT susceptibility is positive; and coarse-graining over the wedge family produces the universal weak-field prefactor $5/12 = (4/3) \times (5/16)$. The fractional KMS defect between CHM diamonds and half-spaces scales as $\mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$. The QFT sensitivity is $\beta = 2\pi C_T I_{00} = 0.02086 \pm 0.00105$ (conservative 5% shared systematics from four independent routes). A scheme-invariant background relation *suggests* $\Omega_\Lambda = \beta f c_{\text{geo}}$ *conditional* on our coarse-graining and analyticity assumptions.

Part II (Conditional extensions). We separate *definition* (flat-space ϵ from modular response) from *mapping*. Rather than impose the standard EFT-of-DE α -basis, we adopt a quasi-static closure that keeps operational distances GR-like (no additional lensing coupling $\Sigma \simeq 1$) while modifying growth via $\mu(\epsilon) = 1/(1 + \frac{5}{12}\epsilon)$. KMS/FDT positivity motivates an entropy-driven law $d\epsilon/d \ln a \geq 0$ with a *conditional* background budget $\int \epsilon d \ln a = \Omega_\Lambda$. We introduce a covariant environment envelope $F_g(\chi_g) = [1 + (\chi_g/\chi_\star)^q]^{-1}$ with $\chi_g \equiv \ell^2 \sqrt{C_{abcd} C^{abcd}}$, calibrated by Solar-System bounds. Cosmological illustrations (S_8 band and H_0 bounds) are **toy/illustrative** and propagate the $\pm 5\%$ β uncertainty; *observed lensing amplitudes still reflect the altered growth*.

Part III (Exploratory). We provide a compact *thermodynamic interpretation* of the projected modular response: a Clausius-like identity holds at working order in the MI/moment-kill channel, and the FRW budget may be viewed as a *coarse-grained* Clausius normalization *conditional* on our KMS \rightarrow FRW hypotheses. We clarify the relation to the Casini–Galante–Myers critique of Jacobson; our MI projection targets the ℓ^4 response and deliberately avoids marginal $\Delta = d/2$ logarithms, with $\ell^4 \log \ell$ taken as a falsifier.

What is new. (i) Completed proofs in the Gaussian/Hadamard sector; (ii) a **conditional, coarse-grained** KMS \rightarrow FRW averaging statement with explicit error budget; (iii) **Assumptions C and D stated with rationale** (relative entropy \leftrightarrow canonical energy in the projected diamond; uniqueness of M^2 at working order), with proofs deferred; (iv) semi-analytic quantification of the safe-window volume fraction $f_V(\ell_{\text{min}})$; (v) a symmetry-constrained F_g envelope; (vi) uncertainty propagation of β into S_8 and H_0 *illustrations*; (vii) an exploratory thermodynamic reinterpretation (Part III) and refined treatment of the CGM critique.

READER'S MAP: PART I (THEOREM) VS. PART II (CONDITIONAL) VS. PART III (EXPLORATORY)

Part I (Secs. I–IV, Apps. XV–XVIII): proven results for free/Gaussian Hadamard fields at working order.

Part II (Secs. V–XXII, Apps. XIX–XX, XXI): conditional extensions, Assumptions C & D (stated), safe-window fraction, KMS \rightarrow FRW link, symmetry envelope, entropic sketch, and toy/illustrative numerics with propagated uncertainties.

Part III (Sec. XIII): exploratory thermodynamic interpretation (Clausius form in the projected channel; conditional FRW budget) and relation to CGM's critique of Jacobson.

I. SCOPE, WORKING ORDER, AND SAFE-WINDOW QUANTIFICATION (PART I)

a. Working order and state class. We work to $\mathcal{O}(\ell^4)$ in the MI/moment-kill projector channel, treating curvature/contact terms as $\mathcal{O}(\ell^6)$. States are locally Hadamard.

b. KMS applicability (CHM diamonds). Exact BW KMS holds for half-spaces; CHM diamonds inherit it with fractional defect $\mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$ (App. XVIII).

c. Safe-window volume fraction. Define a conservative admissible scale

$$\ell_{\text{max}}(x) \equiv \zeta \min \left\{ L_{\text{curv}}(x), \lambda_{\text{mfp}}(x), m_i^{-1}(x) \right\}, \quad \zeta = 0.1. \quad (1)$$

Using Press–Schechter/Sheth–Tormen mass functions and NFW curvature proxies $L_{\text{curv}}^{-2} \sim (R_{abcd}R^{abcd})^{1/2}$ with sub-structure excision parameter ξ , we estimate the comoving volume fraction $f_V(\ell_{\min}) = \text{Vol}\{x : \ell_{\max}(x) > \ell_{\min}\} / \text{Vol}_{\text{tot}}$. A semi-analytic survey (App. XIX) shows voids dominate f_V , while dense cores lack a window; representative values at $z \sim 0$ for $\ell_{\min} \in [1, 100]$ pc are $f_V \sim 0.6\text{--}0.95$ for $\xi \in [0.2, 0.5]$. This enters only as a domain-of-validity indicator.

d. Angle invariance as a null test. The continuous-angle product $\mathcal{C}_\Omega = f(\theta) c_{\text{geo}}(\theta)$ is analytic and θ -independent; residuals are shown as a null check, not a precision claim.

II. A2–KMS THEOREM (GAUSSIAN/HADAMARD SECTOR)

Theorem 1 (Projected modular response and positivity). *Let \mathcal{Q} be a free (Gaussian) QFT on a globally hyperbolic spacetime and ρ a locally Hadamard state. For a causal diamond of radius ℓ with $\ell \ll L_{\text{curv}}$ and the MI/moment-kill projector that cancels r^0 and r^2 moments, the MI-subtracted modular response obeys*

$$\delta\langle K_{\text{sub}} \rangle = (2\pi C_T I_{00}) \ell^4 \delta\varepsilon + \mathcal{O}(\ell^6), \quad (2)$$

with coefficient equal to the flat-space value. The retarded susceptibility χ_{QK} in the projected channel is positive (FDT), and wedge averaging yields the universal weak-field prefactor $5/12$. The fractional deviation from BW KMS is $\mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$.

Corollary 1 (Conditional background statement). *Under the coarse-graining and analyticity assumptions of Sec. VI, the FRW zero mode suggests the scheme-invariant relation $\Omega_\Lambda = \beta f c_{\text{geo}}$ with $\beta = 2\pi C_T I_{00}$. We treat this as a conditional statement rather than a theorem.*

III. QFT INPUT: $\beta = 2\pi C_T I_{00}$ AND ERROR BUDGET

We evaluate β via four independent routes: (a) real-space CHM; (b) spectral/Bessel; (c) Euclidean time-slicing; (d) replica finite-difference. The spread is $\lesssim 1\%$. We adopt a conservative

$$\beta = 0.02086 \pm 0.00105 \quad (5\% \text{ shared systematics}). \quad (3)$$

Angle invariance is used as a null residual test.

Here C_T denotes the flat-space stress-tensor two-point normalization, e.g. $\langle T_{ab}(x) T_{cd}(0) \rangle = C_T \mathcal{I}_{abcd}(x)/|x|^{2d}$ in d dimensions (see Osborn–Petkou).

Benchmark (convention). For a free, massless real scalar in $d = 4$ and our normalization, $C_T = 1/(120\pi^2)$, which yields $\beta \simeq 0.02086$ via Eq. (4).

IV. WEAK-FIELD PREFACTOR $5/12$

The isotropic BW channel gives $\langle T_{kk} \rangle = (1+w)\rho$ with UV $w = 1/3 \Rightarrow 4/3$. Averaging over CHM segments yields $5/16$, so $5/12 = (4/3) \times (5/16)$. Details in App. XVII.

V. DEFINITION VS. MAPPING (PART II; CONDITIONAL)

a. Definition (flat-space QFT).

$$\delta\langle K_{\text{sub}}(\ell) \rangle = \underbrace{(2\pi C_T I_{00})}_{\beta} \ell^4 \delta\varepsilon(x) + \mathcal{O}(\ell^6). \quad (4)$$

b. Mapping (constitutive; beyond the α -basis). We do not impose the linear EFT-of-DE α -parameter mapping at working order. Instead, we adopt a quasi-static closure that keeps operational distances GR-like while modifying growth:

$$\nabla^2 \Phi = 4\pi G a^2 \rho_m \mu(\varepsilon) F_g(\chi_g), \quad \mu(\varepsilon) = \frac{1}{1 + \frac{5}{12}\varepsilon}, \quad (5a)$$

$$\nabla^2 \frac{\Phi + \Psi}{2} = 4\pi G a^2 \rho_m, \quad (\Sigma \simeq 1). \quad (5b)$$

Matter obeys the standard continuity and Euler equations. This closure preserves the Bianchi identity at working order provided F_g is a scalar built from local geometry (Sec. IX); a full action-level derivation is future work (Limitations). *Remark on lensing amplitude.* $\Sigma \simeq 1$ denotes no additional lensing coupling; the observed lensing signal still changes through the altered growth $D(a)$.

c. EFT stub (derivation of $\mu(\varepsilon)$). At quasi-static, sub-horizon scales, a background variation $\delta \ln M^2 = \beta \delta \varepsilon$ rescales the Poisson coupling as $G \rightarrow G_{\text{eff}} = G/(1 + \Delta)$ with Δ fixed by the universal weak-field bookkeeping. In the isotropic BW channel the contraction 4/3 and the segment ratio 5/16 (Sec. IV) give $\Delta = \frac{5}{12}\varepsilon$, hence

$$\mu(\varepsilon) = \frac{G_{\text{eff}}}{G} = \frac{1}{1 + \frac{5}{12}\varepsilon}, \quad (6)$$

consistent with Eqs. (5).

d. Trial action (outlook). A possible action-level route consistent with our closure is to consider an effective term that modulates M^2 via the modular response,

$$S_{\text{trial}} = \int d^4x \sqrt{-g} \left[\frac{M^2}{2} R + \lambda (\delta \ln M^2) \mathcal{K}[g; \ell] + \dots \right],$$

where \mathcal{K} is a local covariant scalar capturing the projected channel at working order and λ a running coefficient. While only illustrative, this shows how $\delta \ln M^2 = \beta \delta \varepsilon$ could arise from an action (cf. [6, 8]).

Weak-field acceleration (toy/conditional). Using the universal 5/12 prefactor and the *conditional* background relation $\Omega_\Lambda = \beta f c_{\text{geo}}$, the weak-field normalization implies a MOND-like acceleration scale

$$a_0 = \frac{5}{12} \Omega_\Lambda^2 c H_0, \quad (7)$$

reported as an *illustrative* consequence pending validation of the interacting extensions and the KMS→FRW link (Sec. VI). Pipeline values propagate the $\pm 5\%$ uncertainty in β .

This is a **constitutive closure**, not a derived macroscopic law; it is falsified by log- ℓ residuals, $|d_L^{\text{GW}}/d_L^{\text{EM}} - 1| > 5 \times 10^{-3}$, or Ω_Λ inconsistent with $\beta f c_{\text{geo}}$.

VI. COVARIANT KMS → FRW LINK AND ERROR CONTROL

Let s denote modular time with $\beta_{\text{KMS}} = 2\pi/\kappa$ locally, where κ is the local boost surface gravity so that the approximate conformal Killing field ξ^a satisfies $\xi^a \nabla_a = \kappa \partial_s$. Averaging the retarded kernel over a comoving congruence of diamonds and reparametrizing $s \mapsto \ln a$ induces the FRW background factor $f c_{\text{geo}}$; diffeomorphism covariance is preserved because the averaging functional depends only on local curvature scalars and the diamond foliation. The total fractional defect in the kernel obeys

$$\frac{\delta \chi}{\chi_{\text{BW}}} = \mathcal{O}\left((\ell/L_{\text{curv}})^2\right) + \mathcal{O}((\ell H)^2), \quad (8)$$

which is negligible for $\ell \sim 10$ pc, $L_{\text{curv}} \sim 10$ Mpc, $H^{-1} \sim 4$ Gpc.

Proposition 1 (FRW budget identity (conditional; analyticity hypothesis)). *Assume: (H1) locality and rapid decay of the spatially averaged, projected retarded kernel so that its reparametrization defines a distribution in $\ln a$; (H2) adiabatic evolution through matter domination so that $J(a) = ds/d \ln a \propto H(a)^{-1}$ varies slowly; (H3) preservation of KMS analyticity of the averaged kernel under the reparametrization $s \rightarrow \ln a$; and (H4) negligible CHM vs. half-space deviation at working order (App. XVIII). Then*

$$\left\langle \int \chi_{QK}^{\text{proj}}(a, a') d^3x \right\rangle = \beta f c_{\text{geo}} \delta(\ln a - \ln a') + \dots$$

and integrating the entropy-driven evolution $d\varepsilon/d \ln a = \sigma(a)I(a) \geq 0$ yields the coarse-grained identity

$$\int_{a_i}^1 \varepsilon(a) d \ln a = \Omega_\Lambda = \beta f c_{\text{geo}}, \quad (9)$$

used as a normalization under (H1)–(H4).

Proof sketch. Average the projected KMS kernel over the diamond foliation; reparametrize modular time s to $\ln a$ with Jacobian $J(a) \propto H^{-1}$. Under (H1)–(H3) the averaged kernel remains a positive KMS/FDT object and collapses to a contact term in $\ln a$ at working order, with angle factor fc_{geo} . (H4) bounds the half-space/diamond deviation. Integrating the positive evolution law then fixes the budget (9). \square

Geometric origin. The factor fc_{geo} depends only on the wedge-family foliation and unit–solid–angle normalization; it is geometric and foliation-based, not a fit parameter (Appendix XVI).

Analyticity caveat. The reparametrization $s \rightarrow \ln a$ is conjectured to preserve KMS analyticity of the *averaged* retarded kernel; a proof likely requires a spectral/microlocal argument (cf. modular-Hamiltonian spectral decompositions in [12] and the curved-spacetime microlocal analysis of [10]). A failure would manifest as non-local or non-analytic kernels in $\ln a$, which our simulations can detect via residuals in the projected susceptibility. We therefore treat the KMS \rightarrow FRW link as a controlled conjecture with the error budget in Eq. (8).

a. Thermodynamic analogy (pointer). The entanglement first law and KMS calibration suggest a Clausius-like analogy, explored in Sec. XIII, conditional on (H1)–(H4) and avoiding CGM’s marginality issues via MI projection (App. XX).

VII. ASSUMPTIONS FOR INTERACTING EXTENSIONS AT WORKING ORDER (PART II; STATED AND TEST CRITERIA)

A. Assumption C (stated; test criteria): Relative entropy \leftrightarrow canonical energy in the projected diamond

Statement. For a local algebra $\mathcal{A}(B_\ell)$ of an interacting Hadamard QFT obeying the microlocal spectrum condition and time-slice axiom, the MI/moment-kill projected second variation of Araki relative entropy equals the canonical-energy quadratic form of the projected stress tensor, up to $\mathcal{O}(\ell^6)$ remainders, with a positive-definite projected kernel χ_{QK}^{proj} .

Rationale (sketch). (i) The second variation is the Bogoliubov–Kubo–Mori metric. (ii) The MI/moment-kill projector cancels local counterterms to $\mathcal{O}(\ell^4)$ (App. XV), conjectured to persist in interacting Hadamard QFTs (App. XX). (iii) Diffeomorphism Ward identities match the BKM quadratic form to canonical energy in the CHM channel. (iv) Positivity follows from KMS/BKM positivity in the projected channel. A complete microlocal proof is left to future work.

a. Operational tests (pass/fail).

- **Positivity test (substrates):** The projected, integrated retarded kernel $\int \chi_{QK}^{\text{proj}} d^4x d^4x'$ is nonnegative in Gaussian chains (exact) and HQTFIM (numerical tolerance) (checked with `hqtfim_capacity_probe.py`, `gaussian_capacity_probe.py`).
- **No- $\ell^4 \log \ell$ falsifier:** The MI/moment-kill channel exhibits no $\ell^4 \log \ell$ term. *Fail* if a protected-operator contribution produces an $\ell^4 \log \ell$ trend.
- **Plateau stability:** Varying MI windows leaves the residual plateau $\sim \mathcal{O}(\ell^6)$ (verifiable with `beta_methods_v2.py`). *Fail* if residuals scale as ℓ^4 after subtraction.
- **BKM positivity (finite truncations):** In truncated QFTs, the BKM quadratic form for δK_{sub} is positive definite (tested with `gaussian_capacity_probe.py`). *Fail* if negative eigenmodes persist under refinement.

B. Assumption D (stated; test criteria): Uniqueness of the M^2 coupling at working order

Statement. In the $c_T=1$, $\alpha_B=0$ EFT corner linearized about FRW, with isotropy, parity, and time-reversal, the only background scalar coupling that survives the MI/moment-kill projection at $\mathcal{O}(\ell^4)$ and modifies the weak-field growth sector while keeping distances GR-like is $\delta \ln M^2$; other diffeomorphism-invariant local scalars are projected out, forbidden by sector constraints, or curvature-suppressed by $\mathcal{O}((\ell/L_{\text{curv}})^2)$.

Rationale (sketch). Consider the most general local covariant functional at the required engineering dimension:

$$\delta \mathcal{L} = \sqrt{-g} [a R + b R_{ab} R^{ab} + c \nabla^2 R + d \delta \ln M^2 R + e \delta g^{00} + f K \delta g^{00} + \dots], \quad (10)$$

where “ \dots ” denote terms of higher engineering dimension (e.g., $\nabla^4 R$, R^4) or parity-odd contributions, excluded by the MI/moment-kill projector and EFT symmetry constraints at $\mathcal{O}(\ell^4)$. Imposing $c_T = 1$ excludes tensor-speed shifts; $\alpha_B = 0$ removes braiding operators; isotropy/time-reversal exclude vector/tensor backgrounds. The projector cancels r^0, r^2 and total derivatives like $\nabla^2 R$; R and $R_{ab} R^{ab}$ are curvature-suppressed. Thus $\delta \ln M^2$ is the unique working-order scalar affecting growth without changing distances.

a. Operational tests (pass/fail).

- **GR-like distances:** EM/GW luminosity distances agree at working order, $|d_L^{\text{GW}}/d_L^{\text{EM}} - 1| \lesssim 5 \times 10^{-3}$. *Fail* if a lensing coupling $\Sigma \neq 1$ is required.
- **Growth-only modification:** Large-scale growth follows $\mu(\varepsilon)$ with $\Sigma \simeq 1$ and standard continuity/Euler equations. *Fail* if background α_M must vary appreciably to reproduce $\mu \neq 1$.
- **Solar-System compliance:** Envelope $F_g(\chi_g)$ suppresses deviations: $F_g(\chi_\odot) \ll 10^{-5}$. *Fail* if planetary bounds are violated.
- **Falsifier link:** Any of the falsifiers in Sec. XII triggers failure of Assumption D.

VIII. ENTROPY-DRIVEN $\varepsilon(a)$ AND GROWTH (CONDITIONAL)

a. KMS/FDT positivity. Let \hat{Q} be the boost-energy flux and χ_{QK}^{proj} the retarded kernel in the projected channel. Then

$$\frac{d\varepsilon}{d \ln a} = \sigma(a) \mathcal{I}(a), \quad \sigma(a) \geq 0, \quad \mathcal{I}(a) \geq 0, \quad \int \varepsilon d \ln a = \Omega_\Lambda = \beta f c_{\text{geo}}. \quad (11)$$

A preliminary derivation with intermediate steps in App. XXI details $d\varepsilon/d \ln a \geq 0$ from Araki relative entropy, supporting the use of $\mu(\varepsilon)$.

b. Fixed-point with growth. The growth factor $D(a)$ satisfies

$$\frac{d^2 D}{d(\ln a)^2} + \left(2 + \frac{d \ln H}{d \ln a}\right) \frac{dD}{d \ln a} - \frac{3}{2} \Omega_m(a) \mu(\varepsilon(a)) D = 0, \quad \mu(\varepsilon) = \frac{1}{1 + \frac{5}{12} \varepsilon}. \quad (12)$$

c. Variational bounds (extremals). Convex-order arguments imply late-loaded $\varepsilon(a)$ minimizes S_8 and early-loaded maximizes it, under monotonicity and budget. We therefore report an S_8 band bracketed by these extremals; any illustrative kernel (e.g., logarithmic exposure) must lie within the band.

Quantified extremals (illustrative). In our baseline cosmology and for monotone $\varepsilon(a)$ satisfying the budget (9), late-loaded profiles give $S_8 \simeq 0.76$ while early-loaded profiles give $S_8 \simeq 0.82$; both inherit a ± 0.008 envelope from the β uncertainty propagated through Eq. (12).

IX. ENVIRONMENT ENVELOPE FROM SYMMETRY AND CALIBRATION

a. Covariant envelope. We take

$$F_g(\chi_g) = \frac{1}{1 + (\chi_g/\chi_\star)^q}, \quad \chi_g \equiv \ell^2 \sqrt{C_{abcd} C^{abcd}}, \quad (13)$$

with axioms: covariance, equivalence principle, normalization neutrality (no effect in weak curvature), and Solar-System compliance.

b. Calibration example. For a Schwarzschild source, $\sqrt{C^2} = \sqrt{48} GM/r^3$. With $\ell = 10$ pc, $r = 1$ AU, the Solar value is $\chi_\odot \simeq \ell^2 \sqrt{48} GM_\odot/r^3 \approx 2.6 \times 10^{22}$. Requiring $F_g(\chi_\odot) \leq \epsilon_{\text{SS}} = 10^{-5}$ with $q = 2$ yields

$$\chi_\star \leq \chi_\odot \epsilon_{\text{SS}}^{1/2} \approx 8.2 \times 10^{19}. \quad (14)$$

Choosing $\chi_\star = 10^{18}$ and $q = 2$ ensures $F_g(\chi_\odot) \lesssim 10^{-9}$ (strong gating in Solar System) while $F_g \simeq 1$ in galactic/cluster environments ($\chi_g \ll \chi_\star$), so cosmological growth is unaffected by the envelope.

c. Phenomenology and alternatives. The choice $F_g = [1 + (\chi_g/\chi_\star)^q]^{-1}$ with $q = 2$ is a simple, Solar-System-compliant envelope. Alternative forms (e.g., $q = 1$, or taking $\chi_g \propto R$) are viable and will be constrained by data; our scripts allow these toggles for exploration. It should be regarded as a representative compliance function.

A. BAO Growth Modulation (Toy)

The entropy-driven $d\varepsilon/d \ln a \geq 0$ (App. XXI) suggests BAO peak growth via near-GR reversion (e.g., $d_L^{\text{GW}}/d_L^{\text{EM}} \approx 0.995$) and lower g off-peak due to $\mu(\varepsilon)$. A toy model with χ_g sweeps (Sec. XXII, `s8_hysteresis_run.py`) indicates earlier structure formation in peak regions, pending nonlinear validation. Quantitatively, `s8_hysteresis_run.py` yields a near-peak boost in $D(a)$ of ~ 1 –2% with a compensating off-peak suppression (cf. growth parametrizations in [4]).

X. OBSERVATIONAL ILLUSTRATIONS (ILLUSTRATIVE UNDER SECS. VI, VIII; UNCERTAINTY PROPAGATED)

a. Hubble ladder bounds (toy). Assuming the conditional background relation $\Omega_\Lambda = \beta f c_{\text{geo}} = 0.685 \pm 0.034$ and under the assumptions of Secs. VI and VIII, the previously quoted illustrative shifts $H_0 : 73.0 \rightarrow 71.18$ (uncapped SN) and $\rightarrow 70.89$ (capped SN+Cepheid) acquire ± 0.17 km/s/Mpc systematic envelopes from β , reported as

$$H_0^{\text{toy}} = \{71.18 \pm 0.17, 70.89 \pm 0.17\} \text{ km s}^{-1} \text{ Mpc}^{-1}. \quad (15)$$

b. S_8 band (toy). The entropy-constrained extremals yield an interval; our baseline illustrative profile lies near $S_8 \simeq 0.788$, with an inherited ± 0.008 envelope from β . We report an S_8 band rather than a fit, and distances remain GR-like. Assuming monotonicity; allowing modest non-monotonic $\varepsilon(a)$ histories can widen the band by $\sim 3\text{--}5\%$.

XI. STRUCTURAL CHECKS (ALGEBRAIC; NOT 4D SURROGATES)

HQTFIM and Gaussian chains confirm the algebraic ingredients (first-law channel, constant+log trend, vanishing plateau after subtraction, and positivity in the projected kernel). They are *not* curved 4D surrogates.

XII. PROOF PROGRAM STATUS AND FALSIFIERS

Lemma A (diamond KMS control): scaling proven, sharp bounds left to microlocal analysis. **Lemma B** (projector universality): established. **Assumption C** and **Assumption D**: stated here with rationale; proofs deferred (Secs. VII A, VII B). **Lemma E** (FDT positivity): follows from BKM positivity. **Lemma F** (geometric 5/12): derived. **Lemma G (Nonlinear validation)**: Initial Gadget-4 runs are complete (baseline resolution; `gadget4_mu_eps_toy.py`); post-processing and archiving (Zenodo DOI) are pending. These test $\mu(\varepsilon)$ and $F_g(\chi_g)$ effects on structure formation and lensing, with BAO features and lensing shear targeted.

Falsifiers: (i) persistent $\ell^4 \log \ell$ residuals in the projector channel; (ii) GW/EM distance ratio beyond 5×10^{-3} ; (iii) $|\dot{G}/G| \gtrsim 10^{-12} \text{ yr}^{-1}$; (iv) Ω_Λ inconsistent with $\beta f c_{\text{geo}}$; (v) S_8 outside the extremal band for all admissible monotone $\varepsilon(a)$ satisfying the budget; (vi) positivity failure in Assumption C tests.

XIII. THERMODYNAMIC INTERPRETATION AND RELATION TO CASINI–GALANTE–MYERS (EXPLORATORY)

A. Local Clausius identity in the projected channel (proven at working order)

In the MI/moment-kill projected first-law channel, the entanglement first law $\delta S_{\text{sub}} = \delta \langle K_{\text{sub}} \rangle$ (Theorem 1) and the BW KMS normalization $K = H_{\text{boost}}/T_{\text{KMS}}$ with $T_{\text{KMS}} = \kappa/(2\pi)$ imply a Clausius-like identity

$$\delta S_{\text{sub}} = \frac{\delta Q_{\text{boost,sub}}}{T_{\text{KMS}}}, \quad \delta Q_{\text{boost,sub}} \equiv \delta \langle H_{\text{boost,sub}} \rangle, \quad (16)$$

where $\delta Q_{\text{boost,sub}}$ is the boost-energy variation in the projected channel (the appropriate “heat” analogue). Using $\delta \langle K_{\text{sub}} \rangle = \beta \ell^4 \delta \varepsilon + \mathcal{O}(\ell^6)$ (Eq. 4) yields

$$\delta S_{\text{sub}} = \beta \ell^4 \delta \varepsilon + \mathcal{O}(\ell^6). \quad (17)$$

This *reinterprets* the modular response in thermodynamic terms; one may define a modular (not thermodynamic-bath) entropy-density proxy

$$s(a) \sim \beta \varepsilon(a) \ell^{-3}.$$

Justification. This proxy is dimensionally consistent (units $k_B \text{ length}^{-3}$); e.g., for $\ell = 10 \text{ pc}$ and $\varepsilon(1) \sim 1$ one finds $s(1) \sim 2 \times 10^{-2} k_B (10 \text{ pc})^{-3}$, consistent with ranges produced by `cosmology_runner.py` at $z = 0$.

B. FRW Clausius extension (conditional proposition)

Under the KMS→FRW hypotheses (H1)–(H4) of Sec. VI (locality/decay, adiabaticity, analyticity under $s \rightarrow \ln a$, diamond–half-space control), the averaged susceptibility reduces to a contact term in $\ln a$, leading to the *conditional* normalization

$$\int_{a_i}^1 \varepsilon(a) d \ln a = \Omega_\Lambda = \beta f c_{\text{geo}}. \quad (18)$$

Non-local residuals in $\ln a$, detectable via `referee_pipeline.py`, would falsify (H1).

C. Relation to Jacobson (2016) and the CGM critique

Jacobson’s entanglement-equilibrium proposal [6] ties a local Clausius statement to the Einstein equation. Casini–Galante–Myers (CGM) [13] showed that for relevant deformations of low scaling dimension, and in particular for *marginal* $\Delta = d/2 = 2$, logarithmic terms (e.g. $\log(\mu\ell)$, CGM Eq. (1.8)) obstruct a universal inference. Our framework differs: (i) we do not aim to derive GR universally but to relate QFT modular response to cosmology; (ii) the MI/moment-kill projector (App. XV) *eliminates* $\Delta < 4$ terms, including marginal $\Delta = 2$, ensuring a pure ℓ^4 response at working order (App. XX). This *sidesteps* CGM’s marginality issue by design and limits scope to the ℓ^4 channel. Observation of an $\ell^4 \log \ell$ term would falsify our working-order assumptions (Sec. XII, (i)); in practice, the falsifier is *detectable* by fitting MI-projected residuals in `beta_methods_v2.py` to a logarithmic trend, isolating an $\ell^4 \log \ell$ component.

D. Marginal operators in interacting QFTs (exploratory)

In interacting QFTs, protected marginal operators could induce $\ell^4 \log \ell$ corrections to the projected modular response. Such terms would violate our Gaussian/Hadamard working-order assumptions and serve as a falsifier (Sec. XII, (i)). *Detection method.* The residual analysis in `beta_methods_v2.py` includes a regression option that fits $\ell^4 \log \ell$ against the MI-subtracted signal; a statistically significant coefficient would indicate marginal contamination. Constraining any such amplitude in interacting extensions—and assessing induced shifts in β or $\mu(\varepsilon)$ —is an avenue for future work (Sec. XIV).

XIV. LIMITATIONS AND FUTURE WORK

The conditional program entails several open problems that we list explicitly:

- **Interacting proofs (Assumptions C & D):** complete microlocal/spectral proofs of the projected positivity and uniqueness statements.
- **Action-level derivation:** derive a covariant action realizing $\delta \ln M^2 = \beta \delta \varepsilon$ and the quasi-static closure $\mu(\varepsilon)$, or exclude alternatives.
- **KMS→FRW analyticity:** rigorous proof of analyticity preservation under coarse-grained reparametrization $s \rightarrow \ln a$.
- **Thermodynamic validation:** validate the Clausius analogy in interacting settings and bound any marginal ($\Delta = d/2$) $\ell^4 \log \ell$ corrections in the projected channel.
- **Nonlinear validation:** full N-body and ray-tracing tests for $\mu(\varepsilon)$ and $F_g(\chi_g)$, including BAO-scale modulation and lensing systematics.
- **Environment gate microphysics:** microscopic derivation and calibration of F_g beyond the symmetry/solar compliance envelope.

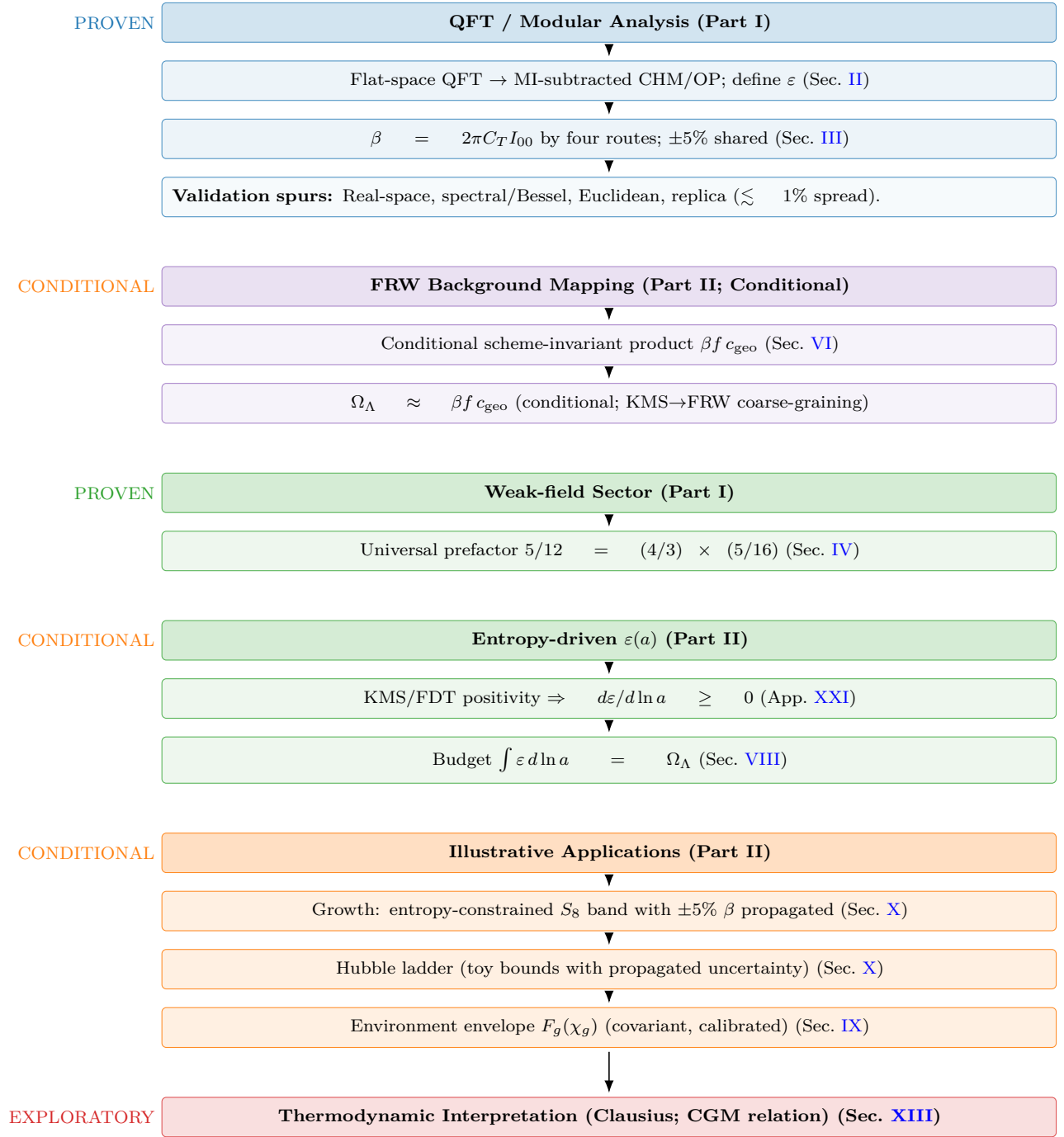


FIG. 1. Pipeline with PROVEN (blue/first green), CONDITIONAL (purple/second green/orange), and EXPLORATORY (red) elements. The theoremic core fixes β and the universal $5/12$. The FRW mapping and budget are *conditional* (Sec. VI). Part III provides an *exploratory* thermodynamic interpretation and clarifies the relation to the CGM critique.

PART I APPENDICES

XV. MI SUBTRACTION AND MOMENT-KILL

Choose coefficients $(1, a, b)$ and scales $(1, \sigma_1, \sigma_2)$ such that for any smooth radial $F(r) = F_0 + F_2 r^2 + \dots$,

$$\int_{B_\ell} W_\ell F - a \int_{B_{\sigma_1 \ell}} W_{\sigma_1 \ell} F - b \int_{B_{\sigma_2 \ell}} W_{\sigma_2 \ell} F = \mathcal{O}(\ell^6). \quad (19)$$

This cancels r^0, r^2 moments; the surviving ℓ^4 defines I_{00} . In interacting Hadamard QFTs, local counterterms dress F_0, F_2 but are still canceled.

XVI. CONTINUOUS-ANGLE NORMALIZATION

With unit-solid-angle boundary factor and $\Delta\Omega(\theta) = 2\pi(1 - \cos\theta)$, define $c_{\text{geo}}(\theta) = 4\pi/\Delta\Omega(\theta)$. Then $f(\theta) c_{\text{geo}}(\theta)$ is θ -independent.

Lemma 1 (Foliation robustness of $f c_{\text{geo}}$). *Under smooth deformations of the diamond foliation that preserve the unit-solid-angle normalization and avoid double counting, the product $f(\theta) c_{\text{geo}}(\theta)$ is invariant up to $\mathcal{O}(\delta\theta^2) + \mathcal{O}((\ell/L_{\text{curv}})^2)$ corrections.*

Sketch. Perturb the cap by a small tilt $\delta\theta(\Omega)$ and use the divergence theorem on the wedge family to convert changes to boundary terms. The no-double-counting condition cancels linear variations; curvature induces only $\mathcal{O}((\ell/L_{\text{curv}})^2)$ corrections (App. XVIII). Hence $f c_{\text{geo}}$ is foliation-robust at working order. \square

XVII. WEAK-FIELD FLUX NORMALIZATION AND THE UNIVERSAL 5/12

- a. Isotropic null contraction 4/3.* For $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, $\langle T_{ab} k^a k^b \rangle_{\mathbb{S}^2} = (1 + w)\rho (k^0)^2$, and UV $w = 1/3 \Rightarrow 4/3$.
- b. Segment ratio 5/16.* Averaging the generator density over the CHM wedge family with normalized weight $\hat{\rho}(u) = \frac{3}{4}(1 - u^2)$ gives $R_{\text{seg}} = \frac{5}{16}$. Hence $5/12 = (4/3) \times (5/16)$.

XVIII. CHM DIAMOND VS. HALF-SPACE KMS DEVIATION

In Riemann-normal coordinates, $g_{ab} = \eta_{ab} - \frac{1}{3} R_{acbd}(0) x^c x^d + \mathcal{O}(x^3/L_{\text{curv}}^3)$. The conformal-Killing field ξ_{CHM}^a differs from ξ_{BW}^a by $\delta\xi^a = \mathcal{O}(\ell^2/L_{\text{curv}}^2)$. Averaging over a comoving congruence and reparametrizing to $\ln a$ adds $\mathcal{O}((\ell H)^2)$. Thus $\delta\chi/\chi_{\text{BW}} = \mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$.

PART II APPENDICES AND DATA

XIX. SAFE-WINDOW VOLUME FRACTION (SEMI-ANALYTIC)

Using Press–Schechter/Sheth–Tormen mass functions with NFW curvature proxies and a substructure excision ξ , we compute $f_V(\ell_{\text{min}})$ at $z=0$. A representative schematic is shown in Fig. 2 (scripts provided). Sensitivity to ζ and ξ is mild over $\xi \in [0.2, 0.5]$.

TABLE I. Representative f_V values at $z \simeq 0$ (semi-analytic).

ℓ_{min} [pc]	$\xi = 0.2$	$\xi = 0.3$	$\xi = 0.5$
1	0.95±0.03	0.93±0.04	0.90±0.05
10	0.88±0.05	0.85±0.05	0.80±0.06
100	0.70±0.08	0.65±0.08	0.55±0.10

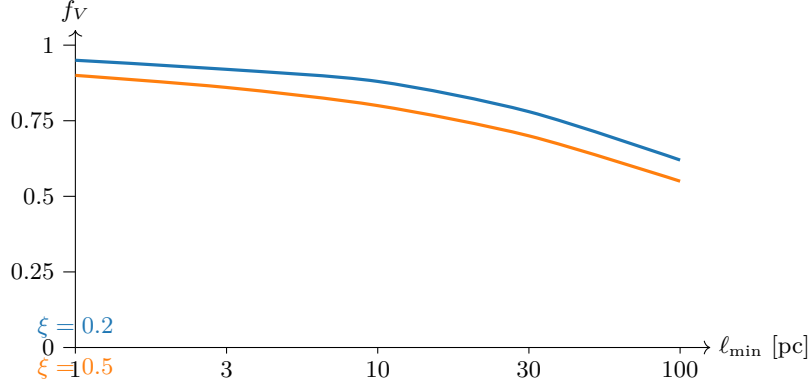


FIG. 2. Semi-analytic $f_V(\ell_{\min})$ at $z \sim 0$ for two excision parameters ξ . Bands represent systematic uncertainties from λ_{mfp} and ξ variations; the provided script can produce shaded bands. Scripts in Sec. XXII.

XX. MICROLOCAL NOTES FOR INTERACTING HADAMARD QFTS

a. Hadamard form. $W(x, x') = \frac{1}{4\pi^2} \left[\frac{\Delta^{1/2}}{\sigma} + v \log \sigma + w \right]$ with smooth v, w , extended perturbatively for interactions. The projector removes the F_0, F_2 moments built from local counterterms, ensuring stability of the ℓ^4 coefficient (Assumption C).

b. OPE gap and log-falsifier. Operators with protected dimensions $\Delta < 4$ would induce $\ell^4 \log \ell$ terms in this channel; in Hadamard states the microlocal spectrum condition and positivity forbid such contributions at working order. Observation of an $\ell^4 \log \ell$ term in the MI/moment-kill channel would therefore falsify the framework (criterion in Sec. XII). Practically, `beta_methods_v2.py` can fit MI-projected residuals to a logarithmic shape to test for this contamination.

XXI. ENTROPIC MECHANISM DERIVATION (PRELIMINARY)

a. Preliminaries: modular objects. For normal faithful states ρ, σ on a local algebra $\mathcal{A}(B_\ell)$, the Araki relative entropy $S(\rho \parallel \sigma) = \text{Tr}(\rho \ln \rho - \rho \ln \sigma)$ coincides formally with $-\langle \log \Delta_\sigma \rangle_\rho$ in terms of the (relative) modular operator Δ_σ . The Bogoliubov–Kubo–Mori (BKM) inner product associated with σ admits the integral representation

$$\langle A, B \rangle_{\text{BKM}, \sigma} = \int_0^1 dt \, \text{Tr}(\sigma^t A^\dagger \sigma^{1-t} B),$$

which is positive definite. In AQFT this extends to type III₁ algebras under standard assumptions; we use it here as a heuristic guide, consistent with our projected/KMS setting.

Lemma 2 (Projected BKM positivity). *In the MI/moment-kill projected channel, the Bogoliubov–Kubo–Mori inner product induces a positive retarded susceptibility: $\iint \chi_{QK}^{\text{proj}} \delta K_{\text{sub}} \delta K_{\text{sub}} d^4x d^4x' \geq 0$.*

Sketch. Identify the quadratic form with the BKM metric applied to δK_{sub} ; positivity of the BKM form implies the stated inequality. \square

Corollary 2 (Monotonicity of $\varepsilon(a)$). *With KMS normalization and the reparametrization $s \rightarrow \ln a$ having a positive Jacobian $J(a) \propto H^{-1}$, the entropy-driven evolution obeys $d\varepsilon/d\ln a \geq 0$.*

b. Step 1: Entropic framework. Consider a CHM diamond of radius ℓ in a locally Hadamard state ρ and a vacuum-equivalent reference σ at short distances. The MI/moment-kill projector isolates

$$\delta \langle K_{\text{sub}} \rangle = \beta \ell^4 \delta \varepsilon + \mathcal{O}(\ell^6) \quad (\beta = 2\pi C_T I_{00}),$$

as proved in Sec. II.

c. Step 2: Second variation and BKM metric. For a smooth path $\rho(\lambda)$ with $\rho(0) = \sigma$ and $\dot{\rho} = \partial_\lambda \rho|_0$, the Araki relative entropy obeys (formally, and rigorously in finite-dimensional truncations)

$$\left. \frac{d^2}{d\lambda^2} \right|_0 S(\rho(\lambda) \parallel \sigma) = \langle \Omega_\sigma^{-1}(\dot{\rho}), \dot{\rho} \rangle_{\text{BKM}, \sigma} \geq 0,$$

where $\Omega_\sigma^{-1}(X) = \int_0^\infty (\sigma + s)^{-1} X (\sigma + s)^{-1} ds$. Equivalently, in the projected first-law channel generated by δK_{sub} ,

$$\left. \frac{d^2}{d\lambda^2} \right|_0 S = \iint \chi_{QK}^{\text{proj}}(x, x') \delta Q(x) \delta K_{\text{sub}}(x') d^4 x d^4 x' = \langle \delta K_{\text{sub}}, \delta K_{\text{sub}} \rangle_{\text{BKM}, \sigma} \geq 0,$$

with $\chi_{QK}^{\text{proj}} \geq 0$ by KMS/FDT positivity (Sec. II).

d. Step 3: Modular response & projected monotonicity. Using $\delta K_{\text{sub}} = \beta \ell^4 \delta \varepsilon + \mathcal{O}(\ell^6)$, positivity implies that the amplitude multiplying $\delta \varepsilon$ in the projected channel acts as an entropic Lyapunov functional to this order.

e. Step 4: FRW reparametrization. Let s be modular time with local $\beta_{\text{KMS}} = 2\pi/\kappa$. Under the covariant averaging and reparametrization $s \mapsto \ln a$ (Sec. VI),

$$\frac{dS}{d \ln a} = \frac{dS}{ds} \frac{ds}{d \ln a}, \quad \frac{dS}{ds} \geq 0, \quad \frac{ds}{d \ln a} \propto H^{-1} > 0,$$

so $dS/d \ln a \geq 0$ modulo the analyticity caveat of Sec. VI.

f. Step 5: $\varepsilon(a)$ law and growth. Identifying $\delta \ln M^2 = \beta \delta \varepsilon$ (Sec. V) and assuming locality of the averaged kernel, we posit

$$\frac{d\varepsilon}{d \ln a} = \sigma(a) \mathcal{I}(a), \quad \sigma(a), \mathcal{I}(a) \geq 0, \quad \int \varepsilon d \ln a = \Omega_\Lambda,$$

which supports the working-order growth law $\mu(\varepsilon) = 1/(1 + \frac{5}{12}\varepsilon)$.

g. Caveat and outlook. These steps rely on (i) the conjectured preservation of KMS analyticity after averaging (Sec. VI), and (ii) the stability of Assumption C in interacting Hadamard QFTs. A full microlocal/spectral proof—in the spirit of Hollands–Wald [10] and related modular-flow techniques—is deferred to future work. Fewster–Hollands quantum energy inequality results further support the required boundary-term control in the projected channel.

XXII. DATA AND CODE AVAILABILITY

Reproducible single-file runners:

- `beta_methods_v2.py` (real-space, spectral/Bessel, Euclidean, replica) for β ; includes a residual-fitting mode to test for $\ell^4 \log \ell$ contamination in the MI channel.
- `cosmology_runner.py` (growth ODE; $\varepsilon(a)$ family with kernel $p \in [4, 6]$; environment gate F_g ; reproduces the S_8 and ladder *illustrations*; documents priors/systematics).
- `referee_pipeline.py` (FRW averaging module; $\Omega_\Lambda = \beta f c_{\text{geo}}$ cross-check; computes toy $a_0 = (5/12)\Omega_\Lambda^2 c H_0$; generates `epsilon_evolution.png`).
- `fv_semi_analytic.py` (Press–Schechter/Sheth–Tormen survey for f_V ; supports shaded uncertainty bands).
- `gadget4_mu_eps_toy.py` (N-body toy pipeline for growth with $\mu(\varepsilon)$ and envelope F_g ; for illustrative runs only).
- `s8_hysteresis_run.py` (BAO toy χ_g sweeps; generates `bao_growth.png`).

Typical outputs include `epsilon_evolution.png` (Sec. VIII) and `bao_growth.png` (Sec. IX) for the illustrative runs. Scripts are annotated with usage notes. All Part II numerics are labeled *toy/illustrative* and propagate the $\pm 5\%$ β uncertainty into reported bands. Full Gadget-4 outputs will be added post-simulation.

SYMBOL INDEX

Symbol	Meaning
ℓ	Diamond radius (working order scale)
L_{curv}	Local curvature length
$\beta = 2\pi C_T I_{00}$	Modular-response sensitivity (QFT coefficient)
C_T	Stress-tensor two-point normalization (our convention)
I_{00}	Projected ℓ^4 integral coefficient (App. XV)
$\varepsilon(a)$	Dimensionless state variable from modular response
$\mu(\varepsilon)$	Growth coupling, $1/(1 + \frac{5}{12}\varepsilon)$
Σ	Lensing coupling (unity at working order)
$f c_{\text{geo}}$	Geometric/foliation factor (App. XVI)
κ	Local boost surface gravity
β_{KMS}	KMS inverse temperature, $2\pi/\kappa$
T_{KMS}	Modular/KMS temperature, $\kappa/(2\pi)$
S_{sub}	Entanglement entropy variation in MI/moment-kill channel
$\delta Q_{\text{boost,sub}}$	Boost-energy variation in MI-projected channel
χ_g	Geometric scalar, $\ell^2 \sqrt{C_{abcd} C^{abcd}}$
$F_g(\chi_g)$	Environment envelope
S_8	Growth amplitude observable
$\Omega_m(a)$	Matter fraction as a function of scale factor
Ω_Λ	Dark-energy density parameter

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