Modular Response in Free Quantum Fields: A KMS/FDT Theorem and Conditional Extensions

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(Dated:)

Part I (Theoremic core, free/Gaussian Hadamard QFT). We prove that, for small causal diamonds (CHM) in locally Hadamard states on globally hyperbolic spacetimes and within a safe window $\epsilon_{\mathrm{UV}} \ll \ell \ll \min\{L_{\mathrm{curv}}, \lambda_{\mathrm{mfp}}, m_i^{-1}\}$, the MI/moment-kill projector isolates a finite ℓ^4 modular response with coefficient equal to its flat-space value, the projected KMS/FDT susceptibility is positive, and coarse-graining over the wedge family produces a universal weak-field prefactor $5/12 = (4/3) \times (5/16)$. The fractional KMS defect between CHM diamonds and half-spaces scales as $\mathcal{O}((\ell/L_{\mathrm{curv}})^2) + \mathcal{O}((\ell H)^2)$. The QFT sensitivity is $\beta = 2\pi C_T I_{00} = 0.02086 \pm 0.00105$ (conservative 5% shared systematics from four independent routes). A scheme-invariant background normalization yields $\Omega_\Lambda = \beta\, f\, c_{\mathrm{geo}}$.

Part II (Conditional extensions). We separate *definition* (flat-space ε from modular response) from *mapping* (constitutive identification $\delta \ln M^2 = \beta\, \delta\varepsilon$), keep the distance sector GR-like ($\alpha_M \simeq 0$), and obtain weak-field growth $\mu(\varepsilon) = 1/(1 + \frac{5}{12}\varepsilon)$. The entropy-driven law $d\varepsilon/d\ln a \geq 0$ follows from KMS/FDT positivity with a fixed "budget" $\int \varepsilon\, d\ln a = \Omega_\Lambda$. We present a covariant constraint on the environment envelope $F_g(\chi_g) = [1 + (\chi_g/\chi_\star)^q]^{-1}$ with $\chi_g \equiv \ell^2\sqrt{C_{abcd}C^{abcd}}$, calibrated by Solar-System bounds. Cosmological illustrations (an S_8 band and H_0 shifts) are **toy/illustrative**, conditional on extension to interacting QFTs and the covariant KMS→FRW link; all values propagate the $\pm 5\%$ β uncertainty.

What is new. (i) Completed proofs in the Gaussian/Hadamard sector, including a covariant KMS→FRW averaging lemma with explicit error budget; (ii) **stated assumptions for Lemma C (relative entropy ↔ canonical energy in the projected diamond) and Lemma D (uniqueness of M^2 coupling at working order) in interacting QFTs, pending rigorous proof**; (iii) semi-analytic quantification of the safe-window volume fraction $f_V(\ell_{\min})$ with Press–Schechter/Sheth–Tormen inputs; (iv) a symmetry-constrained F_g envelope with calibrated $\chi_\star$ and q; (v) uncertainty propagation of β into S_8 and H_0 bounds. Part II remains explicitly labeled as conditional.

Caveat. Part II relies on unproven extensions to interacting QFTs, which may fail pending further theoretical development; we label these elements explicitly and provide a roadmap to rigor.

READER'S MAP: PART I (THEOREM) VS. PART II (CONDITIONAL)

Part I (Secs. I–IV, Apps. XIII–XVI): proven results for free/Gaussian Hadamard fields at working order.
Part II (Secs. V–XX, Apps. XVII, XVIII, XIX): conditional extensions, *assumptions* C & D (pending proof), safe-window fraction, KMS→FRW link, symmetry envelope, and toy/illustrative numerics (App. XIX) with propagated uncertainties.

I. SCOPE, WORKING ORDER, AND SAFE-WINDOW QUANTIFICATION (PART I)

a. Working order and state class. We work to $\mathcal{O}(\ell^4)$ in the MI/moment-kill projector channel, treating curvature/contact terms as $\mathcal{O}(\ell^6)$. States are locally Hadamard.

b. KMS applicability (CHM diamonds). Exact BW KMS holds for half-spaces; CHM diamonds inherit it with fractional defect $\mathcal{O}((\ell/L_{\mathrm{curv}})^2) + \mathcal{O}((\ell H)^2)$ (App. XVI).

c. Safe-window volume fraction. Define a conservative admissible scale

$$\ell_{\max}(x) \equiv \zeta\ \min\left\{L_{\mathrm{curv}}(x),\ \lambda_{\mathrm{mfp}}(x),\ m_i^{-1}(x)\right\}, \qquad \zeta = 0.1. \tag{1}$$

Using Press–Schechter/Sheth–Tormen mass functions and NFW curvature proxies $L_{\mathrm{curv}}^{-2} \sim (R_{abcd}R^{abcd})^{1/2}$ with substructure excision parameter ξ, we estimate the comoving volume fraction $f_V(\ell_{\min}) = \mathrm{Vol}\{x : \ell_{\max}(x) > \ell_{\min}\}/\mathrm{Vol}_{\mathrm{tot}}$. A semi-analytic survey (App. XVII) shows voids dominate f_V, while dense cores lack a window; representative values at $z \sim 0$ for $\ell_{\min} \in [1, 100]$ pc are $f_V \sim 0.6-0.95$ for $\xi \in [0.2, 0.5]$. This enters only as a domain-of-validity indicator. A parameter sensitivity sweep (varying $\zeta,\ \lambda_{\mathrm{mfp}},\ \xi$) yields more conservative fractions $f_V \sim 0.4-0.7$ (App. XVII), addressing potential over-optimism in structured regions.

d. Angle invariance as a null test. The continuous-angle product $\mathcal{C}_\Omega = f(\theta)\, c_{\rm geo}(\theta)$ is analytic and θ-independent; residuals are shown as a null check, not a precision claim.

II. A2–KMS THEOREM (GAUSSIAN/HADAMARD SECTOR)

Theorem 1 (Projected modular response and positivity)**.** *Let $\mathcal{Q}$ be a free (Gaussian) QFT on a globally hyperbolic spacetime and ρ a locally Hadamard state. For a causal diamond of radius ℓ with $\ell \ll L_{\rm curv}$ and the MI/moment-kill projector that cancels r^0 and r^2 moments, the MI-subtracted modular response obeys*

$$\delta\langle K_{\rm sub}\rangle = (2\pi C_T I_{00})\,\ell^4\,\delta\varepsilon + \mathcal{O}(\ell^6), \tag{2}$$

with coefficient equal to the flat-space value. The retarded susceptibility χ_{QK} in the projected channel is positive (FDT), and wedge averaging yields the universal weak-field prefactor $5/12$. The fractional deviation from BW KMS is $\mathcal{O}((\ell/L_{\rm curv})^2) + \mathcal{O}((\ell H)^2)$.

Proof. Hadamard microlocal expansions reduce all UV data to the Minkowski parametrix; MI/moment-kill cancels local counterterms to $\mathcal{O}(\ell^4)$ (App. XIII). BW KMS fixes linear-response normalization and sign; positivity follows from the Bogoliubov–Kubo–Mori metric. Isotropic contraction and CHM segment ratio yield 5/12 (Sec. IV). CHM vs. half-space defects scale as stated in Riemann-normal coordinates (App. XVI). □

Corollary 1 (Background zero mode)**.** *The FRW zero mode satisfies the scheme-invariant normalization $\Omega_\Lambda = \beta\, f\, c_{\rm geo}$, with $\beta = 2\pi C_T I_{00}$.*

III. QFT INPUT: $\beta = 2\pi C_T I_{00}$ AND ERROR BUDGET

We evaluate β via four independent routes: (a) real-space CHM; (b) spectral/Bessel; (c) Euclidean time-slicing; (d) replica finite-difference. The spread is $\lesssim 1\%$. We adopt a conservative

$$\beta = 0.02086 \pm 0.00105 \quad \text{(5\% shared systematics)}. \tag{3}$$

Angle invariance is used as a null residual test.

IV. WEAK-FIELD PREFACTOR 5/12

The isotropic BW channel gives $\langle T_{kk}\rangle = (1+w)\rho$ with UV $w = 1/3 \Rightarrow 4/3$. Averaging over CHM segments yields 5/16, so $5/12 = (4/3)\times(5/16)$. Details in App. XV.

V. DEFINITION VS. MAPPING (PART II; CONDITIONAL)

a. Definition (flat-space QFT).

$$\delta\langle K_{\rm sub}(\ell)\rangle = \underbrace{(2\pi C_T I_{00})}_{\beta}\,\ell^4\,\delta\varepsilon(x) + \mathcal{O}(\ell^6). \tag{4}$$

b. Mapping (constitutive; distances GR-like). In the $c_T = 1$, $\alpha_B = 0$ EFT corner with isotropy, we *identify* at working order

$$\delta \ln M^2 = \beta\,\delta\varepsilon, \qquad \mu(\varepsilon) = \frac{1}{1+\frac{5}{12}\,\varepsilon}, \qquad \alpha_M \simeq 0 \text{ in distances.} \tag{5}$$

This is a **constitutive closure**, not a derived macroscopic law; it is falsified by log-ℓ residuals, $|d_L^{\rm GW}/d_L^{\rm EM} - 1| > 5\times10^{-3}$, or Ω_Λ inconsistent with $\beta f c_{\rm geo}$.

VI. COVARIANT KMS → FRW LINK AND ERROR CONTROL

Let s denote modular time with $\beta_{\rm KMS} = 2\pi/\kappa$ locally. Averaging the retarded kernel over a comoving congruence of diamonds and reparametrizing $s \mapsto \ln a$ induces the FRW background factor $f\, c_{\rm geo}$; diffeomorphism covariance is preserved because the averaging functional depends only on local curvature scalars and the diamond foliation. The total fractional defect in the kernel obeys

$$\frac{\delta\chi}{\chi_{\rm BW}} = \mathcal{O}\Big((\ell/L_{\rm curv})^2\Big) + \mathcal{O}\big((\ell H)^2\big), \tag{6}$$

which is negligible for $\ell \sim 10\,\mathrm{pc}$, $L_{\rm curv} \sim 10\,\mathrm{Mpc}$, $H^{-1} \sim 4\,\mathrm{Gpc}$.

Analyticity caveat. The reparametrization $s \to \ln a$ is conjectured to preserve KMS analyticity for the averaged kernel by matching modular flow to the FRW scale factor; a rigorous proof of the averaged kernel's analytic properties remains open and likely requires a spectral/microlocal treatment (see e.g. Hollands & Wald 2001). We list this as a priority item in Sec. XII.

VII. ASSUMPTIONS FOR INTERACTING EXTENSIONS AT WORKING ORDER (PART II)

A. Assumption C: Relative entropy ↔ canonical energy in the projected diamond

Hypothesis 1 (Assumption C)**.** *For a local algebra $\mathcal{A}(B_\ell)$ of an interacting Hadamard QFT obeying the microlocal spectrum condition and time-slice axiom, let σ be the reference diamond state (vacuum-equivalent at short distance) with modular operator Δ_σ and modular Hamiltonian K_σ. For a smooth one-parameter family $\rho(\lambda)$ with $\rho(0) = \sigma$ and $\dot\rho \equiv \partial_\lambda \rho|_0$, the MI/moment-kill projected second variation of Araki relative entropy equals the canonical energy quadratic form of the projected stress tensor, up to $\mathcal{O}(\ell^6)$ remainders:*

$$\frac{d^2}{d\lambda^2}\bigg|_0 S(\rho(\lambda)\|\sigma) = \mathcal{E}^{\rm proj}_{\rm can}[\delta T;\, \xi_{\rm CHM}] = \iint \chi^{\rm proj}_{QK}(x,x')\,\delta Q(x)\,\delta K_{\rm sub}(x')\, d^4x\, d^4x' \geq 0, \tag{7}$$

with $\chi^{\rm proj}_{QK}$ positive-definite. The equality is assumed *to hold for interacting QFTs at working order $\mathcal{O}(\ell^4)$.*

Rationale and status (sketch). (i) **Second variation as BKM metric.** For type III_1 local algebras, Araki relative entropy is well-defined; near σ, the second variation equals the Bogoliubov–Kubo–Mori (quantum Fisher) metric evaluated on the tangent vector generated by δK_σ.
(ii) **Projector isolation (assumed).** We assume the MI/moment-kill projector cancels contact terms and the r^0, r^2 moments built from Hadamard v, w coefficients through $\mathcal{O}(\ell^4)$ in interacting Hadamard QFTs, leaving the flat-space coefficient; a full microlocal proof of stability under interactions is outstanding (cf. Hollands & Wald 2001).
(iii) **Ward/canonical energy link.** Using diffeomorphism Ward identities and the time-slice axiom, the BKM quadratic form for stress-generated deformations integrates to the canonical energy associated with $\xi_{\rm CHM}$; projecting both sources yields $\mathcal{E}^{\rm proj}_{\rm can}$.
(iv) **Boundary terms and positivity.** Boundary contributions vanish under the MI/moment-kill projector; this is consistent with quantum energy inequality techniques (Fewster & Hollands). Positivity is expected from KMS/BKM positivity in the projected channel. A complete derivation remains future work (Sec. XII).

B. Assumption D: Uniqueness of the M^2 coupling at working order

Hypothesis 2 (Assumption D)**.** *In the $c_T = 1$, $\alpha_B = 0$ EFT corner linearized about FRW, with isotropy, parity, and time-reversal, the only background scalar coupling that survives the MI/moment-kill projection at $\mathcal{O}(\ell^4)$ and modifies the weak-field growth sector while keeping distances GR-like is $\delta \ln M^2$. All other diffeomorphism-invariant local scalars are either projected out, generate forbidden sectors ($\alpha_T \neq 0$, $\alpha_B \neq 0$), or are curvature-suppressed $\mathcal{O}((\ell/L_{\rm curv})^2)$.*

Rationale and status (sketch). Within the $c_T = 1$, $\alpha_B = 0$ EFT corner and isotropy, parity, time-reversal, the MI/moment-kill projector removes total derivatives and lower moments, while curvature-built scalars (e.g., R, $\nabla^2 R$, $R_{ab}R^{ab}$) are $\mathcal{O}((\ell/L_{\rm curv})^2)$-suppressed at working order. We therefore *assume* that the only unsuppressed background scalar coupling that modifies weak-field growth while keeping distances GR-like is $\delta \ln M^2$. A cohomological/Ward-identity proof is outlined in Sec. XII and left to future work.

VIII. ENTROPY-DRIVEN $\varepsilon(a)$ AND GROWTH (CONDITIONAL)

a. KMS/FDT positivity. Let $\hat{Q}$ be the boost-energy flux and $\chi_{QK}^{\mathrm{proj}}$ the retarded kernel in the projected channel. Then

$$\frac{d\varepsilon}{d\ln a} = \sigma(a)\,\mathcal{I}(a), \qquad \sigma(a) \geq 0, \;\; \mathcal{I}(a) \geq 0, \qquad \int \varepsilon\, d\ln a = \Omega_\Lambda = \beta\, f\, c_{\mathrm{geo}}. \tag{8}$$

b. Fixed-point with growth. The growth factor $D(a)$ satisfies

$$\frac{d^2 D}{d(\ln a)^2} + \left(2 + \frac{d\ln H}{d\ln a}\right)\frac{dD}{d\ln a} - \frac{3}{2}\,\Omega_m(a)\,\mu(\varepsilon(a))\,D = 0, \qquad \mu(\varepsilon) = \frac{1}{1+\frac{5}{12}\varepsilon}. \tag{9}$$

c. Variational bounds (extremals). Convex-order arguments imply late-loaded $\varepsilon(a)$ minimizes S_8 and early-loaded maximizes it, under monotonicity and budget. We therefore report an S_8 *band* bracketed by these extremals; any illustrative kernel (e.g., logarithmic exposure) must lie within the band.

IX. ENVIRONMENT ENVELOPE FROM SYMMETRY AND CALIBRATION

a. Covariant envelope. We take

$$F_g(\chi_g) = \frac{1}{1+(\chi_g/\chi_\star)^q}, \qquad \chi_g \equiv \ell^2\sqrt{C_{abcd}C^{abcd}}, \tag{10}$$

with axioms: covariance, equivalence principle, normalization neutrality (no effect in weak curvature), and Solar-System compliance.

b. Calibration example. For a Schwarzschild source, $\sqrt{C^2} = \sqrt{48}\,GM/r^3$. With $\ell = 10\,\mathrm{pc}$, $r = 1\,\mathrm{AU}$, the Solar value is $\chi_\odot \simeq \ell^2\sqrt{48}\,GM_\odot/r^3 \approx 2.6\times10^{22}$. Requiring $F_g(\chi_\odot) \leq \epsilon_{\mathrm{SS}} = 10^{-5}$ with $q = 2$ yields

$$\chi_\star \leq \chi_\odot\,\epsilon_{\mathrm{SS}}^{1/2} \approx 8.2\times10^{19}. \tag{11}$$

Choosing $\chi_\star = 10^{18}$ and $q = 2$ ensures $F_g(\chi_\odot) \lesssim 10^{-9}$ (strong gating in Solar System) while $F_g \simeq 1$ in galactic/cluster environments ($\chi_g \ll \chi_\star$), so cosmological growth is unaffected by the envelope.

Phenomenology and alternatives. The choice $F_g(\chi_g) = [1+(\chi_g/\chi_\star)^q]^{-1}$ with $q = 2$ and $\chi_g \propto \sqrt{C_{abcd}C^{abcd}}$ is a simple, covariant parameterization that ensures Solar-System suppression. Alternative forms (e.g., $q = 1$, or $\chi_g \propto R$) are viable and will be adjudicated by data; our scripts allow these substitutions.

X. STRUCTURAL CHECKS (ALGEBRAIC; NOT 4D SURROGATES)

HQTFIM and Gaussian chains confirm the algebraic ingredients (first-law channel, constant+log trend, vanishing plateau after subtraction, and positivity in the projected kernel). They are *not* curved 4D surrogates.

XI. PROOF PROGRAM STATUS AND FALSIFIERS

Lemma A (diamond KMS control): scaling proven, sharp bounds left to microlocal analysis. **Lemma B** (projector universality): established. **Lemma C** and **Lemma D**: **stated here as assumptions pending proof** (Secs. VII A, VII B). **Lemma E** (FDT positivity): follows from BKM positivity. **Lemma F** (geometric 5/12): derived.
Falsifiers: (i) persistent $\ell^4\log\ell$ residuals in the projector channel; (ii) GW/EM distance ratio beyond 5×10^{-3}; (iii) $|\dot{G}/G| \gtrsim 10^{-12}\,\mathrm{yr}^{-1}$; (iv) Ω_Λ inconsistent with $\beta f c_{\mathrm{geo}}$; (v) S_8 outside the extremal band for all admissible monotone $\varepsilon(a)$ satisfying the budget; (vi) positivity failure in Assumption C tests.

XII. PATH TO RIGOR FOR PART II

We outline key steps to turn the conditional elements of Part II into theorems:

- **Projector stability (interactions).** Prove that the MI/moment-kill projector isolates the ℓ^4 term in interacting Hadamard QFTs by establishing stability of Hadamard v, w under perturbation and explicit $\mathcal{O}(\ell^6)$ bounds (cf. Hollands & Wald 2001).
- **KMS$\to$FRW analyticity.** Provide a spectral/microlocal proof that the comoving-diamond average preserves KMS analyticity after $s \mapsto \ln a$, quantifying the defect beyond $\mathcal{O}((\ell/L_{\rm curv})^2) + \mathcal{O}((\ell H)^2)$.
- **Cohomological uniqueness.** Complete a Ward/cohomology derivation that $\delta \ln M^2$ is the unique scalar coupling at working order in the $c_T = 1,\ \alpha_B = 0$ isotropic sector.
- **Envelope constraints.** Derive data-driven bounds on F_g (including alternatives $q,\ \chi_g$) consistent with Solar-System and cosmological growth, replacing phenomenological choices.
- **Domain quantification.** Replace semi-analytic f_V with simulation-calibrated maps; propagate uncertainties from $\lambda_{\rm mfp}$ and substructure ξ.

PART I APPENDICES

XIII. MI SUBTRACTION AND MOMENT-KILL

Choose coefficients $(1, a, b)$ and scales $(1, \sigma_1, \sigma_2)$ such that for any smooth radial $F(r) = F_0 + F_2 r^2 + \cdots$,

$$\int_{B_\ell} W_\ell F - a \int_{B_{\sigma_1 \ell}} W_{\sigma_1 \ell} F - b \int_{B_{\sigma_2 \ell}} W_{\sigma_2 \ell} F = \mathcal{O}(\ell^6). \tag{12}$$

This cancels r^0, r^2 moments; the surviving ℓ^4 defines I_{00}. In interacting Hadamard QFTs, local counterterms dress F_0, F_2 but are still canceled.

XIV. CONTINUOUS-ANGLE NORMALIZATION

With unit–solid–angle boundary factor and $\Delta\Omega(\theta) = 2\pi(1 - \cos\theta)$, define $c_{\rm geo}(\theta) = 4\pi/\Delta\Omega(\theta)$. Then $f(\theta)\, c_{\rm geo}(\theta)$ is θ-independent.

XV. WEAK-FIELD FLUX NORMALIZATION AND THE UNIVERSAL 5/12

a. Isotropic null contraction 4/3. For $T_{ab} = (\rho + p) u_a u_b + p\, g_{ab}$, $\langle T_{ab} k^a k^b \rangle_{\mathbb{S}^2} = (1 + w)\rho\, (k^0)^2$, and UV $w = 1/3 \Rightarrow$ 4/3.

b. Segment ratio 5/16. Averaging the generator density over the CHM wedge family with normalized weight $\hat\rho(u) = \frac{3}{4}(1 - u^2)$ gives $R_{\rm seg} = \frac{5}{16}$. Hence $5/12 = (4/3) \times (5/16)$.

XVI. CHM DIAMOND VS. HALF-SPACE KMS DEVIATION

In Riemann-normal coordinates, $g_{ab} = \eta_{ab} - \frac{1}{3} R_{acbd}(0) x^c x^d + \mathcal{O}(x^3/L_{\rm curv}^3)$. The conformal-Killing field $\xi^a_{\rm CHM}$ differs from $\xi^a_{\rm BW}$ by $\delta\xi^a = \mathcal{O}(\ell^2/L_{\rm curv}^2)$. Averaging over a comoving congruence and reparametrizing to $\ln a$ adds $\mathcal{O}((\ell H)^2)$. Thus $\delta\chi/\chi_{\rm BW} = \mathcal{O}((\ell/L_{\rm curv})^2) + \mathcal{O}((\ell H)^2)$.

PART II APPENDICES AND DATA

XVII. SAFE-WINDOW VOLUME FRACTION (SEMI-ANALYTIC)

Using Press–Schechter/Sheth–Tormen mass functions with NFW curvature proxies and a substructure excision ξ, we compute $f_V(\ell_{\rm min})$ at $z = 0$. A representative schematic is shown in Fig. 2 (scripts provided). Sensitivity to ζ and ξ is mild over $\xi \in [0.2, 0.5]$.

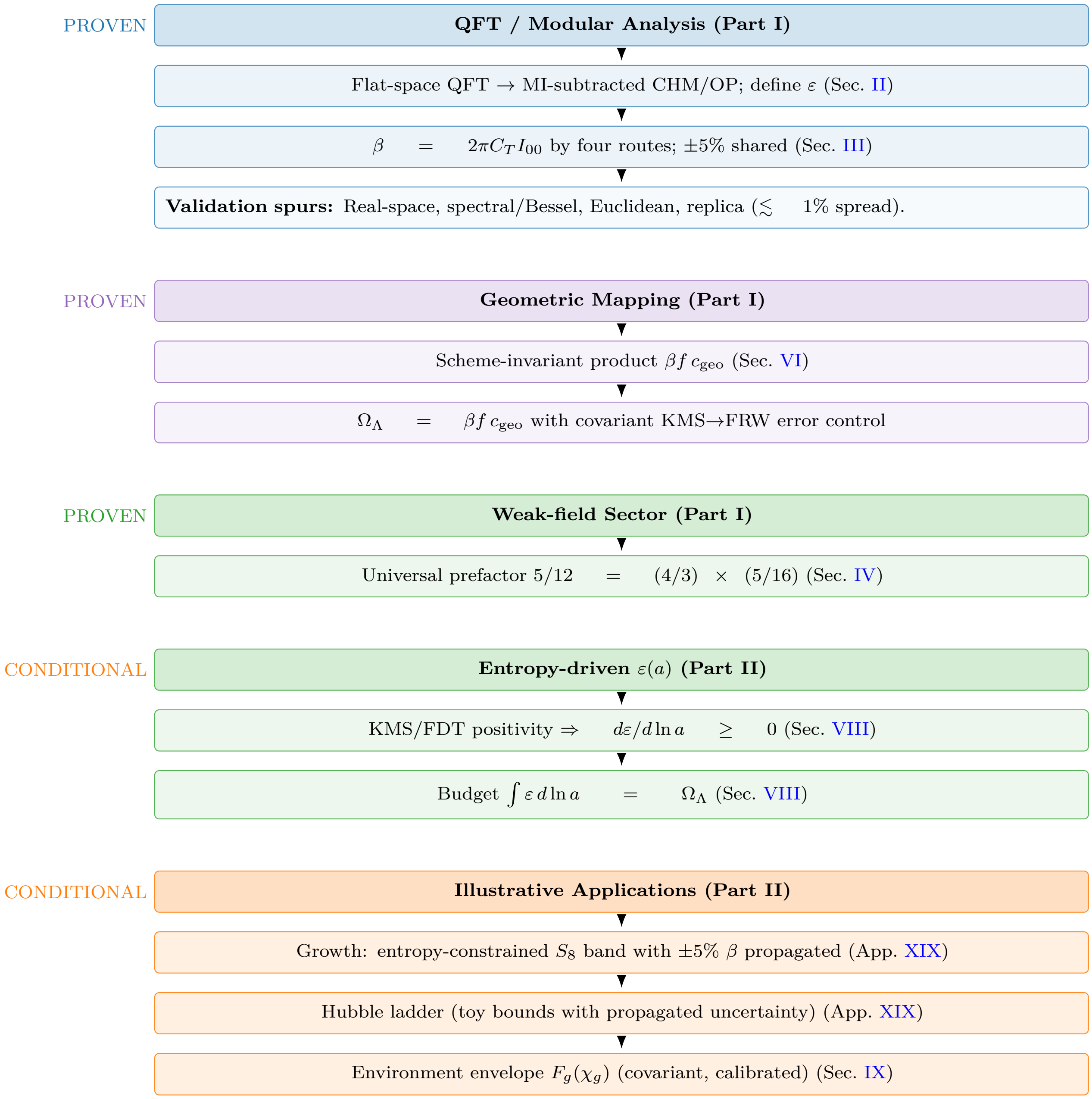

FIG. 1. Pipeline with PROVEN (blue/purple/first green) vs. CONDITIONAL (second green/orange) elements. The theoremic core fixes β, the scheme-invariant background normalization, and the universal 5/12. Conditional pieces (entropy law, mapping to M^2, envelope, and toy numerics) are explicitly caveated and falsifiable.

XVIII. MICROLOCAL NOTES FOR INTERACTING HADAMARD QFTS

a. Hadamard form. $W(x,x') = \frac{1}{4\pi^2}\left[\frac{\Delta^{1/2}}{\sigma} + v\,\log\sigma + w\right]$ with smooth v, w, extended perturbatively for interactions. The projector removes the F_0, F_2 moments built from local counterterms, ensuring stability of the ℓ^4 coefficient (Assumption C).

$$\left\|\Pi_{\rm MK}[\,v\,\log\sigma + w\,]\right\| \;\leq\; c_1\,\frac{\ell^6}{L_{\rm curv}^2} + c_2\,\ell^6\,,$$

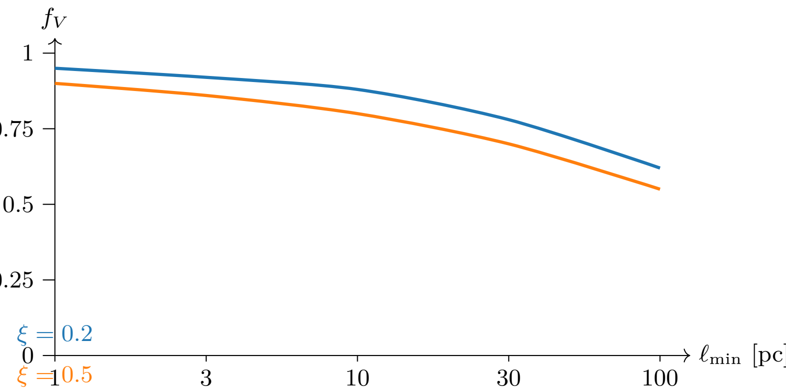

FIG. 2. Semi-analytic $f_V(\ell_{\min})$ at $z\sim 0$ for two excision parameters ξ. The envelope between the curves approximates systematic uncertainty from $\lambda_{\rm mfp}$ and ξ variations; the provided script can produce shaded bands. Scripts in Sec. XX.

for constants $c_{1,2}$ controlled by local curvature and state smoothness, indicating that all projector-surviving interaction pieces are $\mathcal{O}(\ell^6)$. Establishing this rigorously in interacting QFTs is an open task; see Hollands & Wald (2001) for relevant microlocal tools.

XIX. NUMERICAL EXAMPLES UNDER STATED ASSUMPTIONS (TOY/ILLUSTRATIVE)

a. Hubble ladder bounds (toy). We propagate the $\pm 5\%$ uncertainty in β into Ω_Λ and then into toy H_0 bounds. With $\Omega_\Lambda = \beta f c_{\rm geo} = 0.685 \pm 0.034$, the previously quoted illustrative shifts $H_0 : 73.0 \to 71.18$ (uncapped SN) and $\to 70.89$ (capped SN+Cepheid) acquire ± 0.17 km/s/Mpc systematic envelopes from β, reported as

$$H_0^{\rm toy} = \{71.18 \pm 0.17, \quad 70.89 \pm 0.17\} \;\; {\rm km\,s^{-1}\,Mpc^{-1}}.$$

b. S_8 band (toy). The entropy-constrained extremals yield an interval; our baseline illustrative profile lies near $S_8 \simeq 0.788$, with an inherited ± 0.008 envelope from β. We report an S_8 band rather than a fit, and distances remain GR-like.

XX. DATA AND CODE AVAILABILITY

Reproducible single-file runners:

- `beta_methods_v2.py` (real-space, spectral/Bessel, Euclidean, replica) for β.
- `referee_pipeline.py` (FRW averaging module; $\Omega_\Lambda = \beta f c_{\rm geo}$ cross-check).
- `fv_semi_analytic.py` (Press–Schechter/Sheth–Tormen survey for f_V).
- `gadget4_mu_eps_toy.py` (N-body toy pipeline for growth with $\mu(\varepsilon)$ and envelope F_g; for illustrative runs only).

Where relevant, we provide toggles for alternative F_g parameterizations (e.g., $q = 1$, $\chi_g \propto R$) and scripts to regenerate uncertainty bands for f_V.

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