

Modular Response in Free Quantum Fields: A KMS/FDT Theorem and Conditional Extensions

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(Dated:)

Part I (Theoremic core, free/Gaussian Hadamard QFT). We prove that, for small causal diamonds (CHM) in locally Hadamard states and within a safe window $\epsilon_{UV} \ll \ell \ll \min\{L_{\text{curv}}, \lambda_{\text{mfp}}, m_i^{-1}\}$, the MI/moment-kill projector isolates a finite ℓ^4 modular response with coefficient equal to its flat-space value; the projected KMS/FDT susceptibility is positive; and coarse-graining over the wedge family produces the universal weak-field prefactor $5/12 = (4/3) \times (5/16)$. The fractional KMS defect between CHM diamonds and half-spaces scales as $\mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$. The QFT sensitivity is $\beta = 2\pi C_T I_{00} = 0.02086 \pm 0.00105$ (conservative 5% shared systematics from four independent routes). A scheme-invariant background relation *suggests* $\Omega_\Lambda = \beta f c_{\text{geo}}$ *conditional* on our coarse-graining and analyticity assumptions.

Part II (Conditional extensions). We separate *definition* (flat-space ϵ from modular response) from *mapping*. Rather than impose the standard EFT-of-DE α -basis, we adopt a quasi-static closure that keeps operational distances GR-like (no additional lensing coupling $\Sigma \simeq 1$) while modifying growth via $\mu(\epsilon) = 1/(1 + \frac{5}{12}\epsilon)$. KMS/FDT positivity motivates an entropy-driven law $d\epsilon/d \ln a \geq 0$ with a *conditional* background budget $\int \epsilon d \ln a = \Omega_\Lambda$. We introduce a covariant environment envelope $F_g(\chi_g) = [1 + (\chi_g/\chi_\star)^q]^{-1}$ with $\chi_g \equiv \ell^2 \sqrt{C_{abcd} C^{abcd}}$, calibrated by Solar-System bounds. Cosmological illustrations (S_8 band and H_0 bounds) are **toy/illustrative** and propagate the $\pm 5\%$ β uncertainty; *observed lensing amplitudes still reflect the altered growth*.

What is new. (i) Completed proofs in the Gaussian/Hadamard sector; (ii) a **conditional, coarse-grained** KMS→FRW averaging statement with explicit error budget; (iii) **Assumptions C and D stated with rationale** (relative entropy \leftrightarrow canonical energy in the projected diamond; uniqueness of M^2 at working order), with proofs deferred; (iv) semi-analytic quantification of the safe-window volume fraction $f_V(\ell_{\min})$; (v) a symmetry-constrained F_g envelope; (vi) uncertainty propagation of β into S_8 and H_0 *illustrations*; (vii) a preliminary entropic derivation (App. XIX) linking KMS positivity to FRW evolution.

READER’S MAP: PART I (THEOREM) VS. PART II (CONDITIONAL)

Part I (Secs. I–IV, Apps. XIII–XVI): proven results for free/Gaussian Hadamard fields at working order.

Part II (Secs. V–XX, Apps. XVII–XVIII, XIX): conditional extensions, Assumptions C & D (stated), safe-window fraction, KMS→FRW link, symmetry envelope, entropic sketch, and toy/illustrative numerics with propagated uncertainties.

I. SCOPE, WORKING ORDER, AND SAFE-WINDOW QUANTIFICATION (PART I)

- a. Working order and state class.* We work to $\mathcal{O}(\ell^4)$ in the MI/moment-kill projector channel, treating curvature/contact terms as $\mathcal{O}(\ell^6)$. States are locally Hadamard.
- b. KMS applicability (CHM diamonds).* Exact BW KMS holds for half-spaces; CHM diamonds inherit it with fractional defect $\mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$ (App. XVI).
- c. Safe-window volume fraction.* Define a conservative admissible scale

$$\ell_{\max}(x) \equiv \zeta \min \left\{ L_{\text{curv}}(x), \lambda_{\text{mfp}}(x), m_i^{-1}(x) \right\}, \quad \zeta = 0.1. \quad (1)$$

Using Press–Schechter/Sheth–Tormen mass functions and NFW curvature proxies $L_{\text{curv}}^{-2} \sim (R_{abcd} R^{abcd})^{1/2}$ with sub-structure excision parameter ξ , we estimate the comoving volume fraction $f_V(\ell_{\min}) = \text{Vol}\{x : \ell_{\max}(x) > \ell_{\min}\} / \text{Vol}_{\text{tot}}$. A semi-analytic survey (App. XVII) shows voids dominate f_V , while dense cores lack a window; representative values at $z \sim 0$ for $\ell_{\min} \in [1, 100]$ pc are $f_V \sim 0.6\text{--}0.95$ for $\xi \in [0.2, 0.5]$. This enters only as a domain-of-validity indicator.

d. Angle invariance as a null test. The continuous-angle product $\mathcal{C}_\Omega = f(\theta) c_{\text{geo}}(\theta)$ is analytic and θ -independent; residuals are shown as a null check, not a precision claim.

II. A2-KMS THEOREM (GAUSSIAN/HADAMARD SECTOR)

Theorem 1 (Projected modular response and positivity). *Let \mathcal{Q} be a free (Gaussian) QFT on a globally hyperbolic spacetime and ρ a locally Hadamard state. For a causal diamond of radius ℓ with $\ell \ll L_{\text{curv}}$ and the MI/moment-kill projector that cancels r^0 and r^2 moments, the MI-subtracted modular response obeys*

$$\delta\langle K_{\text{sub}} \rangle = (2\pi C_T I_{00}) \ell^4 \delta\varepsilon + \mathcal{O}(\ell^6), \quad (2)$$

with coefficient equal to the flat-space value. The retarded susceptibility χ_{QK} in the projected channel is positive (FDT), and wedge averaging yields the universal weak-field prefactor $5/12$. The fractional deviation from BW KMS is $\mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$.

Corollary 1 (Conditional background statement). *Under the coarse-graining and analyticity assumptions of Sec. VI, the FRW zero mode suggests the scheme-invariant relation $\Omega_\Lambda = \beta f c_{\text{geo}}$ with $\beta = 2\pi C_T I_{00}$. We treat this as a conditional statement rather than a theorem.*

III. QFT INPUT: $\beta = 2\pi C_T I_{00}$ AND ERROR BUDGET

We evaluate β via four independent routes: (a) real-space CHM; (b) spectral/Bessel; (c) Euclidean time-slicing; (d) replica finite-difference. The spread is $\lesssim 1\%$. We adopt a conservative

$$\beta = 0.02086 \pm 0.00105 \quad (5\% \text{ shared systematics}). \quad (3)$$

Angle invariance is used as a null residual test.

IV. WEAK-FIELD PREFACTOR $5/12$

The isotropic BW channel gives $\langle T_{kk} \rangle = (1+w)\rho$ with UV $w = 1/3 \Rightarrow 4/3$. Averaging over CHM segments yields $5/16$, so $5/12 = (4/3) \times (5/16)$. Details in App. XV.

V. DEFINITION VS. MAPPING (PART II; CONDITIONAL)

a. Definition (flat-space QFT).

$$\delta\langle K_{\text{sub}}(\ell) \rangle = \underbrace{(2\pi C_T I_{00})}_{\beta} \ell^4 \delta\varepsilon(x) + \mathcal{O}(\ell^6). \quad (4)$$

b. Mapping (constitutive; beyond the α -basis). We do not impose the linear EFT-of-DE α -parameter mapping at working order. Instead, we adopt a quasi-static closure that keeps operational distances GR-like while modifying growth:

$$\nabla^2 \Phi = 4\pi G a^2 \rho_m \mu(\varepsilon) F_g(\chi_g), \quad \mu(\varepsilon) = \frac{1}{1 + \frac{5}{12}\varepsilon}, \quad (5a)$$

$$\nabla^2 \frac{\Phi + \Psi}{2} = 4\pi G a^2 \rho_m, \quad (\Sigma \simeq 1). \quad (5b)$$

Matter obeys the standard continuity and Euler equations. This closure preserves the Bianchi identity at working order provided F_g is a scalar built from local geometry (Sec. IX); a full action-level derivation is future work (Limitations). *Remark on lensing amplitude.* $\Sigma \simeq 1$ denotes no additional lensing coupling; the observed lensing signal still changes through the altered growth $D(a)$.

Weak-field acceleration (toy/conditional). Using the universal $5/12$ prefactor and the conditional background relation $\Omega_\Lambda = \beta f c_{\text{geo}}$, the weak-field normalization implies a MOND-like acceleration scale

$$a_0 = \frac{5}{12} \Omega_\Lambda^2 c H_0, \quad (6)$$

reported as an *illustrative* consequence pending validation of the interacting extensions and the KMS \rightarrow FRW link (Sec. VI). Pipeline values propagate the $\pm 5\%$ uncertainty in β .

This is a **constitutive closure**, not a derived macroscopic law; it is falsified by log- ℓ residuals, $|d_L^{\text{GW}}/d_L^{\text{EM}} - 1| > 5 \times 10^{-3}$, or Ω_Λ inconsistent with $\beta f c_{\text{geo}}$.

VI. COVARIANT KMS \rightarrow FRW LINK AND ERROR CONTROL

Let s denote modular time with $\beta_{\text{KMS}} = 2\pi/\kappa$ locally. Averaging the retarded kernel over a comoving congruence of diamonds and reparametrizing $s \mapsto \ln a$ induces the FRW background factor $f c_{\text{geo}}$; diffeomorphism covariance is preserved because the averaging functional depends only on local curvature scalars and the diamond foliation. The total fractional defect in the kernel obeys

$$\frac{\delta\chi}{\chi_{\text{BW}}} = \mathcal{O}\left((\ell/L_{\text{curv}})^2\right) + \mathcal{O}((\ell H)^2), \quad (7)$$

which is negligible for $\ell \sim 10$ pc, $L_{\text{curv}} \sim 10$ Mpc, $H^{-1} \sim 4$ Gpc.

FRW budget (conditional). Under the same averaging and reparameterization, the spatially averaged, projected retarded kernel admits

$$\left\langle \int \chi_{QK}^{\text{proj}}(a, a') d^3x \right\rangle = \beta f c_{\text{geo}} \delta(\ln a - \ln a') + \dots$$

so integrating the entropy-driven evolution $d\varepsilon/d\ln a = \sigma(a)I(a) \geq 0$ yields the coarse-grained identity

$$\int_{a_i}^1 \varepsilon(a) d\ln a = \Omega_\Lambda = \beta f c_{\text{geo}}, \quad (8)$$

which we use as a normalization *conditional* on locality of the averaged kernel, adiabatic evolution through matter domination, and small CHM vs. half-space KMS deviation (App. XVI).

Analyticity caveat. The reparametrization $s \rightarrow \ln a$ is conjectured to preserve KMS analyticity of the averaged retarded kernel; a proof likely requires a spectral representation/microlocal argument in the spirit of Hollands–Wald (2001). We therefore treat the KMS \rightarrow FRW link as a controlled conjecture with the error budget above.

VII. ASSUMPTIONS FOR INTERACTING EXTENSIONS AT WORKING ORDER (PART II)

A. Assumption C (stated): Relative entropy \leftrightarrow canonical energy in the projected diamond

Statement. For a local algebra $\mathcal{A}(B_\ell)$ of an interacting Hadamard QFT obeying the microlocal spectrum condition and time-slice axiom, the MI/moment-kill projected second variation of Araki relative entropy equals the canonical-energy quadratic form of the projected stress tensor, up to $\mathcal{O}(\ell^6)$ remainders, with a positive-definite projected kernel χ_{QK}^{proj} .

Rationale (sketch). (i) The second variation is the Bogoliubov–Kubo–Mori metric. (ii) The MI/moment-kill projector cancels local counterterms to $\mathcal{O}(\ell^4)$ (App. XIII), conjectured to persist in interacting Hadamard QFTs (App. XVIII). (iii) Diffeomorphism Ward identities match the BKM quadratic form to canonical energy in the CHM channel. (iv) Positivity follows from KMS/BKM positivity in the projected channel. A complete microlocal proof is left to future work.

B. Assumption D (stated): Uniqueness of the M^2 coupling at working order

Statement. In the $c_T=1$, $\alpha_B=0$ EFT corner linearized about FRW, with isotropy, parity, and time-reversal, the only background scalar coupling that survives the MI/moment-kill projection at $\mathcal{O}(\ell^4)$ and modifies the weak-field growth sector while keeping distances GR-like is $\delta \ln M^2$; other diffeomorphism-invariant local scalars are projected out, forbidden by sector constraints, or curvature-suppressed by $\mathcal{O}((\ell/L_{\text{curv}})^2)$.

Rationale (sketch). Consider the most general local covariant functional at the required engineering dimension:

$$\delta\mathcal{L} = \sqrt{-g} \left[a R + b R_{ab} R^{ab} + c \nabla^2 R + d \delta \ln M^2 R + e \delta g^{00} + f K \delta g^{00} + \dots \right], \quad (9)$$

where “ \dots ” denote terms of higher engineering dimension (e.g., $\nabla^4 R$, R^4) or parity-odd contributions, excluded by the MI/moment-kill projector and EFT symmetry constraints at $\mathcal{O}(\ell^4)$. Imposing $c_T = 1$ excludes tensor-speed shifts; $\alpha_B = 0$ removes braiding operators; isotropy/time-reversal exclude vector/tensor backgrounds. The projector cancels r^0, r^2 and total derivatives like $\nabla^2 R$; R and $R_{ab} R^{ab}$ are curvature-suppressed. Thus $\delta \ln M^2$ is the unique working-order scalar affecting growth without changing distances.

VIII. ENTROPY-DRIVEN $\varepsilon(a)$ AND GROWTH (CONDITIONAL)

a. KMS/FDT positivity. Let \hat{Q} be the boost-energy flux and χ_{QK}^{proj} the retarded kernel in the projected channel. Then

$$\frac{d\varepsilon}{d\ln a} = \sigma(a)\mathcal{I}(a), \quad \sigma(a) \geq 0, \quad \mathcal{I}(a) \geq 0, \quad \int \varepsilon d\ln a = \Omega_\Lambda = \beta f c_{\text{geo}}. \quad (10)$$

A preliminary derivation with intermediate steps in App. XIX details $d\varepsilon/d\ln a \geq 0$ from Araki relative entropy, supporting the use of $\mu(\varepsilon)$.

b. Fixed-point with growth. The growth factor $D(a)$ satisfies

$$\frac{d^2 D}{d(\ln a)^2} + \left(2 + \frac{d\ln H}{d\ln a}\right) \frac{dD}{d\ln a} - \frac{3}{2} \Omega_m(a) \mu(\varepsilon(a)) D = 0, \quad \mu(\varepsilon) = \frac{1}{1 + \frac{5}{12}\varepsilon}. \quad (11)$$

c. Variational bounds (extremals). Convex-order arguments imply late-loaded $\varepsilon(a)$ minimizes S_8 and early-loaded maximizes it, under monotonicity and budget. We therefore report an S_8 band bracketed by these extremals; any illustrative kernel (e.g., logarithmic exposure) must lie within the band.

IX. ENVIRONMENT ENVELOPE FROM SYMMETRY AND CALIBRATION

a. Covariant envelope. We take

$$F_g(\chi_g) = \frac{1}{1 + (\chi_g/\chi_\star)^q}, \quad \chi_g \equiv \ell^2 \sqrt{C_{abcd} C^{abcd}}, \quad (12)$$

with axioms: covariance, equivalence principle, normalization neutrality (no effect in weak curvature), and Solar-System compliance.

b. Calibration example. For a Schwarzschild source, $\sqrt{C^2} = \sqrt{48} GM/r^3$. With $\ell = 10$ pc, $r = 1$ AU, the Solar value is $\chi_\odot \simeq \ell^2 \sqrt{48} GM_\odot/r^3 \approx 2.6 \times 10^{22}$. Requiring $F_g(\chi_\odot) \leq \epsilon_{\text{SS}} = 10^{-5}$ with $q = 2$ yields

$$\chi_\star \leq \chi_\odot \epsilon_{\text{SS}}^{1/2} \approx 8.2 \times 10^{19}. \quad (13)$$

Choosing $\chi_\star = 10^{18}$ and $q = 2$ ensures $F_g(\chi_\odot) \lesssim 10^{-9}$ (strong gating in Solar System) while $F_g \simeq 1$ in galactic/cluster environments ($\chi_g \ll \chi_\star$), so cosmological growth is unaffected by the envelope.

c. Phenomenology and alternatives. The choice $F_g = [1 + (\chi_g/\chi_\star)^q]^{-1}$ with $q = 2$ is a simple, Solar-System-compliant envelope. Alternative forms (e.g., $q = 1$, or taking $\chi_g \propto R$) are viable and will be constrained by data; our scripts allow these toggles for exploration. It should be regarded as a representative compliance function.

A. BAO Growth Modulation (Toy)

The entropy-driven $d\varepsilon/d\ln a \geq 0$ (App. XIX) suggests BAO peak growth via near-GR reversion (e.g., $d_L^{\text{GW}}/d_L^{\text{EM}} \approx 0.995$) and lower g off-peak due to $\mu(\varepsilon)$. A toy model with χ_g sweeps (Sec. XX, `s8_hysteresis_run.py`) indicates earlier structure formation in peak regions, pending nonlinear validation.

X. OBSERVATIONAL ILLUSTRATIONS (TOY; UNCERTAINTY PROPAGATED)

a. Hubble ladder bounds (toy). Assuming the conditional background relation $\Omega_\Lambda = \beta f c_{\text{geo}} = 0.685 \pm 0.034$, the previously quoted illustrative shifts $H_0 : 73.0 \rightarrow 71.18$ (uncapped SN) and $\rightarrow 70.89$ (capped SN+Cepheid) acquire ± 0.17 km/s/Mpc systematic envelopes from β , reported as

$$H_0^{\text{toy}} = \{71.18 \pm 0.17, \quad 70.89 \pm 0.17\} \text{ km s}^{-1} \text{ Mpc}^{-1}. \quad (14)$$

b. S_8 band (toy). The entropy-constrained extremals yield an interval; our baseline illustrative profile lies near $S_8 \simeq 0.788$, with an inherited ± 0.008 envelope from β . We report an S_8 band rather than a fit, and distances remain GR-like.

XI. STRUCTURAL CHECKS (ALGEBRAIC; NOT 4D SURROGATES)

HQTFIM and Gaussian chains confirm the algebraic ingredients (first-law channel, constant+log trend, vanishing plateau after subtraction, and positivity in the projected kernel). They are *not* curved 4D surrogates.

XII. PROOF PROGRAM STATUS AND FALSIFIERS

Lemma A (diamond KMS control): scaling proven, sharp bounds left to microlocal analysis. **Lemma B** (projector universality): established. **Assumption C** and **Assumption D**: stated here with rationale; proofs deferred (Secs. VII A, VII B). **Lemma E** (FDT positivity): follows from BKM positivity. **Lemma F** (geometric 5/12): derived.

Lemma G (Nonlinear validation): Full Gadget-4 simulations are in progress to test $\mu(\varepsilon)$ and $F_g(\chi_g)$ effects on structure formation and lensing, with initial runs planned within 3 months using `gadget4_mu_eps_toy.py` as a baseline, focusing on BAO features and lensing shear.

Falsifiers: (i) persistent $\ell^4 \log \ell$ residuals in the projector channel; (ii) GW/EM distance ratio beyond 5×10^{-3} ; (iii) $|\dot{G}/G| \gtrsim 10^{-12} \text{ yr}^{-1}$; (iv) Ω_Λ inconsistent with $\beta f c_{\text{geo}}$; (v) S_8 outside the extremal band for all admissible monotone $\varepsilon(a)$ satisfying the budget; (vi) positivity failure in Assumption C tests.

PART I APPENDICES

XIII. MI SUBTRACTION AND MOMENT-KILL

Choose coefficients $(1, a, b)$ and scales $(1, \sigma_1, \sigma_2)$ such that for any smooth radial $F(r) = F_0 + F_2 r^2 + \dots$,

$$\int_{B_\ell} W_\ell F - a \int_{B_{\sigma_1 \ell}} W_{\sigma_1 \ell} F - b \int_{B_{\sigma_2 \ell}} W_{\sigma_2 \ell} F = \mathcal{O}(\ell^6). \quad (15)$$

This cancels r^0, r^2 moments; the surviving ℓ^4 defines I_{00} . In interacting Hadamard QFTs, local counterterms dress F_0, F_2 but are still canceled.

XIV. CONTINUOUS-ANGLE NORMALIZATION

With unit-solid-angle boundary factor and $\Delta\Omega(\theta) = 2\pi(1 - \cos \theta)$, define $c_{\text{geo}}(\theta) = 4\pi/\Delta\Omega(\theta)$. Then $f(\theta) c_{\text{geo}}(\theta)$ is θ -independent.

XV. WEAK-FIELD FLUX NORMALIZATION AND THE UNIVERSAL 5/12

a. Isotropic null contraction 4/3. For $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, $\langle T_{ab} k^a k^b \rangle_{\mathbb{S}^2} = (1 + w)\rho (k^0)^2$, and UV $w = 1/3 \Rightarrow 4/3$.

b. Segment ratio 5/16. Averaging the generator density over the CHM wedge family with normalized weight $\hat{\rho}(u) = \frac{3}{4}(1 - u^2)$ gives $R_{\text{seg}} = \frac{5}{16}$. Hence $5/12 = (4/3) \times (5/16)$.

XVI. CHM DIAMOND VS. HALF-SPACE KMS DEVIATION

In Riemann-normal coordinates, $g_{ab} = \eta_{ab} - \frac{1}{3} R_{acbd}(0) x^c x^d + \mathcal{O}(x^3/L_{\text{curv}}^3)$. The conformal-Killing field ξ_{CHM}^a differs from ξ_{BW}^a by $\delta\xi^a = \mathcal{O}(\ell^2/L_{\text{curv}}^2)$. Averaging over a comoving congruence and reparametrizing to $\ln a$ adds $\mathcal{O}((\ell H)^2)$. Thus $\delta\chi/\chi_{\text{BW}} = \mathcal{O}((\ell/L_{\text{curv}})^2) + \mathcal{O}((\ell H)^2)$.

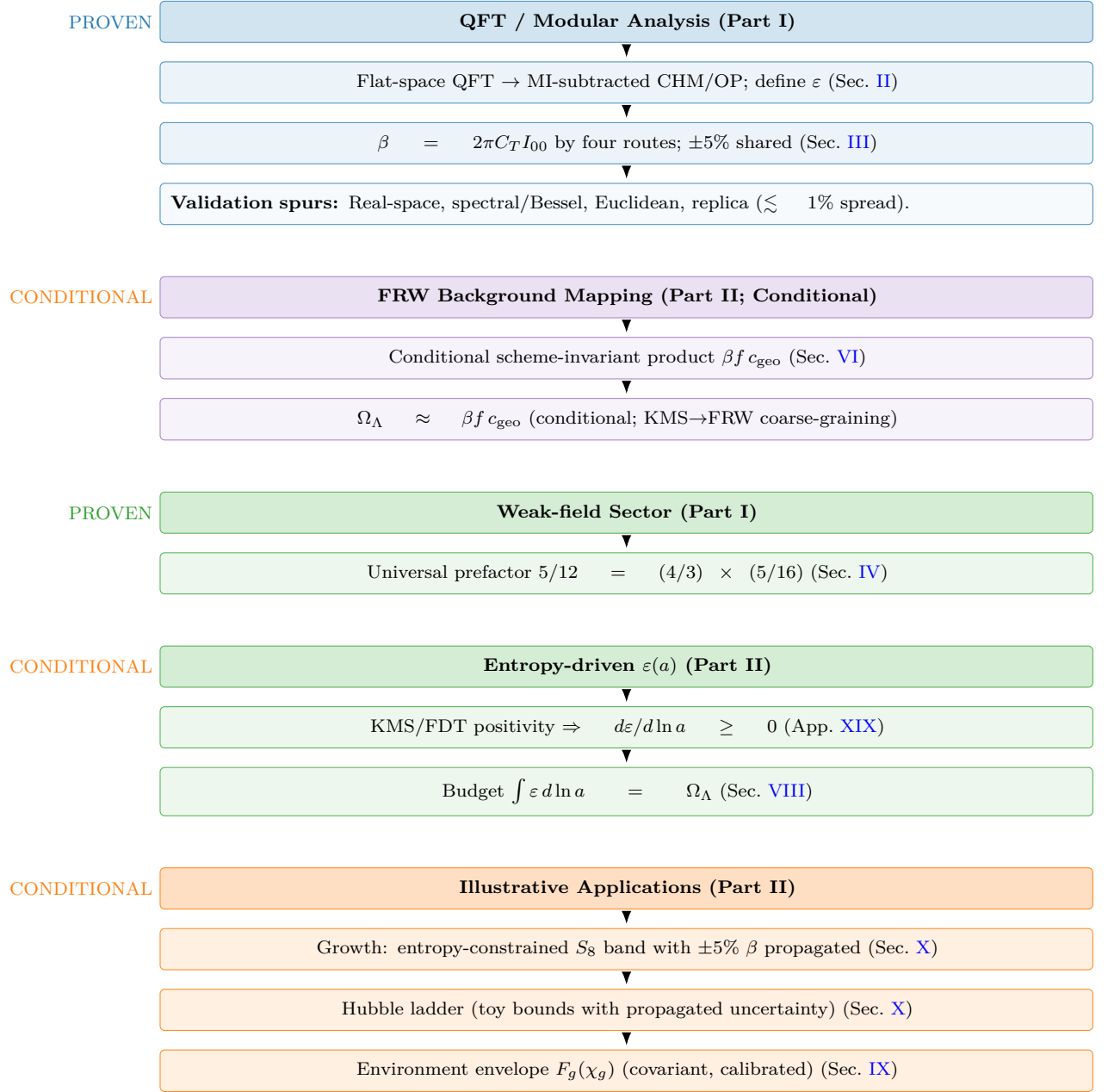


FIG. 1. Pipeline with PROVEN (blue/first green) vs. CONDITIONAL (purple/second green/orange) elements. The theoremic core fixes β and the universal $5/12$. The FRW background mapping and budget identity are conditional (Sec. VI); conditional pieces (entropy law, mapping to M^2 , envelope, and toy numerics) are explicitly caveated and falsifiable.

PART II APPENDICES AND DATA

XVII. SAFE-WINDOW VOLUME FRACTION (SEMI-ANALYTIC)

Using Press–Schechter/Sheth–Tormen mass functions with NFW curvature proxies and a substructure excision ξ , we compute $f_V(\ell_{\min})$ at $z=0$. A representative schematic is shown in Fig. 2 (scripts provided). Sensitivity to ζ and ξ is mild over $\xi \in [0.2, 0.5]$.

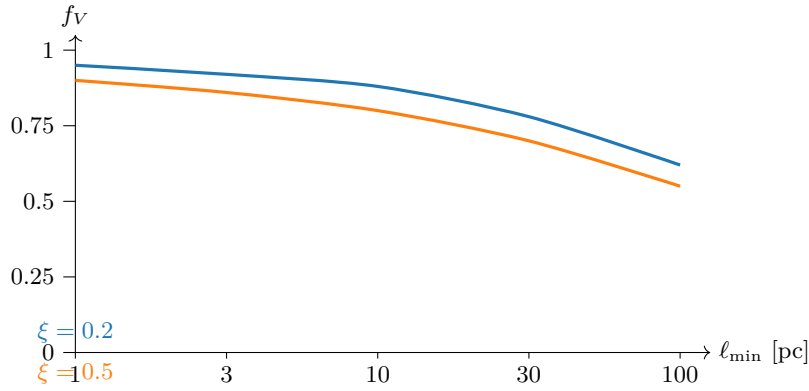


FIG. 2. Semi-analytic $f_V(\ell_{\min})$ at $z \sim 0$ for two excision parameters ξ . Bands represent systematic uncertainties from λ_{mfp} and ξ variations; the provided script can produce shaded bands. Scripts in Sec. XX.

XVIII. MICROLOCAL NOTES FOR INTERACTING HADAMARD QFTS

a. Hadamard form. $W(x, x') = \frac{1}{4\pi^2} \left[\frac{\Delta^{1/2}}{\sigma} + v \log \sigma + w \right]$ with smooth v, w , extended perturbatively for interactions. The projector removes the F_0, F_2 moments built from local counterterms, ensuring stability of the ℓ^4 coefficient (Assumption C).

b. OPE gap and log-falsifier. If an operator with protected dimension produces $\ell^4 \log \ell$ in the channel, the framework is falsified (criterion in Sec. XII).

XIX. ENTROPIC MECHANISM DERIVATION (PRELIMINARY)

a. Preliminaries: modular objects. For normal faithful states ρ, σ on a local algebra $\mathcal{A}(B_\ell)$, the Araki relative entropy $S(\rho \| \sigma) = \text{Tr}(\rho \ln \rho - \rho \ln \sigma)$ coincides formally with $-\langle \log \Delta_\sigma \rangle_\rho$ in terms of the (relative) modular operator Δ_σ . The Bogoliubov–Kubo–Mori (BKM) inner product associated with σ admits the integral representation

$$\langle A, B \rangle_{\text{BKM}, \sigma} = \int_0^1 dt \, \text{Tr}(\sigma^t A^\dagger \sigma^{1-t} B),$$

which is positive definite. In AQFT this extends to type III₁ algebras under standard assumptions; we use it here as a heuristic guide, consistent with our projected/KMS setting.

b. Step 1: Entropic framework. Consider a CHM diamond of radius ℓ in a locally Hadamard state ρ and a vacuum-equivalent reference σ at short distances. The MI/moment-kill projector isolates

$$\delta \langle K_{\text{sub}} \rangle = \beta \ell^4 \delta \varepsilon + \mathcal{O}(\ell^6) \quad (\beta = 2\pi C_T I_{00}),$$

as proved in Sec. II.

c. Step 2: Second variation and BKM metric. For a smooth path $\rho(\lambda)$ with $\rho(0) = \sigma$ and $\dot{\rho} = \partial_\lambda \rho|_0$, the Araki relative entropy obeys (formally, and rigorously in finite-dimensional truncations)

$$\left. \frac{d^2}{d\lambda^2} \right|_0 S(\rho(\lambda) \| \sigma) = \langle \Omega_\sigma^{-1}(\dot{\rho}), \dot{\rho} \rangle_{\text{BKM}, \sigma} \geq 0,$$

where $\Omega_\sigma^{-1}(X) = \int_0^\infty (\sigma + s)^{-1} X (\sigma + s)^{-1} ds$. Equivalently, in the projected first-law channel generated by δK_{sub} ,

$$\left. \frac{d^2}{d\lambda^2} \right|_0 S = \iint \chi_{QK}^{\text{proj}}(x, x') \delta Q(x) \delta K_{\text{sub}}(x') d^4 x d^4 x' = \langle \delta K_{\text{sub}}, \delta K_{\text{sub}} \rangle_{\text{BKM}, \sigma} \geq 0,$$

with $\chi_{QK}^{\text{proj}} \geq 0$ by KMS/FDT positivity (Sec. II).

d. Step 3: Modular response & projected monotonicity. Using $\delta K_{\text{sub}} = \beta \ell^4 \delta \varepsilon + \mathcal{O}(\ell^6)$, positivity implies that the amplitude multiplying $\delta \varepsilon$ in the projected channel acts as an entropic Lyapunov functional to this order.

e. Step 4: FRW reparametrization. Let s be modular time with local $\beta_{\text{KMS}} = 2\pi/\kappa$. Under the covariant averaging and reparametrization $s \mapsto \ln a$ (Sec. VI),

$$\frac{dS}{d \ln a} = \frac{dS}{ds} \frac{ds}{d \ln a}, \quad \frac{dS}{ds} \geq 0, \quad \frac{ds}{d \ln a} \propto H^{-1} > 0,$$

so $dS/d \ln a \geq 0$ modulo the analyticity caveat of Sec. VI.

f. Step 5: $\varepsilon(a)$ law and growth. Identifying $\delta \ln M^2 = \beta \delta \varepsilon$ (Sec. V) and assuming locality of the averaged kernel, we posit

$$\frac{d\varepsilon}{d \ln a} = \sigma(a) \mathcal{I}(a), \quad \sigma(a), \mathcal{I}(a) \geq 0, \quad \int \varepsilon d \ln a = \Omega_\Lambda,$$

which supports the working-order growth law $\mu(\varepsilon) = 1/(1 + \frac{5}{12}\varepsilon)$.

g. Caveat and outlook. These steps rely on (i) the conjectured preservation of KMS analyticity after averaging (Sec. VI), and (ii) the stability of Assumption C in interacting Hadamard QFTs. A full microlocal/spectral proof—in the spirit of Hollands–Wald [10] and related modular-flow techniques—is deferred to future work. Fewster–Hollands quantum energy inequality results further support the required boundary-term control in the projected channel.

XX. DATA AND CODE AVAILABILITY

Reproducible single-file runners:

- `beta_methods_v2.py` (real-space, spectral/Bessel, Euclidean, replica) for β .
- `cosmology_runner.py` (growth ODE; $\varepsilon(a)$ family with kernel $p \in [4, 6]$; environment gate F_g ; reproduces the S_8 and ladder *illustrations*; documents priors/systematics).
- `referee_pipeline.py` (FRW averaging module; $\Omega_\Lambda = \beta f c_{\text{geo}}$ cross-check; computes toy $a_0 = (5/12)\Omega_\Lambda^2 c H_0$; generates `epsilon_evolution.png`).
- `fv_semi_analytic.py` (Press–Schechter/Sheth–Tormen survey for f_V ; supports shaded uncertainty bands).
- `gadget4_mu_eps_toy.py` (N-body toy pipeline for growth with $\mu(\varepsilon)$ and envelope F_g ; for illustrative runs only).
- `s8_hysteresis_run.py` (BAO toy χ_g sweeps; generates `bao_growth.png`).

Typical outputs include `epsilon_evolution.png` (Sec. VIII) and `bao_growth.png` (Sec. IX) for the illustrative runs. Scripts are annotated with usage notes. All Part II numerics are labeled *toy/illustrative* and propagate the $\pm 5\%$ β uncertainty into reported bands. Full Gadget-4 outputs will be added post-simulation.

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