

Homework 2 Problem 1

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1 Prompt

Derive the linearized equations of perturbed motion of an axial symmetric satellite spinning around its major axis of inertial $J_2 > J_1 = J_3$. Assume a constant perturbation torque acting on the spacecraft.

2 Answer

We begin with the equation governing motion:

$$J\dot{\omega} = -\omega \times J\omega + \tau \quad (1)$$

ω is the rotation rate with respect to inertial in the body frame. J is the inertia matrix in the body frame. τ is the input torque in the body frame.

We introduce nominal $\bar{\omega}$ and perturbation $\delta\omega$ components.

$$J(\dot{\bar{\omega}} + \delta\dot{\omega}) = -(\bar{\omega} + \delta\omega) \times J(\bar{\omega} + \delta\omega) + (\bar{\tau} + \delta\tau) \quad (2)$$

We will expand the right side.

$$J(\dot{\bar{\omega}} + \delta\dot{\omega}) = -(\bar{\omega} + \delta\omega) \times J(\bar{\omega} + \delta\omega) + (\bar{\tau} + \delta\tau) \quad (3)$$

$$= -(\bar{\omega} + \delta\omega) \times (J\bar{\omega} + J\delta\omega) + (\bar{\tau} + \delta\tau) \quad (4)$$

$$= -(\bar{\omega} + \delta\omega) \times J\bar{\omega} - (\bar{\omega} + \delta\omega) \times J\delta\omega + (\bar{\tau} + \delta\tau) \quad (5)$$

$$= -(\bar{\omega} \times J\bar{\omega} + \delta\omega \times J\bar{\omega}) - (\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega) + (\bar{\tau} + \delta\tau) \quad (6)$$

$$= -(\bar{\omega} \times J\bar{\omega} + \delta\omega \times J\bar{\omega}) - (\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega) + (\bar{\tau} + \delta\tau) \quad (7)$$

We can eliminate the nominal's evolution with the governing equation.

$$= -(\delta\omega \times J\bar{\omega}) - (\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega) + \delta\tau \quad (8)$$

$$= J\bar{\omega} \times \delta\omega - (\bar{\omega} \times J\delta\omega + \delta\omega \times J\delta\omega) + \delta\tau \quad (9)$$

$$= [J\bar{\omega} \times] \delta\omega - ([\bar{\omega} \times] J\delta\omega + \delta\omega \times J\delta\omega) + \delta\tau \quad (10)$$

$$= ([J\bar{\omega} \times] - [\bar{\omega} \times] J) \delta\omega + \delta\omega \times J\delta\omega + \delta\tau \quad (11)$$

Then we linearize the perturbations about the nominal, ($\delta\omega = 0$).

$$J\delta\dot{\omega} \approx \left. \frac{dJ\delta\dot{\omega}}{d\delta\omega} \right|_{\delta\omega=0} \delta\omega + \delta\tau \quad (12)$$

$$\approx ([J\bar{\omega} \times] - [\bar{\omega} \times] J) \delta\omega + \delta\tau \quad (13)$$

I don't know how to linearize $\delta\omega \times J\delta\omega$ but matlab symbolic toolbox tells me that when it is linearized and evaluated at $\delta\omega = 0$, it becomes zero.

We will now drop the approximate sign for convenience. But note that the approximation is still present.

$$\delta\dot{\omega} = \begin{bmatrix} J_1^{-1} & 0 & 0 \\ 0 & J_2^{-1} & 0 \\ 0 & 0 & J_3^{-1} \end{bmatrix} ([J\bar{\omega} \times] - [\bar{\omega} \times] J) \delta\omega + \delta\tau \quad (14)$$

$$= \begin{bmatrix} 0 & \frac{\bar{\omega}_3(J_2 - J_3)}{J_1} & \frac{\bar{\omega}_2(J_2 - J_3)}{J_1} \\ -\frac{\bar{\omega}_3(J_1 - J_3)}{J_2} & 0 & -\frac{\bar{\omega}_1(J_1 - J_3)}{J_2} \\ \frac{\bar{\omega}_2(J_1 - J_2)}{J_3} & \frac{\bar{\omega}_1(J_1 - J_2)}{J_3} & 0 \end{bmatrix} \delta\omega + J^{-1} \delta\tau \quad (15)$$

At this point, note two things; $J_1 = J_3$ and $\bar{\omega}_1 = \bar{\omega}_3 = 0$.

$$\delta\dot{\omega} = \begin{bmatrix} 0 & 0 & \frac{\bar{\omega}_2(J_2-J_3)}{J_1} \\ 0 & 0 & 0 \\ \frac{\bar{\omega}_2(J_1-J_2)}{J_3} & 0 & 0 \end{bmatrix} \delta\omega + J^{-1}\delta\tau \quad (16)$$

$$\begin{bmatrix} \delta\dot{\omega}_1 \\ \delta\dot{\omega}_2 \\ \delta\dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} \frac{\bar{\omega}_2(J_2-J_3)}{J_1} \delta\omega_3 \\ 0 \\ \frac{\bar{\omega}_2(J_1-J_2)}{J_3} \delta\omega_1 \end{bmatrix} + J^{-1}\delta\tau \quad (17)$$

We can define $k_1 = \frac{J_2-J_3}{J_1}$ and $k_3 = \frac{J_1-J_2}{J_3}$. Then the free response of $\delta\omega$ is given as the following.

$$\delta\omega_1(t) = \delta\omega_1(0) \cos(\Omega t) + \frac{\delta\dot{\omega}_1(0)}{\Omega} \sin(\Omega t) \quad (18)$$

$$\delta\omega_3(t) = \delta\omega_3(0) \cos(\Omega t) + \frac{\delta\dot{\omega}_3(0)}{\Omega} \sin(\Omega t) \quad (19)$$

$$\Omega^2 = k_1 k_3 \bar{\omega}_2^2 \quad (20)$$

I think there could be an error in the above formulation. Try to take the derivative and plug it back in to the equations of motion.

$$\delta\dot{\omega}_1(t) = -\delta\omega_1(0)\Omega \sin(\Omega t) + \delta\dot{\omega}_1(0) \cos(\Omega t) \quad (21)$$

$$\bar{\omega}_2 k_1 \delta\omega_3(t) = -\delta\omega_1(0)\Omega \sin(\Omega t) + \delta\dot{\omega}_1(0) \cos(\Omega t) \quad (22)$$

$$\bar{\omega}_2 k_1 \left(\delta\omega_3(0) \cos(\Omega t) + \frac{\delta\dot{\omega}_3(0)}{\Omega} \sin(\Omega t) \right) = -\delta\omega_1(0)\Omega \sin(\Omega t) + \delta\dot{\omega}_1(0) \cos(\Omega t) \quad (23)$$

$$(24)$$

We know that $\delta\dot{\omega}_1(0) = \bar{\omega}_2 k_1 \delta\omega_3(0)$ and $\delta\dot{\omega}_3(0) = \bar{\omega}_2 k_3 \delta\omega_1(0)$.

$$\bar{\omega}_2 k_1 \left(\delta\omega_3(0) \cos(\Omega t) + \frac{\delta\dot{\omega}_3(0)}{\Omega} \sin(\Omega t) \right) = -\delta\omega_1(0)\Omega \sin(\Omega t) + \delta\dot{\omega}_1(0) \cos(\Omega t) \quad (25)$$

$$\bar{\omega}_2 k_1 \left(\delta\omega_3(0) \cos(\Omega t) + \frac{\bar{\omega}_2 k_3 \delta\omega_1(0)}{\Omega} \sin(\Omega t) \right) = -\delta\omega_1(0)\Omega \sin(\Omega t) + \bar{\omega}_2 k_1 \delta\omega_3(0) \cos(\Omega t) \quad (26)$$

$$\cos \text{ terms: } \bar{\omega}_2 k_1 \delta\omega_3(0) = \bar{\omega}_2 k_1 \delta\omega_3(0) \quad (27)$$

$$\sin \text{ terms: } \bar{\omega}_2 k_1 \frac{\bar{\omega}_2 k_3 \delta\omega_1(0)}{\Omega} = -\delta\omega_1(0)\Omega \quad (28)$$

$$\bar{\omega}_2^2 k_1 k_3 = -\Omega^2 \quad (29)$$

It looks like Renato either had a mistake in his class notes, or I had a mistake in my copying.

We can make a clever substitution.

$$\frac{\delta\dot{\omega}_1(0)}{\Omega} = \frac{\bar{\omega}_2 k_1 \delta\omega_3(0)}{\sqrt{-k_1 k_3 \bar{\omega}_2^2}} \quad (30)$$

$$= \sqrt{-\frac{k_1}{k_3}} \delta\omega_3(0) \quad (31)$$

$$\frac{\delta\dot{\omega}_3(0)}{\Omega} = \frac{\bar{\omega}_2 k_3 \delta\omega_1(0)}{\sqrt{-k_1 k_3 \bar{\omega}_2^2}} \quad (32)$$

$$= \sqrt{-\frac{k_3}{k_1}} \delta\omega_1(0) \quad (33)$$

And now we have an STM.

$$\begin{bmatrix} \delta\omega_1(t) \\ \delta\omega_3(t) \end{bmatrix} = \begin{bmatrix} \cos(\Omega t) & \sqrt{-\frac{k_1}{k_3}} \sin(\Omega t) \\ \sqrt{-\frac{k_3}{k_1}} \sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} \delta\omega_1(0) \\ \delta\omega_3(0) \end{bmatrix} \quad (34)$$

We need to find the particular solution. For linear systems this is an integral. We know that our input torque is constant, so it falls out.

$$\delta\omega(t) = \int_0^t \Phi(t, \sigma) J^{-1} \delta\tau d\sigma \quad (35)$$

$$\delta\omega(t) = \int_0^t \begin{bmatrix} \cos(\Omega(t-\sigma)) & \sqrt{\frac{k_1}{k_3}} \sin(\Omega(t-\sigma)) \\ \sqrt{\frac{k_3}{k_1}} \sin(\Omega(t-\sigma)) & \cos(\Omega(t-\sigma)) \end{bmatrix} d\sigma J^{-1} \delta\tau \quad (36)$$

$$\delta\omega(t) = \begin{bmatrix} -\frac{1}{\Omega} \sin(\Omega(t-\sigma)) & \sqrt{\frac{k_1}{k_3}} \frac{1}{\Omega} \cos(\Omega(t-\sigma)) \\ \sqrt{\frac{k_3}{k_1}} \frac{1}{\Omega} \cos(\Omega(t-\sigma)) & -\frac{1}{\Omega} \sin(\Omega(t-\sigma)) \end{bmatrix} \bigg|_{\sigma=0}^{\sigma=t} J^{-1} \delta\tau \quad (37)$$

$$\delta\omega(t) = \frac{1}{\Omega} \begin{bmatrix} -\sin(\Omega(t-\sigma)) & \sqrt{\frac{k_1}{k_3}} \cos(\Omega(t-\sigma)) \\ \sqrt{\frac{k_3}{k_1}} \cos(\Omega(t-\sigma)) & -\sin(\Omega(t-\sigma)) \end{bmatrix} \bigg|_{\sigma=0}^{\sigma=t} J^{-1} \delta\tau \quad (38)$$

$$\delta\omega(t) = \frac{1}{\Omega} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -\sin(\Omega t) & \sqrt{\frac{k_1}{k_3}} \cos(\Omega t) \\ \sqrt{\frac{k_3}{k_1}} \cos(\Omega t) & -\sin(\Omega t) \end{bmatrix} \right) J^{-1} \delta\tau \quad (39)$$

The general solution is just the sum of the particular and homogeneous solutions.