MATH321-25S1 Rings and Fields

Assignment 1

Due Friday, March 28th, 4pm

Upload your work to Learn. No late work will be accepted. There is a total of **30** marks for this assignment.

Give reasons and show all working. You will lose marks for incomplete solutions or for poorly presented answers.

Problem 1

Let F be a field and

$$S = F \times (F \setminus \{0\}) = \{(a, b) \mid a, b \in F, b \neq 0\}.$$

Define operations in S by $(a, b) + (c, d) = (ad + bc, bd)$, $(a, b)(c, d) = (ac, bd)$. Show that S satisfies the following

(a) Addition is associative. [4 marks]

(b) The element $(0, 1) \in S$ is a neutral element for addition (a "zero"). [3 marks]

(c) Give an example to show that distributivity can fail. (Hint: use the rationals and (1,-1)). [3 marks]

Problem 2

(a) How many roots does $(x - 2)(x - 3) \in \mathbb{Z}_6[x]$ have in $\mathbb{Z}_6$? [3 marks]

(b) Show that there is no monic quadratic polynomial in $\mathbb{Z}_6[x]$ with 5 roots in $\mathbb{Z}_6$. [4 marks]

Problem 3

The polynomial ring $\mathbb{Z}_{25}[x]$ over $\mathbb{Z}_{25}$ has many unusual properties, some of which are explored in this question. These are consequences of the fact that $\mathbb{Z}_{25}$ has zero-divisors.

(a) Show that

$$R = \{ax + b \mid a \in 5\mathbb{Z}_{25}, b \in \mathbb{Z}_{25}\}$$

is a subring of $\mathbb{Z}_{25}[x]$ with 125 elements. [3 marks]

(b) Find a monic quadratic polynomial in $\mathbb{Z}_{25}[x]$ that has 5 distinct zeros and 3 different factorizations into a product of two linear polynomials. [3 marks]

(c) Show that for each $n \in \mathbb{N}$ there is a polynomial in $\mathbb{Z}_{25}[x]$ of degree n that is a unit. [3 marks]

Hint: Find a linear polynomial first that is a unit in R.

(d) $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ also is a commutative ring with 125 elements.

Show that R and $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ are not isomorphic. [4 marks]

Hint: How many units do R and $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ have?