PHILOSOPHICAL TRANSACTIONS.

I. *Mathematical Contributions to the Theory of Evolution.*—VII. *On the Correlation of Characters not Quantitatively Measurable.*

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Received February 7,—Read March 1, 1900.

NOTE.

In August, 1899, I presented a memoir to the Royal Society on the inheritance of coat-colour in the horse and of eye-colour in man, which was read November, 1899, and ultimately ordered to be published in the 'Phil. Trans.' Before that memoir was printed, Mr. YULE'S valuable memoir on Association was read, and, further, Mr. LESLIE BRAMLEY-MOORE showed me that the theory of my memoir as given in § 6 of the present memoir led to somewhat divergent results according to the methods of proportioning adopted. We therefore undertook a new investigation of the theory of the whole subject, which is embodied in the present memoir. The data involved in the paper on coat-colour in horses and eye-colour in man have all been recalculated, and that paper is nearly ready for presentation.* But it seemed best to separate the purely theoretical considerations from their application to special cases of inheritance, and accordingly the old memoir now reappears in two sections. The theory discussed in this paper was, further, the basis of a paper on the Law of Reversion with special reference to the Inheritance of Coat-colour in Basset Hounds recently communicated to the Society, and about to appear in the 'Proceedings.'†

While I am responsible for the general outlines of the present paper, the rough draft of it was taken up and carried on in leisure moments by Mr. LESLIE BRAMLEY-MOORE, Mr. L. N. G. FILON, M.A., and Miss ALICE LEE, D.Sc. Mr. BRAMLEY-MOORE discovered the u-functions; Mr. FILON proved most of their general properties and the convergency of the series; I alone am responsible for sections 4, 5, and 6. Mr. LESLIE BRAMLEY-MOORE sent me, without proof, on the eve of his departure for the Cape, the general expansion for z on p. 26. I am responsible for the present proof and its applications. To Dr. ALICE LEE we owe most of the illustrations and the table on p. 17. Thus the work is essentially a joint memoir in which we have equal part, and the use of the first personal pronoun is due to the fact that the material had to be put together and thrown into form by one of our number.—K. P.

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* Since ordered to be printed in the 'Phil. Trans.'

† Read January 25, 1900. 'Roy. Soc. Proc.,' vol. 66, p. 140.

§ (1.) *On a General Theorem in Normal Correlation.*

Let the frequency surface

$$z = \frac{\mathrm{N}}{2\pi\sqrt{(1-r^2)}\sigma_1\sigma_2} e^{-\frac{1}{2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}-\frac{2rxy}{\sigma_1\sigma_2}\right)},$$

where N = total number of observations,

σ_1, σ_2 = standard deviations of organs x and y,

r = correlation of x and y,

be divided into four parts by two planes at right angles to the axes of x and y at distances h' and k' from the origin. The total volumes or frequencies in these parts will be represented by a, b, c, and d in the manner indicated in the accompanying plan :—

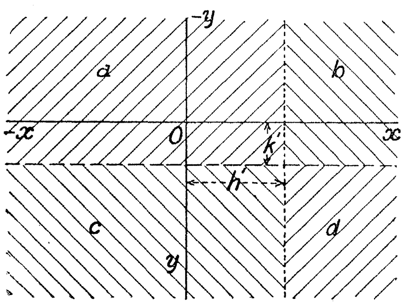


Table of Frequencies

a	b	a + b
c	d	c + d
a + c	b + d	N

Then clearly

$$d = \frac{\mathrm{N}}{2\pi\sqrt{(1-r^2)}\sigma_1\sigma_2}\int_{h'}^{\infty}\int_{k'}^{\infty} e^{-\frac{1}{2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}-\frac{2rxy}{\sigma_1\sigma_2}\right)}\,dx\,dy$$

$$= \frac{\mathrm{N}}{2\pi\sqrt{(1-r^2)}}\int_{h}^{\infty}\int_{k}^{\infty} e^{-\frac{1}{2}\frac{1}{1-r^2}(x^2+y^2-2rxy)}\,dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}),$$

if $h = h'/\sigma_1$ and $k = k'/\sigma_2$.

Further,
$$b + d = \frac{\mathrm{N}}{\sqrt{2\pi}\,\sigma_1}\int_{h'}^{\infty} e^{-\frac{1}{2}\frac{x^2}{\sigma_1^2}}\,dx,$$
$$= \frac{\mathrm{N}}{\sqrt{2\pi}}\int_{h}^{\infty} e^{-\frac{1}{2}x^2}\,dx \quad \text{(ii.)},$$
and
$$c + d = \frac{\mathrm{N}}{\sqrt{2\pi}}\int_{k}^{\infty} e^{-\frac{1}{2}y^2}\,dy \quad \text{(iii.)},$$
$$\frac{(a+c)-(b+d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}}\int_{0}^{h} e^{-\frac{1}{2}x^2}\,dx \quad \text{(iv.)},$$
$$\frac{(a+b)-(c+d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}}\int_{0}^{k} e^{-\frac{1}{2}y^2}\,dy \quad \text{(v.)}.$$

Thus, when a, b, c, and d are known, h and k can be found by the ordinary table of the probability integral, say that of Mr. SHEPPARD ('Phil. Trans.,' A, vol. 192, p. 167, Table VI.*). The limits accordingly of the integral for d in (i.) are known.

Now consider the expression
$$\frac{1}{\sqrt{1-r^2}}\,e^{-\frac{1}{2}\frac{1}{1-r^2}(x^2+y^2-2rxy)} = \mathrm{U}, \text{ say,} \quad \text{(vi.)},$$
and let us expand it in powers of r. Then, if the expansion be
$$\mathrm{U} = e^{-\frac{1}{2}(x^2+y^2)}\left(u_0 + \frac{u_1 r}{\underline{|1}} + \frac{u_2 r^2}{\underline{|2}} + \ldots + \frac{u_n r^n}{\underline{|n}} + \ldots\right) \quad \text{(vii.)},$$
we shall have
$$u_n = e^{\frac{1}{2}(x^2+y^2)}\left(\frac{d^n\mathrm{U}}{dr^n}\right)_{r=0} \quad \text{(viii.)}.$$

Taking logarithmic differentials, we get at once
$$(1-r^2)^2\frac{d\mathrm{U}}{dr} = \{xy + r(1-x^2-y^2) + r^2xy - r^3\}\mathrm{U}.$$

Differentiating n times by LEIBNITZ's theorem, and putting $r = 0$, we have, after some reductions
$$\begin{aligned} u_{n+1} = {} & n(2n-1-x^2-y^2)u_{n-1} \\ & - n(n-1)(n-2)^2 u_{n-3} \\ & + xy\{u_n + n(n-1)u_{n-2}\} \end{aligned} \quad \text{(ix.)}.$$
Hence we find
$$\left.\begin{aligned} u_0 &= 1 \\ u_1 &= xy \\ u_2 &= (x^2-1)(y^2-1) \\ u_3 &= x(x^2-3)\,y(y^2-3) \\ u_4 &= (x^4-6x^2+3)(y^4-6y^2+3) \end{aligned}\right\} \quad \text{(x.)}$$

* See, however, foot-note, p. 5.

Thus the following laws are indicated :—

$$u_n = v_n \times w_n \quad \text{(xi.)},$$

where

$$v_n = xv_{n-1} - (n-1)v_{n-2} \quad \text{(xii.)},$$

$$w_n = yw_{n-1} - (n-1)w_{n-2} \quad \text{(xiii.)}.$$

We shall now show that these laws hold good by induction. Assume

$$u_{n+1} = v_{n+1}w_{n+1} = (xv_n - nv_{n-1})(yw_n - nw_{n-1}).$$

Thus

$$u_{n+1} = xyu_n + n^2u_{n-1} - n(yw_nv_{n-1} + xv_nw_{n-1}).$$

But by (ix.), substituting for u_{n-3} from (xi.) and (xiii.),

$$\begin{aligned}
u_{n+1} = xy\{v_nw_n &+ n(n-1)v_{n-2}w_{n-2}\} + n(2n-1-x^2-y^2)v_{n-1}w_{n-1} \\
&- n(n-1)v_{n-1}w_{n-1} - xyn(n-1)v_{n-2}w_{n-2} \\
&+ n(n-1)(yv_{n-1}w_{n-2} + xv_{n-2}w_{n-1}). \\
= xyv_nw_n &+ n^2v_{n-1}w_{n-1} - n(x^2+y^2)v_{n-1}w_{n-1} \\
&+ n(n-1)(yv_{n-1}w_{n-2} + xv_{n-2}w_{n-1}) \\
= xyv_nw_n &+ n^2v_{n-1}w_{n-1} - n\{yv_{n-1}(yu_{n-1} - \overline{n-1}\,w_{n-2}) \\
&\qquad + xw_{n-1}(xv_{n-1} - \overline{n-1}\,v_{n-2}\} \\
= xyv_nw_n &+ n^2v_{n-1}w_{n-1} - n(yv_{n-1}w_n + xw_{n-1}v_n) \\
= v_{n+1}w_{n+1}&, \text{ as we have seen above.}
\end{aligned}$$

Thus, if the theorem holds for u_n, it holds for u_{n+1}. Accordingly

$$\text{U} = e^{-\frac{1}{2}(x^2+y^2)}\left(1 + \frac{v_1w_1}{\lfloor 1}r + \frac{v_2w_2}{\lfloor 2}r^2 + \ldots + \frac{v_nw_n}{\lfloor n}r^n + \ldots\right) \quad \text{(xiv.)},$$

where the v's and w's are given by (x.), (xii.), and (xiii.).

It is thus clear that $\frac{1}{2\pi}\int_h^\infty \int_k^\infty \text{U}\,dx\,dy$ consists of a series of which the general term is

$$\frac{1}{\lfloor n}\text{V}_n\text{W}_n r^n$$

where

$$\text{V}_n = \frac{1}{\sqrt{2\pi}}\int_h^\infty e^{-\frac{1}{2}x^2}v_n dx$$

$$\text{W}_n = \frac{1}{\sqrt{2\pi}}\int_k^\infty e^{-\frac{1}{2}y^2}w_n dy.$$

It remains to find these integrals.

The general form of v_n is given by

$$v_n = x^n - \frac{n(n-1)}{2\lfloor 1}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2\lfloor 2}x^{n-4} - \&\text{c.} \quad \text{(xv.)}.$$

For this obviously gives (x.). Assume it true for v_{n-1} and v_{n-2}, then

$$\begin{aligned} xv_{n-1} - (n-1)v_{n-2} &= x^n - \frac{(n-1)(n-2)}{2\,\underline{|1}} x^{n-2} + \frac{(n-1)(n-2)(n-3)(n-4)}{2^2\,\underline{|2}} x^{n-4} - \dots \\ &\quad - (n-1)x^{n-2} + \frac{(n-1)(n-2)(n-3)}{2\,\underline{|1}} x^{n-4} - \dots \\ &= x^n - \frac{n(n-1)}{2\,\underline{|1}} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2\,\underline{|2}} x^{n-4} - \dots \\ &= v_n. \end{aligned}$$

Thus the expression (xv.) is shown to hold by induction, the general terms being

$$(-1)^r \frac{(n-1)(n-2)\dots(n-2r+1)}{2^{r-1}\,\underline{|r-1}} \left(\frac{n-2r}{2r} + 1\right) x^{n-2r}$$
$$= (-1)^r \frac{n(n-1)(n-2)\dots(n-2r+1)}{2^r\,\underline{|r}} x^{n-2r},$$

or the general term in u_n.

We notice at once that

$$\frac{dv_n}{dx} = nv_{n-1} \quad \dots \dots \dots \quad \text{(xvi.)}.$$

Thus, by (xii.)

$$v_n = xv_{n-1} - \frac{dv_{n-1}}{dx}.$$

Multiply by $e^{-\frac{1}{2}x^2}$ and integrate

$$\int e^{-\frac{1}{2}x^2} v_n\, dx = \int xe^{-\frac{1}{2}x^2} v_{n-1}\, dx - \int e^{-\frac{1}{2}x^2} \frac{dv_{n-1}}{dx}\, dx.$$

Integrating the latter integral by parts, we have

$$\int v_n e^{-\frac{1}{2}x^2} dx = -e^{-\frac{1}{2}x^2} v_{n-1},$$

or

$$V_n = \frac{1}{\sqrt{2\pi}} \int_h^\infty v_n e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2} (v_{n-1})_{x=h}.$$

Now $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2}$ can be found from any table of the ordinates of the normal curve, *e.g.*, Mr. SHEPPARD's, 'Phil. Trans.,' A, vol. 192, p. 153, Table I.* We shall accordingly put

$$H = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2}, \qquad K = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2} \quad \dots \dots \quad \text{(xvii.)},$$

and look upon H and K as known quantities.

* For our present purposes the differences of Mr. SHEPPARD'S tables are occasionally too large, but the following series give very close results :—

Let
$$\chi_1 = \sqrt{\frac{\pi}{2}} \frac{(a+c)-(b+d)}{N} = \int_0^h e^{-\frac{1}{2}x^2} dx \text{ by (iv.)},$$
$$\chi_2 = \sqrt{\frac{\pi}{2}} \frac{(a+b)-(c+d)}{N} = \int_0^k e^{-\frac{1}{2}y^2} dy \text{ by (v.)}.$$

Further, let us write $(v_{n-1})_{x=h}$ as $\bar{v}_{n-1}$, and similarly $(w_{n-1})_{y=k}$ as $\bar{w}_{n-1}$. Thus

$$\mathrm{V}_n = \mathrm{H}\,.\,\bar{v}_{n-1}, \qquad \mathrm{W}_n = \mathrm{K}\,.\,\bar{w}_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xviii.}).$$

We have then from (i.)

$$\begin{aligned}\frac{d}{\mathrm{N}} &= \frac{1}{2\pi}\int_h^\infty\int_k^\infty \mathrm{U}\,dx\,dy \\ &= \frac{1}{2\pi}\int_h^\infty\int_k^\infty e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy + \overset{\infty}{\underset{1}{\mathrm{S}}}\left(\frac{r^n}{\lfloor n}\mathrm{HK}\,\bar{v}_{n-1}\bar{w}_{n-1}\right) \\ &= \frac{(b+d)(c+d)}{\mathrm{N}^2} + \overset{\infty}{\underset{1}{\mathrm{S}}}\left(\frac{r^n}{\lfloor n}\mathrm{HK}\,\bar{v}_{n-1}\bar{w}_{n-1}\right)\end{aligned}$$

by (ii.) and (iii.).

Or, remembering that $\mathrm{N} = a + b + c + d$, we can write this

$$\begin{aligned}\frac{ad - bc}{\mathrm{N}^2\mathrm{HK}} &= \overset{\infty}{\underset{1}{\mathrm{S}}}\left(\frac{r^n}{\lfloor n}\bar{v}_{n-1}\bar{w}_{n-1}\right) \\ &= r + \frac{r^2}{2}hk + \frac{r^3}{6}(h^2-1)(k^2-1) + \frac{r^4}{24}h(h^2-3)k(k^2-3) \\ &\quad + \frac{r^5}{120}(h^4-6h^2+3)(k^4-6k^2+3) \\ &\quad + \frac{r^6}{720}h(h^4-10h^2+15)k(k^4-10k^2+15) \\ &\quad + \frac{r^7}{5040}(h^6-15h^4+45h^2-15)(k^6-15k^4+45k^2-15) \\ &\quad + \frac{r^8}{40320}h(h^6-21h^4+105h^2-105)k(k^6-21k^4+105k-105) + \text{, \&c.}\end{aligned}$$

$$. \quad . \quad . \quad . \quad . \quad (\text{xix.}).$$

Then

$$h = \chi_1 + \frac{1}{\lfloor 3}\chi_1^3 + \frac{7}{\lfloor 5}\chi_1^5 + \frac{127}{\lfloor 7}\chi_1^7 + \dots$$

$$\frac{1}{\mathrm{H}} = \sqrt{2\pi}\left(1 + \frac{1}{\lfloor 2}\chi_1^2 + \frac{7}{\lfloor 4}\chi_1^4 + \frac{127}{\lfloor 6}\chi_1^6 + \dots\right),$$

and

$$k = \chi_2 + \frac{1}{\lfloor 3}\chi_2^3 + \frac{7}{\lfloor 5}\chi_2^5 + \frac{127}{\lfloor 7}\chi_2^7 + \dots$$

$$\frac{1}{\mathrm{K}} = \sqrt{2\pi}\left(1 + \frac{1}{\lfloor 2}\chi_2^2 + \frac{7}{\lfloor 4}\chi_2^4 + \frac{127}{\lfloor 6}\chi_2^6 + \dots\right).$$

These follow from the considerations that if

$$\chi_1 = \sqrt{2\pi}\,\phi_1, \qquad \chi_2 = \sqrt{2\pi}\,\phi_2,$$

$$\frac{d\phi_1}{dh} = \mathrm{H}, \qquad \frac{d\phi_2}{dk} = \mathrm{K},$$

$$\frac{d\mathrm{H}}{d\phi_1} = -h, \qquad \frac{d\mathrm{K}}{d\phi_2} = -k,$$

whence it is easy to find the successive differentials of h with regard to ϕ_1 and k with regard to ϕ_2, and then obtain the above results by MACLAURIN'S theorem. There is, of course, no difficulty in calculating H and K from (xvii.) directly. That method was adopted in the numerical illustrations.

Here the left-hand side is known, and since h and k are known, we can find the coefficients of any number of powers of r so soon as the first two have been found, from (xii.) and (xiii.).

Accordingly the correlation can be found if we have only made a grouping of our frequencies into the four divisions, a, b, c, and d.

If h and k be zero, we have from (xvii.) and (iv.)

$$\mathrm{H} = \mathrm{K} = \frac{1}{\sqrt{2\pi}}$$

$$a + c = b + d = \tfrac{1}{2}\mathrm{N}.$$

The right-hand side of (xix.) is now

$$r + \frac{1}{\underline{|3}} r^3 + \quad . \quad . \quad .$$

or equal to $\sin^{-1} r$.

Hence

$$r = \sin 2\pi \frac{(ad - bc)}{\mathrm{N}^2}$$

$$= \cos \pi \frac{b}{a + b} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xx.}),$$

which agrees with a result of Mr. SHEPPARD's, 'Phil. Trans.,' A, vol. 192, p. 141. We have accordingly reached a generalised form of his result for *any* class-index whatever. Clearly, also, r being known, we can at once calculate the frequency of pairs of organs with deviations as great as or greater than h and k.

§ (2.) *Other Series for the Determination of r.*

For many purposes the series (xix.) is sufficiently convergent to give r for given h and k with but few approximations, but we will now turn to other developments.

We have by (vii.)

$$\int_0^r \mathrm{U}\, dr = e^{-\frac{1}{2}(x^2+y^2)} \left(u_0 r + \frac{u_1 r^2}{\underline{|2}} + \ldots + u_n \frac{r^{n+1}}{\underline{|n+1}} + \ldots \right).$$

Put $x = h$, $y = k$, and write for brevity

$$\epsilon = \frac{ad - bc}{\mathrm{N}^2\mathrm{HK}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxi.}).$$

It follows at once from (xix.) that

$$\epsilon = e^{\frac{1}{2}(h^2+k^2)} \int_0^r \mathrm{U}\, dr$$

$$= e^{\frac{1}{2}(h^2+k^2)} \int_0^r \frac{1}{\sqrt{1-r^2}} e^{-\frac{1}{2}\frac{1}{1-r^2}(h^2+k^2-2rhk)}\, dr$$

$$\left.\begin{aligned} &= e^{\frac{1}{2}h^2}\int_0^\theta e^{-\frac{1}{2}(h\tan\theta - k\sec\theta)^2}\,d\theta \\ &= e^{\frac{1}{2}h^2}\int_0^\theta e^{-\frac{1}{2}(k\tan\theta - h\sec\theta)^2}\,d\theta \end{aligned}\right\} \quad \ldots\ldots \quad \text{(xxii.)},$$

if $r = \sin\theta$.

Now either of the quantities under the sign of integration in (xxii.) can be expanded in powers of θ by MACLAURIN's theorem. Thus let

$$\begin{aligned} \chi &= e^{-\frac{1}{2}(k\tan\theta - h\sec\theta)^2} \\ &= \chi_0 + \left(\frac{d\chi}{d\theta}\right)_0\theta + \left(\frac{d^2\chi}{d\theta^2}\right)_0\frac{\theta^2}{\lfloor 2} + \ldots + \left(\frac{d^n\chi}{d\theta^n}\right)_0\frac{\theta^n}{\lfloor n} + \ldots \end{aligned}$$

Then

$$\epsilon = e^{\frac{1}{2}h^2}\left(\chi_0\theta + \left(\frac{d\chi}{d\theta}\right)_0\frac{\theta^2}{\lfloor 2} + \left(\frac{d^2\chi}{d\theta^2}\right)_0\frac{\theta^3}{\lfloor 3} + \ldots + \left(\frac{d^n\chi}{d\theta^n}\right)_0\frac{\theta^{n+1}}{\lfloor n+1} + \ldots\right),$$

and it remains to find $\left(\frac{d^n\chi}{d\theta^n}\right)_0$.

Now $$\log\chi = -\tfrac{1}{2}(k\tan\theta - h\sec\theta)^2.$$

Hence

$$\cos^3\theta\frac{d\chi}{d\theta} = -\chi[(h^2+k^2)\sin\theta - hk(\tfrac{3}{2} - \tfrac{1}{2}\cos 2\theta)].$$

Differentiating $\overline{n-1}$ times by LEIBNITZ's theorem, and putting $\theta = 0$,

$$\begin{aligned} 4\left(\frac{d^n\chi}{d\theta^n}\right)_0 - 4hk\left(\frac{d^{n-1}\chi}{d\theta^{n-1}}\right) + \ldots \\ + \frac{(n-1)\ldots(n-r+1)}{\lfloor r-1}\left[\cos\frac{r\pi}{2}\left\{\frac{n-r}{r}(3+3^r) - 4(h^2+k^2)\right\}\right. \\ \left. + \sin\frac{r\pi}{2}2^r hk\right]\left(\frac{d^{n-r}\chi}{d\theta^{n-r}}\right)_0 + \ldots = 0 \quad \ldots\ldots \quad \text{(xxiii.)}. \end{aligned}$$

Clearly $\chi_0 = e^{-\frac{1}{2}h^2}$, then we rapidly find

$$\begin{aligned} \left(\frac{d\chi}{d\theta}\right)_0 &= hk\,e^{-\frac{1}{2}h^2} \\ \left(\frac{d^2\chi}{d\theta^2}\right)_0 &= -(h^2+k^2-h^2k^2)e^{-\frac{1}{2}h^2} \\ \left(\frac{d^3\chi}{d\theta^3}\right)_0 &= hk\,\{h^2k^2 - 3(h^2+k^2) + 5\}. \end{aligned}$$

Or, finally

$$\epsilon = \theta + \tfrac{1}{2}hk\theta^2 - (h^2+k^2-h^2k^2)\frac{\theta^3}{6} + hk\{h^2k^2 - 3(h^2+k^2) + 5\}\frac{\theta^4}{24} + \ldots \quad \text{(xxiv.)},$$

where more terms if required can be found by (xxiii.). If θ be fairly small, θ^5 will be negligible. Or if h and k be small, the lowest term in the next factor will be $h^2 + k^2$, and this into $\theta^5/\underline{|5}$ is generally quite insensible. Very often two or three terms on the right-hand side of (xxiv.) give quite close enough values of θ, and accordingly of $r = \sin\theta$. (xxiv.) is clearly somewhat more convergent than (xix.) if h and k are, as usually happens, less than unity.

Returning now to (xix.), let us write it

$$\epsilon = f(r, h, k).$$

This is the equation that must be solved for r. Suppose r_0 a root of this when we retain only few terms on the right, say a root of the quadratic

$$\epsilon = r + \tfrac{1}{2}hkr^2.$$

Then if $r = r_0 + \rho$,

$$\epsilon = f(r_0, h, k) + \rho f'(r_0, h, k) + \tfrac{1}{\underline{|2}}\rho^2 f''(r_0, h, k) + \&c.$$

Hence $\rho = \dfrac{\epsilon - f(r_0hk)}{f'(r_0hk)}$ to a third approximation

$$= -\frac{\frac{1}{6}(h^2-1)(k^2-1)r_0^{\,3}}{\dfrac{1}{\sqrt{1-r_0^{\,2}}}\,e^{-\frac{1}{2}\frac{1}{1-r_0^2}(h^2+k^2-2r_0hk)}} \text{ nearly} \quad \ldots\ldots \quad \text{(xxv.)},$$

which gives us a value of ρ which, substituted in ρ^2 in the above equation, introduces only terms of the 6th order in r_0.

Another integral expression for ϵ of Equation (xxi.) may here be noticed :

$$\epsilon = e^{\frac{1}{2}(h^2+k^2)}\int_0^r \frac{dr}{\sqrt{1-r^2}}\, e^{-\frac{1}{2}\frac{1}{1-r^2}(h^2+k^2-2rhk)}.$$

Put $h = \dfrac{1}{\sqrt{2}}(\beta+\gamma)$, $k = \dfrac{1}{\sqrt{2}}(\beta-\gamma)$.

Hence

$$\epsilon = e^{\frac{1}{2}(\beta^2+\gamma^2)}\int_0^r \frac{dr}{\sqrt{1-r^2}}\, e^{-\frac{1}{2}\left(\frac{\beta^2}{1+r}+\frac{\gamma^2}{1-r}\right)}$$

$$= e^{\frac{1}{4}(\beta^2+\gamma^2)}\int_0^r \frac{dr}{\sqrt{1-r^2}}\, e^{-\frac{1}{4}\left(\beta^2\frac{1-r}{1+r}+\gamma^2\frac{1+r}{1-r}\right)}.$$

Let $\tan 2\phi = \dfrac{1-r}{1+r}$, or, $r = \cos 2\,\phi$.

Therefore

$$\epsilon = 2e^{\frac{1}{4}(\beta^2+\gamma^2)}\int_\phi^{45^\circ} e^{-\frac{1}{4}(\beta^2\tan^2\phi+\gamma^2\cot^2\phi)}\,d\phi,$$

$$= 2e^{\frac{1}{4}(\beta^2+\gamma^2)}\int_1^v e^{-\frac{1}{4}\left(\frac{\beta^2}{v^2}+\gamma^2v^2\right)}\frac{dv}{1+v^2}$$

where $v = \cot\phi$ and is > 1.

It seems possible that interesting developments for ϵ might be deduced from this integral expression.

§ (3.) *To show that the Series for r is Convergent if $r < 1$, whatever be the Values of h and k.*

Write the series in the form of p. 6, *i.e.* :—

$$\epsilon = \overset{n}{\underset{1}{\mathrm{S}}} \frac{r^n}{\underline{|n}} \bar{v}_{n-1} \bar{w}_{n-1}.$$

Now

$$\left.\begin{aligned} \bar{v}_{n+1} &= h\bar{v}_n - n\bar{v}_{n-1} \\ \bar{w}_{n+1} &= k\bar{w}_n - n\bar{w}_{n-1} \end{aligned}\right\} \text{by (xii.) and (xiii.).}$$

From these we deduce

$$\bar{v}_{n+1} = \{h^2 - (2n-1)\}\,\bar{v}_{n-1} - (n-1)(n-2)\,\bar{v}_{n-3}$$
$$\bar{w}_{n+1} = \{k^2 - (2n-1)\}\,\bar{w}_{n-1} - (n-1)(n-2)\,\bar{w}_{n-3}$$

Now let
$$s_n = \bar{v}_{n-1} r^{\frac{1}{2}n} \Big/ \{\underline{|n}\}^{\frac{1}{2}}, \quad t_n = \bar{w}_{n-1} r^{\frac{1}{2}n} \Big/ \{\underline{|n}\}^{\frac{1}{2}}.$$

Then we find

$$s_{n+2} = \frac{h^2 - (2n-1)}{\sqrt{(n+1)(n+2)}}\, s_n r - \sqrt{\frac{(n-1)(n-2)^2}{n(n+1)(n+2)}}\, s_{n-2} r^2,$$

$$t_{n+2} = \frac{k^2 - (2n-1)}{\sqrt{(n+1)(n+2)}}\, t_n r - \sqrt{\frac{(n-1)(n-2)}{n(n+1)(n+2)}}\, t_{n-2} r^2.$$

Thus, when n is large, we find the ratio of successive terms s_{n+2}/s_n or t_{n+2}/t_n is given by ρ, where

$$\rho = -2r - r^2/\rho \text{ or, } \rho = -r.$$

The ultimate ratio of $s_{n+2}\, t_{n+2}$ to $s_n\, t_n$ is accordingly given by r^2, but this is the ratio of alternate terms of the original series. The original series thus breaks up into two series, one of odd and one of even powers of r. Both these series are absolutely convergent whatever h and k be, having an ultimate convergence ratio of r^2

§ (4.) *To find the Probable Error of the Correlation Coefficient as Determined by the Method of this Memoir.*

Given a division of the total frequency N into a, b, c, d groups, where $a + b + c + d = \mathrm{N}$, then the probable error of any one of them, say a, is $\cdot 67449\, \sigma_a$, where*

$$\sigma_a = \sqrt{\frac{a(\mathrm{N}-a)}{\mathrm{N}}} \quad \ldots\ldots\ldots\ldots \quad \text{(xxvi.)}.$$

Let $b + d = n_1$, $c + d = n_2$, then

* The standard deviation of an event which happens np times and fails nq times in n trials is well known to be $\sqrt{npq}$. The probable errors here dealt with are throughout, of course, those arising from different samples of the same general population.

$$\sigma_{n_1} = \sqrt{\frac{n_1(\mathrm{N} - n_1)}{\mathrm{N}}} \qquad \sigma_{n_2} = \sqrt{\frac{n_2(\mathrm{N} - n_2)}{\mathrm{N}}} \quad . \quad . \quad . \quad \text{(xxvii.)}.$$

To obtain r_{cd} we have, if $\delta\eta$ denotes an error in any quantity η,

$$\delta c + \delta d = \delta n_2,$$

$$\therefore \ \sigma_c^2 + \sigma_d^2 + 2\sigma_c \sigma_d r_{cd} = \sigma_{n_2}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxviii.)},$$

by squaring, summing for all possible variations in c and d, and dividing by the total number of variations.

Hence, substituting the values of the standard deviations as found above, we deduce

$$\sigma_c \sigma_d r_{cd} = -cd/\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxix.)}.$$

In a similar manner

$$\delta n_1 \delta d = \delta b \, \delta d + (\delta d)^2,$$

$$\sigma_d \sigma_{n_1} r_{dn_1} = \sigma_b \sigma_d r_{bd} + \sigma_d^2$$

$$\sigma_d \sigma_{n_1} r_{dn_1} = d\,(a + c)/\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxx.)}.$$

and

$$\sigma_d \sigma_{n_2} r_{dn_2} = d\,(a + b)/\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxi.)},$$

Now

$$n_1 = \frac{\mathrm{N}}{\sqrt{2\pi}} \int_h^\infty e^{-\frac{1}{2}x^2} \, dx,$$

$$\therefore \ \delta n_1 = -\frac{\mathrm{N}}{\sqrt{2\pi}} \, e^{-\frac{1}{2}h^2} \delta h = -\mathrm{NH}\delta h.$$

Thus

$$\sigma_{n_1} = \mathrm{NH}\,\sigma_h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxii.)},$$

and similarly

$$\sigma_{n_2} = \mathrm{NK}\,\sigma_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxiii.)}.$$

Hence the probable error of h

$$= \frac{\cdot 67449}{\mathrm{H}\sqrt{\mathrm{N}}} \sqrt{\frac{(b + d)\,(a + c)}{\mathrm{N}^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxiv.)},$$

and of k

$$= \frac{\cdot 67449}{\mathrm{K}\sqrt{\mathrm{N}}} \sqrt{\frac{(c + d)\,(a + b)}{\mathrm{N}^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxv.)}.$$

They can be found at once, therefore, when H and K have been found from an ordinate table of the exponential curve, and a, b, c, d are given. We have thus the probable error of the means as found from any double grouping of observations.

Next, noting that

$$\delta n_1 \, \delta n_2 = \mathrm{N}^2\mathrm{HK} \; \delta h \, \delta k,$$

we have

$$\sigma_{n_1} \sigma_{n_2} r_{n_1 n_2} = \mathrm{N}^2\mathrm{HK} \; \sigma_h \sigma_k r_{hk},$$

or

$$r_{n_1 n_2} = r_{hk}.$$

But
$$\delta n_1 \delta n_2 = (\delta b + \delta d)(\delta c + \delta d),$$

$$\sigma_{n_1}\sigma_{n_2} r_{n_1 n_2} = \sigma_b \sigma_c r_{bc} + \sigma_b \sigma_d r_{bcd} + \sigma_c \sigma_d r_{cd} + \sigma_d^2,$$

$$= \frac{ad - bc}{\mathrm{N}} \quad \text{(xxxvi.)},$$

therefore

$$\sigma_h \sigma_k r_{hk} = \frac{ad - bc}{\mathrm{NHK}} \quad \text{(xxxvii.)}.$$

$$r_{hk} = \frac{ad - bc}{\sqrt{(b + d)(a + c)(c + d)(a + b)}} \quad \text{(xxxviii.)}.$$

This is an important result; it expresses the correlation between errors in the position of the means of the two characters under consideration. But if the probabilities were independent there could be no such correlation. Thus r_{hk} might be taken as a measure of divergence from independent variation. We shall return to this point later.

Since $\delta n_1 = -\mathrm{HN}\delta h$, we have $\delta n_1 \delta d = -\mathrm{HN}\delta d \delta h$, whence we easily deduce

$$r_{dn_1} = -r_{dh} \quad \text{(xxxix.)}.$$

Similarly
$$r_{dn_2} = -r_{dk} \quad \text{(xl.)}.$$

Now d is a function of r, h, and k. Hence if $d = f(r, h, k)$,

$$\delta d = \frac{df}{dr}\delta r + \frac{df}{dh}\delta h + \frac{df}{dk}\delta k$$

$$= \gamma_0 \delta r + \gamma_1 \delta h + \gamma_2 \delta k \quad \text{(xli.)}.$$

Whence transposing, squaring, summing, and dividing by the total number of observations, we find

$$\gamma_0^2 \sigma_r^2 = \sigma_d^2 + \gamma_1^2 \sigma_h^2 + \gamma_2^2 \sigma_k^2 - 2\gamma_1 \sigma_d \sigma_h r_{dh} - 2\gamma_2 \sigma_d \sigma_k r_{dk} + 2\gamma_1 \gamma_2 \sigma_h \sigma_k r_{hk}$$

$$= \sigma_d^2 + \left(\frac{\gamma_1}{\mathrm{HN}}\right)^2 \sigma_{n_1}^2 + \left(\frac{\gamma_2}{\mathrm{KN}}\right)^2 \sigma_{n_2}^2 + 2\left(\frac{\gamma_1}{\mathrm{HN}}\right)\sigma_d \sigma_{n_1} r_{dn_1}$$
$$+ 2\left(\frac{\gamma_2}{\mathrm{HN}}\right)\sigma_d \sigma_{n_2} r_{dn_2} + \frac{2\gamma_1\gamma_2}{\mathrm{N^2HK}}\sigma_{n_1}\sigma_{n_2} r_{n_1 n_2} \quad \text{(xlii.)}.$$

Substituting the values of the standard deviations and correlations as found above, we have

$$\sigma_r^2 = \frac{1}{\mathrm{N}\gamma_0^2}\left\{ d(a + b + c) + \left(\frac{\gamma_2}{\mathrm{HN}}\right)^2 (a + b)(d + c) + \left(\frac{\gamma_1}{\mathrm{KN}}\right)^2 (a + c)(d + b) \right.$$
$$\left. + \frac{2\gamma_1\gamma_2}{\mathrm{NHK}}(ad - bc) + \frac{2\gamma_2}{\mathrm{HN}} d(b + a) + \frac{2\gamma_1}{\mathrm{KN}} d(c + a) \right\} \quad \text{(xliii.)}.$$

It remains now to determine γ_0, γ_1, and γ_2.

By Equation (i.)

$$d = f(r, h, k) = \frac{\mathrm{N}}{2\pi\sqrt{1-r^2}} \int_h^\infty \int_k^\infty e^{-\frac{1}{2}\frac{x^2+y^2-2xy}{1-r^2}}\, dx\, dy,$$

$$\gamma_1 = \frac{df}{dh} = -\frac{\mathrm{N}}{2\pi\sqrt{1-r^2}} \int_k^\infty e^{-\frac{1}{2(1-r^2)}(h^2+y^2-2rhy)}\, dy$$

$$= -\frac{\mathrm{N}}{2\pi\sqrt{1-r^2}} e^{-\frac{1}{2}h^2} \int_k^\infty e^{-\frac{1}{2}\frac{(y-rh)^2}{1-r^2}}\, dy$$

$$= -\mathrm{H} \frac{\mathrm{N}}{\sqrt{2\pi}} \int_{\beta_2}^\infty e^{-\frac{1}{2}z^2}\, dz \quad \text{(xliv.)},$$

where

$$\beta_2 = \frac{k - rh}{\sqrt{1-r^2}}.$$

Thus

$$\gamma_1 / (\mathrm{NH}) = -\left(\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}z^2}\, dz - \frac{1}{\sqrt{2\pi}} \int_0^{\beta_2} e^{-\frac{1}{2}z^2}\, dz\right)$$

$$= \psi_2 - \tfrac{1}{2} \quad \text{(xlv.)}.$$

Similarly

$$\gamma_2 / (\mathrm{NK}) = \psi_1 - \tfrac{1}{2} \quad \text{(xlvi.)}.$$

Here

$$\psi_1 = \frac{1}{\sqrt{2\pi}} \int_0^{\beta_1} e^{-\frac{1}{2}z^2}\, dz, \qquad \psi_2 = \frac{1}{\sqrt{2\pi}} \int_0^{\beta_2} e^{-\frac{1}{2}z^2}\, dz \quad \text{(xlvii.)},$$

where

$$\beta_1 = \frac{h - rk}{\sqrt{1-r^2}}, \qquad \beta_2 = \frac{k - rh}{\sqrt{1-r^2}} \quad \text{(xlviii.)},$$

and thus ψ_1 and ψ_2 can be found at once from the tables when β_1 and β_2 are found from the known values of r, h, k.

Lastly, we have from Equation (xxi.)

$$\frac{ad - bc}{\mathrm{N}^2\mathrm{HK}} = e^{\frac{1}{2}(h^2+k^2)} \int_0^r \mathrm{U}\, dr,$$

or

$$\frac{d}{\mathrm{N}} = \frac{(d+b)(d+c)}{\mathrm{N}^2} + \frac{1}{2\pi} \int_0^r \mathrm{U}\, dr.$$

Thus*

$$\gamma_0 = df/dr = \frac{1}{2\pi} \mathrm{NU},$$

$$\gamma_0 / \mathrm{N} = \chi_0,$$

where

$$\chi_0 = \frac{1}{2\pi} \frac{1}{\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}(h^2+k^2-2rhk)} \quad \text{(xlix.)}$$

a value which can again be found as soon as r, h, k are known. $\gamma_0 = \chi_0 \mathrm{N}$ is clearly the ordinate of the frequency surface corresponding to $x = h$, $y = k$.

Substituting in Equation (xliii.) we have, after some reductions,

* By Equations (ii.) and (iii.), $d + b$ and $d + c$ are independent of r.

$$\text{Probable error of } r = \cdot 67449\sigma_r$$

$$= \frac{\cdot 67449}{\sqrt{\mathrm{N}\chi_0}} \left\{ \frac{(a+d)(c+b)}{4\mathrm{N}^2} + \psi_2{}^2 \frac{(a+c)(d+b)}{\mathrm{N}^2} + \psi_1{}^2 \frac{(a+b)(d+c)}{\mathrm{N}^2} \right.$$

$$\left. + 2\psi_1\psi_2 \frac{ad-bc}{\mathrm{N}^2} - \psi_2 \frac{ab-cd}{\mathrm{N}^2} - \psi_1 \frac{ac-bd}{\mathrm{N}_2} \right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{l.}),$$

where χ_0, ψ_1, and ψ_2 are readily found from Equations (xlix.), (xlvii.), and (xlviii.). Thus the probable error of r can be fairly readily found. It must be noted in using this formula, that a is the quadrant in which the mean falls, so that h and k are both *positive* (see fig., p. 2). In other words, we have supposed $a + c > b + d$ and $a + b > c + d$. Our lettering must always be arranged so as to suit this result before we apply the above formula.

§ (5.) *To Find a Physical Meaning for the Series in r, or for the ϵ of Equation* (xxi.).

Return to the original distribution $\dfrac{a \mid b}{c \mid d}$ of p. 2. If the probabilities of the two characters or organs were quite independent, we should expect the distribution

$$\begin{array}{c|c} \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} & \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} \\ \hline \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} & \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} \end{array}$$

Now re-arranging our actual data we may put it thus:

$$\frac{a \mid b}{c \mid d} = \begin{array}{c|c} \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} + \dfrac{ad-bc}{\mathrm{N}} & \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} - \dfrac{ad-bc}{\mathrm{N}} \\ \hline \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} - \dfrac{ad-bc}{\mathrm{N}} & \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} + \dfrac{ad-bc}{\mathrm{N}} \end{array}$$

Accordingly correlation denotes that $\dfrac{ad-bc}{\mathrm{N}}$ has been transferred from each of the second and fourth compartments, and the same amount added to each of the first and third compartments. If $\eta = (ad - bc)/\mathrm{N}^2$, then η is the *transfer per unit of the total frequency.* The magnitude of this transfer is clearly a measure of the divergence of the statistics from independent variation. It is physically quite as significant as the correlation coefficient itself, and of course much easier to determine. It must vanish with the correlation coefficient. We see from (xxi.) that

$$\eta = \epsilon \times \mathrm{HK},$$

or we have an interpretation for the series in r of (xix.).

Now, obviously any function of η, just like η itself, would serve as a measure of the divergence from perfectly independent variation. It is convenient to choose a function which shall lie arithmetically between 0 and 1.

Now consider what happens in the case of perfect correlation, *i.e.*, all the observations fall into a straight line. Hence if $ad > bc$, either b or c is zero, for a straight line cannot cut all four compartments, and a and d are obviously positive. Thus c and b can only be zero if $\eta = (c + d)(a + c)/N^2$ or $(a + b)(b + d)/N^2$. In order that b should be zero, it is needful that h and k, as given by (iv.) and (v.), should be positive or $a + c > b + d$, $a + b > c + d$, and the mean fall under the 45° line through the vertical and horizontal lines dividing the table into four compartments, *i.e.*, $h > k$. These conditions would be satisfied if $ad > bc$ and $a > d$, $c > b$. Now suppose our four-compartment table arranged so that

$$ad > bc, \quad a > d, \quad c > b,$$

and consider the function

$$Q_1 = \sin\frac{\pi}{2}\frac{\eta}{(a + b)(b + d)/N^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{li.}),$$

or

$$Q_1 = \sin\frac{\pi}{2}\frac{ad - bc}{(a + b)(b + d)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lii.}).$$

This function vanishes if $\eta = 0$, and it further = unity if $b = 0$. Thus it agrees at the limits 0 and 1 with the value of the correlation coefficient. Again, when h and k are both zero, $a = d$, $b = c$, and $Q_1 = \sin\frac{\pi}{2}\frac{a - b}{a + b}$, is thus r by (xx.). Hence we have found a function which vanishes with r and equals unity with r, while it is also equal to r if the divisions of the table be taken through the medians.

Now, I take it that these are very good conditions to make for any function of a, b, c, d which is to vanish with the "transfer," and to serve as a measure of the degree of dependent variability, or what Mr. YULE has termed the degree of "association." Mr. YULE has selected for his coefficient of association the expression

$$Q_2 = \frac{ad - bc}{ad + bc} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{liii.}).$$

This vanishes with the transfer, equals unity if b or c be zero, and minus unity if a or d be zero. The latter is, of course, unnecessary if we agree to arrange a, b, c, d so that ad is always greater than bc. Now it is clear that Q_2 possesses a great advantage over Q_1 in rapidity of calculation, but the coefficient of correlation is also a coefficient which measures the association, and it is a great advantage to select one which agrees to the closest extent with the correlation, for then it enables us to determine other important features of the system.

If we do not make all the above conditions, we easily obtain a number of coefficients which would vanish with the transfer. Thus for example the correlation r_{hk} of Equation (xxxviii.) is such an expression.* It has the advantage of a symmetrical form, and has a concise physical meaning. It does not, however, become unity when

* In fact (xxxvii.) gives us $\epsilon = \sigma_h\sigma_k r_{hk}$.

either, but not both, b and c vanish, nor does it, unless we multiply it by $\pi/2$ and take its sine, equal the coefficient of correlation when $a = d$ and $b = c$.

Again, we might deduce a fairly simple approximation to the coefficient of correlation from the Equation (xxiv.) for θ, using only its first few terms. Thus we find

$$\text{Sin } 2\pi \frac{ad - bc}{\mathrm{N}^2\{1 - \frac{1}{2}(\chi_1^2 + \chi_2^2)\} + \pi\chi_1\chi_2(ad - bc)} \quad . \quad . \quad . \quad . \quad . \quad \text{(liv.)},$$

where

$$\chi_1 = \sqrt{\frac{\pi}{2}}\frac{(a + c) - (b + d)}{\mathrm{N}},$$

$$\chi_2 = \sqrt{\frac{\pi}{2}}\frac{(a + b) - (c + d)}{\mathrm{N}},$$

as an expression which vanishes with the transfer, and will be fairly close to the coefficient of correlation. It is not, however, exactly unity when either b or c is zero. But without entering into a discussion of such expressions, we can write several down which fully satisfy the three conditions :—

(i.) Vanishing with the transfer.

(ii.) Being equal to unity if b or $c = 0$.

(iii.) Being equal to the correlation for median divisions.

Such are, for example :—

$$\mathrm{Q}_3 = \sin\frac{\pi}{2}\frac{\sqrt{ad} - \sqrt{bc}}{\sqrt{ad} - \sqrt{bc}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lv.)},$$

$$\mathrm{Q}_4 = \sin\frac{\pi}{2}\frac{1}{1 + \dfrac{2bc \quad \mathrm{N}}{(ad - bc)(b + c)}}, \; ad > bc \quad . \quad . \quad . \quad . \quad \text{(lvi.)},$$

$$\mathrm{Q}_5 = \sin\frac{\pi}{2}\frac{1}{\sqrt{1 + \kappa^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lvii.)},$$

where

$$\kappa^2 = \frac{4abcd\,\mathrm{N}^2}{(ad - bc)^2(a + d)(b + c)}.$$

Only by actual examination of the numerical results has it seemed possible to pick out the most efficient of these coefficients. Q_1 was found of little service. The following table gives the values of Q_2, Q_3, Q_4, and Q_5 in the case of fifteen series selected to cover a fairly wide range of values :—

No.	r.	h.	k.	Q_2.	Q_3.	Q_4.	Q_5.
1	·5939 ± ·0247	− ·0873	− ·4163	·7067	·6054	·6168	·6100
2	·5557 ± ·0261	− ·4189	− ·4163	·6688	·5657	·5405	·5570
3	·5529 ± ·0247	− ·0873	− ·0012	·6828	·5809	·5699	·5813
4	·5264 ± ·0264	+ ·2743	+ ·3537	·6345	·5331	·5200	·5283
5	·5213 ± ·0294	+ ·6413	+ ·6966	·6530	·5511	·4878	·5160
6	·5524 ± ·0307	+ 1·0234	+ ·3537	·7130	·6118	·6169	·6138
7	·5422 ± ·0288	+ ·6463	+ ·5828	·6693	·5673	·5136	·5452
8	·2222 ± ·0162	+ ·3190	+ ·3190	·2840	·2268	·2164	·2251
9	·3180 ± ·0361	+ ·1381	+ ·0696	·3959	·3185	·3176	·3183
10	·5954 ± ·0272	+ 1·5114	+ ·7414	·7860	·7100	·6099	·6803
11	·4708 ± ·0292	+ ·0865	− ·0054	·5692	·4712	·4720	·4715
12	·2335 ± ·0335	+ ·0405	+ ·0054	·2996	·2385	·2385	·2385
13	·2451 ± ·0205	+ ·2707	+ ·0873	·3103	·2473	·2456	·2470
14	·1002 ± ·0394	+ ·4557	+ ·1758	·1311	·1032	·0993	·1029
15	·6928 ± ·0164	+ ·5814	+ ·5814	·8032	·7108	·6699	·6897

Now an examination of this table shows that notwithstanding the extreme elegance and simplicity of Mr. YULE's coefficient of association Q_2, the coefficients Q_3, Q_4, and Q_5, which satisfy also his requirements, are much nearer to the values assumed by the correlation. I take this to be such great gain that it more than counterbalances the somewhat greater labour of calculation. If we except cases (6) and (10), in which h or k take a large value exceeding unity, we find that Q_3, Q_4, and Q_5 in the fifteen cases hardly differ by as much as the probable error from the value of the correlation. If we take the mean percentage error of the difference between the correlation and these coefficients, we find

Mean difference of $Q_2 = 24·38$ per cent.
" " $Q_3 = 3·95$ "
" " $Q_4 = 2·94$ "
" " $Q_5 = 2·72$ "

Thus although there is not much to choose between Q_4 and Q_5, we can take Q_5 as a good measure of the degree of independent variation.

The reader may ask: Why is it needful to seek for such a measure? Why cannot we always use the correlation as determined by the method of this paper? The answer is twofold. We want first to save the labour of calculating r for cases where the data are comparatively poor, and so reaching a fairly approximate result rapidly. But labour-saving is never a wholly satisfactory excuse for adopting an inferior method. The second and chief reason for seeking such a coefficient as Q lies in the fact that all our reasoning in this paper is based upon the normality of the frequency. We require to free ourselves from this assumption if possible, for the difficulty, as is exemplified in Illustration V. below, is to find material which actually obeys within the probable errors any such law. Now, by considering the coefficient of regression, $r\sigma_1/\sigma_2 = S(xy)/(N\sigma_1\sigma_2)$, as the slope of the line which best fits the series of points determined as the means of arrays of x for given values of y, we have once and for all freed ourselves from the difficulties attendant upon assuming normal frequency. We become indifferent to the deviations from that law, merely observing how closely or not our means of arrays fall on a line. When we are not given arrays but gross grouping under certain divisions, we have seen that the "transfer" is also a physical quantity of a significance independent of normality. We want accordingly to take a function which vanishes with the transfer, and does not diverge widely from the correlation in cases that we can test. Here the correlation is not taken as something peculiar to normal distributions, but something significant for all distributions whatever. Such a function of a suitable kind appears to be given by Q_5.

§ 6. *On the "Excess" and its Relation to Correlation and Relative Variability.*

There is another method of dealing with the correlation of characters for which we cannot directly discover a quantitative scale which deserves consideration. It is capable of fairly wide application, but, unlike the methods previously discussed, it requires the data to be collected in a special manner. It has the advantage of not applying only to the normal surface of frequency, but to any surface which can be converted into a surface of revolution by a slide and two stretches.

It is well known that not only the normal curve but the normal surface has a type form from which all others can be deduced by stretching or stretching and sliding. Thus in 1895 the Cambridge Instrument Company made for the instrument room at University College, London, a "biprojector," an instrument for giving arbitrary stretches in two directions at right angles to any curve. In this manner by the use of type-templates we were able to draw a variety of curves with arbitrary parameters, *e.g.*, all ellipses from one circle, parabolas from one parabola, normal curves from one normal curve template. Somewhat later Mr. G. U. YULE commenced a model of a normal frequency surface on the BRILL system of interlaced curves. This, by the variable amount of slide given to its two rectangular systems of normal curves, illustrated the changes from zero to perfect correlation. This model was exhibited at a College *soirée* in June, 1897. Geometrically this property has been taken by Mr. W. F. SHEPPARD as the basis of his valuable paper on correlation in the 'Phil. Trans.,' A, vol. 192, pp. 101–167. It is a slight addition to, and modification of, his results that I propose to consider in this section.

The equation to the normal frequency surface is, as we have seen in § 1,

$$z = \frac{N}{2\pi\sigma_1\sigma_1\sqrt{1-r^2}} \text{ expt. } \left\{-\tfrac{1}{2}\left(\frac{x^2}{\sigma_1{}^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2{}^2}\right)\frac{1}{1-r^2}\right\}$$

Now write $x/(\sigma_1\sqrt{1-r^2}) = x'$, $y/\sigma_2 = y'$. This is merely giving the surface two uniform stretches (or squeezes) parallel to the coordinate axes. We have for the frequency of pairs lying between x, $x + \delta x$, and y, $\delta + \delta y$,

$$z\delta x\delta y = \frac{\mathrm{N}}{2\pi}\,\delta x'\delta y' \text{ expt.}\left\{-\tfrac{1}{2}\left(\left(x' - \frac{ry'}{\sqrt{1-r^2}}\right)^2 + y'^2\right)\right\}.$$

Now give the area a uniform slide parallel to the axis of x defined by $r/\sqrt{1-r^2}$ at unit distance from that axis. This will not change the basal unit of area $\delta\alpha = \delta x'\delta y'$, and analytically we may write

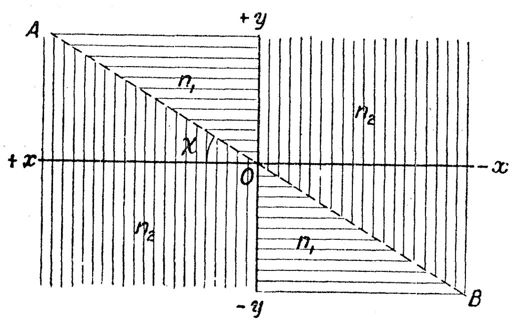
$$\mathrm{X} = x' - y'r/\sqrt{1-r^2}, \qquad \mathrm{Y} = y', \qquad \mathrm{R}^2 = \mathrm{X}^2 + \mathrm{Y}^2.$$

Whence we find
$$z\delta x\delta y = \frac{\mathrm{N}}{2\pi}\,\delta\alpha \text{ expt.}\left(-\tfrac{1}{2}\mathrm{R}^2\right).$$

This is the mechanical changing of the YULE-BRILL model analytically represented. The surface is now one of revolution, and the proof would have been precisely the same if we had written in the above results any function f, instead of the exponential.* It is easy to see that any volume cut off by two planes through the axis of the surface is to the whole volume as the angle between the two planes is to four right angles. Further the corresponding volumes of this surface and the original surface are to each other as unity to the product of the two stretches. Lastly, any plane through the z-axis of the original solid remains a plane through the z-axis after the two stretches and the slide. These points have all been dealt with by Mr. SHEPPARD (p. 101 *et seq.*, *loc. cit.*). I will here adopt his notation $r = \cos \mathrm{D}$, and term with him D the *divergence*. Thus cot D is (in the language of the theory of strain) the slide, and D is the angle between the strained positions of the original x and y directions. Now consider any plane which makes an angle χ with the plane of xz before strain. Then, since the contour lines of the correlation surface are ellipses, the volumes of the surface upon the like shaded opposite angles of the plan diagram below will be equal; and if they be n_1 and n_2, then $n_1 + n_2 = \frac{1}{2}\mathrm{N}$. If n_1' and n_2' be the volumes after strain, then by what precedes we shall have

$$n_1 = \sigma_1\sigma_2\sqrt{1-r^2} \times n_1', \qquad n_2 = \sigma_1\sigma_2\sqrt{1-r^2} \times n_2',$$

and
$$(n_2 - n_1)/(n_1 + n_2) = (n_2' - n_1')/(n_1' + n_2').$$



* The generalisation is not so great as might at first appear, for I have convinced myself that this property of conversion into a surface of revolution by stretches and slides does not hold for actual cases of markedly skew correlation.

Now n_1' and n_2' will be as the angles between the strained positions of the planes bounding n_1 and n_2. Ox does not change its direction. Oy is turned through an angle $\pi/2 - \mathrm{D}$ clockwise, and χ becomes χ'', say. Hence

$$n_1' : n_2' :: \frac{\pi}{2} - \chi'' + \frac{\pi}{2} - \mathrm{D} : \frac{\pi}{2} + \chi'' - \frac{\pi}{2} + \mathrm{D}.$$

or

$$(n_2' - n_1')/(n_2' + n_1') = \frac{2}{\pi}(\chi'' + \mathrm{D}) - 1.$$

Let us write $\mathrm{E}_1 = 2(n_2 - n_1)$ and term it the *excess* for the y-character for the line AB. Then we easily find:

$$\tan\left(\frac{\mathrm{E}_1}{\mathrm{N}}\frac{\pi}{2} + \frac{\pi}{2}\right) = \tan(\chi'' + \mathrm{D}) = \frac{\cot\chi'' + \cot\mathrm{D}}{\cot\chi''\cot\mathrm{D} - 1} \quad . \quad . \quad . \quad . \quad \text{(lviii.)}.$$

It remains to determine $\tan\chi''$ and substitute. The stretches alter $\tan\chi$ into $\tan\chi'$, such that

$$\tan\chi' = \frac{\sigma_1\sqrt{1-r^2}}{\sigma_2}\tan\chi.$$

Further, by the slide

$$\cot\chi'' = \cot\chi' - \cot\mathrm{D} = \frac{\sigma_2}{\sigma_1\sqrt{1-r^2}}\cot\chi - \cot\mathrm{D}.$$

Hence we have by (lviii.) above

$$-\cot\left(\frac{\mathrm{E}_1}{\mathrm{N}}\frac{\pi}{2}\right) = \frac{\sigma_2}{\sigma_1\sqrt{1-r^2}}\cot\chi \Big/ \left(\frac{\sigma_2}{\sigma_1\sqrt{1-r^2}}\cot\chi\cot\mathrm{D} - \cot^2\mathrm{D} - 1\right),$$

or,

$$-\tan\left(\frac{\mathrm{E}_1}{\mathrm{N}}\frac{\pi}{2}\right) = \cot\mathrm{D} - \frac{\sigma_1}{\sigma_2}\frac{\tan\chi}{\sin\mathrm{D}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lix.)}.$$

Now the excess E_1 is the difference of the frequencies in the sum of the strips of the volume made by planes parallel to the plane yz on the two sides of the plane ABz (defined by χ), taken without regard to sign. For on one side of the mean yy this is $n_2 - n_1$, and on the other $-(n_1 - n_2)$. Hence we have this definition of E_1, the *column excess* for any line through the mean of a correlation table: *Add up the frequencies above and below the line in each column and take their differences without regard to sign, and their sum is the column excess.*

If we are dealing with an actual correlation table and not with a method of collecting statistics, then care must be taken to properly proportion the frequencies in the column in which the mean occurs, and also in the groups which are crossed by the line. It is the difficulty of doing this satisfactorily, especially if the grouping, as in eye and coat colour, is large and somewhat rough, that hinders the effective use of the method, if the statistics have not been collected *ad hoc*.

Now let E_2 be the *row excess* for the line AB, defined in like manner, then we have in the same way

$$-\tan\left(\frac{E_2}{N}\frac{\pi}{2}\right) = \cot D - \frac{\sigma_2}{\sigma_1}\frac{\cot\chi}{\sin D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lix.}^{\text{bis}}).$$

Now eliminate σ_2/σ_1 between (lix.) and (lix.$^{\text{bis}}$); then

$$\left(\tan\left(\frac{E_1}{N}\frac{\pi}{2}\right) + \cot D\right)\left(\tan\left(\frac{E_2}{N}\frac{\pi}{2}\right) + \cot D\right) = \frac{1}{\sin^2 D}.$$

Whence we deduce

$$\cot D = \cot\frac{E_1 + E_2}{N}\frac{\pi}{2},$$

and, therefore,

$$r = \cos D = \cos\frac{E_1 + E_2}{N}\frac{\pi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lx.}).$$

Substituting for D in (lix.) we find further

$$\frac{\sigma_1}{\sigma_2} = \cot\chi\cos\left(\frac{E_2}{N}\frac{\pi}{2}\right)\bigg/\cos\left(\frac{E_1}{N}\frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxi.}).$$

Thus Equations (lx.) and (lxi.) give the coefficient of correlation and the relative variability of the two characters. The latter is, I believe, quite new, the former novel in form.

If we call m_1 the frequency in the angle χ (AOx of the figure above), then it is easy to see that $E_1 = 2(n_2 - n_1) = N - 4n_1$, and similarly $E_2 = N - 4m_1$. Thus $(E_1 + E_2)/N = 2(N - 2(n_1 + m_1))/N$. But $n_1 + m_1$ is the frequency in the first quadrant. This Mr. SHEPPARD terms P, while that in the second he terms R. We have thus $(E_1 + E_2)/N = 2R/(R + P)$, or

$$r = \cos\frac{R}{R + P}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxii.}),$$

i.e., Mr. SHEPPARD's fundamental result* ('Phil. Trans.,' A, vol. 192, p. 141).

We can, of course, get Mr. SHEPPARD's result directly if we put $\chi = 0$, when we have at once $E_1 = 2(R - P)$, $E_2 = N = 2(R + P)$, and the result follows.

Equation (lxi.) may also be written in the form

$$\frac{\sigma_1}{\sigma_2} = \cot\chi\sin\left(\frac{m_1}{N}2\pi\right)\bigg/\sin\left(\frac{n_1}{N}2\pi\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxiii.}).$$

If we put $\chi = 0$, then m_1 becomes zero, and the right-hand side of (lxiii.) is indeterminate. If we proceed, however, to the limit by evaluating the frequency in an indefinitely thin wedge of angle χ, we reach merely the identity $\sigma_1/\sigma_2 = \sigma_1/\sigma_2$. Hence there is no result corresponding to (lxi.) to be obtained by taking Mr. SHEPPARD's case of $\chi = 0$.

The following are the values of the probable errors of the quantities involved:—

* In the actual classification of data (lx.) and (lxii.) suggest quite different processes. We can apply (lx.) where (lxii.) is difficult or impossible, *e.g.*, correlation in shading of birds' eggs from the same clutch.

$$\text{Probable error of } E_1 = \cdot 67449\sqrt{N(1 - E_1^2/N^2)} \quad . \quad . \quad . \quad . \quad . \quad \text{(lxiv.).}$$

$$\text{,, \quad ,, \quad } E_2 = \cdot 67449\sqrt{N(1 - E_2^2/N^2)} \quad . \quad . \quad . \quad . \quad . \quad \text{(lxv.).}$$

$$\text{Correlation between errors in } E_1 \text{ and } E_2 = -\sqrt{\frac{(1 - E_1/N)(1 - E_2/N)}{(1 + E_1/N)(1 + E_2/N)}} \quad . \quad . \quad \text{(lxvi.).}$$

$$\text{Probable error in } r = \frac{\cdot 67449 \sin D\sqrt{D(\pi - D)}}{\sqrt{N}} \quad . \quad . \quad . \quad . \quad . \quad \text{(lxvii.),}$$

where $D = \frac{E_1 + E_2}{N}\frac{\pi}{2}$ (*cf.* SHEPPARD, *loc. cit.*, p. 148).

Probable error in ratio $\sigma_1/\sigma_2 =$

$$\frac{\cdot 67449}{\sqrt{N}}\frac{\sigma_1}{\sigma_2}\frac{\pi}{2}\left\{\left(1 - \frac{E_1^2}{N^2}\right)\tan^2\left(\frac{E_1}{N}\frac{\pi}{2}\right) + \left(1 - \frac{E_2^2}{N_2}\right)\tan^2\left(\frac{E_2}{N}\frac{\pi}{2}\right)\right.$$
$$\left. + 2\left(1 - \frac{E_1}{N}\right)\left(1 - \frac{E_2}{N}\right)\tan\left(\frac{E_1}{N}\frac{\pi}{2}\right)\tan\left(\frac{E_2}{N}\frac{\pi}{2}\right)\right\}^{\frac{1}{2}} \quad . \quad . \quad \text{(lxviii.).}$$

The application of the method here discussed to statistics without quantitative scale can now be indicated. If the characters we are dealing with have the same scale, although it be unknown, then, if the quantitative *order* be maintained, *i.e.*, individuals arranged in order of lightness or darkness of coat or eye-colour, the diagonal line on the table at 45° will remain unchanged, however we may suppose parts of the scale to be distorted, for the distortion will be the same at corresponding points of both axes. Further, if we suppose the mean of the two characters to be the same, this 45° line will pass through that mean, and will serve for the line AB of the above investigation. In this case we must take $\tan\chi = 1$, and consequently (lxi.) becomes

$$\sigma_1/\sigma_2 = \cos\left(\frac{E_2}{N}\frac{\pi}{2}\right)\bigg/\cos\left(\frac{E_1}{N}\frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxix.).}$$

We can even, when the mean is a considerable way off the 45° line, get, in some cases, good results. Thus, the correlation in stature of husband and wife worked out by the ordinary product moment process is ·2872. But in this case $E_1 = 382{\cdot}062$ $E_2 = 806{\cdot}425$, and this gives the correlation ·2994. On the other hand, the actual ratio of variabilities is 1·12, while (lxix.) makes it 2·76! This arises from the fact that the errors in E_1 and E_2, due to the mean being off the 45° line, tend to cancel in $E_1 + E_2$, but tend in directly opposite directions in the ratio of the cosines. Similarly the correlation between father and son works out ·5666, which may be compared with the values given in Illustration V. below, ranging from ·5198 to ·5939. Again, correlation in eye-colour between husband and wife came out by the excess process ·0986, and by the process given earlier in the present Memoir ·1002. But all these are favourable examples, and many others gave much worse results. We ought really only to apply it to find σ_1/σ_2 when the means are on the 45° line, as in the correlation of the same character in brethren, and even in this case the statistics ought to be collected *ad hoc, i.e.*, we ought to make a very full quantitative order, and then notice for each individual case the number above and below the type. For example, suppose we had a diagram of some twenty-five to thirty eye tints in order (*e.g.*, like BERTRAND's), then we take any individual, note his tint, and observe how many relatives of a particular class—brethren or cousins, say—have lighter and how many darker eyes; the difference of the two would be the excess for this individual. The same plan would be possible with horses' coat-colour and other characters. After trying the plan of the excesses on the data at my disposal for horses' coat-colour and human eye-colour (which were not collected *ad hoc*), I abandoned it for the earlier method of this Memoir; for, the classification being in large groups, the proportioning of the excess (as well as the differences in the means) introduced too great errors for such investigations.

§ 7. *On a Generalisation of the Fundamental Theorem of the Present Memoir.*

If we measure deviations in units of standard deviations, we may take for the equation to the correlation surface for n variables

$$z = \frac{N}{(2\pi)^{\frac{1}{2}n}\sqrt{R}} e^{-\frac{1}{2}\left\{S_1\left(\frac{R_{ss}}{R}x_s^2\right) + 2S_2\left(\frac{R_{ss'}x_s x_{s'}}{R}\right)\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxx.)},$$

where

$$R = \begin{vmatrix} 1 & r_{12} & r_{13} \dots\dots\dots\dots & r_{1n} \\ r_{21} & 1 & r_{23} \dots\dots\dots\dots & r_{2n} \\ r_{31} & r_{32} & 1 \dots\dots\dots\dots & r_{3n} \\ \dots & \dots & \dots\dots\dots\dots & \dots \\ \dots & \dots & \dots\dots\dots\dots & \dots \\ \dots & \dots & \dots\dots\dots\dots & \dots \\ \dots & \dots & \dots\dots\dots\dots & \dots \\ r_{n-1,1} & r_{n-1,2} & r_{n-1,3} \dots 1 & r_{n-1,n} \\ r_{n,1} & r_{n,2} & r_{n,3} \dots r_{n-1,n} & 1 \end{vmatrix}$$

and R_{pq} is the minor obtained by striking out the pth row and qth column. r_{pq} is, of course, the correlation between the pth and qth variables, and equals r_{qp}. S_1 denotes a summation for s from 1 to n, and S_2 a summation of every possible pair out of the n quantities 1 to n.

Now take the logarithmic differential of z with regard to r_{pq}. We find

$$\frac{1}{z}\frac{dz}{dr_{pq}} = -\frac{1}{2R}\frac{dR}{dr_{pq}} - \tfrac{1}{2}S_1\left\{\frac{d}{dr_{pq}}\left(\frac{R_{ss}}{R}\right)x_s^2\right\} - S_2\left\{\frac{d}{dr_{pq}}\left(\frac{R_{ss'}}{R}\right)x_s x_{s'}\right\}$$

$$= -\frac{R_{pq}}{R} + S_1\left(\frac{R_{ps}\,R_{qs}}{R^2}x_s^2\right) + S_2\left(\frac{R_{ps}\,R_{qs'} + R_{ps'}\,R_{qs}}{R^2}x_s x_{s'}\right)$$

For

$$dR/dr_{pq} = 2R_{pq}$$

and, generally, whether s is or is not $= s'$, or these are or are not $= p$ and q, we have

$$\frac{d}{dr_{pq}}\left(\frac{R_{ss'}}{R}\right) = -\frac{R_{ps}R_{qs'} + R_{ps'}R_{qs}}{R^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \text{(lxxi.)}.$$

This follows thus:

$$\frac{d}{dr_{pq}}\left(\frac{R_{ss'}}{R}\right) = \frac{1}{R}\frac{dR_{ss'}}{dr_{pq}} - \frac{R_{ss'}}{R^2}\frac{dR}{dr_{pq}} = \frac{1}{R}\frac{dR_{ss'}}{dr_{pq}} - \frac{2R_{ss'}R_{pq}}{R^2},$$

or we have to show that

$$\begin{aligned}\frac{dR_{ss'}}{dr_{pq}} &= \frac{2R_{ss'}R_{pq} - R_{ps}R_{qs'} - R_{ps'}R_{qs}}{R} \\ &= \frac{R_{ss'}R_{pq} - R_{ps}R_{qs'}}{R} + \frac{R_{ss'}R_{pq} - R_{ps'}R_{qs}}{R} \\ &= {}_{pq}R_{ss'} + {}_{qp}R_{ss'}\end{aligned}$$

where ${}_{pq}R_{ss'}$ is the minor corresponding to the term r_{pq} in $R_{ss'}$, and ${}_{qp}R_{ss'}$ the minor corresponding to the term r_{qp}.* But this last result is obvious because $R_{ss'}$ only contains r_{pq} in two places, *i.e.*, as r_{pq} and r_{qp}.

Putting $s = s'$, we have the other identity required above, *i.e.*,

$$\frac{d}{dr_{pq}}\left(\frac{R_{ss}}{R}\right) = -\frac{2R_{ps}R_{qs}}{R^2}.$$

Returning now to the value for $\frac{1}{z}\frac{dz}{dr_{pq}}$ on the previous page, we see that the two sum terms may be expressed as a product, or we may put

$$\frac{1}{z}\frac{dz}{dr_{pq}} = -\frac{R_{pq}}{R} + S_1\left(\frac{R_{ps}}{R}x_s\right) \times S_1\left(\frac{R_{qs}}{R}x_s\right).$$

Now write

$$z = \frac{N}{(2\pi)^{\frac{1}{2}n}\sqrt{R}}e^{-\phi}.$$

Then $$\frac{d\phi}{dx_p} = S_1\left(\frac{R_{ps}}{R}x_s\right), \quad \frac{d\phi}{dx_q} = S_1\left(\frac{R_{qs}}{R}x_s\right) \quad \text{and} \quad \frac{d^2\phi}{dx_p\,dx_q} = \frac{R_{pq}}{R}.$$

Hence $$\frac{1}{z}\frac{dz}{dr_{pq}} = -\frac{d^2\phi}{dx_p\,dx_q} + \frac{d\phi}{dx_p}\frac{d\phi}{dx_q}.$$

Now differentiate $\log z$ with regard to x_p. Then

$$\frac{dz}{dx_p} = -z\frac{d\phi}{dx_p}$$

* See also SCOTT, 'Theory of Determinants,' p. 59.

$$\frac{d^2z}{dx_p dx_q} = -z\frac{d^2\phi}{dx_p dx_q} - \frac{dz}{dx_q}\frac{d\phi}{dx_p}$$

$$\frac{1}{z}\frac{d^2z}{dx_p dx_q} = -\frac{d^2\phi}{dx_p dx_q} + \frac{d\phi}{dx_p}\frac{d\phi}{dx_q}$$

Thus finally

$$\frac{dz}{dr_{pq}} = \frac{d^2z}{dx_p dx_q} \quad \text{(lxxii.)}.$$

In other words, the operator d/dr_{pq} acting on z can always be replaced by the operator $d^2/dx_p dx_q$.

Let $d/d\rho_{pq}$ denote the effect of applying the operator d/dr_{pq} to z, and putting r_{pq} zero after *all* differentiations have been performed, then the effect of this operator will be the same as if we used $d^2/dx_p dx_q$ on z, putting r_{pq} zero *before* differentiation. Generally, let F be any series of operations like d/dr_{pq}, then we see that

$$\mathrm{F}\left(\frac{d}{dr_{pq}}, \frac{d}{dr_{p'q'}}, \frac{d}{dr_{p''q''}} \cdots\cdots\right) z$$
$$= \mathrm{F}\left(\frac{d^2}{dx_p dx_q}, \frac{d^2}{dx_{p'} dx_{q'}}, \frac{d^2}{dx_{p''} dx_{q''}}, \cdots\cdots\right) \frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}n}} e^{-\frac{1}{2}\mathrm{S}_1(x_s^2)}.$$

Now let F be the function which gives the operation of expanding z by MACLAURIN'S theorem in powers of the correlation coefficients, *i.e.*,

$$\mathrm{F} = e^{\mathrm{S}_2\left(r_{ss'}\frac{d}{d\rho_{ss'}}\right)},$$

then

$$z = e^{\mathrm{S}_2\left(r_{ss'}\frac{d}{d\rho_{ss'}}\right)} z = \frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}n}} e^{\mathrm{S}_2\left(r_{ss'}\frac{d}{dx_s dx_{s'}}\right)} e^{-\frac{1}{2}\mathrm{S}_1(x_s^2)}.$$

This is the generalised form of result (xiv.) reached above.

Now let

$$z_0 = \frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}n}} e^{-\frac{1}{2}\mathrm{S}_1(x_s^2)},$$

then z_0 is the ordinate of a frequency surface of the nth order, in which the distribution of the n variables is absolutely independent. We have accordingly the extremely interesting geometrical interpretation that the operator

$$e^{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)},$$

applied to a surface of frequency for n independent variables converts it into a surface of frequency for n dependent variables, the correlation between the sth and s'th variables being $r_{ss'}$.*

* I should like to suggest to the pure mathematician the interest which a study of such operators would have, and in particular of the generalised form of projection in hyperspace indicated by them.

Expanding, we have

$$z = z_0 + \mathrm{S}_2\left(r_{rs'} \frac{d^2}{dx_s dx_{s'}}\right) z_0 + \frac{1}{\underline{|2}} \left\{\mathrm{S}_2\left(r_{ss'} \frac{d^2}{dx_s dx_{s'}}\right)\right\}^2 z_0$$
$$+ \ldots + \frac{1}{\underline{|m}} \left\{\mathrm{S}_2\left(r_{ss'} \frac{d^2}{dx_s dx_{s'}}\right)\right\}^m z_0 + \ldots \quad \text{(lxxiii.)}.$$

Our next stage is to evaluate the operation

$$\mathrm{S}_2\left(r_{ss'} \frac{d^2}{dx_s dx_{s'}}\right)^m z_0.$$

Let us put

$$_sv_1 = x_s, \qquad _sv_2 = x_s^2 - 1, \qquad _sv_3 = x_s(x_s^2 - 3),$$

and $_sv_p$ = the pth function of x_s as defined by (xv.).

Let ϵ_s be a symbol such that ϵ_s^p represents $_sv_p$. Then we shall show that

$$\mathrm{S}_2\left(r_{ss'} \frac{d^2}{dx_s dx_{s'}}\right)^m z_0 = z_0 \left\{\mathrm{S}_r\left(r_{ss'}\epsilon_s\epsilon_{s'}\right)\right\}^m \quad \ldots \quad \text{(lxxiv.)}.$$

We shall prove this by induction.

By (xii.)

$$_sv_{m+1} = x_s \, _sv_m - m \, _sv_{m-1},$$

or

$$\epsilon_s^{m+1} = x_s \, \epsilon_s^m - m \, \epsilon_s^{m-1},$$

and by (xvi.)

$$\frac{d_sv_m}{dx_s} = m \, _sv_{m-1}, \quad \text{or} \quad \frac{d\epsilon_s^m}{dx_s} = m \, \epsilon_s^{m-1}.$$

Now, let $\chi(\epsilon_s)$ be any function of ϵ_s

$$= \mathrm{S}(\mathrm{A}_q\epsilon_s^q),$$

if we suppose it can be expanded in powers of ϵ_s.

Then

$$\begin{aligned}
\frac{d}{dx_s}\chi(\epsilon_s) &= \mathrm{S}\left(\mathrm{A}_q \frac{d}{dx_s}(\epsilon_s^q)\right) \\
&= \mathrm{S}(\mathrm{A}_q \, q\epsilon_s^{q-1}) \\
&= \mathrm{S}(\mathrm{A}_q(x_s \, \epsilon_s^q - \epsilon_s^{q+1})) \\
&= x_s\mathrm{S}(\mathrm{A}_q \, \epsilon_s^q) - \epsilon_s \, \mathrm{S}(\mathrm{A}_q \, \epsilon_s^q) \\
&= (x_s - \epsilon_s)\,\chi(\epsilon_s) \quad \ldots \quad \text{(lxxv.)}.
\end{aligned}$$

Similarly $$\frac{d^2}{dx_s dx_{s'}}\chi(\epsilon_s, \epsilon_{s'}) = (x_s - \epsilon_s)(x_{s'} - \epsilon_{s'})\,\chi(\epsilon_s, \epsilon_{s'}) \quad \ldots \quad \text{(lxxvi.)}.$$

Now suppose that

$$\left\{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\} z_0 = z_0\{\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^m,$$

then

$$\left\{S_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^{m+1} z_0 = S_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right) z_0 U,$$

where U stands for $\{S_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^m$.

Hence, remembering that $dz_0/dx_s = -z_0 x_{s'}$,

$$\begin{aligned}\left\{S_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^{m+1} z_0 &= z_0 S_2(r_{ss'}x_s x_{s'})U + z_0 S_2\left(r_{ss'}\frac{d^2 U}{dx_s dx_{s'}}\right) \\ &\quad - z_0 S_2\left(r_{ss'}\left(x_s \frac{dU}{dx_{s'}} + x_{s'}\frac{dU}{dx_s}\right)\right) \\ &= z_0 S_2(r_{ss'}x_s x_{s'})U + z_0 S_2\{r_{ss'}(x_s - \epsilon_s)(x_{s'} - \epsilon_{s'})\}U \\ &\quad - z_0 S_2\{r_{ss'}\{x_s(x_{s'} - \epsilon_{s'}) + x_{s'}(x_s - \epsilon_s)\}U \\ &= z_0 S_2(r_{ss'}\epsilon_s\epsilon_{s'})U \\ &= z_0\{S_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^{m+1},\end{aligned}$$

which had to be proved.

But it is easy to show by simple differentiation that

$$S_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right) z_0 = z_0\, S_2(r_{ss'}\, {}_s v_1\, {}_{s'}v_1) = z_0\, S_2(r_{ss'}\epsilon_s\epsilon_{s'})$$

$$\begin{aligned}\left\{S_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^2 z_0 &= z_0\, S_2(r^2_{ss'}\, {}_s v_2\, {}_{s'}v_2 + 2r_{pq}r_{ss'}\, {}_p v_1\, {}_q v_1\, {}_s v_1\, {}_{s'}v_1 + 2r_{ps}r_{ss'}\, {}_{s'}v_2\, {}_p v_1\, {}_{s'}v_1) \\ &= z_0\, \{S_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^2 \quad \text{(lxxvii.)}.\end{aligned}$$

Hence the theorem is generally true.

Thus we conclude that

$$z = z_0\left[1 + S_2(r_{ss'}\epsilon_s\epsilon_{s'}) + \frac{1}{\underline{|2}}\left\{S_2(r_{ss'}\epsilon_s\epsilon_{s'})\right\}^2 + \ldots + \frac{1}{\underline{|m}}\left\{S_2(r_{ss'}\epsilon_s\epsilon_{s'})\right\}^m + \ldots\right] \quad \text{(lxxviii.)}.$$

It is quite straightforward, if laborious, to write down the expansion for any number of variables.

Now let Q be the total frequency of complices of variables with x_1 lying between h_1 and ∞, x_2 between h_2 and ∞, . . . x_s between h_s and ∞, . . . x_n between h_n and ∞; and let Q_0 be the frequency of such complices if there were no correlations. Then

$$Q = \int_{h_1}^{\infty}\int_{h_2}^{\infty} \ldots \int_{h_s}^{\infty} \ldots \int_{h_n}^{\infty} z\, dx_1\, dx_2 \ldots dx_s \ldots dx_n$$

$$Q_0 = \int_{h_1}^{\infty}\int_{h_2}^{\infty} \ldots \int_{h_s}^{\infty} \ldots \int_{h_n}^{\infty} z_0\, dx_1\, dx_2 \ldots dx_s \ldots dx_n$$

Now let

$$\beta_s H_s = \frac{1}{\sqrt{2\pi}}\int_{h_s}^{\infty} e^{-\frac{1}{2}x_s^2} dx_s$$

where

$$H_s = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxix.)}.$$

We have $\quad Q_0 = N\beta_1\beta_2 \, . \, . \, \beta_s \, . \, . \, \beta_n \, H_1 \, H_2 \, . \, . \, H_s \, . \, . \, H_n.$

But by (xviii.)

$$\frac{1}{\sqrt{2\pi}} \int_{h_s}^{\infty} {}_s v_p \, e^{-\frac{1}{2}x_s^2} \, dx_s = H_s \, {}_s\bar{v}_{p-1} = H_s\beta_s \frac{{}_s\bar{v}_{p-1}}{\beta_s},$$

where

$${}_s\bar{v}_{p-1} = \left[{}_s v_{p-1}\right]_{x_s = h_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxx.)},$$

and as above,

$$\beta_s = \int_{h_s}^{\infty} e^{-\frac{1}{2}x_s^2} dx_s \Big/ e^{-\frac{1}{2}h_s^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxxi.)}.$$

Thus

$$\frac{1}{(\sqrt{2\pi})^n} \int_{h_1}^{\infty} \int_{h_2}^{\infty} \cdots \int_{h_s}^{\infty} \cdots \int_{h_n}^{\infty} {}_s v_p \, {}_{s'} v_{p'} \, {}_{s''} v_{p''} \cdots e^{-\frac{1}{2}(x_1^2 + x_2^2 + \, . \, . \, . \, + x_s^2 + \, . \, . \, . \, + x_n^2)} \, dx_1 dx_2 \ldots dx_s \ldots dx_n$$

$$= H_1 H_2 \ldots H_s \ldots H_n \beta_1\beta_2 \ldots \beta_s \ldots \beta_n \frac{{}_s\bar{v}_{p-1}}{\beta_s} \frac{{}_{s'}\bar{v}_{p'-1}}{\beta_{s'}} \frac{{}_{s''}\bar{v}_{p''-1}}{\beta_{s''}} \ldots,$$

or

$$\int_{h_1}^{\infty} \int_{h_2}^{\infty} \cdots \int_{h_s}^{\infty} \cdots \int_{h_n}^{\infty} z_0 \Pi({}_s v_p) \, dx_1 dx_2 \ldots dx_s \ldots dx_n = Q_0 \Pi \left(\frac{{}_s\bar{v}_{p-1}}{\beta_s}\right)$$

$$. \quad . \quad . \quad . \quad . \quad \text{(lxxxii.)}.$$

where Π denotes a product of ${}_s v_p$ for any number of v's with any s and p. The rule, therefore, is very simple. We must expand the value of z in v's as given by (lxxviii.) above, then the multiple integral of this will be obtained by lowering every v's right-hand subscript by unity (remembering that ${}_s v_0 = 1$), and further dividing by the β of the left-hand subscript. The general expression up to terms of the fourth order has been written down; it involves thirty-four sums, each represented by a type term All these would only occur in the case of the correlation of eight organs, or when we have to deal with twenty-eight coefficients of correlation. Such a number seems beyond our present power of arithmetical manipulation, so that I have not printed the general expressions. At the same time, the theory of multiple correlation is of such great importance for problems of evolution, in which over and over again we have three or four correlated characters to deal with,* that it seems desirable to place on record the expansion for these cases. I give four variables up to the fourth and three variables up to the fifth order terms. Afterwards I will consider special cases.

* In my memoir on Prehistoric Stature I have dealt with five correlated organs, *i.e.*, ten coefficients. In some barometric investigations now in hand we propose to deal with at least fifteen coefficients, while Mr. BRAMLEY-MOORE, in the correlation of parts of the skeleton, has, in a memoir not yet published, dealt with between forty and fifty cases of four variables or six coefficients.

*Value of the Quadruple Integral in the Case of Four Variables.**

$$
\begin{aligned}
\frac{Q - Q_0}{Q_0} = {} & \frac{r_{12}}{\beta_1\beta_2} + \frac{r_{13}}{\beta_1\beta_3} + \frac{r_{14}}{\beta_1\beta_4} + \frac{r_{23}}{\beta_2\beta_3} + \frac{r_{24}}{\beta_2\beta_4} + \frac{r_{34}}{\beta_3\beta_4} \\
& + \frac{1}{|\underline{2}} \left\{ \frac{r_{12}^2}{\beta_1\beta_2} v_1'v_1'' + \frac{r_{13}^2}{\beta_1\beta_3} v_1'v_1''' + \frac{r_{14}^2}{\beta_1\beta_4} v_1'v_1^{\text{iv}} + \frac{r_{23}^2}{\beta_2\beta_3} v_1''v_1''' + \frac{r_{24}^2}{\beta_2\beta_4} v_1''v_1^{\text{iv}} \right. \\
& + \frac{r_{34}^2}{\beta_3\beta_4} v_1'''v_1^{\text{iv}} + \frac{2r_{12}r_{13}}{\beta_1\beta_2\beta_3} v_1' + \frac{2r_{12}r_{14}}{\beta_1\beta_2\beta_4} v_1' + \frac{2r_{13}r_{14}}{\beta_1\beta_3\beta_4} v_1' \\
& + \frac{2r_{12}r_{23}}{\beta_1\beta_2\beta_3} v_1'' + \frac{2r_{13}r_{23}}{\beta_1\beta_2\beta_3} v_1''' + \frac{2r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} + \frac{2r_{12}r_{34}}{\beta_1\beta_2\beta_3\beta_4} \\
& + \frac{2r_{13}r_{34}}{\beta_1\beta_3\beta_4} v_1''' + \frac{2r_{14}r_{34}}{\beta_1\beta_3\beta_4} v_1^{\text{iv}} + \frac{2r_{23}r_{34}}{\beta_2\beta_3\beta_4} v_1''' \\
& + \frac{2r_{24}r_{34}}{\beta_2\beta_3\beta_4} v_1^{\text{iv}} + \frac{2r_{12}r_{24}}{\beta_1\beta_2\beta_4} v_1'' + \frac{2r_{13}r_{24}}{\beta_1\beta_2\beta_3\beta_4} \\
& \left. + \frac{2r_{14}r_{24}}{\beta_1\beta_2\beta_4} v_1^{\text{iv}} + \frac{2r_{23}r_{24}}{\beta_2\beta_3\beta_4} v_1'' \right\} \\
+ \frac{1}{|\underline{3}} & \left\{ \frac{r_{12}^3}{\beta_1\beta_2} v_2'v_2'' + \frac{r_{13}^3}{\beta_1\beta_3} v_2'v_2''' + \frac{r_{14}^3}{\beta_1\beta_4} v_2'v_2^{\text{iv}} + \frac{r_{23}^3}{\beta_1\beta_3} v_2''v_2''' + \frac{r_{24}^3}{\beta_2\beta_4} v_2''v_2^{\text{iv}} \right. \\
& + \frac{r_{34}^3}{\beta_3\beta_4} v_2'''v_2^{\text{iv}} + \frac{3r_{12}^2 r_{13}}{\beta_1\beta_2\beta_3} v_2'v_1'' + \frac{3r_{12}r_{13}^2}{\beta_1\beta_2\beta_3} v_2'v_1''' + \frac{3r_{12}^2 r_{14}}{\beta_1\beta_2\beta_4} v_2'v_1''' \\
& + \frac{3r_{12}r_{14}^2}{\beta_2\beta_2\beta_4} v_2'v_1^{\text{iv}} + \frac{3r_{13}^2 r_{14}}{\beta_1\beta_2\beta_4} v_2'v_1''' + \frac{3r_{13}r_{14}^2}{\beta_1\beta_3\beta_4} v_2'v_1^{\text{iv}} \\
& + \frac{3r_{12}^2 r_{23}}{\beta_1\beta_2\beta_3} v_2''v_1' + \frac{3r_{12}r_{23}^2}{\beta_1\beta_2\beta_3} v_2''v_1''' + \frac{3r_{13}^2 r_{23}}{\beta_1\beta_2\beta_3} v_2'''v_1' \\
& + \frac{3r_{13}r_{23}^2}{\beta_1\beta_2\beta_3} v_2'''v_1'' + \frac{3r_{14}^2 r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1^{\text{iv}} + \frac{3r_{14}r_{23}^2}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1''' \\
& + \frac{3r_{12}^2 r_{24}}{\beta_1\beta_2\beta_4} v_2''v_1' + \frac{3r_{24}^2 r_{12}}{\beta_1\beta_2\beta_4} v_2''v_1^{\text{iv}} + \frac{3r_{13}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''' \\
& + \frac{3r_{13}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1^{\text{iv}} + \frac{3r_{14}^2 r_{24}}{\beta_1\beta_2\beta_4} v_2^{\text{iv}}v_1' + \frac{3r_{14}r_{24}^2}{\beta_1\beta_2\beta_4} v_2^{\text{iv}}v_1'' \\
& + \frac{3r_{23}^2 r_{24}}{\beta_2\beta_3\beta_4} v_2''v_1''' + \frac{3r_{23}r_{24}^2}{\beta_2\beta_3\beta_4} v_2''v_1^{\text{iv}} + \frac{3r_{34}r_{12}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1'' \\
& + \frac{3r_{34}^2 r_{12}}{\beta_1\beta_2\beta_3\beta_4} v_1'''v_1^{\text{iv}} + \frac{3r_{13}^2 r_{34}}{\beta_1\beta_2\beta_4} v_2'''v_1' + \frac{3r_{13}r_{34}^2}{\beta_1\beta_3\beta_4} v_2'''v_1^{\text{iv}} \\
& + \frac{3r_{14}^2 r_{34}}{\beta_1\beta_3\beta_4} v_2^{\text{iv}}v_1' + \frac{3r_{14}r_{34}^2}{\beta_1\beta_3\beta_4} v_2^{\text{iv}}v_1''' + \frac{3r_{23}^2 r_{34}}{\beta_2\beta_3\beta_4} v_2'''v_1''
\end{aligned}
$$

* To simplify the notation, v_s', v_s'', v_s''' v_s^{iv} have been used for ${}_1\bar{v}_s$, ${}_2\bar{v}_s$, ${}_3\bar{v}_s$, ${}_4\bar{v}_s$.

$$+ \frac{3r_{23}r_{34}^2}{\beta_2\beta_3\beta_4} v_2{}'''v_1{}^{\text{iv}} + \frac{3r_{24}^2 r_{34}}{\beta_2\beta_3\beta_4} v_2{}^{\text{iv}}v_1{}'' + \frac{3r_{24}r_{34}^2}{\beta_2\beta_3\beta_4} v_2{}^{\text{iv}}v_1{}'''$$

$$+ \frac{6r_{12}r_{13}r_{14}}{\beta_1\beta_2\beta_3\beta_4} v_2{}' + \frac{6r_{12}r_{13}r_{23}}{\beta_1\beta_2\beta_3} v_1{}'v_1{}''v_1{}''' + \frac{6r_{12}r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''$$

$$+ \frac{6r_{13}r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''' + \frac{6r_{12}r_{13}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}'' + \frac{6r_{12}r_{14}r_{24}}{\beta_1\beta_2\beta_4} v_1{}'v_1{}''v_1{}^{\text{iv}}$$

$$+ \frac{6r_{12}r_{14}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}^{\text{iv}} + \frac{6r_{12}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2{}'' + \frac{6r_{13}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1{}''v_1{}'''$$

$$+ \frac{6r_{14}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1{}''v_1{}^{\text{iv}} + \frac{6r_{12}r_{13}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''' + \frac{6r_{12}r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}^{\text{iv}}$$

$$+ \frac{6r_{12}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''' + \frac{6r_{13}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2{}''' + \frac{6r_{14}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'''v_1{}^{\text{iv}}$$

$$+ \frac{6r_{12}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}^{\text{iv}} + \frac{6r_{13}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'''v_1{}^{\text{iv}} + \frac{6r_{14}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2{}^{\text{iv}}$$

$$\left. + \frac{6r_{23}r_{24}r_{34}}{\beta_2\beta_3\beta_4} v_1{}''v_1{}'''v_1{}^{\text{iv}} + \frac{6r_{13}r_{14}r_{34}}{\beta_1\beta_3\beta_4} v_1\, v_1{}'''v_1{}^{\text{iv}} \right\}$$

$$+\frac{1}{|\underline{4}}\left\{ \frac{r_{12}^4}{\beta_1\beta_2} v_3{}'v_3{}'' + \frac{r_{13}^4}{\beta_1\beta_3} v_3{}'v_3{}''' + \frac{r_{14}^4}{\beta_1\beta_4} v_3{}'v_3{}^{\text{iv}} + \frac{r_{23}^4}{\beta_2\beta_3} v_3{}''v_3{}''' + \frac{r_{24}^4}{\beta_2\beta_4} v_3{}''v_3{}^{\text{iv}} + \frac{r_{34}^4}{\beta_3\beta_4} v_3{}'''v_3{}^{\text{iv}} \right.$$

$$+ \frac{4r_{12}^3 r_{13}}{\beta_1\beta_2\beta_3} v_3\, v_2{}'' + \frac{4r_{12}r_{13}^3}{\beta_1\beta_2\beta_3} v_3{}'v_2{}''' + \frac{4r_{12}^3 r_{14}}{\beta_1\beta_2\beta_4} v_3{}'v_2{}'' + \frac{4r_{12}r_{14}^3}{\beta_1\beta_2\beta_4} v_3{}'v_2{}^{\text{iv}}$$

$$+ \frac{4r_{13}^3 r_{14}}{\beta_1\beta_3\beta_4} v_3{}'v_3{}''' + \frac{4r_{13}r_{14}^3}{\beta_1\beta_3\beta_4} v_3{}'v_2{}^{\text{iv}} + \frac{4r_{12}^3 r_{23}}{\beta_1\beta_2\beta_3} v_3{}''v_2{}' + \frac{4r_{12}r_{23}^3}{\beta_1\beta_2\beta_3} v_3{}''v_2{}'''$$

$$+ \frac{4r_{13}^3 r_{23}}{\beta_1\beta_2\beta_3} v_3{}'''v_2{}' + \frac{4r_{13}r_{23}^3}{\beta_1\beta_2\beta_3} v_3{}'''v_2{}'' + \frac{4r_{14}^3 r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2{}'v_2{}^{\text{iv}} + \frac{4r_{14}r_{23}^3}{\beta_1\beta_2\beta_3\beta_4} v_2{}''v_2{}'''$$

$$+ \frac{4r_{12}^3 r_{24}}{\beta_1\beta_2\beta_4} v_2{}'v_3{}'' + \frac{4r_{12}r_{24}^3}{\beta_1\beta_2\beta_4} v_3{}''v_2{}^{\text{iv}} + \frac{4r_{13}^3 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2{}'v_2{}''' + \frac{4r_{13}r_{24}^3}{\beta_1\beta_2\beta_3\beta_4} v_2{}'\, v_2{}^{\text{iv}}$$

$$+ \frac{4r_{14}^3 r_{24}}{\beta_1\beta_2\beta_4} v_2{}'v_3{}^{\text{iv}} + \frac{4r_{14}r_{24}^3}{\beta_1\beta_2\beta_4} v_2{}''v_3{}^{\text{iv}} + \frac{4r_{23}^3 r_{24}}{\beta_2\beta_3\beta_4} v_3{}''v_2{}''' + \frac{4r_{23}r_{24}^3}{\beta_2\beta_3\beta_4} v_3{}''v_2{}^{\text{iv}}$$

$$+ \frac{4r_{12}^3 r_{34}}{\beta_1\beta_3\beta_4} v_2{}'v_2{}'' + \frac{4r_{12}r_{34}^3}{\beta_1\beta_2\beta_3\beta_4} v_2{}'''v_2{}^{\text{iv}} + \frac{4r_{13}^3 r_{34}}{\beta_1\beta_3\beta_4} v_3{}'''v_2{}' + \frac{4r_{13}r_{34}^3}{\beta_1\beta_3\beta_4} v_3{}'''v_2{}^{\text{iv}}$$

$$+ \frac{4r_{14}^3 r_{34}}{\beta_1\beta_3\beta_4} v_2{}'v_3{}^{\text{iv}} + \frac{4r_{14}r_{34}^3}{\beta_1\beta_3\beta_4} v_2{}'''v_3{}^{\text{iv}} + \frac{4r_{23}^3 r_{34}}{\beta_2\beta_3\beta_4} v_2{}''v_3{}'' + \frac{4r_{23}r_{34}^3}{\beta_2\beta_3\beta_4} v_3{}'''v_2{}^{\text{iv}}$$

$$+ \frac{4r_{24}^3 r_{34}}{\beta_1\beta_3\beta_4} v_2{}''v_3{}^{\text{iv}} + \frac{4r_{24}r_{34}^3}{\beta_2\beta_3\beta_4} v_2{}'''v_3{}^{\text{iv}} + \frac{6r_{12}^2 r_{13}^2}{\beta_1\beta_2\beta_3} v_3{}'v_1{}''v_1{}''' + \frac{6r_{12}^2 r_{14}^2}{\beta_1\beta_2\beta_4} v_3{}'v_1{}''v_1{}^{\text{iv}}$$

$$+ \frac{6r_{13}^2 r_{14}^2}{\beta_1\beta_3\beta_4} v_3{}'v_1{}'''v_1{}^{\text{iv}} + \frac{6r_{12}^2 r_{23}^2}{\beta_1\beta_2\beta_3} v_1{}'v_3{}''v_1{}''' + \frac{6r_{13}^2 r_{23}^2}{\beta_1\beta_2\beta_3} v_1{}'v_1{}''v_3{}''' + \frac{6r_{14}^2 r_{23}^2}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''v_1{}'''v_1{}^{\text{iv}}$$

$$+ \frac{6r_{12}^2 r_{24}^2}{\beta_1\beta_2\beta_4} v_1{}'v_3{}''v_1{}^{\text{iv}} + \frac{6r_{13}^2 r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''v_1{}'''v_1{}^{\text{iv}} + \frac{6r_{14}^2 r_{24}^2}{\beta_1\beta_2\beta_4} v_1{}'v_1{}''v_3{}^{\text{iv}} + \frac{6r_{23}^2 r_{24}^2}{\beta_2\beta_3\beta_4} v_3{}'\, v_1{}'''v_1{}^{\text{iv}}$$

$$+ \frac{6r_{12}^2 r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_1'''v_1^{\text{iv}} + \frac{6r_{13}^2 r_{34}^2}{\beta_1\beta_3\beta_4} v_1'v_3'''v_1^{\text{iv}} + \frac{6r_{14}^2 r_{34}^2}{\beta_1\beta_3\beta_4} v_1'v_1'''v_3^{\text{iv}} + \frac{6r_{23}^3 r_{34}^2}{\beta_2\beta_3\beta_4} v_1''v_3'''v_1^{\text{iv}}$$

$$+ \frac{6r_{24}^2 r_{34}^2}{\beta_2\beta_3\beta_4} v_1''v_1'''v_3^{\text{iv}} + \frac{12r_{12}^2 r_{13}r_{14}}{\beta_1\beta_2\beta_3\beta_4} v_3'v_1'' + \frac{12r_{12}r_{13}^2 r_{14}}{\beta_1\beta_2\beta_3\beta_4} v_3'v_1''' + \frac{12r_{12}r_{13}r_{14}^2}{\beta_1\beta_2\beta_3\beta_4} v_3'v_1^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{13}r_{23}}{\beta_1\beta_2\beta_3} v_2'v_2''v_1''' + \frac{12r_{12}r_{13}^2 r_{23}}{\beta_1\beta_2\beta_3} v_2'v_1''v_2''' + \frac{12r_{12}r_{13}r_{23}^2}{\beta_1\beta_2\beta_3} v_1'v_2''v_2'''$$

$$+ \frac{12r_{12}^2 r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2'' + \frac{12r_{12}r_{14}^2 r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1^{\text{iv}} + \frac{12r_{12}r_{14}r_{23}^2}{\beta^1\beta_2\beta_3\beta_4} v_1'v_2''v_1'$$

$$+ \frac{12r_{13}^2 r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2'' + \frac{12r_{13}r_{14}^2 r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1'''v^{\text{iv}} + \frac{12r_{13}r_{14}r_{23}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2'''v_1''$$

$$+ \frac{12r_{12}^2 r_{13}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2'' + \frac{12r_{12}r_{13}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1''' + \frac{12r_{12}r_{13}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_1^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{14}r_{24}}{\beta_1\beta_2\beta_4} v_2'v_2''v_1^{\text{iv}} + \frac{12r_{12}r_{14}^2 r_{24}}{\beta_1\beta_2\beta_4} v_2'v_1''v_2^{\text{iv}} + \frac{12r_{12}r_{14}r_{24}^2}{\beta_1\beta_2\beta_4} v_1'v_2''v_2^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{14}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1'''v_1^{\text{iv}} + \frac{12r_{13}r_{14}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2^{\text{iv}} + \frac{12r_{13}r_{14}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2^{\text{iv}}v_1''$$

$$+ \frac{12r_{12}^2 r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_3'' + \frac{12r_{12}r_{23}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_3''v_1''' + \frac{12r_{12}r_{23}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_3''v_1^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_2''' + \frac{12r_{13}r_{23}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2''' + \frac{12r_{13}r_{23}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_2''v_1'''v_1^{\text{iv}}$$

$$+ \frac{12r_{14}^2 r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_2^{\text{iv}} + \frac{12r_{14}r_{23}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_1'''v_1^{\text{iv}} + \frac{12r_{14}r_{23}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{13}r_{34}}{\beta_1\beta_2\beta_3\beta_3} v_2'v_1''v_2''' + \frac{12r_{12}r_{13}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2''' + \frac{12r_{12}r_{13}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2'''v_1^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1^{\text{iv}} + \frac{12r_{12}r_{14}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2^{\text{iv}} + \frac{12r_{12}r_{14}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1'''v_2^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{14}r_{34}}{\beta_1\beta_3\beta_4} v_2'v_2'''v_1^{\text{iv}} + \frac{12r_{13}r_{14}^2 r_{34}}{\beta_1\beta_3\beta_4} v_2'v_1'''v_2^{\text{iv}} + \frac{12r_{13}r_{14}r_{34}^2}{\beta_1\beta_3\beta_4} v_1'v_2'''v_2^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2''v_1''' + \frac{12r_{12}r_{23}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2''' + \frac{12r_{12}r_{23}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1''v_2'''v_1^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_3^{\text{iv}} + \frac{12r_{13}r_{23}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_3''' + \frac{12r_{13}r_{23}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_3'''v_1^{\text{iv}}$$

$$+ \frac{12r_{14}^2 r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1'''v_2^{\text{iv}} + \frac{12r_{14}r_{23}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_2'''v_1^{\text{iv}} + \frac{12r_{14}r_{23}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_2'''v_2^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2''v_1^{\text{iv}} + \frac{12r_{12}r_{24}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2^{\text{iv}} + \frac{12r_{12}r_{24}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1'''v_2^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2'''v_1^{\text{iv}} + \frac{12r_{13}r_{24}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1'''v_2^{\text{iv}} + \frac{12r_{13}r_{24}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_2'''v_2^{\text{iv}}$$

$$+ \frac{12r_{14}^2 r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_3^{\text{iv}} + \frac{12r_{14}r_{24}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_3^{\text{iv}} + \frac{12r_{14}r_{24}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'''v_3^{\text{iv}}$$

$$+ \frac{12r_{23}^2 r_{24}r_{34}}{\beta_2\beta_3\beta_4} v_2''v_2'''v_1^{\text{iv}} + \frac{12r_{23}r_{24}^2 r_{34}}{\beta_2\beta_3\beta_4} v_2''v_1'''v_2^{\text{iv}} + \frac{12r_{23}r_{24}r_{34}^2}{\beta_2\beta_3\beta_4} v_1''v_2'''v_2^{\text{iv}}$$

$$+ \frac{24r_{12}r_{13}r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1''' + \frac{24r_{12}r_{13}r_{14}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1^{\text{iv}} + \frac{24r_{12}r_{13}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2''v_1'''$$

$$+ \frac{24r_{12}r_{14}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2''v_1^{\text{iv}} + \frac{24r_{13}r_{14}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_1'''v_1^{\text{iv}} + \frac{24r_{12}r_{13}r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1'''v_1^{\text{iv}}$$

$$+ \frac{24r_{12}r_{13}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_2''' + \frac{24r_{12}r_{14}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_1'''v_1^{\text{iv}} + \frac{24r_{13}r_{14}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2'''v_1^{\text{iv}}$$

$$+ \frac{24r_{12}r_{13}r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1'''v_1^{\text{iv}} + \frac{24r_{12}r_{14}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_2^{\text{iv}} + \frac{24r_{13}r_{14}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1'''v_2^{\text{iv}}$$

$$+ \frac{24r_{12}r_{23}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_1'''v_1^{\text{iv}} + \frac{24r_{13}r_{23}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_2'''v_1^{\text{iv}} + \frac{24r_{14}r_{23}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1'''v_2^{\text{iv}}\Bigg\} \quad . \quad . \quad . \quad . \quad . \text{(lxxxiii.)}.$$

In the case of three variables, we must cancel in the above expression all terms involving β_4. Thus we shall have 3 instead of 6 first order terms, 6 instead of 21 second order terms, 10 instead of 56 third order terms, and 15 instead of 126 fourth order terms—a much more manageable series.

I give below the extra term necessary for calculating the value of $(Q - Q_0)/Q_0$ as far as the fifth order terms in the case of three variables.

Fifth Order Terms for Three Variables.

$$\frac{1}{\underline{|5}}\Bigg\{\frac{r_{23}^5}{\beta_2\beta_3} v_4''v_4''' + \frac{r_{31}^5}{\beta_3\beta_1} v_4'''v_4' + \frac{r_{12}^5}{\beta_1\beta_2} v_4'v_4'' + \frac{5r_{31}^4 r_{12}}{\beta_1\beta_2\beta_3} v_3'''v_4'$$

$$+ \frac{5r_{31}^4 r_{23}}{\beta_1\beta_2\beta_3} v_3'v_4''' + \frac{5r_{12}^4 r_{23}}{\beta_1\beta_2\beta_3} v_3'v_4'' + \frac{5r_{12}^4 r_{31}}{\beta_1\beta_2\beta_3} v_4'v_3''$$

$$+ \frac{5r_{23}^4 r_{31}}{\beta_1\beta_2\beta_3} v_3''v_4''' + \frac{5r_{23}^4 r_{12}}{\beta_1\beta_2\beta_3} v_4''v_3''' + \frac{10r_{31}^3 r_{12}^2}{\beta_1\beta_2\beta_3} v_4'v_1''v_2'''$$

$$+ \frac{10r_{31}^3 r_{23}^2}{\beta_1\beta_2\beta_3} v_2'v_1'v_4''' + \frac{10r_{12}^3 r_{23}^2}{\beta_1\beta_3\beta_2} v_2'v_4''v_1''' + \frac{10r_{12}^3 r_{31}^2}{\beta_1\beta_2\beta_3} v_4'v_2''v_1'''$$

$$+ \frac{10r_{23}^3 r_{31}^2}{\beta_1\beta_2\beta_3} v_1'v_2''v_4''' + \frac{10r_{23}^3 r_{12}^2}{\beta_1\beta_2\beta_3} v_1'v_4''v_2''' + \frac{20r_{23}^3 r_{31}r_{12}}{\beta_1\beta_2\beta_3} v_1'v_3''v_3'''$$

$$+ \frac{20r_{31}^3 r_{12}r_{23}}{\beta_1\beta_2\beta_3} v_3'v_1''v_3''' + \frac{20r_{12}^3 r_{23}r_{31}}{\beta_1\beta_2\beta_3} v_3'v_3''v_1''' + \frac{30r_{23}r_{31}^2 r_{12}^2}{\beta_1\beta_2\beta_3} v_3'v_3''v_2'''$$

$$+ \frac{30r_{31}r_{12}^2 r_{23}^2}{\beta_1\beta_2\beta_3} v_2'v_3''v_2''' + \frac{30r_{12}r_{23}^2 r_{31}^2}{\beta_1\beta_2\beta_3} v_2'v_2''v_3'''\Bigg\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \text{(lxxxiv.)}.$$

A numerical illustration of these formulæ will be given in the latter part of this Memoir. It will, however, be clear that what we want are tables of $\log\left(\frac{{}_s\bar{v}_p}{\beta_s}\right)$, including $\log\left(\frac{{}_s\bar{v}_0}{\beta_s}\right)$ or $\log\left(\frac{1}{\beta_s}\right)$ for a series of values of h. Such tables would render the computation of $\frac{Q - Q_0}{Q_0}$ fairly direct and rapid; they could be fairly easily calculated from existing tables for the ordinate and area of the normal curve, and I hope later to find some one willing to undertake them.

Meanwhile let us look at special cases. In the first place, suppose, in the case of three variables, that the division of the groups is taken at the mean, *i.e.*, $h_1 = h_2 = h_3 = 0$. Then we have

$$\beta_1 = \beta_2 = \beta_3 = \int_0^\infty e^{-\frac{1}{2}x^2}\,dx = \sqrt{\frac{\pi}{2}}.$$

$$v_1' = v_1'' = v_1''' = 0$$

$$v_2' = v_2'' = v_2''' = -1$$

$$v_3' = v_3'' = v_3''' = 0$$

$$v_4' = v_4'' = v_4''' = 3.$$

Hence we have

$$\begin{aligned} Q &= \int_0^\infty\int_0^\infty\int_0^\infty z\,dx_1\,dx_2\,dx_3 = Q_0\left\{1 + \frac{2}{\pi}(r_{12} + r_{13} + r_{23})\right\} \\ &\quad + \frac{1}{\lfloor 3}\left\{\frac{2}{\pi}(r_{12}{}^3 + r_{13}{}^3 + r_{23}{}^3)\right\} + \frac{1}{\lfloor 5}\left\{\frac{2}{\pi}9\,(r_{23}{}^5 + r_{31}{}^5 + r_{12}{}^5)\right\} + \dots \\ &= Q_0\left\{1 + \frac{2}{\pi}(\sin^{-1} r_{12} + \sin^{-1} r_{13} + \sin^{-1} r_{23})\right\} \quad \text{(lxxxv.)}. \end{aligned}$$

Let $r_{12} = \cos D_{12}$, $r_{13} = \cos D_{13}$, $r_{23} = \cos D_{23}$, and let E be the spherical excess of the spherical triangle whose angles are the divergences D_{12}, D_{13}, D_{23}. Then we have

$$\frac{Q - Q_0}{Q_0}\frac{\pi}{2} = \frac{3\pi}{2} - D_{12} - D_{13} - D_{23} = \frac{\pi}{2} - E.$$

Or:

$$\sin\frac{Q - Q_0}{Q_0}\frac{\pi}{2} = \cos E \quad \text{(lxxxvi.)}.$$

Now take the case of four variables. Here we have

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \sqrt{\frac{\pi}{2}}$$

$$v_2' = v_2'' = v_2''' = v_2^{\text{iv}} = 1$$

$$v_4' = v_4'' = v_4''' = v_4^{\text{iv}} = 3,$$

and all the odd v's zero. Hence

$$\begin{aligned}\frac{Q-Q_0}{Q} &= \frac{2}{\pi}(r_{12}+r_{13}+r_{14}+r_{23}+r_{24}+r_{34})+\left(\frac{2}{\pi}\right)^2(r_{14}r_{23}+r_{12}r_{34}+r_{13}r_{24})\\
&+\frac{2}{\pi}\frac{1}{\underline{|3}}(r_{12}^3+r_{13}^3+r_{14}^3+r_{23}^3+r_{24}^3+r_{34}^3)+\left(\frac{2}{\pi}\right)^2(r_{12}r_{13}r_{14}+r_{12}r_{23}r_{24}\\
&+r_{13}r_{23}r_{34}+r_{14}r_{24}r_{34})+\left(\frac{2}{\pi}\right)^2\frac{1}{\underline{|3}}(r_{14}^3r_{23}+r_{14}r_{23}^3+r_{13}r_{24}^3+r_{13}^3r_{24}+r_{12}^3r_{34}\\
&+r_{12}r_{34}^3)+\left(\frac{2}{\pi}\right)^2\frac{1}{\underline{|2}}(r_{12}^2r_{14}r_{23}+r_{13}^2r_{14}r_{23}+r_{12}^2r_{13}r_{24}+r_{13}r_{14}^2r_{24}+r_{13}r_{23}^2r_{24}\\
&+r_{14}r_{23}r_{24}^2+r_{12}r_{13}^2r_{34}+r_{12}r_{14}^2r_{34}+r_{12}r_{23}^2r_{34}+r_{14}r_{23}r_{34}^2+r_{12}r_{24}^2r_{34}\\
&+r_{13}r_{24}r_{34}^2)+\quad\ldots\ldots\quad \text{(lxxxvii.)}.\end{aligned}$$

This is the correct value including terms of the fourth order, but to this order of approximation we can throw it into a much simpler form. Let $r_{ss'}=\sin\delta_{ss'}$, then

$$\begin{aligned}\frac{Q-Q_0}{Q_0}\frac{\pi}{2} &= \sin^{-1}r_{12}+\sin^{-1}r_{13}+\sin^{-1}r_{14}+\sin^{-1}r_{23}+\sin^{-1}r_{24}+\sin^{-1}r_{34}\\
&+\frac{2}{\pi}(\sin^{-1}r_{12}\sin^{-1}r_{13}\sin^{-1}r_{14}+\sin^{-1}r_{12}\sin^{-1}r_{23}\sin^{-1}r_{24}\\
&+\sin^{-1}r_{13}\sin^{-1}r_{23}\sin^{-1}r_{34}+\sin^{-1}r_{14}\sin^{-1}r_{24}\sin^{-1}r_{34})\\
&+\frac{2}{\pi}[\sin^{-1}r_{14}\sin^{-1}r_{23}\{(1-r_{12}^2)(1-r_{13}^2)(1-r_{24}^2)(1-r_{34}^2)\}^{-\frac{1}{2}}\\
&\quad+\sin^{-1}r_{12}\sin^{-1}r_{34}\{(1-r_{13}^2)(1-r_{14}^2)(1-r_{23}^2)(1-r_{24}^2)\}^{-\frac{1}{2}}\\
&\quad+\sin^{-1}r_{13}\sin^{-1}r_{24}\{(1-r_{12}^2)(1-r_{14}^2)(1-r_{23}^2)(1-r_{34}^2)\}^{-\frac{1}{2}}]\\
&=\delta_{12}+\delta_{13}+\delta_{14}+\delta_{23}+\delta_{24}+\delta_{34}\\
&+\frac{2}{\pi}(\delta_{12}\delta_{13}\delta_{14}+\delta_{12}\delta_{23}\delta_{24}+\delta_{13}\delta_{23}\delta_{34}+\delta_{14}\delta_{24}\delta_{34})\\
&+\frac{2}{\pi}\left(\frac{\delta_{14}\delta_{23}\cos\delta_{14}\cos\delta_{23}+\delta_{12}\delta_{34}\cos\delta_{12}\cos\delta_{34}+\delta_{13}\delta_{24}\cos\delta_{13}\cos\delta_{24}}{\cos\delta_{12}\cos\delta_{13}\cos\delta_{14}\cos\delta_{23}\cos\delta_{24}\cos\delta_{34}}\right)\\
&\quad\ldots\ldots\quad\text{(lxxxviii.)}.\end{aligned}$$

We may write this
$$\sin\frac{Q-Q_0}{Q_0}\frac{\pi}{2}=\cos E' \quad\ldots\ldots\quad \text{(lxxxix.)}$$
where

$$\begin{aligned}E' &= \frac{\pi}{2}-\delta_{12}-\delta_{13}-\delta_{14}-\delta_{23}-\delta_{24}-\delta_{34}\\
&-\frac{2}{\pi}(\delta_{12}\delta_{13}\delta_{14}+\delta_{12}\delta_{23}\delta_{24}+\delta_{13}\delta_{23}\delta_{24}+\delta_{14}\delta_{24}\delta_{34})\\
&-\frac{2}{\pi}\left(\frac{\delta_{14}\delta_{23}\cos\delta_{14}\cos\delta_{23}+\delta_{12}\delta_{34}\cos\delta_{12}\cos\delta_{34}+\delta_{13}\delta_{24}\cos\delta_{13}\cos\delta_{24}}{\cos\delta_{12}\cos\delta_{13}\cos\delta_{14}\cos\delta_{23}\cos\delta_{24}\cos\delta_{34}}\right)\end{aligned}$$

The expressions E and E′ of (lxxxvi.) and (lxxxix.) are of considerable interest, for they enable us to express the area of a spherical triangle in three-dimensioned space, and (up to the above degree of approximation) the volume of a "tetrahedron" on a "sphere" in hyperspace of four dimensions. In fact, the whole theory of hyperspace "spherical trigonometry" needs investigation in relation to the properties of multiple correlation.

In our illustrations (viii.) and (ix.) will be found examples of the above formulæ applied to important cases in triple and quadruple correlation in the theory of heredity. I consider that the formulæ above given will cover numerous novel applications, for many of which greater simplicity will be introduced owing to the choice of special values for the h's or for the correlation coefficients.

(8.) *Illustrations of the New Methods.*

Illustration I. Inheritance of Coat-colour in Horses.—The following represents the distribution of sires and fillies in 1050 cases of thoroughbred racehorses, the grouping being made into all coat-colour classed as "bay and darker," "chesnut and lighter":—

	Colour.	Sires.		
		Bay and darker.	Chesnut and lighter.	
Fillies.	Bay and darker . . .	631	125	756
	Chesnut and lighter .	147	147	294
		778	272	1050

a	b	$a + b$
c	d	$c + d$
$a + c$	$b + d$	N

Then we require the correlation between sire and filly in the matter of coat-colour, and also the probable error of its determination.

We have from (iv.) and (v.)

$$\alpha_1 = \frac{(a + c) - (b + d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}} \int_0^h e^{-\frac{1}{2}x^2} dx = \cdot 481{,}905,$$

$$\alpha_2 = \frac{(a + b) - (c \div d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}} \int_0^k e^{-\frac{1}{2}y^2} dy = \cdot 440{,}000.$$

Hence from the probability integral tables

$$h = \cdot 64630, \qquad k = \cdot 58284.$$

We have then: $\log \mathrm{HK} = 1\cdot 037{,}3514$ by (xvii.), Thence $\epsilon = \dfrac{ad - bc}{N^2HK} = \cdot 619{,}068$ from (xxi.).

Calculating out the coefficients of the series in r in (xix.) we find

$$\cdot 619{,}068 = r + \cdot 188{,}345r^2 + \cdot 064{,}0814r^3 + \cdot 107{,}8220r^4 + \cdot 005{,}9986r^5 + \cdot 067{,}2682r^6 + \&c.$$

Neglecting powers of r above the second, we find by solving the quadratic and taking the positive root

$$r = \cdot 5600.$$

Solving by two approximations the sextic we finally determine

$$r = \cdot 5422,$$

correct, I think, to four places of figures.

Turning now to the probable error as given by Equation (l.), I find

$$h^2 + k^2 - 2rhk = \cdot 348{,}924,$$

and from (xlix.)

$$\log \chi_0 = \bar{1}\cdot 170{,}0947.$$

Further : $\dfrac{k - rh}{\sqrt{1 - r^2}} = \cdot 275{,}642$, $\quad \dfrac{h - rk}{\sqrt{1 - r^2}} = \cdot 393{,}078.$

Hence from (xlvii.) and (xlviii.) we find

$$\psi_1 = \frac{1}{\sqrt{2\pi}}\int_0^{\cdot 393{,}078} e^{-\frac{1}{2}z^2 dz}, \qquad \psi_2 = \frac{1}{\sqrt{2\pi}}\int_0^{\cdot 275{,}642} e^{-\frac{1}{2}z^2 dz},$$

and by means of the probability integral table

$$\psi_1 = \cdot 108{,}884, \qquad \psi_2 = \cdot 152{,}865.$$

By substituting in (l.), we find

$$\text{probable error of } r = \cdot 0288.$$

From (xxxiv.) and (xxxv.) we find

$$\text{p.e. of } h = \cdot 0282. \qquad \text{p.e. of } k = \cdot 0278.$$

Thus, finally, we may sum up our results

$$h = \cdot 6463 \pm \cdot 0282, \qquad k = \cdot 5828 \pm \cdot 0278,$$
$$r = \cdot 5422 \pm \cdot 0288.$$

The probable error of this r, if we had been able to find it from the product moment, would have been $\cdot 0147$, or only about one-half its present value.

Illustration II.—Our analysis opens a large field suggested by the following problem :—*What is the chance that an exceptional man is born of an exceptional father?*

Of course much depends on how we define "exceptional," and any numerical measure of it must be quite arbitrary. As an illustration, let us take a man who possesses a character only possessed by one man in twenty as exceptional. For example, only one man in twenty is more than 6 feet 1·2 inches in height, and such a stature may be considered "exceptional." In a class of twenty students we generally find one of "exceptional" ability, and so on. Accordingly we have classed fathers and sons who possess characters only possessed by one man in twenty as exceptional. We first determine h and k, so that the tail of the frequency curve cut off is $\frac{1}{20}$ of its whole area. This gives us $h = k = 1{\cdot}64485$.

Next we determine $\text{HK} = \dfrac{1}{2\pi}e^{-\frac{1}{2}(h^2+k^2)}$, and find $\log \text{HK} = \bar{2}{\cdot}026{,}8228$.

Then we calculate the coefficients of the various powers of r in (xix.). We find

$$\log \tfrac{1}{2}hk = {\cdot}131{,}2225.$$

$$\log \tfrac{1}{6}(h^2-1)(k^2-1) = \bar{1}{\cdot}685{,}5683.$$

$$\log \frac{hk}{24}(h^2-3)(k^2-3) = \bar{3}{\cdot}990{,}1176.$$

$$\log \tfrac{1}{120}(h^4-6h^2+3)(k^4-6k^2+3) = \bar{1}{\cdot}464{,}4772.$$

$$\log \frac{hk}{720}(h^4-10h^2+15)(k^4-10k^2+15) = \bar{2}{\cdot}925{,}6367.$$

It remains to determine what value we shall give to r, the paternal correlation. It ranges from ·3 to ·5 for my own measurements as we turn from blended to exclusive inheritance. Taking these two extreme values we find

$$\frac{ad-bc}{\text{N}^2} = {\cdot}0046344 \quad \text{or} \quad {\cdot}0096779.$$

But $\dfrac{ad-bc}{\text{N}^2} = \dfrac{d}{\text{N}} - \dfrac{(d+b)(d+c)}{\text{N}^2}$, and the second term is the chance of exceptional fathers with exceptional sons, when variation is independent, *i.e.*, when there is no heredity, $= \frac{1}{20} \times \frac{1}{20} = {\cdot}0025$.

Thus $\qquad d/\text{N} = {\cdot}007134 \quad \text{or} \quad {\cdot}012178;$

accordingly $\qquad b/\text{N} = {\cdot}042866 \quad \text{or} \quad {\cdot}037822.$

Hence we conclude that of the 5 per cent. of exceptional men ·71 per cent. in the first case, and 1·22 per cent. in the second case, are born of exceptional fathers, and 4·29 per cent. in the first case and 3·78 per cent. in the second case of non-exceptional fathers. In other words, out of 1000 men of mark we may expect 142 in the first case, 244 in the second, to be born of exceptional parents, while 858 in the first and 756 in the second are born of undistinguished fathers. In the former case the odds are about 6 to 1, in the latter 3 to 1 against a distinguished son having a distinguished father. This result confirms what I have elsewhere stated, that we trust to the great mass of our population for the bulk of our distinguished men. On the other hand it does not invalidate what I have written on the importance of creating good stock, for a good stock means a bias largely above that due to an exceptional father alone.

In addition to this the $\frac{1}{20}$ of the population forming the exceptional fathers produce 142 or 244 exceptional sons to compare with the 858 or 756 exceptional sons produced by the $\frac{19}{20}$ of the population who are non-exceptional. That is to say that the *relative* production is as 142 to 45·2, or 244 to 39·8, *i.e.*, in the one case as more than 3 to 1, in the other case as more than 6 to 1. *In other words, exceptional fathers produce exceptional sons at a rate* 3 *to* 6 *times as great as non-exceptional fathers.* It is only because exceptional fathers are themselves so rare that we must trust for the bulk of our distinguished men to the non-exceptional class.

Illustration III. Heredity in Coat-colour of Hounds.—To find the correlation in coat-colour between Basset hounds which are half-brethren, say, offspring of the same dam.

Here the classification is simply into lemon and white (lw) and lemon, black and white or tricolour (t),

The following is the table for 4172 cases :—

Colour.	t.	lw.	Totals.
t.	1766	842	2608
lw.	842	722	1564
Totals	2608	1564	4172

Proceeding precisely in the same way as in the first illustration we find :

$$\alpha_1 = \alpha_2 = \cdot 25024$$
$$h = k = \cdot 318{,}957$$
$$\log \mathrm{KH} = \bar{1}\cdot 157{,}6378$$
$$\epsilon = \cdot 226{,}234.$$

It will be sufficient now to go to r^4. We have

$$\cdot 226{,}234 = r + \cdot 050{,}867\, r^2 + \cdot 134{,}480\, r^3 + \cdot 035{,}587\, r^4.$$

The quadratic gives $r = \cdot 2237$. Using the NEWTONIAN method of approximating to the root we find

$$r \doteq \cdot 2222.$$

Summing up as before, after finding the probable errors, we have

$$h = k = \cdot 3190 \pm \cdot 0133,$$
$$r = \cdot 2222 \pm \cdot 0162.$$

Illustration IV. Inheritance of Eye-colour in Man.—To find the correlation in eye-colour between a maternal grandmother and her granddaughter. Here the classification is into eyes described as grey or lighter, and eyes described as dark grey or darker.*

	Tint.	Maternal grandmother.		Totals.
		Grey or lighter.	Dark grey or darker.	
Granddaughter.	Grey or lighter	254	136	390
	Dark grey or darker . .	156	193	349
	Totals	410	329	739

As before, we find

$$\alpha_1 = \cdot 109{,}607, \qquad \alpha_2 = \cdot 055{,}480,$$
$$h = \cdot 138{,}105, \qquad k = \cdot 069{,}593,$$
$$\log \mathrm{HK} = \bar{1}\cdot 196{,}6267,$$
$$\epsilon = \cdot 323{,}760.$$

Series for r up to r^4

$$\cdot 323{,}760 = r + \cdot 004{,}806 r^2 + \cdot 162\cdot 696 r^3 + \cdot 000{,}358 r^4.$$

The quadratic gives $r = \cdot 3233$, and the biquadratic

$$r = \cdot 3180,$$

the value of the term in r^4 being $\cdot 000{,}00366$, so that higher terms may be neglected.

Determining the probable errors as in Illustration I., we sum up :—

* According to Mr. GALTON'S classification, the first group contains eyes described as light blue, blue, dark blue, blue-green, grey; and the second eyes described as dark grey, hazel, light brown, brown, dark brown, verv dark brown, black.

$$h = \cdot 1381 \pm \cdot 0312,$$
$$k = \cdot 0696 \pm \cdot 0311,$$
$$r = \cdot 3180 \pm \cdot 0361.$$

Illustration V. Inheritance of Stature.—The following data have been found for the inheritance of stature between father and son from my Family Data cards, 1078 cases:—

Mean stature of father	. . . .	67″·698
,, son	. . . . .	68″·661
Standard deviation of father	. .	2″·7048
,, son	. . .	2″·7321

Correlation $= \cdot 5198 \pm \cdot 0150$.

Now for purposes of comparison of methods the correlation has been determined for this material from various groupings of fathers and sons:—

(A.) *Fathers.*

	Class.	Below 67″·5.	Above 67″·5.	Totals.
Sons.	Below 67″·5 . .	269·25	95·75	365
	Above 67″·5 . .	232·25	480·75	713
	Totals . . .	501·5	576·5	1078

(B.) *Fathers.*

	Class.	Below 66″·5.	Above 66″·5.	Totals.
Sons.	Below 67″·5 . .	211·25	153·75	365
	Above 67″·5 . .	152·75	560·25	713
	Totals . . .	364	714	1078

(C.)

Fathers.

Sons. Class.	Below 67″·5.	Above 67″·5.	Totals.
Below 68·5″ . .	356·25	182·25	538·5
Above 68·5″ . .	145·25	394·25	539·5
Totals . . .	501·5	576·5	1078

(D.)

Fathers.

Sons. Class.	Below 68″·5.	Above 68″·5.	Totals.
Below 69″·5 . .	506	182	688
Above 69″·5 . .	149·5	240·5	390
Totals . . .	655·5	422·5	1078

(E.)

Fathers.

Sons. Class.	Below 69″·5.	Above 69″·5.	Totals.
Below 70″·5 . .	669	147	816
Above 70″·5 . .	128	134	262
Totals . . .	797	281	1078

(F.)

Fathers.

Sons. Class.	Below 70″·5.	Above 70″·5.	Totals.
Below 69″·5 . .	641·25	46·75	688
Above 69″·5 . .	271·75	118·25	390
Totals . . .	913	165	1078

TABLE of Results.

Classificatiom.	Correlation.	Mean of sons.	Mean of fathers.
		k.	h.
A	·5939 ± ·0247	68″·64 (− ·416,32)	67″·74 (− ·087,00)
B	·5557 ± ·0261	68″·64 (− ·416,32)	67″·63 (− ·418,86)
C	·5529 ± ·0247	68″·50 (− ·001,16)	67″·74 (− ·087,30)
D	·5264 ± ·0264	68″·53 (·353,71)	67″·77 (·274,30)
E	·5213 ± ·0294	68″·60 (·696,57)	67″·76 (·641,30)
F	·5524 ± ·0307	68″·53 (·353,71)	67″·73 (1·023,44)

Now these results are of quite peculiar interest. They show us :—

(i.) That the probable error of r, as found by the present method, increases with h and k. But the increase is not very rapid, so that the probable errors of the series range only between ·025 and ·031. Hence while it is an advantage, it is not a very great advantage, to take the divisions of the groups near the medians. It is an advantage which may be easily counterbalanced by some practical gain in the method of observation when the division is not close to the medians.

(ii.) While the probable error, as found from the present method of calculation, is 1·5 to 2 times the probable error as found from the product moment, it is by no means so large as to seriously weigh against the new process, if the old is unavailable. It is quite true that the results given by the present process for six arbitrary divisions differ very considerably among themselves. But a consideration of the probable errors shows that the differences are sensibly larger than the probable error of the differences, even in some case double; hence it is not the method but the assumption of normal correlation for such distributions which is at fault. As we shall hardly get a better variable than stature to hypothesise normality for, we see the weakness of the position which assumes without qualification the generality of the GAUSSIAN law of frequency.

(iii.) We cannot assert that the smaller the probable error the more nearly will the correlation, as given by the present process, agree with its value as found by the product moment. If we did we should discard ·5213, a very accordant result, in favour of ·5529, or even ·5939. The fact is that the higher the correlation the lower, *ceteris paribus*, the probable error, and this fact may obscure the really best result. Judging by the smallness of h and k and of the probable error, we should be inclined to select C or the value ·5529. This only differs from ·5198 by slightly more than the probable error of the difference (·033 as compared with ·029); but since both are found from the *same* statistics, and not from different samplings ot the same population, this forms sufficient evidence in itself of want of normality. The approximate character of all results based on the theory of normal frequency must be carefully borne in mind; and all we ought to conclude from the present data for inheritance of stature from father to son would be that the correlation $= \cdot 55 \pm \cdot 015$, while the product moment method would tell us more definitely that its value was $\cdot 52 \pm \cdot 015$. There is no question that the latter method is the better, but this does not hinder the new method from being extremely serviceable; for many cases it is the only one available.

Illustration VI. Effectiveness of Vaccination.—To find the correlation between strength to resist small-pox and the degree of effective vaccination.

We have in the earlier illustrations chosen cases in which in all probability a scale of character might possibly, if with difficulty, be determined. In the present case, the relationship is a very important one, but a quantitative scale is hardly discoverable. Nevertheless, it is of great interest to consider what results flow from the application of our method. We may consider our two characters as strength to resist the ravages of small-pox and as degree of effective vaccination. No quantitative scales are here available; all the statistics provide are the number of recoveries and deaths from small-pox, and the absence or presence of a definite vaccination cicatrix. Taking the Metropolitan Asylums Board statistics for the epidemic of 1893, we have the table given below, where the cases of "no evidence" have been omitted. Proceeding in the usual manner we find

$$\alpha_1 = \cdot 86929 \qquad \alpha_2 = \cdot 54157$$
$$h = 1\cdot 51139 \qquad k = \cdot 74145$$
$$\epsilon = \cdot 782454.$$

Hence the equation for r is

$$\cdot 782{,}454 = r + \cdot 560{,}310 r^2 - \cdot 096{,}378 r^3 + \cdot 081{,}881 r^4 - \cdot 000{,}172 r^5 - \cdot 040{,}059 r^6$$

whence $r = \cdot 5954$.

Summing up we have, after calculating the probable errors,

$$h = 1\cdot 5114 \pm \cdot 0287,$$
$$k = \cdot 7414 \pm \cdot 0205,$$
$$r = \cdot 5954 \pm \cdot 0272.$$

Strength to resist Small-pox when incurred.

	Cicatrix.	Recoveries.	Deaths.	Total.
Degree of effective Vaccination.	Present	1562	42	1604
	Absent	383	94	477
	Total	1945	136	2081

We see accordingly that there is quite a large correlation between recovery and the presence of the cicatrix. The two things are about as closely related as a child to its "mid-parent." While the correlation is very substantial and indicates the protective character of vaccination, even after small-pox is incurred, it is, perhaps, smaller than some over-ardent supporters of vaccination would have led us to believe.

Illustration VII. Effectiveness of Antitoxin Treatment.—To measure quantitatively the effect of antitoxin in diphtheria cases.

In like manner we may find the correlation between recovery and the administration of antitoxin in diphtheria cases. The statistics here are, however, somewhat difficult to obtain in a form suited to our purpose. The treatment by antitoxin began in the Metropolitan Asylums Board hospitals in 1895, but the serum was then administered only in those cases which gave rise to anxiety. Hence we cannot correlate recovery and death with the cases treated or not treated in that year, for those who were likely to recover were not dosed. In the year 1896 the majority of the cases were, on the contrary, treated with antitoxin, and those not treated were the slight cases of very small risk; hence, again, we are in great difficulties in drawing up a table.* Further, if we compare an antitoxin year with a non-antitoxin year, we ought to compare the cases treated with antitoxin in the former year with those which would probably have been treated with it in the latter year. Lastly, the dosage, nature of cases treated, and time of treatment have been modified by the experience gained, so that it seems impossible to club a number of years together, and so obtain a satisfactorily wide range of statistics. In 1897, practically all the laryngeal cases were treated with antitoxin. Hence the best we can do is to compare the laryngeal cases in two years, one before and one after the introduction of antitoxin. The numbers available are thus rather few, but will help us to form some idea of the correlation. I take the following data from p. 8 of the Metropolitan Asylums Board 'Report upon the Use of Antitoxic Serum for 1896':—

Laryngeal cases.	Recoveries.	Deaths.	Totals.
With antitoxin, 1896	319	143	462
Without antitoxin, 1894 . . .	177	289	466
Totals	496	432	928

* When a new drug or process is introduced the medical profession are naturally anxious to give every patient the possible benefit of it, and patients of course rush to those who first adopt it. But if the real efficiency of the process or drug is to be measured this is very undesirable. No definite data by which to measure the effectiveness of the novelty are thus available.

Here I find $r = \cdot 4708 \pm \cdot 0292$.

A further table is of interest :—

Laryngeal cases.	Requiring tracheotomy.	Not requiring it.	Totals.
Without antitoxin, 1894 . . .	261	205	466
With antitoxin, 1896	188	274	462
Totals	449	479	928

In this case we have $r = \cdot 2385 \pm \cdot 0335$.

Lastly, I have drawn up a third table :—

Total Infantile Cases, Ages 0—5 years.

	Recovery.	Death.	Totals.
With antitoxin, 1896 . . .	912	434	1346
Without antitoxin, 1894 . .	615	556	1171
Totals	1527	990	2517

Here we have* $r = \cdot 2451 \pm \cdot 0205$.

The three coefficients are all sensible as compared with their probable errors, and that between the administration of antitoxin and recovery in laryngeal cases is substantial. But the relationship is by no means so great as in the case of vaccination, and if its magnitude justifies the use of antitoxin, even when balanced against other ills which may follow in its train, it does not justify the sweeping statements of its effectiveness which I have heard made by medical friends. It seems until wider statistics are forthcoming a case for cautiously feeling the way forward rather than for hasty generalisations.

Illustration VIII. Effect on Produce of Superior Stock.—To find the effect of superiority of stock on percentage goodness of produce.

To illustrate this and also the formula (lxxxiii.) for six correlation coefficients, we will investigate the effect of selecting sire, dam, and one grandsire on the produce when there

* The values of r for all the three cases of this Illustration were determined with great ease from Equation (xxiv.).

is selective pairing of dam and sire. We will suppose grandsire, dam, and sire to be above the average, and investigate what proportion of the produce will be above the average. As numbers very like those actually occurring in the case of dogs, horses, and even men, we may take

Correlation of grandsire and offspring . = ·25
,, sire or dam and offspring = ·5 in both cases
,, sire and grandsire . . . = ·5
Selective mating for sire and dam. . . = ·2

We will suppose zero correlation between paternal grandsire and dam, although with selective mating this may actually exist.* We have then the following system :—

$$r_{14} = \cdot 25, \quad r_{24} = \cdot 5, \quad r_{34} = \cdot 5, \quad r_{23} = \cdot 2, \quad r_{12} = \cdot 5, \quad r_{13} = 0.$$

Hence, substituting these values in (lxxxvii.), we find—after some arithmetic :

$$(\mathrm{Q} - \mathrm{Q}_0)/\mathrm{Q}_0 = 1 \cdot 4851.$$

But Q_0 is the chance of produce above the average if there were no heredity between grandsire, sire, and dam, and no assortative mating.

Hence it equals $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\mathrm{N} = \dfrac{\mathrm{N}}{16} \quad \therefore \mathrm{Q} = \cdot 1553\,\mathrm{N}.$

Or, of the produce ·5 N above the average, ·1553 N instead of ·0625 N are born of the superior stock owing to inheritance, &c. In other words, out of the ·5 N above the average, ·1553 N are produced by the stock in sire, dam, and grandsire above the average, or by ·1827 of the total stock.† The remaining ·8173 only produce ·3447 N, or the superior stock produces produce above the average at over twice the rate of the inferior stock. Absolutely, the inferior stock being seven times as numerous produces about seven-tenths of the superior offspring.

Illustration IX. Effect of Exceptional Parentage.—Chance of an exceptional man being born of exceptional parents.

Let us enlarge the example in Illustration II., and seek the proportion of exceptional men, defined as one in twenty, born of exceptional parents in a community with assortative mating.

* A correlation, if there be substantial selective mating, may exist between a man and his mother-in-law. Its rumoured absence, if established scientifically, would not, however, prove the non-existence of selective mating, for A may be correlated with B and C, but these not correlated with each other.

† The proportion of pairs of parents associated with a grandsire above the average was found by putting ·5, ·2, and 0 for the three correlation coefficients in (lxxxv.). In comparing with Illustration II., the reader must remember we there dealt with an exceptional father, 1 in 20, here only with relatives above the average—a very less stringent selection.

Here we take for father and son $r_{12} = \cdot 5$, for mother and son $r_{13} = \cdot 5$, and for assortative mating, $r_{23} = \cdot 2$.

We have then to apply the general formulæ (lxxxiii.) and (lxxxiv.) for the case of three variables. We have

$$h_1 = h_2 = h_3 = 1 \cdot 64485$$
$$\beta_1 = \beta_2 = \beta_3 = \cdot 484{,}795$$
$$v_1' = v_1'' = v_1''' = 1 \cdot 644{,}850$$
$$v_2' = v_2'' = v_2''' = 1 \cdot 705{,}532$$
$$v_3' = v_3'' = v_3''' = - \cdot 484{,}356$$
$$v_4' = v_4'' = v_4''' = - 5 \cdot 913{,}290$$

Whence, after some arithmetical reduction, we find

$$(Q - Q_0)/Q_0 = 20 \cdot 0389.$$

But $Q_0 = \frac{1}{20} \times \frac{1}{20} \times \frac{1}{20} N = \frac{1}{8000} N$. Hence $Q = \cdot 00263$ N.

We must now distinguish between the absolute and relative production of exceptional men by exceptional and non-exceptional parents. The exceptional pairs of parents are obtained by (xix.), whence we deduce, putting $r = \cdot 2$, $h = k = 1 \cdot 64485$,

$$\frac{ad - bc}{N^2} = \frac{d}{N} - \frac{(d + b)(d + c)}{N^2} = \frac{d}{N} - \frac{1}{400} = \cdot 002745.$$

Whence the number of pairs of parents, both exceptional

$$= \cdot 005245 \, N.$$

Thus, ·005245 N pairs of exceptional parents produce ·00263 N exceptional sons, and ·994755 N pairs of parents, non-exceptional in character, produce ·04737 N exceptional sons, *i.e.*, the remainder of the $\frac{1}{20}$ N. The rates of production are thus as ·5014 to ·0476. Or: *Pairs of exceptional parents produce exceptional sons at a rate more than ten times as great as pairs of non-exceptional parents.* At the same time, eighteen times as many exceptional sons are born to non-exceptional as to exceptional parents, for the latter form only about $\frac{1}{2}$ per cent. of the community.

The reader who will carefully investigate Illustrations II., VIII., and IX. will grasp fully why so many famous men are born of undistinguished parents, but will, at the same time, realise the overwhelming advantage of coming of a good stock.

U.

V.

Z.

ERRATUM.

Page 14, line 2. *For* $\frac{\cdot 67449}{\sqrt{N\chi_0}}$ *read* $\frac{\cdot 67449}{\sqrt{N}\chi_0}$.

HARRISON AND SONS, PRINTERS IN ORDINARY TO HER MAJESTY, ST. MARTIN'S LANE, LONDON, W.C.