Introduction to Physics of Complex Systems

Problem sheet 2
hand in until noon, 06.11.2023

Dr. Philip Bittihn, Dr. Komal Bhattacharyya

Problems 1-3 are homework to be handed in until noon, 06.11.2022. Problems 4 and 5 are for discussion in the tutorials in the week of November 6.

Problem 1: Fixed points and stability — 3x1 points

Analyze the following systems graphically. In each case, draw the graphs and indicate the direction of the flow on the real line. Use this to find all the fixed points and classify their stability. (Hint: When $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) - g(x)$, sketch the graphs of $f(x)$ and $g(x)$ on the same axes, and look for intersections. You can also use this to decide when the right-hand side is positive or negative and consequently determine the stability properties of the fixed points.)

a) $\frac{\mathrm{d}x}{\mathrm{d}t} = x - x^2$

b) $\frac{\mathrm{d}x}{\mathrm{d}t} = x - \cos x$

c) $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - 2\cos x$

Problem 2: Saddle-node bifurcation — 3x2 points

For the following systems, show that a saddle-node bifurcation occurs at some critical value of r. You can decide whether to do this fully analytically (determining fixed points and their stability as a function of r) or by arguing about the shape of the right-hand side function and how r alters it. In any case, sketch the qualitatively different right-hand sides that occur as r is varied (one below, one at, one above the bifurcation). Always determine the critical value of r analytically.

a) $\dot{x} = 1 + rx + x^2$

b) $\dot{x} = r - \cosh x$

c) $\dot{x} = r + x - \ln(1+x), \quad x > -1$

Hint: To determine the critical value of r analytically, it is not necessary to calculate the fixed points analytically for arbitrary parameters values (which is not even possible in some of the cases above), if you already know that the bifurcation occurs. Remember that the stability properties change at the bifurcation point, meaning that, for a system $\dot{x} = f(x; p)$ with some parameter p, both $f(x^*; p^*) = 0$ (fixed point) and $f'(x^*; p^*) = 0$ (stability change) must hold at the bifurcation point x^*, p^* simultaneously.

Problem 3: Non-dimensionalization — 3 points + 3 bonus points

a) Show that the logistic growth equation $\dot{N} = \mu N(1 - \frac{N}{K})$ for $\mu, K > 0$ can be non-dimensionalized to $\dot{n} = n(1-n)$ by rescaling N and time (do not forget that $\dot{N} = \frac{\mathrm{d}N}{\mathrm{d}t}$ has units of time in the denominator). Indicate how the new coordinates $n = \alpha N$, and $t' = \gamma t$ are related to the original ones by specifying the factors α, γ in terms of the original parameters. What do you learn from the non-dimensionalized form of the equation? (Hint: How does the qualitative behavior change upon parameter changes in μ and K?) (1 point)

b) Show that the harmonic oscillator from the lecture

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2, \tag{1b}$$

$c, k, m > 0$, can be non-dimensionalized to $\dot{y}_1 = py_2$, $\dot{y}_2 = -y_1 - y_2$ with just one dimensionless parameter p by rescaling x_1, x_2 and time. Indicate how the new coordinates $y_1 = \alpha x_1$, $y_2 = \beta x_2$ and $t' = \gamma t$ are related to the original ones by specifying the factors α, β, γ and the parameter p in terms of the original parameters c, k, m. (2 points)

c) The dimensionless parameter p above does not change when c is replaced by $-c$ (if this is not the case, check the lecture slides for the correct dependence of p on c). Therefore, the behavior of the non-dimensionalized system is independent of the sign of c. On the other hand, c is the friction coefficient, so a negative friction should lead to growing instead of decaying oscillations. How can both be true? (1 bonus point)

d) Solve both the system (1) and the non-dimensionalized version numerically using the odeint function. Use the transformation factors α, β, γ to transform your chosen initial condition for the full system to an initial condition for the non-dimensionalized one. In the same way, plot the trajectory of the full system in the y_1–y_2 space on top of the non-dimensionalized trajectory. They should be identical (you can use markers instead of a line for one of them for better visibility). (2 bonus points)

Problem 4: Some short questions

a) Explain the terms "phase space", "fixed point", and "bifurcation".

b) Draw an example of a dynamics with two fixed points in one dimension. How many possibilities are there for their stability?

c) Can there be a bifurcation where one fixed point (with defined stability) splits into two fixed points (with defined stability)? Why/why not?

d) In what contexts have you heard the terms "tipping point" and "hysteresis"? Can you relate these to bifurcations and illustrate them using a "made-up" bifurcation diagram?

Problem 5: Discrete-time dynamics

In the lecture, we focus on continuous-time dynamical systems. However, dynamical systems can also be discrete-time maps. For example

$$x_{n+1} = rx_n(1 - x_n) \tag{2}$$

describes a mapping from one time point n to the next one, $n + 1$. This system is famous in nonlinear dynamics and called the *logistic map*, in analogy to the logistic growth law from problem 3a). Usually, $0 \le r \le 4$ is assumed, so values from $[0, 1]$ are mapped back to $[0, 1]$.

a) In a continuous-time system $\dot{x} = f(x)$, a fixed point x_0 is defined by the condition $f(x_0) = 0$. What is the criterion for a fixed point in a discrete-time system $x_{n+1} = f(x_n)$?

b) A fixed point x_0 in a continuous-time system is linearly stable if $f'(x_0) < 0$. Can you explain why, in a discrete-time system, the corresponding criterion is $|f'(x_0)| < 1$? Hint: Consider again the evolution of a small perturbation using a Taylor expansion.

c) Use the two criteria from above to determine the fixed points of the logistic map (2) and determine their stability. What happens when r crosses the value 1 from below?

The logistic map is famous because, despite its simplicity, it can display a variety of behaviors depending on the parameter r, even chaos. We are not going to investigate this here, but if you like, you can try it numerically by iteratively applying the map to a starting value. Increase r to see the behavior changing.