

# Maximum-likelihood estimates and likelihood-ratio criteria for multivariate elliptically contoured distributions

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## ABSTRACT

For a class of multivariate elliptically contoured distributions the maximum-likelihood estimators of the mean vector and covariance matrix are found under certain conditions. Likelihood-ratio criteria are obtained for a class of null hypotheses. These have the same form as in the normal case.

## RÉSUMÉ

On donne des conditions permettant de calculer les estimateurs du maximum de vraisemblance du vecteur des moyennes et de la matrice des covariances d'une famille de lois multidimensionnelles dont les contours sont elliptiques. On élabore, à partir du rapport de vraisemblance, des critères permettant de tester une certaine classe d'hypothèses. Ces critères ont la même forme que dans le cas normal.

## 1. INTRODUCTION

Elliptically contoured distributions, which constitute a generalization of the normal distribution, may be determined by the same parameters. We show that in a certain class of these distributions the estimator of the mean vector is the estimator under normality and the estimator of the covariance matrix is a constant multiple (depending on the family) of the estimator under normality. The likelihood-ratio criteria for testing certain hypotheses are identical to those for normality.

An  $N$ -dimensional random vector  $\mathbf{x}$  is distributed according to a (univariate) elliptically contoured distribution with parameters  $\boldsymbol{\mu}$  and  $\mathbf{V} (\geq \mathbf{0})$  if its characteristic function has the form  $\exp(i\mathbf{t}'\boldsymbol{\mu})\phi(\mathbf{t}'\mathbf{V}\mathbf{t})$ . If the distribution has a density with  $\mathbf{V} > \mathbf{0}$ , it can be written in the form

$$|\mathbf{V}|^{-1/2} g[(\mathbf{x} - \boldsymbol{\mu})' \mathbf{V}^{-1} (\mathbf{x} - \boldsymbol{\mu})]. \quad (1)$$

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An  $n \times p$  random matrix

$$\mathbf{X} = (\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(n)})' \quad (2)$$

is distributed according to a multivariate elliptically contoured distribution with parameters  $\boldsymbol{\mu}_{(j)}$  and  $\boldsymbol{\Sigma}_{(j)} (\geq \mathbf{0})$ ,  $j = 1, \dots, n$ , if its characteristic function has the form

$$\exp\left(i \sum_{j=1}^n \mathbf{t}_{(j)}' \boldsymbol{\mu}_{(j)} \phi(\mathbf{t}_{(1)}' \boldsymbol{\Sigma}_{(1)} \mathbf{t}_{(1)} + \dots + \mathbf{t}_{(n)}' \boldsymbol{\Sigma}_{(n)} \mathbf{t}_{(n)})\right). \quad (3)$$

If  $\mathbf{X}$  has a density, it has the form

$$\prod_{j=1}^n |\boldsymbol{\Sigma}_{(j)}|^{-1/2} g\left[\sum_{i=1}^n (\mathbf{x}_{(i)} - \boldsymbol{\mu}_{(i)})' \boldsymbol{\Sigma}_{(i)}^{-1} (\mathbf{x}_{(i)} - \boldsymbol{\mu}_{(i)})\right]. \quad (4)$$

This multivariate distribution can be written as a univariate distribution by defining  $N = np$ ,

$$\begin{aligned} \mathbf{x} = \text{vec } \mathbf{X}' &= (\mathbf{x}'_{(1)}, \dots, \mathbf{x}'_{(n)})', & \boldsymbol{\mu} &= (\boldsymbol{\mu}'_{(1)}, \dots, \boldsymbol{\mu}'_{(n)})', \\ \mathbf{V} &= \text{diag}(\boldsymbol{\Sigma}_{(1)}, \dots, \boldsymbol{\Sigma}_{(n)}), \end{aligned} \quad (5)$$

where the last matrix is an  $np \times np$  matrix with diagonal blocks  $\boldsymbol{\Sigma}_{(1)}, \dots, \boldsymbol{\Sigma}_{(n)}$  and off-diagonal blocks of  $\mathbf{0}$ 's. Various multivariate distributions can be written as special cases of the univariate distribution with  $(\boldsymbol{\mu}, \mathbf{V})$  restricted to appropriate sets  $\Omega$ . These distributions have been studied by many authors, among whom are Thomas (1970), Dawid (1977), Chmielewski (1980), Fraser and Ng (1980), Jensen and Good (1981), Kariya (1981), Eaton (1983), and Fang and Chen (1984).

## 2. MAIN RESULTS

**THEOREM 1.** *Let  $\Omega$  be a set in the space of  $(\boldsymbol{\mu}, \mathbf{V})$ ,  $\mathbf{V} > \mathbf{0}$ , such that if  $(\boldsymbol{\mu}, \mathbf{V}) \in \Omega$  then  $(\boldsymbol{\mu}, c\mathbf{V}) \in \Omega$  for all  $c > 0$ . Suppose  $g$  is such that  $g(\mathbf{x}'\mathbf{x})$  is a density in  $\mathbb{R}^N$  and  $y^{N/2} g(y)$  has a finite positive maximum  $y_g$ . Suppose that on the basis of an observation  $\mathbf{x}$  from  $|\mathbf{V}|^{-1/2} g[(\mathbf{x} - \boldsymbol{\mu})' \mathbf{V}^{-1} (\mathbf{x} - \boldsymbol{\mu})]$  the MLEs under normality  $(\tilde{\boldsymbol{\mu}}, \tilde{\mathbf{V}}) \in \Omega$  exist and are unique and that  $\tilde{\mathbf{V}} > \mathbf{0}$  with probability 1. Then the MLEs for  $g$  are*

$$\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}, \quad \hat{\mathbf{V}} = \frac{N}{y_g} \tilde{\mathbf{V}}, \quad (6)$$

and the maximum of the likelihood is

$$|\hat{\mathbf{V}}|^{-1/2} g(y_g). \quad (7)$$

*Proof.* Let  $\mathbf{B} = |\mathbf{V}|^{-1/N} \mathbf{V}$  and

$$d = (\mathbf{x} - \boldsymbol{\mu})' \mathbf{V}^{-1} (\mathbf{x} - \boldsymbol{\mu}) = \frac{(\mathbf{x} - \boldsymbol{\mu})' \mathbf{B}^{-1} (\mathbf{x} - \boldsymbol{\mu})}{|\mathbf{V}|^{1/N}}. \quad (8)$$

Then  $(\boldsymbol{\mu}, \mathbf{B}) \in \Omega$  and  $|\mathbf{B}| = 1$ . The likelihood is

$$|(\mathbf{x} - \boldsymbol{\mu})' \mathbf{B}^{-1} (\mathbf{x} - \boldsymbol{\mu})|^{-N/2} d^{N/2} g(d). \quad (9)$$

Under normality  $g(d) = (2\pi)^{-N/2} e^{-d/2}$ , and the maximum of (9) is attained at  $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}}$ ,  $\mathbf{B} = \tilde{\mathbf{B}} = |\tilde{\mathbf{V}}|^{-1/N} \tilde{\mathbf{V}}$ , and  $d = N$ . In general the maximum of (9) is attained at  $\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}$ ,  $\hat{\mathbf{B}} = \tilde{\mathbf{B}}$ , and  $\hat{d} = y_g$ . Then the MLE of  $\mathbf{V}$  is

$$\hat{\mathbf{V}} = |\hat{\mathbf{V}}|^{1/N} \hat{\mathbf{B}} = \frac{|\hat{\mathbf{V}}|^{1/N}}{|\tilde{\mathbf{V}}|^{1/N}} \tilde{\mathbf{V}}. \quad (10)$$

Use of (8) to evaluate  $|\hat{\mathbf{V}}|^{1/N}$  and  $|\tilde{\mathbf{V}}|^{1/N}$  yields (6). Substitution of these values into (9) with use of (8) yields (7). Q.E.D.

**COROLLARY 1.** *Let the conditions of Theorem 1 hold, and let  $\omega \subset \Omega$  be a set such that if  $(\boldsymbol{\mu}, \mathbf{V}) \in \omega$  then  $(\boldsymbol{\mu}, c\mathbf{V}) \in \omega$  for all  $c > 0$ . The likelihood-ratio criterion for testing the null hypothesis  $(\boldsymbol{\mu}, \mathbf{V}) \in \omega$  is  $|\tilde{\mathbf{V}}_\Omega|^{1/2}/|\hat{\mathbf{V}}_\omega|^{1/2}$ , where  $\tilde{\mathbf{V}}_\Omega$  and  $\hat{\mathbf{V}}_\omega$  are the MLEs of  $\mathbf{V}$  in  $\Omega$  and  $\omega$ , respectively, under normality.*

Some sufficient conditions for  $y^{N/2} g(y)$  to have a finite positive maximum are given in the following lemmas.

**LEMMA 1.** *Suppose that  $g(\mathbf{x}'\mathbf{x})$  is a density for  $\mathbf{x} \in \mathbb{R}^N$  such that  $g(y)$  is continuous ( $0 \leq y < \infty$ ) and decreasing for  $y$  sufficiently large. Then the function*

$$h(y) = y^{N/2} g(y), \quad y \geq 0, \quad (11)$$

*has a maximum at some finite  $y_g > 0$ .*

*Proof.* Since  $g(\mathbf{x}'\mathbf{x})$  is a density,

$$1 = \int_{\mathbb{R}^N} g(\mathbf{x}'\mathbf{x}) d\mathbf{x} = \frac{\pi^{N/2}}{\Gamma(N/2)} \int_0^\infty y^{(N/2)-1} g(y) dy \quad (12)$$

by use of the transformation to spherical coordinates (e.g., Anderson 1984, Chapter 7, Problem 4). Then

$$\begin{aligned} 2^{-N/2} (2y)^{N/2} g(2y) &= y^{N/2} g(2y) \leq y^{(N/2)-1} \int_y^{2y} g(z) dz \\ &\leq \int_y^{2y} z^{(N/2)-1} g(z) dz \rightarrow 0 \end{aligned} \quad (13)$$

as  $y \rightarrow \infty$ ; that is,  $h(y) \rightarrow 0$  as  $y \rightarrow \infty$ . Furthermore,  $h(0) = 0$  because  $h(y)/y$  has a finite integral over  $[0, \infty)$ . Since  $h(y)$  is nonnegative and continuous, it has a maximum in  $(0, \infty)$ . Q.E.D.

**LEMMA 2.** *Suppose that  $g(y)$  is continuous ( $0 \leq y < \infty$ ) such that  $g(\mathbf{x}'\mathbf{x})$  is a density for  $\mathbf{x} \in \mathbb{R}^N$  and that  $\mathcal{E}\mathbf{x}'\mathbf{x} < \infty$ . Then  $h(y)$  has a maximum at some finite  $y_g > 0$ .*

*Proof.* Since  $\mathcal{E}\mathbf{x}'\mathbf{x} < \infty$ , the integral of  $h(y)$  over  $[0, \infty)$  exists and  $h(y) \rightarrow 0$  as  $y \rightarrow \infty$ . Since  $h(0) = 0$  and  $h(y)$  is nonnegative and continuous, it has a maximum in  $(0, \infty)$ . Q.E.D.

**LEMMA 3.** *Let the conditions of Corollary 1 hold. Suppose  $\Omega = \Omega_m \times \Omega_v$  and  $\omega = \omega_m \times \omega_v$ , such that  $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \omega_m$  implies  $\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1 \in \omega_m$  and  $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \Omega_m$  implies  $\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1 \in \Omega_m$ . If the MLEs of  $\boldsymbol{\mu}, \mathbf{B}$  based on the observation  $\mathbf{x}$  are  $\hat{\boldsymbol{\mu}}_\omega, \hat{\mathbf{B}}_\omega$  in  $\omega$  and  $\hat{\boldsymbol{\mu}}_\Omega, \hat{\mathbf{B}}_\Omega$  in  $\Omega$ , then the MLEs based on  $\mathbf{x} + \boldsymbol{\mu}^*$  for  $\boldsymbol{\mu}^* \in \omega_m$  are  $\hat{\boldsymbol{\mu}}_\omega + \boldsymbol{\mu}^*, \hat{\mathbf{B}}_\omega$  in  $\omega$ , and for  $\boldsymbol{\mu}^* \in \Omega_m$  are  $\hat{\boldsymbol{\mu}}_\Omega + \boldsymbol{\mu}^*, \hat{\mathbf{B}}_\Omega$  in  $\Omega$ . Similarly, if  $c\boldsymbol{\mu} \in \omega_m$  for all  $c$  and for all  $\boldsymbol{\mu} \in \omega_m$ , and  $c\boldsymbol{\mu} \in \Omega_m$  for all  $c$  and for all  $\boldsymbol{\mu} \in \Omega_m$ , then  $[\hat{\boldsymbol{\mu}}_\omega(c\mathbf{x}), \hat{\mathbf{B}}_\omega(c\mathbf{x})] = [c\hat{\boldsymbol{\mu}}_\omega(\mathbf{x}), \hat{\mathbf{B}}_\omega(\mathbf{x})]$  and  $[\hat{\boldsymbol{\mu}}_\Omega(c\mathbf{x}), \hat{\mathbf{B}}_\Omega(c\mathbf{x})] = [c\hat{\boldsymbol{\mu}}_\Omega(\mathbf{x}), \hat{\mathbf{B}}_\Omega(\mathbf{x})]$ .*

*Proof.* The estimate  $[\hat{\boldsymbol{\mu}}_\omega(\mathbf{x}), \hat{\mathbf{B}}_\omega(\mathbf{x})]$  based on  $\mathbf{x}$  minimizes  $(\mathbf{x} - \boldsymbol{\mu})' \mathbf{B}^{-1}(\mathbf{x} - \boldsymbol{\mu})$  for  $\boldsymbol{\mu} \in \omega_m$  and  $\mathbf{B} \in \omega_v$ ,  $|\mathbf{B}| = 1$  (and does not depend on  $g$ ). The estimate  $[\hat{\boldsymbol{\mu}}_\omega(\mathbf{x} + \boldsymbol{\mu}^*), \hat{\mathbf{B}}_\omega(\mathbf{x} + \boldsymbol{\mu}^*)]$  based on  $\mathbf{x} + \boldsymbol{\mu}^*$  minimizes  $(\mathbf{x} + \boldsymbol{\mu}^* - \boldsymbol{\mu})' \mathbf{B}^{-1}(\mathbf{x} + \boldsymbol{\mu}^* - \boldsymbol{\mu}) = [\mathbf{x} - (\boldsymbol{\mu} - \boldsymbol{\mu}^*)]' \mathbf{B}^{-1}[\mathbf{x} - (\boldsymbol{\mu} - \boldsymbol{\mu}^*)]$  for  $\boldsymbol{\mu} \in \omega_m$  and  $\mathbf{B} \in \omega_v$ ,  $|\mathbf{B}| = 1$ , that is, for  $\boldsymbol{\mu} - \boldsymbol{\mu}^* \in \omega_m$ . Then  $\hat{\boldsymbol{\mu}}_\omega(\mathbf{x} + \boldsymbol{\mu}^*) = \hat{\boldsymbol{\mu}}_\omega(\mathbf{x}) + \boldsymbol{\mu}^*$  and  $\hat{\mathbf{B}}_\omega(\mathbf{x} + \boldsymbol{\mu}^*) = \hat{\mathbf{B}}_\omega(\mathbf{x})$ . The same argument holds for  $\Omega$ , and similar arguments for estimates based on  $c\mathbf{x}$ . Q.E.D.

COROLLARY 2. Let the conditions of Lemma 3 hold. The LRC based on  $c\mathbf{x} + \boldsymbol{\mu}^*$  is the same as the LRC based on  $\mathbf{x}$ .

THEOREM 2. Let the conditions of Lemma 3 hold. Suppose the distribution of

$$\lambda(\mathbf{x}) = \frac{(\mathbf{x} - \hat{\boldsymbol{\mu}}_\Omega)' \hat{\mathbf{B}}_\Omega^{-1} (\mathbf{x} - \hat{\boldsymbol{\mu}}_\Omega)}{(\mathbf{x} - \hat{\boldsymbol{\mu}}_\omega)' \hat{\mathbf{B}}_\omega^{-1} (\mathbf{x} - \hat{\boldsymbol{\mu}}_\omega)} \quad (14)$$

under normality does not depend on  $(\boldsymbol{\mu}, \mathbf{V}) \in \omega_m \times \omega_v$ . Then the distribution of  $\lambda(\mathbf{x})$  for arbitrary  $g(\cdot)$  does not depend on  $g(\cdot)$  and does not depend on  $(\boldsymbol{\mu}, \mathbf{V}) \in \omega_m \times \omega_v$ .

*Proof.* The conditions imply  $\boldsymbol{\theta} \in \omega_m$ . Then for  $\boldsymbol{\mu} = \boldsymbol{\theta}$  and  $\mathbf{V} \in \omega_v$ ,  $\mathbf{x}$  has the distribution of  $R\mathbf{V}^{1/2} \mathbf{U}^{(N)}$ , where  $R^2$  has the density  $[\pi^{N/2}/\Gamma(N/2)] r^{(N/2)-1} g(r)$  and  $\mathbf{U}^{(N)}$  has the uniform distribution on the unit sphere in  $\mathbb{R}^N$  independent of  $R$ . Since  $\lambda(\mathbf{x}) = \lambda(c\mathbf{x})$  for all  $c$ , the distribution of  $\lambda(R\mathbf{V}^{1/2} \mathbf{U}^{(N)})$  for any  $g$  (including normality) is the distribution of  $\lambda(\mathbf{V}^{1/2} \mathbf{U}^{(N)})$ . The latter does not depend on  $\mathbf{V} \in \omega_v$  under normality, nor therefore on  $\mathbf{V}$  for any  $g$ . Q.E.D.

### 3. EXAMPLES

EXAMPLE 1. Suppose  $\mathbf{X}$  has the density (4) with  $\boldsymbol{\mu}_{(1)} = \cdots = \boldsymbol{\mu}_{(n)} = \boldsymbol{\nu}$  say,  $\boldsymbol{\Sigma}_{(1)} = \cdots = \boldsymbol{\Sigma}_{(n)} = \boldsymbol{\Sigma}$  say, and  $n > p$ . Under normality the MLEs of  $\boldsymbol{\nu}$  and  $\boldsymbol{\Sigma}$  are  $\hat{\boldsymbol{\nu}} = \bar{\mathbf{x}}$  and  $\hat{\boldsymbol{\Sigma}} = (1/n)\mathbf{W}$ , where

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{j=1}^n \mathbf{x}_{(j)}, \quad \mathbf{W} = \sum_{j=1}^n (\mathbf{x}_{(j)} - \bar{\mathbf{x}})(\mathbf{x}_{(j)} - \bar{\mathbf{x}})'. \quad (15)$$

The assumption  $n > p$  implies  $\Pr\{\mathbf{W} > \mathbf{0}\} = 1$ . In general the MLEs of  $\boldsymbol{\nu}$  and  $\boldsymbol{\Sigma}$  are  $\hat{\boldsymbol{\nu}} = \bar{\mathbf{x}}$  and  $\hat{\boldsymbol{\Sigma}} = (p/y_g)\mathbf{W}$ .

EXAMPLE 2. (Testing lack of correlation between sets of variates). Suppose  $\mathbf{X}$  has the distribution in Example 1. Partition  $\boldsymbol{\nu}$ ,  $\boldsymbol{\Sigma}$ , and  $\mathbf{W}$  as follows:

$$\boldsymbol{\nu} = \begin{Bmatrix} \boldsymbol{\nu}(1) \\ \vdots \\ \boldsymbol{\nu}^{(q)} \end{Bmatrix}, \quad \boldsymbol{\Sigma} = \begin{Bmatrix} \boldsymbol{\Sigma}_{11} & \cdots & \boldsymbol{\Sigma}_{1q} \\ \vdots & & \vdots \\ \boldsymbol{\Sigma}_{q1} & \cdots & \boldsymbol{\Sigma}_{qq} \end{Bmatrix}, \quad \mathbf{W} = \begin{Bmatrix} \mathbf{W}_{11} & \cdots & \mathbf{W}_{1q} \\ \vdots & & \vdots \\ \mathbf{W}_{q1} & \cdots & \mathbf{W}_{qq} \end{Bmatrix}. \quad (16)$$

By Corollary 1, the LRC for testing  $H: \boldsymbol{\Sigma}_{ij} = \mathbf{0}, i \neq j, i, j = 1, \dots, q$ , is  $|\mathbf{W}| \prod_{j=1}^q |\mathbf{W}_{jj}|^{-1}]^{n/2}$ , the same as in the normal case.

EXAMPLE 3. (Testing the hypothesis of equality of covariances). Let  $\mathbf{X}$  have the density (4) with  $\boldsymbol{\mu}_{(j)} = \boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma}_{(j)} = \boldsymbol{\Sigma}_i, j = \bar{n}_{i-1} + 1, \dots, \bar{n}_i, i = 1, \dots, q$ , where  $\bar{n}_0 = 0, \bar{n}_i = n_1 + \dots + n_i, n_i > p, i = 1, \dots, q, \bar{n}_q = n$ , and

$$H: \boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_q. \quad (17)$$

Let  $\mathbf{x} = \text{vec } \mathbf{X}', \boldsymbol{\mu} = (\boldsymbol{\mu}^{(1)'}, \dots, \boldsymbol{\mu}^{(1)'}, \dots, \boldsymbol{\mu}^{(q)'}, \dots, \boldsymbol{\mu}^{(q)'})'$ , and  $\mathbf{V} = \text{diag}(\boldsymbol{\Sigma}_1, \dots, \boldsymbol{\Sigma}_1, \dots, \boldsymbol{\Sigma}_q, \dots, \boldsymbol{\Sigma}_q)$ ; apply Theorem 1 and its corollary to find that the LRC for testing  $H$  is

$$\prod_{i=1}^q \left[ \frac{|\mathbf{W}_i|^{n_i/2}}{\left| \prod_{j=1}^q \mathbf{W}_j \right|^{n_i/2}} \left( \frac{n}{n_i} \right)^{pn_i/2} \right], \quad (18)$$

where

$$\bar{\mathbf{x}}^{(i)} = \frac{1}{n_i} \sum_{j=\bar{n}_{i-1}+1}^{\bar{n}_i} \mathbf{x}_{(j)}, \quad i = 1, \dots, q, \quad (19)$$

$$\mathbf{W}_i = \sum_{j=\bar{n}_{i-1}+1}^{\bar{n}_i} (\mathbf{x}_{(j)} - \bar{\mathbf{x}}^{(i)}) (\mathbf{x}_{(j)} - \bar{\mathbf{x}}^{(i)})', \quad i = 1, \dots, q \quad (20)$$

(Anderson 1984, Chapter 10).

By Theorem 2 the null distributions of the LRCs in Examples 2 and 3 are independent of  $g$ . Similarly, the LRCs for testing the hypothesis that a mean vector is equal to the zero vector, testing equality of several means, testing equality of several means and covariance matrices, and testing the general linear hypothesis in the multiple regression model (studied also by Thomas 1970) have null distributions independent of  $g$ 's satisfying the conditions of Theorem 2. However, there are some LRCs whose distributions depend on  $g$ . For example, the LRCs in single samples for testing  $H: \Sigma = \Sigma_0$ , where  $\Sigma_0 > \mathbf{0}$  is given, and testing  $H: \Sigma = \Sigma_0$  and  $\nu = \nu_0$ , where  $\Sigma_0 > \mathbf{0}$  and  $\nu_0$  are given, have null distributions depending on  $g$ .

The invariance or uniqueness of the null distributions of most of the above tests have been discussed by Dawid (1977), Fraser and Ng (1980), Jensen and Good (1981), Kariya (1981), Eaton (1983), Eaton and Kariya (1984), and Fang and Chen (1984).

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