CS188, Winter 2017
Problem Set 0: Math Prerequisites
Due Jan 19,2017

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1 Problem 1

(a) **Solution:**

$$\frac{\partial y}{\partial x} = -xe^{-x}\sin z + \sin z e^{-x} = e^{-x}\sin z(1-x)$$

2 Problem 2

Solution: Solution to problem 2

(a) Problem 2a

Solution: $y^T z = 1 * 2 + 3 * 3 = 11$

(b) Problem 2b

Solution:

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} * \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 * \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 * \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

(c) Problem 2c

Solution: X is invertible since the two vectors given by its columns are linearly independent. The inverse can be calculated with the well-known formula for the inverse of a 2 x 2 invertible matrix:

$$X^{-1} = \frac{1}{ad - bc} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2} * \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$$

(d) Problem 2d

Solution: The rank is the number of leading ones in reduced-row echelon form. Equivalently, it is the number of linearly independent rows (or columns) in the matrix, which is 2 as stated above. So rank(X) = 2.

3 Problem 3

Solution: Solution to problem 3

(a) Problem 3a

Solution: $\mu_{sample} = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1+1+0+1+0}{5} = 0.6$

Problem 3b

Solution: $\sigma^2_{sample} = \frac{1}{N-1}\sum_{i=1}^{N})x_i - \mu_{sample}) = 0.3$

Problem 3c

Solution: Since each particular sample is independent of the other, we have

$$P = P(1) * P(1) * P(0) * P(1) * P(0) = \frac{1}{2^5} = \frac{1}{32}$$

Problem 3d

Solution: Let P(S) denote the probability of sample S. Let p = P(X = 1). Then, one has:

$$P(S) = p^3(1-p)^2$$

.

We want to find p such that the function is maximized (and p is in between 0 and 1 otherwise P(S) = 0), so we take the derivative and set it equal to zero:

$$\frac{\partial P(S)}{\partial p} = 5p^4 - 8p^3 + 3p^2 = 0$$

$$5p^2 - 8p + 3 = 0$$

Solving for p, we obtain

$$p = \frac{3}{5}$$

Problem 3e

Solution:

$$P(X = T|Y = b) = \frac{P(X = T, Y = b)}{P(Y = b)} = \frac{.1}{.25} = 0.4$$

4 Problem 4

Solution: Solution to problem 4

(a) Problem 4a

Solution: False.

(b) Problem 4b

Solution: True.

(c) Problem 4c

Solution: False.

(d) Problem 4d

Solution: False

(e) Problem 4e

Solution: True.

5 Problem 5

Solution: Solution to problem 5

(a) Problem 5a

Solution: v

(b) Problem 5b

Solution: iv

(c) Problem 5c

Solution: ii

(d) Problem 5d

Solution: i

(e) Problem 5e

Solution: iii

6 Problem 6

Solution: Solution to problem 6

(a) Problem 6a

Solution:

$$mean : p$$

$$variance : p(1-p)$$

(b) Problem 6b

Solution:

$$Var[X] = E[(X - E[X])^2] = \sigma^2$$

For 2X, We know that

$$E[2X] = 2E[X]$$

So by substitution

$$Var[2X] = E[(2X - 2E[2X])^2] = 4E[(X - E[X])^2] = 4\sigma^2$$

which is the variance of 2X. This makes sense since the spread of the data becomes twice the original, increasing the standard deviation by 2, and the variance is the standard deviation's square.

For X + 2, we know that

$$E[X + 2] = E[X] + E[2] = E[X] + 2$$

Therefore,

$$Var[X+2] = E[(X+2-E[X+2])^2] = E[(X+2-E[X]-2)^2] = E[(X-E[X])^2] = \sigma^2$$

so the variance does not change if we add 2. This makes sense since the overall spread of the data is the same

7 Problem 7

Solution: Solution to problem 7

(a) Problem 7a

Solution: i. Both. The functions differ by a constant, so their asymptotic growth rate is the same.

$$f(n) = \Theta(g(n))$$

ii.

$$g(n) = O(f(n))$$

. An exponential function grows asymptotically faster than a polynomial function.
iii.

$$g(n) = O(f(n))$$

. f(n) grows asymptotically faster than g(n). This can be seen by taking the limit as n becomes large:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} (\frac{3}{2})^n = \infty$$

(b) Problem 7b

Solution:
find-transition-index(arr, i, j):
if i is greater or equal to j, return -1 (no index found)
let mid = i + floor((j-i)/2)
if arr[mid] == 0 and arr[mid + 1] == 1: return mid
if arr[mid] == 0 and arr[mid + 1] == 0: return find-transition-index(arr, mid+1, j)
else return find-transition-index(arr,i,mid)

The algorithm is correct since it picks the middle index at each step, and sees if there is a transition. If not, half the problem space can be discarded based on the values of arr[mid] and arr[mid+1], and it recursively searches the correct half until a transition is found or i exceeds j, implying no transition exists.

Looking at the algorithm, we see that the problem size reduces by

half each time find-transition-index is called. Therefore, we have the following recurrence:

$$T(n) = T(n/2) + O(1)$$

with O(1) base cases. This recurrence solves to O(n log n).

8 Problem 8

Solution: Solution to problem 8

(a) Problem 8a

Solution:

$$E[XY] = \sum_{i=1}^{N} x_i y_i p(x_i, y_i)$$

By the product rule,

$$p(x_i, y_i) = p(x_i|y_i)p(y)$$

but since x, y are independent,

$$p(x_i|y_i) = p(x_i)$$

Therefore,

$$p(x_i, y_i) = p(x_i)p(y_i)$$

and

$$E[XY] = \sum_{i=1}^{N} x_i p(x_i) y_i p(y_i) = \sum_{i=1}^{N} x_i p(x_i) * \sum_{i=1}^{N} y_i p(y_i) = E[X] * E[Y]$$

(b) Problem 8b

Solution:

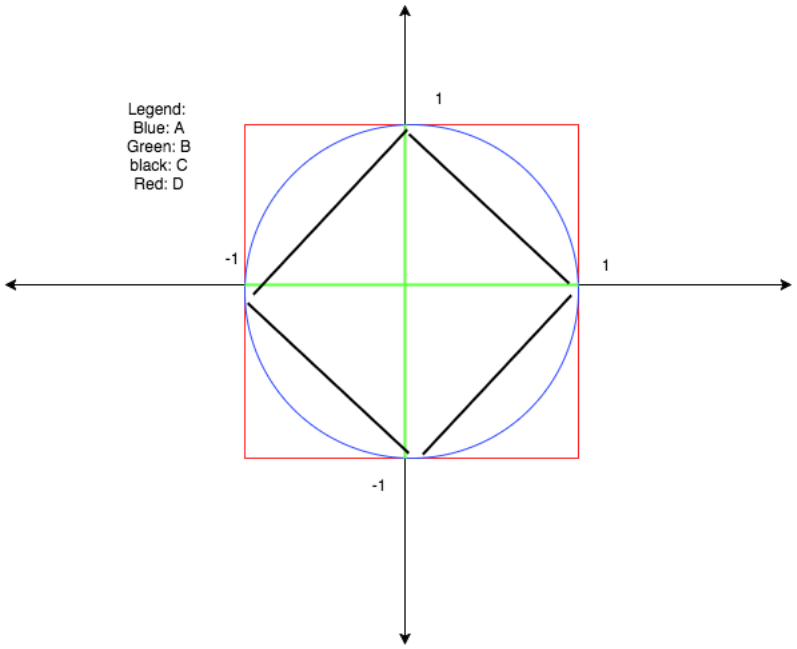
i. Assuming a fair die, by the law of large numbers, the number of times a 3 shows up should be close to 1000.

ii. The central limit theorem says that the distribution of X converges weakly to the Gaussian as n tends to infinity.

9 Problem 9

Solution: Solution to problem 9

(a) Problem 9a



(b) Problem 9b

Solution: i. An eigenvalue is a scalar lambda such that

$$A(v) = \lambda v$$

where A is a square matrix and v is a vector.
An eigenvector is a (nonzero) vector v who's direction does not change when the linear transformation A is applied to it. In other words, v is an eigenvector of a square matrix A if the product A*v is a scalar multiple of v.

ii.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$det(A - \lambda I) = det(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}) = (2-\lambda)^2 - 1 = 0$$

$$\lambda_1 = 3, \lambda_2 = 1$$

For the corresponding eigenvector v_1 of λ_1, we want to find a vector in the nullspace of $A - 3I$:

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} * v_1 = 0$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For the corresponding eigenvector v_2 of λ_2, we want to find a vector in the nullspace of $A - I$:

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} * v_2 = 0$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

iii. We show this by induction:
Base Case: k = 1: we have $Av = \lambda v$ for n eigenvalues and n eigenvectors. This is given.
Inductive Hypothesis: Assume the n eigenvalues of A^{k-1} take the form λ^{k-1}. That is, assume that there are n distinct eigenvalues and eigenvectors that satisfy $A^{k-1}v = \lambda^{k-1}v$
Inductive Step: We show that the hypothesis holds for j = k, namely, that the eigenvalues of A^k fulfill $A^k v = \lambda^k v$ where λ and v represent the n eigenvalues and eigenvectors of A. First, we notice

$$A^k v = AA^{k-1}v$$

From the inductive hypothesis, we know

$$A^{k-1}v = \lambda^{k-1}v$$

Therefore,

$$A^k v = A\lambda^{k-1} v = \lambda^{k-1} Av$$

Since $Av = \lambda v$,

$$A^k v = \lambda^{k-1}\lambda v = \lambda^k v$$

Showing what we wanted. Since we showed this for any eigenvalue corresponding to its eigenvector, it applies for all n eigenvalues and eigenvectors of A.

(c) Problem 9c

Solution:

i.

$$\frac{\partial(a^T x)}{\partial x} = a^T$$

ii.

$$\frac{\partial(x^T Ax)}{\partial x} = A^T x + x^T A$$

$$\frac{\partial^2(x^T Ax)}{\partial x^2} = \frac{\partial(A^T x + x^T A)}{\partial x} = A^T + A = 2A$$

(d) Problem 9d **Solution:**

i. Consider two points, x_1 and x_2 on the line. This implies that

$$w^T x_1 + b = w^T x_2 + b = 0$$

$$w^T x_1 = w^T x_2$$

$$w^T(x_1 - x_2) = 0$$

We notice that the vector given by $v = x_1$ - x_2 is a vector parallel to our line. Since the dot product of w^T and a vector $v = x_1 - x_2$ is zero, the vector w must be orthogonal to v, implying that w is orthogonal to $w^T x + b = 0$.

ii. First, we let x* be the point on the line that is closest to the origin. Then, one has $x* = min a^T a$ given the constraint $w^T a + b = 0$. Taking the derivative and using Lagrange multipliers, one obtains:

$$2x* = \lambda w \longrightarrow x* = \frac{\lambda}{2} w$$

Plugging the value in the constraint, we have

$$w^T a + b = 0$$

$$w^T(\frac{\lambda}{2}w) + b = 0$$

$$\lambda = \frac{-2b}{w^T w}$$

$$x* = \frac{-bw}{w^t w}$$

Having calculate $x*$, we can now calculate its distance from the origin:

$$d = \sqrt{(x*)^T x*} = \sqrt{(\frac{-b}{w^T w})^2 w^T w} = \frac{b}{w^T w}\sqrt{w^T w} = \frac{b}{||w||}$$

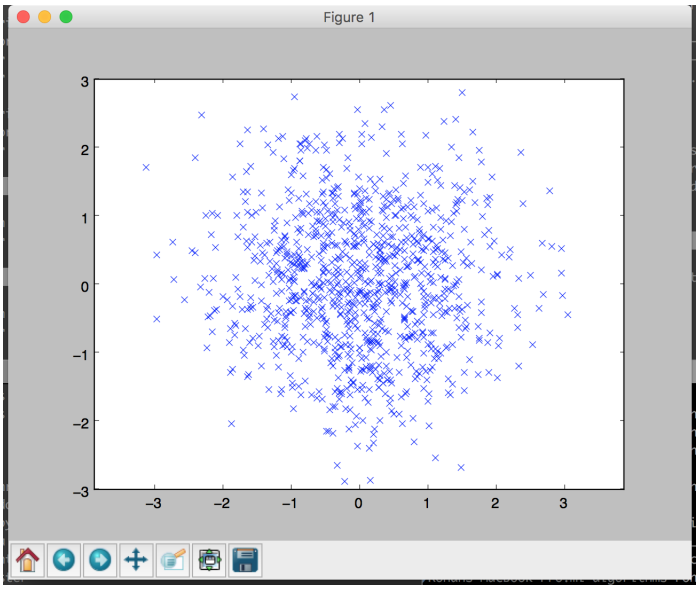
as expected.

10 Problem 10

Solution: Solution to problem 10

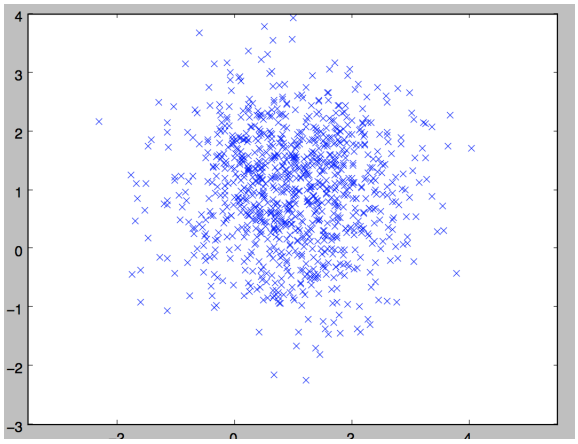
i. Problem 10a

Solution:



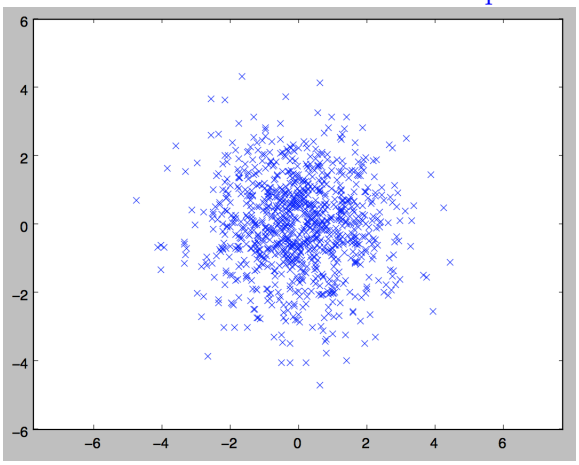
ii. Problem 10b

Solution: The distribution is centered around 1, not zero.



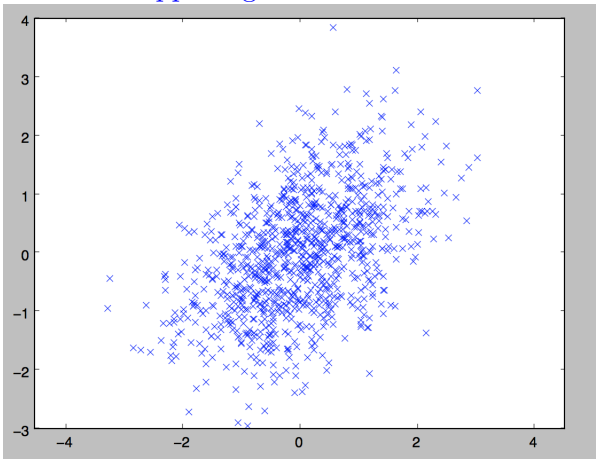
iii. Problem 10c

Solution: The data becomes more spread out.



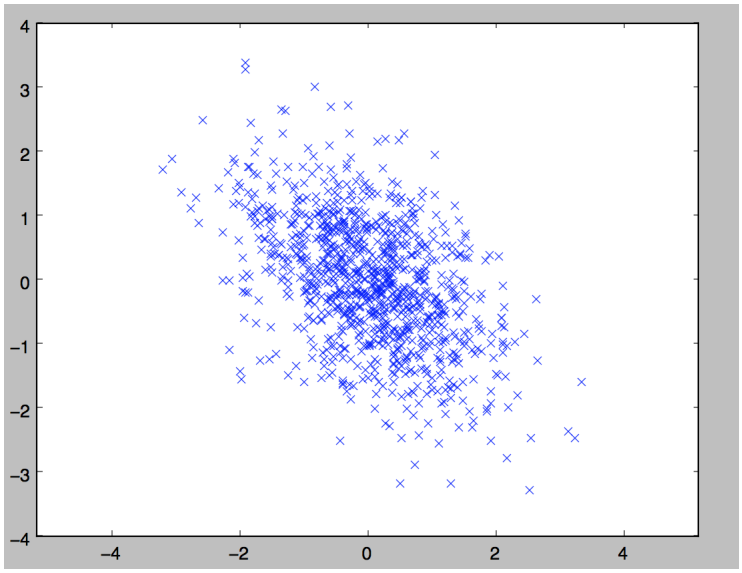
iv. Problem 10d

Solution: The data become skewed, stretching from the lower left to the upper right.



v. Problem 10e

Solution: The data become skewed so that it stretches from the upper left to the bottom right.


4
3
2
1
0
−1
−2
−3
−4
−4
−2
0
2
4

11 Problem 11

Solution: Solution to problem 11

(a) Problem 11

Solution:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

$$det(A - \lambda I) = det(\begin{bmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}) = (1-\lambda)(3-\lambda), \lambda_{largest} = 3$$

Find a vector v in the nullspace of A - 3I:

$$(A - 3I)v = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} (v) = 0$$

$$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

12 Problem 12

Solution: Solution to problem 12

(a) Problem 12a

Solution: Name: The MNIST Database of Handwritten Digits

(b) Problem 12b

Solution: Data set is available at http://yann.lecun.com/exdb/mnist/

(c) Problem 12c

Solution: The dataset contains images of handwritten data as well as labels that tell us which number the handwritten digit corresponds to. Each training example is a pair (image, label) and the features of this training set are the individual pixels in the image, and we wish to predict the number that the image represents.

(d) Problem 12d

Solution: There are 60,000 training examples in the dataset (ie, 60,000 (image, label) pairs) as well as a testing set of 10,000 examples that can be used to evaluate your model after it is trained.

(e) Problem 12e

Solution: Each training example contains a 28 x 28 black and white image. Therefore, there are $28^2 = 784$ features for each training example, where each feature is either 0 or 1, representing the pixel at that position. Even though the image is a square, when a model sees the features, they are generally flattened out to a 784 dimensional vector. We are then left with a 784 dimensional vector of numbers which we can feed through models (artificial neural networks for example) to find a function that maps images to numbers.