Homework 2 Problem 1

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1 Prompt

Derive the linearized equations of perturbed motion of an axial symmetric satellite spinning around its major axis of inertial $J_2 > J_1 = J3$. Assume a constant perturbation torque acting on the spacecraft.

2 Answer

We begin with the equation governing motion:

$$J\dot{\omega} = -\omega \times J\omega + \tau \tag{1}$$

ω is the rotation rate with respect to inertial in the body frame. J is the inertia matrix in the body frame. τ is the input torque in the body frame.

We introduce nominal $\bar{\cdot}$ and perturbation $\delta\cdot$ components.

$$J\left(\dot{\bar{\omega}} + \delta\dot{\omega}\right) = -\left(\bar{\omega} + \delta\omega\right) \times J\left(\bar{\omega} + \delta\omega\right) + \left(\bar{\tau} + \delta\tau\right) \tag{2}$$

We will expand the right side.

$$\begin{aligned}
J\left(\dot{\bar{\omega}} + \delta\dot{\omega}\right) &= -\left(\bar{\omega} + \delta\omega\right) \times J\left(\bar{\omega} + \delta\omega\right) + \left(\bar{\tau} + \delta\tau\right) && (3)\\
&= -\left(\bar{\omega} + \delta\omega\right) \times \left(J\bar{\omega} + J\delta\omega\right) + \left(\bar{\tau} + \delta\tau\right) && (4)\\
&= -\left(\bar{\omega} + \delta\omega\right) \times J\bar{\omega} - \left(\bar{\omega} + \delta\omega\right) \times J\delta\omega + \left(\bar{\tau} + \delta\tau\right) && (5)\\
&= -\left(\bar{\omega} \times J\bar{\omega} + \delta\omega \times J\bar{\omega}\right) - \left(\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega\right) + \left(\bar{\tau} + \delta\tau\right) && (6)\\
&= -\left(\bar{\omega} \times J\bar{\omega} + \delta\omega \times J\bar{\omega}\right) - \left(\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega\right) + \left(\bar{\tau} + \delta\tau\right) && (7)
\end{aligned}$$

We can eliminate the nominal's evolution with the governing equation.

$$\begin{aligned}
&= -\left(\delta\omega \times J\bar{\omega}\right) - \left(\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega\right) + \delta\tau && (8)\\
&= J\bar{\omega} \times \delta\omega - \left(\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega\right) + \delta\tau && (9)\\
&= \left[J\bar{\omega}\times\right]\delta\omega - \left(\left[\bar{\omega}\times\right]J\delta\omega + \delta\omega \times J\delta\omega\right) + \delta\tau && (10)\\
&= \left(\left[J\bar{\omega}\times\right] - \left[\bar{\omega}\times\right]J\right)\delta\omega + \delta\omega \times J\delta\omega + \delta\tau && (11)
\end{aligned}$$

Then we linearize the peturbations about the nominal, ($\delta\omega = 0$).

$$\begin{aligned}
J\delta\dot{\omega} &\approx \left.\frac{dJ\delta\dot{\omega}}{d\delta\omega}\right|_{\delta\omega=0}\delta\omega + \delta\tau && (12)\\
&\approx \left(\left[J\bar{\omega}\times\right] - \left[\bar{\omega}\times\right]J\right)\delta\omega + \delta\tau && (13)
\end{aligned}$$

I don't know how to linearize $\delta\omega \times J\delta\omega$ but matlab symbolic toolbox tells me that when it is linearized and evaluated at $\delta\omega = 0$, it becomes zero.

We will now drop the approximate sign for convenience. But note that the approximation is still present.

$$\begin{aligned}
\delta\dot{\omega} &= \begin{bmatrix} J_1^{-1} & 0 & 0 \\ 0 & J_2^{-1} & 0 \\ 0 & 0 & J_3^{-1} \end{bmatrix}\left(\left[J\bar{\omega}\times\right] - \left[\bar{\omega}\times\right]J\right)\delta\omega + \delta\tau && (14)\\
&= \begin{bmatrix} 0 & \frac{\bar{\omega}_3(J_2-J_3)}{J_1} & \frac{\bar{\omega}_2(J_2-J_3)}{J_1} \\ -\frac{\bar{\omega}_3(J_1-J_3)}{J_2} & 0 & -\frac{\bar{\omega}_1(J_1-J_3)}{J_2} \\ \frac{\bar{\omega}_2(J_1-J_2)}{J_3} & \frac{\bar{\omega}_1(J_1-J_2)}{J_3} & 0 \end{bmatrix}\delta\omega + J^{-1}\delta\tau && (15)
\end{aligned}$$

At this point, note two things; $J_1 = J_3$ and $\bar{\omega}_1 = \bar{\omega}_3 = 0$.

$$\delta\dot{\omega} = \begin{bmatrix} 0 & 0 & \frac{\bar{\omega}_2(J_2-J_3)}{J_1} \\ 0 & 0 & 0 \\ \frac{\bar{\omega}_2(J_1-J_2)}{J_3} & 0 & 0 \end{bmatrix} \delta\omega + J^{-1}\delta\tau \tag{16}$$

$$\begin{bmatrix} \delta\dot{\omega}_1 \\ \delta\dot{\omega}_2 \\ \delta\dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} \frac{\bar{\omega}_2(J_2-J_3)}{J_1}\delta\omega_3 \\ 0 \\ \frac{\bar{\omega}_2(J_1-J_2)}{J_3}\delta\omega_1 \end{bmatrix} + J^{-1}\delta\tau \tag{17}$$

We can define $k_1 = \frac{J_2-J_3}{J_1}$ and $k_3 = \frac{J_1-J_2}{J_3}$. Then the free response of $\delta\omega$ is given as the following.

$$\delta\omega_1(t) = \delta\omega_1(0)\cos(\Omega t) + \frac{\delta\dot{\omega}_1(0)}{\Omega}\sin(\Omega t) \tag{18}$$

$$\delta\omega_3(t) = \delta\omega_3(0)\cos(\Omega t) + \frac{\delta\dot{\omega}_3(0)}{\Omega}\sin(\Omega t) \tag{19}$$

$$\Omega^2 = k_1 k_3 \bar{\omega}_2^2 \tag{20}$$

I think there could be an error in the above formulation. Try to take the derivative and plug it back in to the equations of motion.

$$\delta\dot{\omega}_1(t) = -\delta\omega_1(0)\Omega\sin(\Omega t) + \delta\dot{\omega}_1(0)\cos(\Omega t) \tag{21}$$

$$\bar{\omega}_2 k_1 \delta\omega_3(t) = -\delta\omega_1(0)\Omega\sin(\Omega t) + \delta\dot{\omega}_1(0)\cos(\Omega t) \tag{22}$$

$$\bar{\omega}_2 k_1 \left(\delta\omega_3(0)\cos(\Omega t) + \frac{\delta\dot{\omega}_3(0)}{\Omega}\sin(\Omega t)\right) = -\delta\omega_1(0)\Omega\sin(\Omega t) + \delta\dot{\omega}_1(0)\cos(\Omega t) \tag{23}$$

$$\tag{24}$$

We know that $\delta\dot{\omega}_1(0) = \bar{\omega}_2 k_1 \delta\omega_3(0)$ and $\delta\dot{\omega}_3(0) = \bar{\omega}_2 k_3 \delta\omega_1(0)$.

$$\bar{\omega}_2 k_1 \left(\delta\omega_3(0)\cos(\Omega t) + \frac{\delta\dot{\omega}_3(0)}{\Omega}\sin(\Omega t)\right) = -\delta\omega_1(0)\Omega\sin(\Omega t) + \delta\dot{\omega}_1(0)\cos(\Omega t) \tag{25}$$

$$\bar{\omega}_2 k_1 \left(\delta\omega_3(0)\cos(\Omega t) + \frac{\bar{\omega}_2 k_3 \delta\omega_1(0)}{\Omega}\sin(\Omega t)\right) = -\delta\omega_1(0)\Omega\sin(\Omega t) + \bar{\omega}_2 k_1 \delta\omega_3(0)\cos(\Omega t) \tag{26}$$

$$\text{cos terms: } \bar{\omega}_2 k_1 \delta\omega_3(0) = \bar{\omega}_2 k_1 \delta\omega_3(0) \tag{27}$$

$$\text{sin terms: } \bar{\omega}_2 k_1 \frac{\bar{\omega}_2 k_3 \delta\omega_1(0)}{\Omega} = -\delta\omega_1(0)\Omega \tag{28}$$

$$\bar{\omega}_2^2 k_1 k_2 = -\Omega^2 \tag{29}$$

It looks like Renato either had a mistake in his class notes, or I had a mistake in my copying.

We can make a clever substitution.

$$\frac{\delta\dot{\omega}_1(0)}{\Omega} = \frac{\bar{\omega}_2 k_1 \delta\omega_3(0)}{\sqrt{-k_1 k_3}\bar{\omega}_2} \tag{30}$$

$$= \sqrt{-\frac{k_1}{k_3}}\delta\omega_3(0) \tag{31}$$

$$\frac{\delta\dot{\omega}_3(0)}{\Omega} = \frac{\bar{\omega}_2 k_3 \delta\omega_1(0)}{\sqrt{-k_1 k_3}\bar{\omega}_2} \tag{32}$$

$$= \sqrt{-\frac{k_3}{k_1}}\delta\omega_1(0) \tag{33}$$

And now we have an STM.

$$\begin{bmatrix} \delta\omega_1(t) \\ \delta\omega_3(t) \end{bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sqrt{-\frac{k_1}{k_3}}\sin(\Omega t) \\ \sqrt{-\frac{k_3}{k_1}}\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} \delta\omega_1(0) \\ \delta\omega_3(0) \end{bmatrix} \tag{34}$$

We need to find the particular solution. For linear systems this is an integral. We know that our input torque is constant, so it falls out.

$$\delta\omega(t) = \int_0^t \Phi(t,\sigma) J^{-1} \delta\tau d\sigma \tag{35}$$

$$\delta\omega(t) = \int_0^t \begin{bmatrix} \cos(\Omega(t-\sigma)) & \sqrt{\frac{k_1}{k_3}} \sin(\Omega(t-\sigma)) \\ \sqrt{\frac{k_3}{k_1}} \sin(\Omega(t-\sigma)) & \cos(\Omega(t-\sigma)) \end{bmatrix} d\sigma J^{-1} \delta\tau \tag{36}$$

$$\delta\omega(t) = \begin{bmatrix} -\frac{1}{\Omega} \sin(\Omega(t-\sigma)) & \sqrt{\frac{k_1}{k_3}} \frac{1}{\Omega} \cos(\Omega(t-\sigma)) \\ \sqrt{\frac{k_3}{k_1}} \frac{1}{\Sigma} \cos(\Omega(t-\sigma)) & -\frac{1}{\Omega} \sin(\Omega(t-\sigma)) \end{bmatrix} \Bigg|_{\sigma=0}^{\sigma=t} J^{-1} \delta\tau \tag{37}$$

$$\delta\omega(t) = \frac{1}{\Omega} \begin{bmatrix} -\sin(\Omega(t-\sigma)) & \sqrt{\frac{k_1}{k_3}} \cos(\Omega(t-\sigma)) \\ \sqrt{\frac{k_3}{k_1}} \cos(\Omega(t-\sigma)) & -\sin(\Omega(t-\sigma)) \end{bmatrix} \Bigg|_{\sigma=0}^{\sigma=t} J^{-1} \delta\tau \tag{38}$$

$$\delta\omega(t) = \frac{1}{\Omega} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -\sin(\Omega t) & \sqrt{\frac{k_1}{k_3}} \cos(\Omega t) \\ \sqrt{\frac{k_3}{k_1}} \cos(\Omega t) & -\sin(\Omega t) \end{bmatrix} \right) J^{-1} \delta\tau \tag{39}$$

The general solution is just the sum of the particular and homogeneous solutions.