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Trigonometric tables: explicating their construction principles in China

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Abstract The trigonometric table and its construction principles were introduced to China as part of calendar reform, spear-headed by Xu Guangqi (1562–1633) in the late 1620s to early 1630s. Chinese scholars attempted and succeeded in uncovering how the construction principles were established in the seventeenth century and then in the eighteenth century expanded to include more algorithms to compute the values of trigonometric lines. Successful as they were in discoursing the construction principles, most Chinese scholars did not actually construct trigonometric tables anew. In the early nineteenth century, a revolutionary approach was developed, which resembles computing a finite sum of power series to trigonometric functions of an arbitrary arc less than a one-half circle. Though hailed by many modern historians as Chinese achievements in developing "infinite series" of trigonometric functions, this approach was viewed by the actors at the time as a quick means to construct trigonometric tables. Interestingly, even with these "quick" methods, no trigonometric table was constructed. Besides the fact that constructing a trigonometric table afresh is a time-consuming business, the classification of the trigonometric table and their construction principles into different genres of knowledge by scholars offers an additional explanation of drastically uneven treatment of trigonometric tables and their construction principles.

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1 Introduction

It is said that one of the defining characteristics of trigonometry as a subject is "its *systematic* ability to convert back and forth between measures of angles and of lengths."[1] If such a view is to be taken, considering the fact that trigonometric functions of an angle (or an arc on a circle with a predetermined radius) in the early modern period were expressed in terms of the length of associated line segments,[2] one would logically reach the conclusion that one of the ultimate goals of trigonometry was to construct trigonometric tables, which juxtapose measures of angles and the lengths of the associate line segments.

The story of trigonometry in China begins with the Jesuits at the beginning of the seventeenth century.[3] Though overtly simplified, the Jesuits' entries into the Chinese courts were often depicted as victorious, based on "exactitude in the calculation of celestial phenomena, especially solar and lunar eclipses."[4] In the midst of trigonometry, the mathematical basis for astronomy and calendric science stood a powerful computational tool, trigonometric tables, which has not been systematically discussed in the historiography in English.[5] It might not be feasible, nor is it necessarily advantageous to assess how much of the Jesuit's accurate prediction should attribute to the

[1] "What distinguishes trigonometry as a subject in its own right is its *systematic* ability to convert back and forth between measures of angles and of lengths"; see Van Brummelen (2009, 9).

[2] In traditional Chinese mathematics, the concept of an angle was described in terms of the arc subtended by the angle on a circle. The discussions in this paper do not make a point to distinguish them.

[3] Before the arrival of the Jesuits, the Chinese had obtained certain "proto-trigonometric" results in astronomy, calendrical science, and land surveying, including an eighth-century "tangent table." Trigonometry as an independent subject never developed in China prior to the Jesuit's presence in China. For a survey of the results obtained by the Chinese before 1600s, see Chen (2010, 68–72).

[4] Based on the analysis of the degree of precision of both the calculations and the observations of luni-solar eclipses, Lü (2007) discussed the competition among the European, Chinese, and Islamic methods in astronomical predictions in late Ming China (1368–1644) and reached the conclusion that Chongzhen Emperor's (reigned 1627–1644) decision of going forward with the calendar reform by adopting the European method was less due to its superiority of calculation and prediction, but more out of the frustration with the failure of the Chinese method. After the fall of Ming dynasty, in a contest of the solar eclipse prediction recorded in 1644, the Western method and its observation tools won by achieving a perfect match (*yan* 驗) of the predictions (*yüce* 預測) of the first contact, the totality, and the fourth contact with the observed times. See Huang (1990).

[5] In the historiography of trigonometry in Chinese, Li (1927) discusses the various trigonometric tables and certain aspects of trigonometry that appeared in mathematical treatises published between the seventeenth and nineteenth century in China. Yan (1958) describes the Chinese achievements in spherical trigonometry and emphasizes that these results were obtained without utilizing plane or spherical triangles. Bai Shangshu identifies some of the European sources for the trigonometric treatise *Dace* 大測 (Grand Measure) in Bai (1963) and for *Celiang quanyi* 測量全義 (Complete Principles of Measurement) in Bai (1984). Chu Ping-yi examines the role played by the computational device such as trigonometric tables in the scientific development. See Chu (1999). Dong Jie in his doctoral thesis analyzes the trigonometric treatises composed in late Ming and early Qing China between 1630s and 1722. He focuses on the reasoning and proofs of trigonometric properties, including the construction principles of trigonometric tables; see Dong (2011). In the literature in English, Isaia Iannoccone discussed *the Table of Eight Lines dividing the Circle*, an early trigonometric table introduced by the Jesuits into China in 1630s and identified its European source; see Iannaccone (1998). Cullen (1982) describes the first Chinese table of tangents, its construction, the accuracy of its entries, and its origin. Needham's scant discussion of trigonometry in China focuses on

computational prowess of the trigonometric tables; nevertheless, the story of trigonometric tables and how they were "domesticated" by Chinese scholars in the seventeenth to the early nineteenth century through the discussions of their principles of construction is intellectually illuminating and interesting in its own right.

This paper will examine the trigonometric tables and analyze the treatises explaining the principles of construction by the Jesuits as well as by Chinese scholars from 1630s to 1850s. The tables and/or the accompanying treatises chosen for analysis have certain characteristics of "firsts" in China: the first complete trigonometric table, the first treatise discussing trigonometry, the first logarithmic trigonometric table, the first treatise that explicitly describes the Chinese counterpart to trigonometric tables, the first treatise providing complete geometric constructions and "proofs" to the construction principles, and last but not the least, the first trigonometric table constructed from anew by following the construction principles by a Chinese scholar. These texts are summarized in Table 1.

In addition, we will also examine the trigonometric table and the construction principles included in the compendium, *Yüzhi shuli jingyun* 御製數理精蘊 (the Essence of Numbers and their Principles, Imperially Composed), one of the editorial projects taken by the court following Kangxi Emperor's 康熙 (1654–1722) decrees toward the end of his reign. Other trigonometric tables or treatises discussing properties that can be construed as construction principles did exist in China. They are excluded in our analysis because in most cases, the trigonometric tables and the construction principles do not appear together.[6]

In the year of 1839, a treatise by Ming Antu 明安圖 (1692–1763),[7] *Geyuan milü jiefa* 割圓密率捷法 (Quick Methods for the Circle's Division and Precise Lü), was published posthumously. This treatise utilizes a series of similar isosceles triangles to obtain equalities of numerous ratios and to derive algorithms that resemble the finite sum of the series expansion of trigonometric lines. These algorithms compute the length of the trigonometric lines directly from the given measure of an (acute) arc and

Footnote 5 continued
the work of Guo Shoujing 郭守敬 (1231–1316), a thirteenth-century astronomer, and contains none of the development after the arrival of the Jesuits; see Needham (1959, 108–110).

6 Wang Xichan 王錫闡 (1628–1682) and Dai Zhen 戴震 (1724–1777) each discussed in their respective trigonometric treatise the properties which form the basis of constructing trigonometric tables. Neither of them included trigonometric tables in his treatise; more importantly, they did not present these properties in the context of constructing trigonometric tables. Consequently, their treatises are not included in the following analysis. Among the tables discussed, Xue's table is logarithmic. A few other regular or logarithmic trigonometric tables during the period are still extant. Two notable ones are: *Cuiwei shangfang shuxue* 翠薇山房數學 (Mathematics of Cuiwei Hillhouse, 1815) by Zhang Zuonan 張作楠 (1772–1850) and Jiang Lintai 江臨泰 (late eighteenth century), and *Baxian jianbiao* 八綫簡表 (A Simple Table of Eight Lines) compiled and published by Jia Buwei 賈步緯 (1840–1903) in 1903. The former was simplified from that in *Numbers and Principles* and both have no accompanying construction principles.

7 Ming Antu is also known as Minggantu in the literature. No currently known documentation, Chinese or Mongolian, indicates Ming's birth and death years. Most scholars in China follow the practice started by Li Di, who used 1692 and 1763 as Ming's birth and death years, respectively.

Table 1 Trigonometric tables, accompanying treatises, authors/compilers, years of composition or publication

Trigonometric tables and the accompanying treatises containing construction principles	Compiler/translator/author	Date
Geyuan baxian biao 割圓八線表 (Table of Eight Lines Dividing the Circle), *Dace* 大測 (Grand Measure)	Johann Schreck (1576–1630), Giacomo Rho (1593–1638), Adam Schall von Bell (1592–1666), Johann Schreck (1576–1630)	1630s
Bili sixian xinbiao 比例四線新表 (the New Logarithmic Table of Four Lines), *Zhengxian bu* 正弦部 (Section on Sines)	Nicolas Smogulecki (1609–1656) Xue Fengzuo 薛鳳祚 (1600–1680)	1662
Tianhu xiangxian biao 天弧象限表 (the Table of Celestial Arcs [in the] Quadrant)	Li Zijin 李子金 (1622–1701)	1673
Jie baxian geyuan zhi gen 解八線割圓之根 (Explicating the Origin of [how] Eight Lines Divides the Circle)	[Mei Wending 梅文鼎 (1633–1721) and] Yang Zuomei 楊作枚 (fl. Around 1700)[a]	1723
Yixian biao 一線表 (the Table of One Line)	An Qingqiao 安清翹 (1756–1829)	1819

[a] This is a treatise in the collection of Mei Wending's works in astronomy and mathematics, *Lisuan quanshu* 曆算全書 (Complete Writings on Mathematics and Calendric Astronomy). The current evidence shows that the treatise was composed by Yang Zuomei. There will be more discussions on its authorship later. Since it is part of the collection of Mei's works, we list Mei as a co-author. In most copies extant in China, the compendium contains only the construction principles; the story of the trigonometric table in Mei's *Lisuan quanshu* will, however, prove interesting to be included

vice versa.[8] Two decades before the publication of Ming's treatise, some version(s) of Ming's manuscript was circulated among a small circle of scholars, some of who also developed and published similar algorithms independently.[9] Chinese scholars at the time mostly considered these algorithms as a quick way to constructing trigonometric tables, instead of replacing them.

The analysis below will show that none of the trigonometric tables that appeared in China between 1630 and 1850 was constructed from scratch, with one possible exception, the *Table of One Line*. Some are translated and modified European tables,

[8] For the discussion of Ming Antu's treatise, see Luo (1988, 1990) among numerous articles in Chinese on the topic of "power series" in traditional Chinese mathematics. For scholarships in European languages, see Jami (1990, 1988). For a brief discussion on similar isosceles triangles on a circle, see Engelfriet (1998, 415–419). Martzloff (1997, 353–361) describes the development of "infinite series" in China based on Ming's work. There one can also learn of various European treatises on the topic of infinite series that were available at the Jesuit library in seventeenth- to eighteenth-century Beijing.

[9] Dong Youcheng 董祐誠 (1791–1823) claimed that he read Ming's algorithms without any accompanying explication or diagrams in 1819 and independently derived similar algorithms based on the same principle, similar isosceles triangles on a circle; see Dong Youcheng (1821, preface) and Tian (2005, 307).

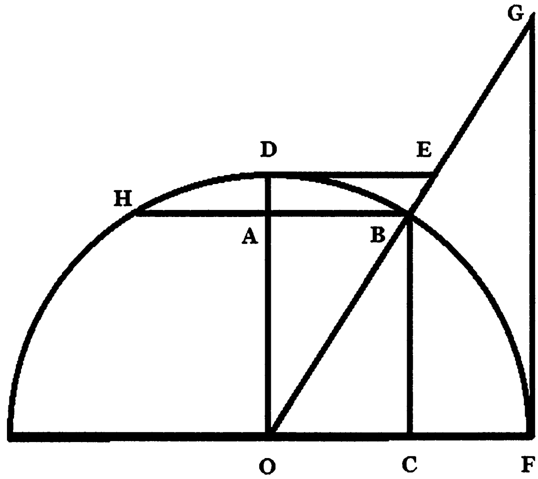


Fig. 1 Geometric definitions of trigonometric lines

some an augmentation to an old table, and others are simplification of or direct copy of translated tables. In general, most treatises are detailed in disseminating the principles and yet scant in actual application of them to constructing tables. Such an uneven treatment of construction principles and the actual edification of the table highlight the different natures of various kinds of "scientific knowledge" in China. It also raises the issue of classifications of knowledge perceived by the actors in Qing China. In the end, we will discuss the actors' views on the functionality of the tables in contrast to that of their construction principles and position them in the dichotomy of *li* 理 (principles) and *ji* 技 (skills) in the Chinese tradition.

Most of the trigonometric terms coined in the translated treatises are still in use today. Their definitions in the seventeenth century are given in Fig. 1 and Table 2; the line segments are associated with the arc *DB* (or angle *DOB*) on the circle centered at O (Fig. 1). Strictly speaking, the lengths of these line segments are not the values of the corresponding modern trigonometric functions. Rather, they are equal to the products of the radius of the circle and the modern function values.[10] In this paper, however, whenever we refer to these trigonometric terms in words, e.g., the sine of an arc, we mean the length of the line segment for the appropriate arc, while the expressions $\sin(\alpha)$ or $\tan(\alpha)$ retain the modern numerical values for the arc α.

2 The "first" trigonometric table introduced by the Jesuits

Even though there is a record of a collection of tangent lengths in the Jesuit translated texts published earlier,[11] our analysis of trigonometric tables and texts begin with

[10] In short, the sine length of an arc α is equal to $r \sin(\alpha)$, where r is the radius of the circle on which the arc α is located. In the seventeenth century, the trigonometric line segments were defined for arcs between 0° and 180°; for an obtuse arc, the trigonometric segments for its supplementary arc are used. In particular, all trigonometric values are positive.

[11] The earliest record of Jesuit-introduced trigonometric tables is a table of the length of shadows in *Biaodu Shuo* 表度說 (On [using the] Table and Measure) by Sabatino de Ursis (1575–1620). This table records shadow lengths from 0° to 90° with a 10-min increment. It can be construed as a table of tangent lengths.

Table 2 Chinese terms for trigonometric lines with their modern names

Segment	Translated terms	Modern terms
DE	*Zhengqie* 正切	Tangent
OE	Zhengge 正割	Secant
FG	*Yüqie* 餘切	Cotangent
OG	*Yüge* 餘割	Cosecant
AB	*Zhengxian* 正弦	Sine
BC (=*AO*)	*Yüxian* 餘弦	Cosine
AD	*Zhengshi* 正矢	Versine
CF	*Yüshi* 餘矢	Co-versine

the *Table of Eight Lines Dividing the Circle* (hereafter, the *Table of Eight Lines*) and the treatise *Grand Measure*, which discusses the principles of construction. Both are part of the *Chongzhen lish* 崇禎曆書 (Astronomical Compendium of the Chongzhen Reign, hereafter *Chongzhen Astronomical Compendium*).[12] The *Table of Eight Lines* is identified as a translated and simplified version of a European table.[13] In spite of the name, the *Table of Eight Lines* contains a table of only six lines (equivalent to the modern-day six trigonometric functions) for the arcs ranging from 0° to 45°, with a 1-min increment.[14] The values in the tables are whole numbers; the values for the sine and cosine lines have 6 or fewer digits and the rest 9 or fewer. The table explicitly states that the radius of the circle is one hundred thousand in length.[15] Figure 2 is the first page in the *Table of Eight Lines* for the trigonometric lines of arcs measured 0′ to 30′.[16] The entries in the top row from the right to the left are 0 (degree), sine, tangent line, secant line, cosine, cotangent line, cosecant line, and an empty space. The bottom row reads from right to left, an empty space, cosine, cotangent line, cosecant

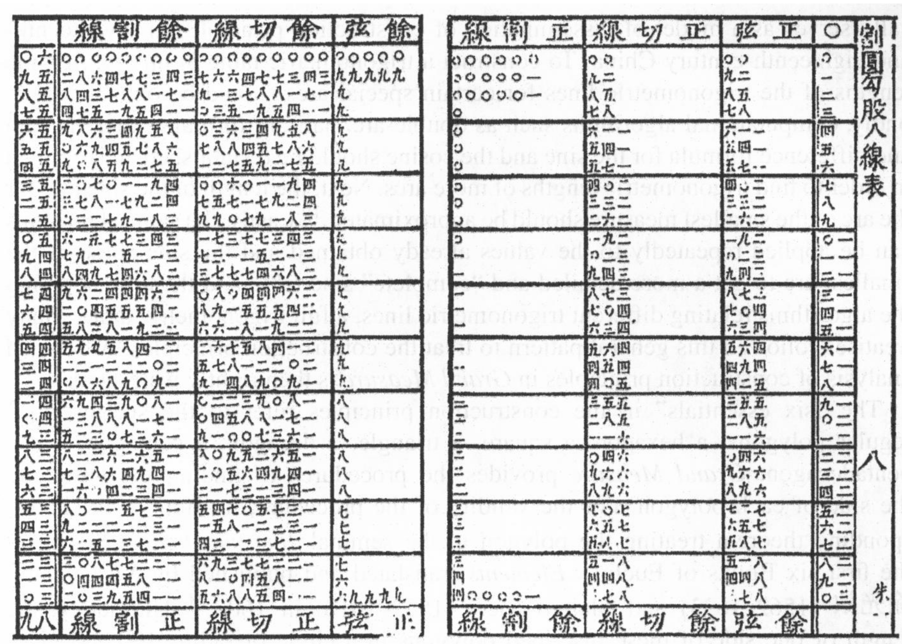
[12] This compendium was later presented to the Qing court by Adam Schall von Bell under the name *Xiyang xinfa lishu* 西洋新法曆書 (Astronomical Compendium according to the new Western methods). It was further revised by Ferdinand Verbiest (1623–1688). In addition to corrections on the content including numerical values, the title was re-fashioned as *Xinfa lishu* 新法曆書 (Astronomical Compendium according to the new methods) to de-emphasize the Western, i.e., foreign, contribution. Eventually, it was included in *Siku quanshu* 四庫全書 (The Complete Texts of the Four Repositories) as *Xinfa suanshu* 新法算書 (Mathematical Compendium according to the new methods).

[13] See Iannaccone (1998).

[14] The "eight lines" in the title refer to the eight trigonometric lines in Table 2. The texts before the table explains that the versine and coversine lines are simply the differences between the radius and the cosine and sine lines, and therefore, their lengths are not needed in the table. See Schreck et al. (1631, 1206).

[15] The explanatory texts before the table indicate that the radius used in more detailed tables (*da biao* 大表, literally great tables) might have eight, nine, or even ten digits, but the current (smaller) table has only six digits to expedite calculation; see Schreck et al. (1631, 1206).

[16] This page in fact are two leaves from two different pages, 8b and 9a, in Schreck et al. (1631, 1208–1209). In the traditional thread-bound format, they are seen at the same time when the treatise is opened to that page.


割圓勾股八線表

餘割線 | 餘切線 | 餘弦 | 正割線 | 正切線 | 正弦 | 〇

正割線 | 正切線 | 正弦 | 餘割線 | 餘切線 | 餘弦 | 八十九

Fig. 2 First page of the trigonometric values in *Table of Eight Lines*

line, sine, tangent line, secant line, and 89 (degrees). In the first column on the right under 0 (degree) are the measures of the arcs from $0'$ to $30'$. In the last column on the left under the empty space, the measures are listed in the decreasing order, from (89°) $60'$ to $30'$. This arrangement reduces the bulk of the table in half.[17] This cost-saving arrangement also becomes a standard practice for all the trigonometric tables in China afterward.

It is worth noting that the order of the digits of the numerical values from the single to the tens, and on, is from left to right, contrary to the modern convention from right to left. For example, the value of $\sin(30')$ in the second column on the right and last numerical value at the bottom is 873 in the modern form although it is written horizontally from left to right *san qi ba* 三七八 (three seven eight).

Grand Measure as the first treatise of trigonometry in China was translated from various European sources.[18] Its treatment of construction principles, *liuzong sanyao erjianfa* 六宗三要二簡法 (six essential [lengths of the sides of inscribed regular polygons in a circle], three important [methods], and two simple methods), basi-

[17] The *Table of Eight Lines* includes texts explaining how to read and use the trigonometric values of arcs greater than 45°; see Schreck et al. (1631, 1205–1206).

[18] According to Bai Shangshu, the content of *Grand Measure* was taken from Bartholomaeus Pitiscus's (1561–1613) *Trigonometriae* and Simon Stevin's (1548–1620) *Memoires Mathématiques*; see Bai (1963). *Memoires Mathématiques* is a French translation of Stevin's *Hypomnemata Mathematica*. Both treatises were present in Beijing in early 1600s according to the catalogue of the North Church (*Beitang*, 北堂). Bai did not explain how he determined that the content of *Grand Measure* was taken from *Memoires Mathématiques* instead of *Hypomnemata Mathematica*.

cally serves as a model of dissemination of construction principles in seventeenth- and eighteenth-century China: To construct a trigonometric table *in theory*, first the lengths of the trigonometric lines for certain special arcs should be obtained. Secondly, computational algorithms such as double-arc, half-arc formulas, and the sum and difference formula for the sine and the cosine should be (established and) applied in order to find trigonometric lengths of more arcs. Next, the length of the sine line for the arc of the smallest measure should be approximated.[19] Lastly, the same algorithms can be applied repeatedly to the values already obtained and the sine value of the smallest arc to get a more detailed and "complete" sine table and then, by applying the algorithms relating different trigonometric lines, a full trigonometric table. Many treatises followed this general pattern to treat the construction principles. A detailed analysis of construction principles in *Grand Measure* is then rightly warranted.

The "six essentials" in the construction principles refer to the six inscribed regular polygons: a hexagon, a square, a triangle, a decagon, a pentagon, and a pentadecagon. *Grand Measure* provides the procedures of finding the length of the side of each polygon. For the validity of the procedure, it refers to the corresponding theorem treating the polygon in the seminal *Jihe yuanben* 幾何原本,[20] the first six Books of Euclid's *Elements* translated and published by Xu Guangqi 徐光啟 (1562–1633) and Matteo Ricci (1552–1610) in 1607.[21] Ultimately, the length of one side of these inscribed polygons in a circle of radius 100,000 gives rise to the length of the sine lines for the arcs of 30°, 45°, 60°, 18°, 36°, and 12°.

The three important and the two simple methods are presented in *Grand Measure* as algorithms, the conventional way mathematical properties were recorded in treatises, and as equalities of line segments (Table 3).[22] As an example of an equality of line segments, the third important method (3-3), a version of the half-arc formula for the sine lines, is described as follows:

[19] Granted, almost all numerical values in the table are approximation, being finite decimal expressions for irrational numbers. Chinese scholars during Ming dynasty differentiated two types of approximate values. Obtained through "exact" procedures, the first kind consists of values which can be made "precise" if the computations can be carried on indefinitely to get infinite decimal expressions. These values were not considered by the scholars at the time as "approximation." The values of the other kind are, on the contrary, obtained through approximation procedures; they carry negligent discrepancy. The sine values for the special arcs belong to the former while the value of $\sin(1')$ is of the latter kind, to be discussed below.

[20] For the discussion on the translation and the content of *Jihe yuanben*, see Engelfriet (1998, 105–286).

[21] For the hexagon, Proposition 15 in Book Four of *Jihe yuanben* is cited; for the square, Proposition 6 of Book Four; for the equilateral triangle, Proposition 12 of Book Thirteen; for the decagon, Proposition 9 of Book Thirteen; for the pentagon, Proposition 10 of Book Thirteen; and lastly for the pentadecagon, Proposition 16 of Book Four, which is for the construction of the inscribed regular pentadecagon, not the procedure of finding the length of one side. The procedure of finding the length is developed directly for the pentadecagon, based on certain figures in *Grand Measure*. See Schreck (1631, 1185–1187). It seems an oversight to cite Book Thirteen of *Jihe yuanben* in the text since only the first six Books were translated into Chinese.

[22] For (3-1), (3-2), and (3-3), see Schreck (1631, 1188–1189); for (2-1) and (2-2); see Schreck (1631, 1189–1191). For more detailed discussion on the simple methods, see Li and Du (1987, 463–470) and Chu (1999, 63).

Table 3 Three essential and two simple methods in *Grand Measure*

$(r\sin(\theta))^2 + (r\cos(\theta))^2 = r^2$		(3-1)
$\left(2r\sin\left(\frac{\theta}{2}\right)\right)^2 = (r\sin\theta)^2 + (r - r\cos(\theta))^2$		(3-3)
$r\sin\theta = r\sin(60^\circ + \theta) - r\sin(60^\circ - \theta)$		(2-1)
Given quantities	Desired quantities	
$r\sin(\alpha), r\cos(\alpha)$	$r\sin(2\alpha)$	(3-2)
$r\sin(\alpha), r\cos(\alpha), r\sin(\beta), r\cos(\beta)$	$r\sin(\alpha+\beta), r\sin(\alpha-\beta)$	(2-2)

> The square of the chord of an arc is equal to the sum of the square of its sine and that of its versed sine.[23]

The algorithms are rather straightforward; for example, the first method of (2-2):

> Take the sine of the first arc to multiply the cosine of the second arc. Take the sine of the second arc to multiply the cosine of the first arc. The two products are added together as the dividend. Divide [the dividend] with the divisor, the radius, to get the sine of the arc which is the sum of the two arcs.[24]

These five methods were explained with concrete examples. The term, *lun yue* 論曰 (literally, discussion says), which *Jihe yuanben* designates as the beginning of a proof, is not used consistently. With the exception of the property (2-2), the explanations of these methods amount to proving these properties.[25]

Having introduced the construction principles of the trigonometric tables, *Grand Measure* then demonstrates how sine lengths of new arcs can be obtained by applying the three important methods to the values of the sine lines from the six essentials. The illustration utilizes the sine value of 12°: starting with $\sin(12^\circ)$, apply the half-arc formula (3-3) repeatedly to get the sine values of $6^\circ, 3^\circ, 1^\circ 30'$, and $45'$; then apply Pythagorean Theorem (3-1) to these values to get sine lengths of $84^\circ, 87^\circ, 88^\circ 30', 89^\circ 15'$. Next apply the equation (3-3) again to $\sin(84^\circ)$ to get the sine values of $42^\circ, 21^\circ, 10^\circ 30'$, and $5^\circ 15'$. The list of sine values is obtained by the application of the equations (3-3) and (3-1) in turn. The list of sine values in *Grand*

[23] The text reads, "各弧之全弦上方與其正半弦上偕其矢上兩方并等"; see (1631, 1189). The chord of an arc is equal to twice of the sine line of one-half of the arc. For example, in Fig. 1, the chord of the arc BDH is the line segment BAH, which is equal to twice of the line segment *AB*, which is the sine line for the arc BD, which is one-half of the arc BDH.

[24] The text reads, "以前弧之正弦乘後弧之餘弦, 以後弧之正弦乘前弧之餘弦各得數并之以為實. 以半徑為法而一, 得兩弧相加為總弧之正弦"; see Schreck (1631, 1189).

[25] See Tian (2005, 278–282) and Dong Jie (2011, 16–21).

Measure obtained from $\sin(12°)$ consists of the arcs of the following 49 measures[26]:

$$12°, 6°, 3°, 1°30', 45', 84°, 87°, 88°30', 89°15', 42°, 21°, 10°30', 5°15', 43°30', 21°45', 44°15', 48°, 69°, 79°30', 84°45', 46°30', 68°15', 45°45', 24°, 34°30', 17°15', 39°45', 23°15', 66°, 55°30', 72°45', 50°15', 66°45', 33°, 16°30', 8°15', 27°45', 57°, 73°30', 81°45', 62°15', 28°30', 14°15', 36°45', 61°30', 75°45', 53°15', 30°45', 59°15'.$$

As the measures of the special arcs are all multiple of 3, the grid of sine lines obtained by applying the important and simple methods, consists of arcs of the measure in the modern form $\frac{3m}{2^n}$ where m is a positive integer and n any integer. To build a "complete" trigonometric table, it is necessary to find a (approximated) value of the sine line of the smallest measure, i.e., $r\sin(1')$ in this case, in the table.[27] In *Grand Measure*, the explanation was made explicit as follows.

The equation (3-3) can be applied repeatedly to $\sin(3°)$ to get $\sin(45')$, while the lengths of sine lines for the arcs of $1'$ to $44'$ remain to be found. After further applications of equation (3-3) to $\sin(45')$, the lengths of $\sin(11'15'')$ and $\sin(22'30'')$ are obtained, 32,724.5 and 65,449, respectively. It is observed that proportionality exists between these small arcs and their sine lines:

$$22'30'' = 2 \times 11'15'' \text{ and } \sin(22'30'') = 65{,}449 = 2 \times 32{,}724.5 = 2 \times \sin(11'15'')$$

That is, when arcs are "small" enough, the sine line of the arc is very close to the arc itself and "they more or less merge together as one line."[28] Consequently, the rule of three can be applied to $22'30''$ and $10'$ to get $\sin(10') = 29{,}088$ and then to $10'$ and $1'$ to get $\sin(1') \approx 2909$.[29] After the value of $\sin(1')$ is found, the method (2-2) can be applied repeatedly to find the length of the sine lines for arcs of $2'$ through $15'$ and then to arcs of other measures up to 90°. In theory, a sine table of arcs from 0° to 90° with a $1'$-increment can be constructed from the ground up by following the principles codified in the *Six Essentials, Three Important, and Two Simple Methods*.

26 Curiously, the sine value of 78°, the complementary arc of 12°, is not included in the list and consequently, the sine lengths of 39°, 19°30′, and 9°45′ are also missing from the list.

27 This technical difficulty is wide-spread across cultures when it comes to constructing trigonometric tables. For the effort to find better approximation of $\sin(1°)$ in Islamic trigonometry, see Van Brummelen (2009, 140–149). This fact was overlooked by many historians of Chinese mathematics. For example, Tian (2005) and Li (1927) did not explore this aspect in their discussion of construction principles of trigonometric tables.

28 The text reads, "...略似相合为一線矣也"; see Schreck (1631, 1193).

29 In the text, the value of $\sin(1')$ is described as 2909 *ruo* 弱 (less), i.e., less than 2909' see Schreck (1631, 1193).

The text then demonstrates the ways of finding the values of tangents and secants via the rule of three. Take the tangent line as an example[30]:

$$r\cos(A){:}r\sin(A) = r{:}r\tan(A).$$

With a known sine length of an arc, the cosine line can be found by applying Pythagorean theorem, the equation (3-1); therefore, an application of the rule of three will produce the length of the tangent line.[31] The secant lines can be constructed similarly via the rule of three:

$$r\cos(A){:}r = r{:}r\sec(A).$$

When different astronomical and calendrical systems compete for dominance and seek "integration or appropriation," the issue of measuring units comes to the fore. The measuring system in the traditional Chinese astronomy for circular arcs, roughly 365.25 *du* 度 ([Chinese] degree) as the measure for a full circle, poses an obstacle for the early seventeenth-century Chinese scholars who were well versed in the traditional approach and yet unfamiliar with the Western method, to utilize the newly available powerful computational tool, the trigonometric table. The Jesuits were well aware of the peril. The texts before the table indicate that there was (or should be) a conversion table (*tonglü biao* 通率表), to convert the measures in the traditional system to the new ones (and presumably vice versa), so as to accommodate measures from different systems.[32] Such a table, if existed in the early seventeenth century, is no longer extant. The degree to which the incongruent systems of measures posed such an obstacle that prevented the adaptation of trigonometric tables by the Chinese astronomers following the traditional approach remained to be studied;[33] nevertheless, the integration of Chinese and Western methods in calendric science Xu Guangqi advocated in his memorials and treatises did not materialize.

The terms, *Six Essentials, Three Important, and Two Simple Methods*, became synonymous with the construction principles of trigonometric tables in China even

30 The Chinese text describes these four quantities as "four *lü*." It is understood that when four quantities are described as four *lü*, they satisfy the required equality of ratios. See Schreck (1631, 1193). The concept of *lü* can be seen as early as in the *Nine Chapters of the Mathematical Art* (Jiuzhang suanshu 九章算術); see Chemla and Guo (2004, 956–959).

31 The discussion of the tangents can be seen in Schreck (1631, 1193); the secants Schreck (1631, 1193–1194).

32 The texts in the *Table of Eight Lines* describe the measures of the 365.25 *du* as *tian du* 天度 (celestial degree) and the Jesuit system of 360° as *ping du* 平度. See Schreck et al. (1631, 1208).

33 Chu (1999) argues that the two incongruous measures of angles prevented scholars following the traditional approach from piecemeal utilizing trigonometric tables in their computation. The effort by the Jesuits to provide a conversion table certainly acknowledges this obstacle and, at the same time, also suggests that such a conversion table should smooth the adaptation. Chu seemingly misplaces the blame on the incongruous measures as the sole or at least the main reason that the calendar reform in the 1630s resulted in an all-or-nothing attitude when it came to adopting the Western astronomic system. Our discussion in a later section will show the incongruity of measures poses much smaller obstacles in the efforts to "integrate" different systems.

when more essentials and/or methods were added to the collection. As well known, it is prohibitively time-consuming to construct a trigonometric table afresh.[34] It is no surprise that the Jesuits involved in the calendar reform in 1630s did not construct a trigonometric table afresh and opted to translate and simplify an existing table. As we will see below that "plagiarizing" or modifying an existing table becomes the norm for most of the tables that appear in Qing China. Only after more advanced algorithms such as one calculating the sine length of the one-fifth of an arc became available did a table constructed anew became more plausible.

3 The first logarithmic trigonometric tables in China

The *Table of Eight Lines* in the *Chongzhen Astronomical Compendium*, in which the cosmological model was Tychonic, was introduced by the Jesuits at the court while the first logarithmic trigonometric table was introduced into China by a Jesuit far away from Beijing, Nicolas Smogulecki, who criticized the Tychonic system and offered an ambiguously alternative,[35] with the aid from his Chinese disciple and collaborator Xue Fengzuo. *Bili sixian xinbiao* 比例四線新表 (The New Logarithmic Table of Four Lines, hereafter the *Table of Four Lines*), part of Xue's voluminous collectanea *Lixue huitong* 曆學會通 (An Integration of Calendric Studies), is the very first logarithmic trigonometric table in China.[36] In its preface dated 1662, Xue stated that the logarithmic approach could transform the troublesome multiplication/division into addition/subtraction and therefore expedite the necessary computations in calendar-making.[37] The implication is clear that this new table is an *improvement* over the one previously introduced (i.e., the Table of Eight Lines).

The *Table of Four Lines* consists, as its title indicates, of values for four trigonometric lines: sine, cosine, tangent, and cotangent, with the measures of the arcs ranging from 0° to 45°, with a 1-*fen* 分 ([Chinese] minute = 0.01°) increment. Once again, the

[34] Georg Joachim Rhaeticus (1514–1576) spent 12 years constructing a trigonometric table, but did not succeed. His disciple Valentinus Otto (1550–1605) finished his work and published it in 1596, cited from Chu (1999).

[35] Smogulecki was heavily influenced by the Copernican astronomer Philippe van Lansberge (1561–1632). In fact, Smogulecki adapted Lansberge's astronomy to China, characterizing its Chinese version as *zhenyuan* 眞原 (true principles) against the Chinese version of Tychonic astronomy contained in the *Chongzhen Astronomical Compendium*. The competing astronomical systems in mid-seventeenth-century China are quite numerous and complex. According to Xue, there are the Old Chinese Method, the New Chinese Method, the Chinese-Islamic System from the Western Areas, the Current Western Method, and the New Western Method, the last being one introduced by Smogulecki. Xue's intention was to achieve "integration" of all these systems. For Xue's career as an astronomer after he met Smogulecki and brief discussion on these systems, see Shi (2007, 68–70).

[36] The term *bili* 比例 literally means proportionality, but Smogulecki and Xue used it to differentiate their logarithmic trigonometric table from the regular *Table of Eight lines*. For the introduction of logarithmic tables into China, see Han (1992, 110–111).

[37] The text reads, "對數者，苦(若)乘除之煩，變為加減，用之作曆省易無訛者也." (The logarithm [approach can] transform the troublesome multiplication and division into addition and subtraction; [when] applied in calendar-making, [it is] simplistic and easy without errors. See Smogulecki and Xue (1662, 6:638).

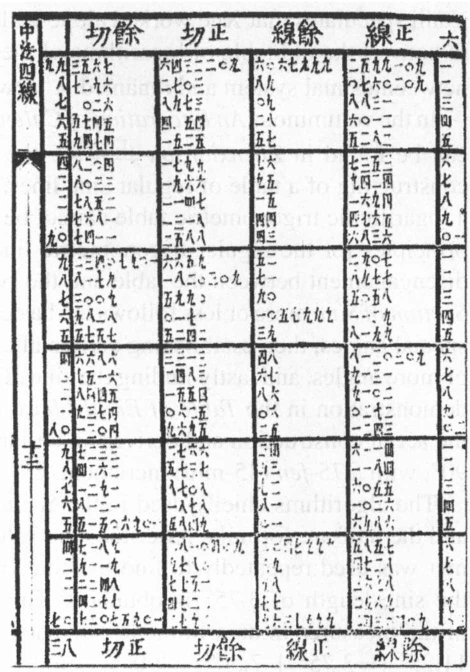


Fig. 3 Page 13 in *Table of Four Lines*

format of the table allows the values for arcs in the range between 45° and 90° to be read from the same table. Xue states that in order to improve the old [Western] method (i.e., the one introduced in *Grand Measure*) and achieve integration of the Western and Chinese methods, the *Table of Four lines* changes the measuring system slightly, from 1° equal to 60 min to 1° equal to 100 *fen*.[38] The entries in the table have seven digits. Figure 3 is the right leaf of the page 13 from the *Table of Four Lines*.[39] The entries in the top row from the right to the left are: 6 (degrees), sine line, cosine line, tangent, and cotangent.[40] In the first column, the measures of the arcs are from 1 *fen* to 30 *fen* and in the last column, the measures of arcs from (83°) 99 *fen* to 70 *fen* in a descending order. The way the numerical digits should be read is the same as in the *Table of Eight Lines*.

One possible European source for the *Table of Four Lines* is A. Vlacq's 1628 logarithmic trigonometric table, *Canon Triangvlorvm, Sive Tabvla Artificialivm, Sinnum, Tangentium & Secantium.* Since Vlacq's table is sexagesimal, additional work is needed to convert the entries to Xue's centesimal one. Based on Xue's preface, Shi

[38] See Smogulecki and Xue (1662, 6:638). In traditional Chinese method, one Chinese degree (*du*) is equal to 100 Chinese minutes (*fen*). Thus, it can be construed as an effort to "integrate" (*huitong* 會通) the Western and Chinese methods that Xue's adopted 360 (Western) degrees as the measure for the full circle and set 1° equal to 100 *fen*. It should be noted that Xue' *fen* is not equal to the traditional *fen* in the traditional Chinese system.

[39] See Smogulecki and Xue (1662, 6:646).

[40] The Chinese names here for sine and cosine in the *Table of Four Lines* are *zheng xian* 正線 and *yü xian* 餘線, slightly different from those used in the *Table of the Eight Lines*.

Yunli speculates that Xue worked alone without Smogulecki's guidance or assistance to convert the old table of logarithms of the sexagesimal trigonometric lines into the new centesimal system and renamed it "new table" (*Xinbiao* 新表).[41]

In the voluminous *An Integration of Calendrical Studies*, the construction principles can be found in *Zhenxian bu* 正弦部 (the *Section of Sine*), in which a process of constructing of a table of regular sine lines, not logarithmic, is discussed. Obviously a logarithmic trigonometric table cannot be established by applying the construction principles for the regular trigonometric lines. Xue did not reconcile or explain the disengagement between the table and the construction principles in the treatise. The *Section of Sine* more or less follows the basic pattern of first finding the sine lengths for special angles, then establishing certain algorithms that can calculate the sine lengths of more angles, and lastly finding the sine of 1°. After criticizing the old "incomplete" demonstration in the *Table of Eight Lines*, the *Section on Sine* explicitly carries out the act of construction and derives an un-tabulated list of 120 sine lengths from 0° to 90°, with a 75-*fen* (45-min) increment.[42]

The algorithms chiefly used in the *Section on Sine* are the Pythagorean Theorem and the half-arc formula. After the sine values of for 30° and 45° are found, the former was used repeatedly to find the sine values for 15°, 75°, 7.5°, and 3.75°. Once the sine length of 3.75° is obtained, Xue then describes the lengths of sine lines of the measures in an arithmetic sequence with the common difference 3.75°, from 11.25° ($=3.75^\circ + 7.50^\circ$) to 90°:

$$11.25^\circ, 18.75^\circ, 22.50^\circ, 26.25^\circ, 30^\circ, 33.75^\circ, 37.50^\circ, 41.25^\circ, 45^\circ, 48.75^\circ, 52.50^\circ,$$
$$56.25^\circ, 60^\circ, 63.75^\circ, 67.50^\circ, 71.25^\circ, 75^\circ, 78.75^\circ, 82.50^\circ, 86.25^\circ, \text{ and } 90^\circ.$$

The sine length of an entry in the list is presented as obtained from the sine lengths of the previous entry and 3.75°. Take 18.75° as an example. The text describes the procedure as "add another [3.75° to the 15° to] get 18.75°. Apply the aforementioned algorithm [to] get the sine [line of 18.75°] 3,214,395." The confusing part is that no algorithm similar to (2-2) in the two simple methods can be found in the treatise.[43] Interestingly, the sine lengths for these angles can actually be obtained through a half-arc algorithm and the Pythagorean Theorem. For example, $\sin(11.25^\circ)$ and $\sin(78.75^\circ)$ are obtained in the following manner: Since 11.25° is one-half of 22.50°, which is one-half of 45°, the sine length of which is known, the half-arc algorithm can be applied twice to obtain that of 11.25°, and consequently, the sine value of 78.75° can be found through Pythagorean Theorem.

41 See Shi (2007, 111). For the reconstruction of the *Table of Four Lines* and the claimed European source, see Roegel (2011).

42 The text reads, "今舊法割圓表久鏽行世, 而獨於取正弦之法闕畧不全. 學者求其法而不可得," (Now the old methods and trigonometric tables [have] long [been] shining and available to the world, but [certain] processes of finding sine [lines] are missing, truncated, and incomplete. Scholars [wanted to] obtain them, but could not). See [Smogulecki and Xue, 6:627].

43 The text read, "再加為一八度七五分，用上法得正弦三二一四三九五." Curiously, "the aforementioned algorithm" mentioned in the text can only be either the Pythagorean Theorem or the half-arc formula for sines. See Smogulecki and Xue (1662, 6:631).

The *Section on Sine* then constructed geometric figures to find the sine lengths for 36° and 12°.[44] By applying the two algorithms repeatedly, the sine values can be computed for the following measures, from that of 36°[45]:

$$36°, 54°, 18°, 9°, 4.50°, 2.25°, 72°, 81°, 85.50°, 87.75°, 27°, 13.50°, 6.75°, 63°, 67.50°, 83.25°, 40.50°, 20.25°, 49.50°, 69.75°, 42.75°, 47.25°, 31.50°, 15.75°, 58.50°, 74.25°, 38.25°, 51.75°, 24.75°, 65.25°, 29.25°, \text{ and } 60.75°;$$

and from 12°,

$$12°, 6°, 3°, 1.50°, 0.75°, 78°, 84°, 87°, 88.50°, 89.25°, 39°, 19.50°, 9.75°, 51°, 70.50°, 80.25°, 42°, 21°, 10.50°, 5.25°, 48°, 69°, 79.50°, 84.75°, 43.50°, 21.75°, 46.50°, 69.25°, 44.25°, 45.75°, 25.50°, 12.75°, 64.50°, 77.25°, 35.25°, 54.75°, 24°, 66°, 34.50°, 17.25°, 55.50°, 72.75°, 39.75°, 50.25°, 23.25°, 66.75°, 32.25°, 57.75°, 33°, 16.50°, 57°, 73.50°, 8.25°, 81.75°, 27.75°, 62.25°, 28.50°, 14.25, 61.50°, 75.75°, 36.75°, 53.25°, 30.75°, \text{ and} 59.25°.$$

The order of the measures follows a logical pattern. The algorithms used are the half-arc algorithm and Pythagorean Theorem, as indicated earlier. Starting with 36°, the half-arc algorithm can be applied repeated to get the sine lengths for 18°, 9°, 4.50°, and 2.25°. Next, by performing the Pythagorean Theorem on 18°, 9°, 4.50°, and 2.25°, the sine lines of 72°, 81°, 85.50°, and 87.75° can also be found; still the half-arc formula to 54°, the sine values for 27°, 13.50°, and 6.75° are obtained. Furthermore, Pythagorean Theorem can be performed again to get the sine values of their respective complementary arcs. This pattern determines which measures to be included and their respective position in the list. With these sine lengths, Xue effectively constructs an un-tabulated list of sine values from 0° to 90° with a 0.75° increment.[46]

Next, Xue approximates sin(1°) via a geometric figure in which sine line of 1° is juxtaposed with the sine lines of 1.50° and 0.75° whose values have been found earlier. Xue posits from the figure that the true value of the sine line of 1° is between

$$\frac{\sin(0.75°)}{3} + \sin(0.75°) = (0.)174528 \text{ and}$$
$$\frac{\sin(1.50°) - \sin(0.75°)}{3} + \sin(0.75°) = (0.)174520.$$

44 The approach to finding the sine line of including the figure is the same as the one used by Ptolemy (90 CE–168 CE). See Thomas (2000, 429–431). This is cited from Dong (2011, 60).

45 The geometric figure for finding 36° and the sine values for the measures in the list from 36° can be seen in Smogulecki and Xue (1662, 6:632–633). The geometric figure to find 12° and the sine values for the measures from 12° can be seen in Smogulecki and Xue (1662, 6:633–634).

46 Dong Jie tabulates these sine values and considers that Xue has a regular sine table. See Dong (2011, 190–191).

Therefore, an approximation of the sine value of 1° is taken to be the average of the two numbers, 174,524.[47] Applying the half-arc algorithm to this value, Xue obtains the sine value of 0.5°; however, sin(0.25°) is found by the rule of three, 0.75°: sin(0.75°) ≈ 0.25°: sin(0.25°).[48] The first three quantities are known, and therefore, the fourth can be found.

Most of the content of the *Section of Sine* can be found in the section titled "*De Frabeica Sinuum* (On constructing Sine)" in Simon Stevin's (1548–1620) *Hypomnemata Mathematica*, with the exception of the manner and the order of the measures in the list from sin(3.75°).[49] It is worth noting that all the geometric figures for finding sine values of special arcs in the *Section of Sine* are identical to those in *De Frabeica Sinuum*;[50] moreover, the manner and order in which the sine values are obtained in the lists from sin(36°) and sin(12°) discussed earlier, are evocative of the discussion of the same values in *Hypomnemata Mathematica*. While the sine values in the *Section of Sine* have seven significant digits, the values of in *Hypomnemata Mathematica* have nine digits.

The disconnectedness between the construction principles and the trigonometric table is much more pronounced between the *Table of Four Lines* and *Section of Sine*. The *Section on Sine* is placed right before the *Table of Four Lines* in the compendium even though the logarithmic trigonometric table cannot, as emphasized earlier, be constructed from the principles discussed in the *Section on Sine*. Secondly, the reason for including the procedure of finding sin(1°) is somewhat dubious. In *Grand Measure*, the approximate value of sine length of 1′ is given because it is the smallest measure of arcs and therefore crucial in constructing a table; however, 1° is not the smallest measure in the un-tabulated list, which is 0.75°, nor is it in the logarithmic trigonometric table, which is 0.01°. In the discussion of finding the sine length of 0.25°, the rule of three is applied to 0.25°, 0.75°, and sin(0.75°), the last of which was obtained independent of sin(1°). Consequently, sin(1°) has no crucial role in constructing a trigonometric table in Xue's approach (nor does 0.25° for that matter). This observation suggests that the *Section on Sine* probably has more than one European source.

In spite of the flaws, Xue's efforts to explicitly elucidate a concrete way to construct a sine table should be lauded. While the construction principles in *Grand Measure* along with the sine values of special angles and the smallest measure 1′ provide an atlas

[47] Dong Jie points out a discrepancy here in the computation: If sin(1.50°) is taken as the previously calculated value (0.)261869, then the lesser value $\frac{\sin(1.50°)-\sin(0.75°)}{3}+\sin(0.75°)$ becomes (0.)174553.67, which is in fact greater than the greater value 174,528. This does not seem a typographic error. See Dong Jie (2011, 60–61) for a more detailed discussion and figures. Such a discrepancy suggests that these values and the discussion of finding sin(1°) might be taken from a source different from the one from which the values of other sine lines come.

[48] See Smogulecki and Xue (1662, 6:633–634).

[49] While the *Section of Sine* presents sine values of the measures from the list of sin(3.75°) in the order of an arithmetic progression, *Hypomnemata Mathematica* groups the sine value of a measure with that of its complementary arc, followed by the half of the first entry and then its complementary arc, etc. The discussion of sin(36°) and the measures in its list can be found in Stevin (1605–1608, 7–9), that of sin(3.75°) in Stevin (1605–1608, 9), and that of sin(12°) in Stevin (1605–1608, 9–11).

[50] The figures in the *Section of Sine* are identical to those in Stevin (1605–1608, 7–9) except for the labels for the points.

for interested scholars to build a trigonometric table, Xue's *Section on Sine* actually leads its readers to tread on a path to an un-tabulated list of sine values. A treatise such as this serves an educational purpose by making a difficult process more accessible to a broader audience.

4 The Table of Celestial Arcs

As the Jesuit-introduced trigonometry became more accessible and accepted to certain circles of the educated Chinese, treatises authored by the Chinese on the subject also emerged. Next, we will turn to a trigonometric table included in a treatise by an obscure scholar with limited impact.[51] The importance of such a treatise lies in the fact that the author explicitly identifies the pre-Jesuit Chinese "counterpart" to trigonometric tables from the West, which in turn helps shed more light on the reason why a true "integration" of the Chinese and Western methods in astronomical computation at the end of Ming dynasty was not and could not have been realized.

Tianhu xiangxian biao 天弧象限表 (the Table of Celestial Arcs [in a] Quadrant, hereafter the *Table of Celestial Arcs*) is a work of Li Zijin 李子金, prefaced in 1683. Gao Honglin concludes, based on the fact that Li referred to the *Table of Celestial Arcs* in his *Suanfa tongyi* 算法通義 (Comprehensive Meaning of Computation) published in 1673 that the *Table of Celestial Arcs* or a version of its manuscript must have been completed prior to 1673.[52]

Li also adopts the practice that one (Western) degree is equal to 100 *fen* as did Xue. The *Table of Celestial Arcs* consists of the lengths for sine and cosine lines from $0°$ to $45°$, with a 10-*fen* ($= 6'$)-increment. The radius of the circle is 100,000, and therefore, almost all entries in the table have five or less significant digits. Figure 4 shows a page from the treatise. The first column from the right lists top down 0 [degree], 0 [*fen*], 10 [*fen*], ..., till 100 [*fen*] ($=1°$); the first row, reading from the right, 0 (degree), sine, cosine, empty space, 1°, sine, cosine, and another empty space, where the two empty spaces are headings for arc measures listed in an decreasing order, 90° to 89 and 89° to 88, respectively, for the cosine and sine length of the complementary arcs. Upon comparison, the values in the *Tables of Celestial Arcs* are identical to those in the *Table of Eight Lines* for $6', 12', \ldots, 54'$ of each degree.[53] Li in fact stated in his preface that his table is simplification of the *Table of Eight Lines*.[54]

51 The copy of the *Table of Celestial Arcs* consulted for this article is the one in the library of the Institute of History of Natural Science, Chinese Academy of Sciences, Beijing. It is a hand-copied version donated by Li Yan without pagination. According to Gao Honglin, there is only one extant print copy kept by the author's (Li Zijin's) descendant in Henan province; see Gao (1993). Due to the fact that the existing treatises by scholars well versed in astronomy and mathematic in the eighteenth and nineteenth centuries hardly referred to this work, I speculate that Li Zijin's trigonometric treatise had limited impact.

52 See Gao (1993).

53 Gao (1993) also mentions this fact.

54 The text reads, "天弧象限表者, 乃本西洋之割圓八線表, 而變通其數, 省約其文者也 (The *Table of Celestial Arc* is obtained] based on the *Table of Eight Lines Dividing the Circle* from Western ocean [by]

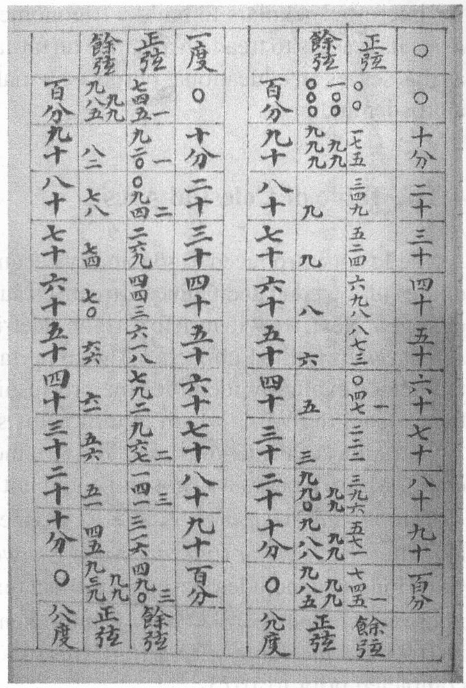


Fig. 4 A page in the hand-copied *Table of Celestial Arcs*

The accompanying text in the *Table of Celestial Arcs* did not discuss its construction principles. Prior to listing the values in the table, Li describes various ways of finding trigonometric lines provided the values for other trigonometric lines of the same arc are given; in addition, Li uses an example of trigonometric lines with concrete lengths to illustrate these procedures. Li also states without justification the properties of triangles, equivalent to the modern-day the Law of Sine, the side-angle-side postulate, and angle-side-angle postulate. All three were demonstrated with examples with concrete lengths or measures for angles.[55]

Trigonometry and the eight lines, Li contends, are developed by "Western Confucians (*xiru* 西儒)"[56] out of the desire to explore the relation between arcs on a circle and their chords while the traditional Chinese methods of *licha* 立差 (cubic difference), *pingcha* 平差 (square difference), and *dingcha* 定差 (corrected difference)[57] for astro-

Footnote 54 continued
expediently changing its numbers and shortening its texts.)" See the preface of Li (1673). Our interpretation of "expediently changing its numbers" is that Li adopts the practice that a degree is equal to 100 *fen*.

55 See Li (1673).

56 That Li used "Western Confucian" to refer to the Jesuits indicates that he was not hostile toward them; there is no indication, however, that Li was a Chinese convert to Catholicism. It is not clear whether he had direct contacts with the Jesuits.

57 The translation for *licha*, *pingcha*, and *dingcha* (*ting cha* in Sivin's book) is taken from Sivin (2009). They are the third-order, second-order, and first-order finite difference for a sequence of astronomical measures. These differences were used for interpolation in astronomical computation. For the text in Season-Granting

nomical computation serve the same purpose.[58] In short, these pre-Jesuit methods can be construed as a Chinese "counterpart" as computational tools in calendar-making. At the same time, such a technical detail also shed light on the reason why the computationally powerful trigonometric tables cannot be appropriated piecemeal into the Chinese system to expedite complex computations.

As the computations in the traditional Chinese approach to astronomy are intricately embedded in the form of algorithms that directly manipulate the observed data, a practitioner of traditional approach can perform computations following the prescription in the algorithms without understanding which line segments might be equivalent to trigonometric lines of certain arcs. That is, there is no obvious part in the process where certain computations can be replaced by consulting a trigonometric table. Consequently, in terms of computation, it is either a wholesale adoption of the Jesuit system along with the trigonometric tables or none at all. From this perspective, Li's or Xue's adopting the practice "1° being equal to 100 *fen*" only gives an illusion of integration of the Chinese and Western methods.

5 A thorough explanation and "proofs" of construction principles

Lisuan quanshu 曆算全書 (Complete Writings on Mathematics and Calendric Astronomy, hereafter the *Complete Writings*), the collectanea of works by one of the most prolific scholars well versed in mathematics and astronomy during Qing China (1644–1911), Mei Wending, was published posthumously in 1723.[59] In this collection, the construction principles of trigonometric tables are explicated in greater details compared to prior treatises on the same subject, while the trigonometric table was seemingly and peculiarly "absent."

In the 1723 printing of the *Complete Writings*, the table of content indicates that a trigonometric table (*geyuan baxian zhi biao* 割圜八線之表, a table of eight lines dividing the circle) will be published later (*xuchu* 續出); see the first column in Fig. 5a. The edition published in 1859, however, does not include a trigonometric table either. Moreover, the space in the table of content where the characters of "a trigonometric table, one *juan*" are supposed to be is left blank, while the characters for "published later" remain; see the first column in Fig. 5b. According to Kobayashi Tatsuhiko, a copy of the 1724 printing of the *Complete Writings*, including a trigonometric table,

Footnote 57 continued

calendar system that describes how these differences were used and Sivin's interpretation, see Sivin (2009, 414–416).

58 The text reads, "古人但以立差平差定差三率...其法與弧背求弦之數暗相符合 ([The] ancients only used the three quantities, the cubic difference, square difference, and corrected difference, . . .; these [three] methods implicitly match the numbers of finding [the length of] the chord from [the measure of] the arc." See Li (1673, preface).

59 Current studies of Mei Wending's works are mostly based on individual treatises in *Complete Writings*. For the scope of Mei's works, see Yan (1989). For the studies on various treatises in *Complete Writings*, see Footnote 3 in Jami (2012, 25).

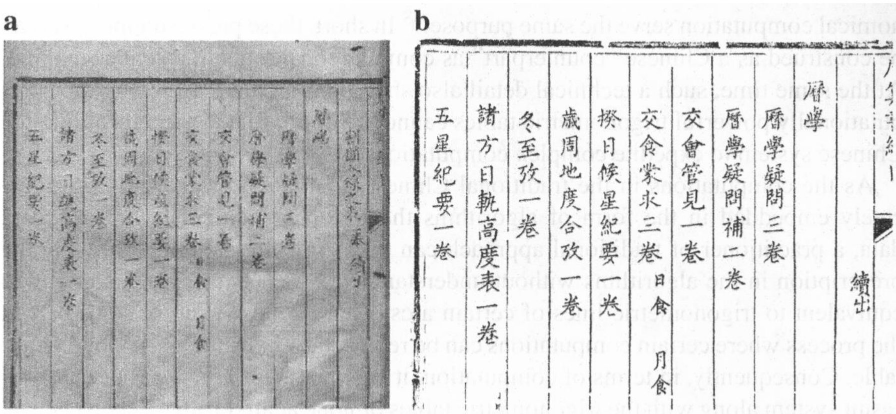


Fig. 5 **a** The Table of Content in the 1723 edition. **b** The Table of Content in the 1859 edition

is extant in Japan. Upon comparison, that trigonometric table is identical to the *Table of Eight Lines* discussed earlier.[60]

The construction principles in the *Complete Writings* are explained in *Jie baxian geyuan zhi gen* 解八線割圓之根 (Explicating the Origin of [how] Eight Lines Divides the Circle, hereafter *Explicating the Origin*). Compared to the explanation in earlier works, the elucidation can qualify as either explicit geometric constructions or proofs for desired properties. The existing evidence shows that *Explicating the Origin* was composed by Yang Zuomei although Mei had an unpublished manuscript that might or might not have been its basis.[61] *Explicating the Origin* singles out the insufficient explanation of construction principles in *Grand Measure* and Xue's *Section of Sine* as one of the reasons for its composition.[62] Its title rightfully and adequately reflects its purpose.

60 An extant Japanese translation of the *Complete Writings* was based on the 1724 printing. See Kobayashi (2003, 255–262). According to Feng Lishen, the trigonometric table in the 1724 printing is identical to the *Table of Eight Lines*. See Feng (2009, 177–186).

61 In Mei's records and notes of his own treatises and manuscripts, there is an entry of a manuscript titled *Zhengxian jianfa bu* 正弦簡法補 (Supplement to the Simple Methods of Sine); see Mei (1703, 47b–48a). *Explicating the Origin*, however, is not listed among the manuscripts. In the table of content of the 1724 printing, it is indicated that *Explicating the Origin* was written by Yang Zuomei (*Yang zuo* 楊作, composed by Yang); see Kobayashi (2003, 255). The front page of the treatise also indicates that *Complete Writings* was edited and supplemented (*ding bu* 訂補) by Yang. Mei Wending's grandson Mei Juecheng was not satisfied with *Complete Writings* and composed a manuscript documenting the numerous errors and criticizing the inept choice of the title. In particular, Mei Juecheng emphasized that *Explicating the Origin* was not a work of his grandfather and it should not have been included in the collection; see Mei (1739). I thank Professor Han Qi for this piece of evidence.

62 According to *Explicating the Origin*, *Grand Measure* did not provide the construction of the inscribed pentagon and decagon, and Xue Fengzuo's *Section of Sine* provided the algorithms without any furthering and illumination (*faming* 發明). *Grand Measure* refers to Book 13 of Euclid's *Elements* for the construction for those two regular polygons; however, Book 13 was not translated into Chinese. To be fair, Xue did provide certain explanation to the computation. See Yang (1723, 1b) for the criticism.

The sequential presentation of the material in *Explicating the Origin* roughly follows that in *Grand Measure*, the sine values of special arcs are constructed and calculated first, followed then by the discussions and "proofs" of the algorithms for computing the general sine values from those of the special arcs. The first six inscribed regular polygons constructed in *Explicating the Origin* are exactly the "six essentials (*liuzong*)" in *Grand Measure*; the treatise makes an effort to provide geometric construction before computing the length of the side for each of the inscribed regular polygons. In addition, *Explicating the Origin* also provides the construction of an inscribed regular nonagon (nine-sided) as well as the calculation of chord length of an arc of 40°. Even though a crucial step is absent in the provided construction, the calculation of the chord of 40° is nevertheless accurate with the aid of the value of $\sin(1^\circ)$ and some algorithms which appear later in the treatise.[63] The lengths of the sides of these inscribed regular polygons are called in the treatise as *biaogen* 表根 (origin of the table).

Along with the sine lengths of 30°, 45°, 60°, 18°, 36°, 12°, and 20° (from the sides of the 7 inscribed regular polygons), *Explicating the Origin* also describes two ways of finding the value of $\sin(1^\circ)$. The first approach is the one employed by Xue in the *Section of Sine*; consequently, the value of $\sin(1^\circ)$ obtained is exactly as in Xue's, 174,524. The other utilizes the rule of three to find an approximated value of $\sin(15')$ 436,326, based on the proportional relation of the sine lines for 22′30′ and 11′15′ as in *Grand Measure*. Then, the algorithm to find the sine of the sum of two arcs is applied to the sine and cosine lines of 45′ and 15′ to get $\sin(1^\circ)$ as 1,745,236,145. It was commented in the treatise that this value is comparable to Xue's but seems more accurate.[64]

The seven methods to find general sine and cosine values are described in *Explicating the Origin* as "methods of constructing the table" (*zuobiao zhi fa* 做表之法).[65] The Method Three is identical to the equation (2-1), describing the equality of two expressions. The Method Seven is a special case of Ptolemy's Theorem, in which one side of the inscribed quadrilateral is a diameter of the circle: If four of the three sides and two diagonals are known, the length of the last side can be found. The rest of the methods are actually algorithms. We summarize and contrast them with their counterparts in *Grand Measure* in Table 4.

63 The missing step is concerned with the construction of a line segment by connecting a point with another point of an already-constructed line segment. The prescription did not specify which point on the segment should be chosen and seemingly implied that the choice should not matter, which proves to be to the contrary in later construction. Assuming the construction of a regular nonagon, however, the length of its side can be calculated correctly, by applying Pythagorean Theorem to the length of a side of a regular decagon and the value of $r\sin(1^\circ)$, which the treatise provides much later. See Yang (1723, 15a–17b).

64 The value of $\sin(45')$ is obtained following Xue's approach to $\sin(1^\circ)$. Pythagorean Theorem is then applied to find $\cos(45')$. The treatise commented that the proportionality of the arcs and the corresponding sine lines holds for arcs smaller than 20′ and that even though there are discrepancies in the proportionality computation, they cannot be detected by [current] computational method. See Yang (1723, 18).

65 Technically, there are more than seven algorithms. For example, the first method contains algorithms of finding, given the sine value of an arc, the cosine of the same arc, and the sine and cosine of its half-arc. See Yang (1723, 20a).

Table 4 Comparison of computational methods in *Explicating the Origin* and in *Grand Measure*

Given quantities	Desired quantities	Counterparts in *Grand Measure*	
$r\sin(\alpha)$	$r\cos(\alpha), r\sin\left(\frac{\alpha}{2}\right), r\cos\left(\frac{\alpha}{2}\right)$	(3-1), (3-3)	M-1
$r\sin(\alpha), r\cos(\alpha)$	$r\sin(2\alpha), \quad r\cos(2\alpha)$	(3-2), (3-1)	M-2
$r\sin(\alpha), r\cos(\alpha), r\sin(\beta), r\cos(\beta)$	$r\sin\left(\frac{90-(\alpha+\beta)}{2}\right)$		M-4
$r\sin(\alpha)$	$\mathrm{vers}(2\alpha), r\cos(2\alpha)$		M-5
$r\sin(\alpha), r\cos(\alpha), r\sin(\beta), r\cos(\beta)$	$r\sin(\alpha+\beta), r\sin(\alpha-\beta)$	(2-2)	M-6
$r\sin\theta = r\sin(60^\circ+\theta) - r\sin(60^\circ-\theta)$		(2-1)	M-3
A special case of Ptolemy's Theorem			M-7

In discussing these methods, the explanation amounts to providing "proofs" although it does not stress establishing the validity of the methods.[66] The geometric figures and the proof for M-6 [or (2-2) in *Grand* Measure] are elucidated in detail in *Explicating the Origin*.[67]

No trigonometric table was produced following *Explicating the Origin*, and the table included in the 1724 printing was the same as the *Table of Eight Lines*. These observations reinforce our perception of disengagement between the principles of construction and the actual acts of construction. The situation will change, however, for the next two trigonometric tables, which were amended or constructed anew following the discussion of additional properties in the construction principles.

6 Trigonometric tables imperially composed

Yüzhi shuli jingyun 御製數理精蘊 (the Essence of Numbers and Their Principles Imperial Composed, hereafter *Numbers and Principles*) was among the numerous editorial projects taken by the court following Kangxi Emperor's (1654–1722) decrees toward the end of his reign.[68] Printed a few months before Kangxi's death in 1722, this compendium was the "final outcome of Kangxi's lifelong pursue of mathemat-

[66] The text mistakenly describes Method Four as among those in *Grand Measure* although there is no direct counterpart for Method Four. See Yang (1723, 23b). Method Seven is also in *Grand Measure*, but not as part of construction principles of trigonometric tables; see Schreck (1631, 1178).

[67] Upon comparison, the figures and explanation for Method Six in *Explicating the Origin* are identical to those in Wang Xichan's *Yuanjie* 圜解 (The Circle Explicated). See Dong Jie (2011, 145).

[68] For a description of the general editorial projects following Kangxi's decree, see Jami (2012, 267–280). *Yuzhi lüli yuanyuan* 御製律曆淵源 (Origins of Pitch Pipes and the Calendar Imperial Composed) is the result of the project on mathematics, astronomy, and music; *Numbers and Principles* is one of the three parts. The other two are *Lixiang kaocheng* 曆象考成 (Thorough Investigation of Astronomical Phenomena) and *Lü lü zhengyi* 律呂正義 (Correct Interpretation of the [standard] Pitch Pipes). For the description of the compilation of *Numbers and Principles* and its related personnel and offices, see Han (1992, 1999, 2014, 2015).

Fig. 6 Trigonometric table in *Numbers and Principles*

ics."[69] It contains almost all things mathematical known in early eighteenth-century China,[70] including a trigonometric table and its principles of construction. Backed by the resources at the court, the trigonometric table in the *Numbers and Principles* has a 10-s increment, the smallest among the extant tables before 1800 in China. The values in the trigonometric tables in *Numbers and Principles* render the radius of the circle to be 10,000,000. The table contains the lengths of six trigonometric lines from 0° to 45°. Figure 6 shows the upper half of the first page of the trigonometric table.[71]

Reading from right to left, the headings in the top row of Fig. 6 are, 0°, sine, tangent line, secant line, second (and) minute. The characters for "second" and "minute" are in the same register. It is worth pointing out that the numerical values in the table should be read from left to right, contrary to that in the *Table of Eight Lines*, from right to left. For example, the value of $\sin(4')$, the last entry in the column of sine (the second column) in Fig. 6, is 一一六三六 (one one six three six from left to right), same as in the modern form while the entry recorded in the *Table of Eight Lines* for $\sin(4')$ is 六一一 (six one one from left to right) although it means 116 in the modern form. It is

69 The general belief used to be that *Numbers and Principles* was completed and published in 1723. Based on a memorial to Kangxi Emperor, Han Qi concludes that bronze movable type edition of *Numbers and Principles* was printed in 1722 prior to Kangxi's death. Private communication with Han Qi. For certain editions of *Numbers and Principles*, see Fan (1999). The quote is taken from Jami (2012, 8).

70 One exception is the subject of symbolic algebra. While cossic algebra (*Jiegenfang* 借根方 in Chinese) takes up 6 chapters (*juan* 卷) in *Numbers and Principles*, symbolic algebra (*Aerrebala xin fa* 阿爾熱巴拉新法)) was not included. See Han (1999, 2007), and Jami (2012, 294–304). For a detailed discussion of the construction of *Numbers and Principles*, see Han (1991) and Jami (2012, 315–384); in particular, for the list of subjects contained in the compendium; see Jami (2012, 317–319).

71 See *Shuli jingyun biao* (*baxian biao* 1:1a). The trigonometric table, logarithmic trigonometric table, and logarithmic table in *Numbers and Principles* are grouped together in *Shuli jingyuan biao*, 數理精蘊表 (Tables in the *Essence of Numbers and Their Principles*). In this section for tables, certain explanatory texts are also included. Placed before the trigonometric table, *Baxian biaoshuo* 八線表說 (Discussions of the *Table of Eight Lines*) has its own pagination separated from that of the table. For the discussion of the tables in *Numbers and Principles*, see Han (2014).

not clear when, how, and why the new way of recording numerical values horizontally occurred.

The trigonometric table in *Numbers and Principles* is not a direct translation from a European source. The texts preceding the table describe that an old table from the West has the radius 100,000 while a new table has the radius 10,000,000 and the values are listed to the minute.[72] Moreover, due to the fact that the interpolated approximation values for tangent and secant lines are not "precise" (*mihe* 密合) for arcs greater than 60°, it explains, the compilers use certain methods to find the trigonometric values for every 10 s.[73] It is worth noting that this is the only instance of texts we know in which the precision of the values in the trigonometric table is described as an issue. It is not clear whether such a statement was made based on experiential evidences.[74] The construction principles are mainly for obtaining values for sine and cosine lines; other trigonometric lines are obtained through the basic relations between sine/cosine lines and others.

The task of amending a trigonometric table is a considerable feat in itself. The number of entries is 2700 ($=45 \times 60$) in the original table with a 1-min increment; to amend such a table, additional 13,500 ($=45 \times 60 \times 5$) entries should be added to obtain the desired 10-s increment. Without the resources at the court, such a task might take years to complete by any individual.

The sixteenth chapter (*juan* 卷) of the second part of *Numbers and Principles* deals with topics that fall under "division of the circle (*geyuan* 割圓),"[75] among which are the definitions of the eight lines (*geyuan baxian* 割圓八線), the construction principles of trigonometric tables (i.e., *liuzong sanyao, erjianfa*, the same terms in *Grand Measure*), the fundamental relations between trigonometric lines (*baxian xiangqiu* 八線相求),

[72] It is possible that the old table from the West refers to the *Table of Eight Lines*. Several trigonometric tables were made for Kangxi Emperor's personal use: a table of sine, tangent, and secant lines and the logarithmic values of sine and tangent lines in *Shubiao* 數表 (Numerical Tables), and a table of the logarithmic values of the sine, cosine, tangent, and cotangent lines in *Dushubiao* 度數表 (Tables of Degrees and Numbers), and a table of sine, tangent, and secant lines in the *Yuzhi shubiao jingxiang* 御製數表精詳 (Essential Details of Numerical Tables Imperially Composed). Upon comparison, the values in the trigonometric table in *Details of Numerical Tables* are the same as those in the regular trigonometric table in the *Numerical Tables*, which uses 100,000 as the radius. They are different from the *Table of Eight Lines*. It is not known whether these tables were translated or constructed from scratch. For discussions, see Han (1991, 1997). Jami (2012, 309) also has a short discussion on *Details of Numerical Tables*. For these tables, see Shubiao (2000), *Dushubiao*, and Yüzhi shubiao jingxiang (2000). I thank Professor Han Qi for showing me the former two tables.

[73] The text reads, "西洋舊表設半徑為十萬...又有新表設半徑為一千萬, 逐分列表. 用中比例以求秒數只可用於正弦餘弦. 若切線割線至六十度以後...用中比例尚不能密合, 又用本法細推每十秒. 遞析求零秒則用比例所差 無多." See *Shuli jingyun biao* (*baxian biao shuo* 1:1a). I have not been able to identify the source of the "new table" although it might be one brought by the French Jesuits to China in 1690s.

[74] The less-accurate nature for the tangent and secant values for arcs closer to 90° is expected. These values are obtained by dividing sine values or the radius with the cosine values. When the measure of the arc is closer to 90°, the values for cosine lines are closer to 0. Consequently, any small discrepancy in cosine is magnified tremendously in the tangent and secant lengths.

[75] Two *juan* discusses the knowledge that fall under "division of the circle," the fifteenth and sixteenth *juan* of Part II of *Numbers and Principles*. The fifteenth discusses the methods found in traditional Chinese mathematics (*Yüzhi shuli jingyun*, 3:526–544), while the methods in the sixteenth are considered Western (*Yüzhi shuli jingyun*, 3:545–565).

and the prescriptions of finding the trigonometric lines for (all) arcs in a quadrant (*qiu xiangxiannei gexian zongfa* 求象限內各線總法). The definition of the trigonometric lines is broadened to include obtuse arcs, whose measures are between 90° and 180°. The eight trigonometric lines for obtuse arcs are in effect the ones for their supplementary arcs. Emphasis is again placed on the sine line, the text explains, since other lines are derived from sine lines (*zhuxian jie youci ersheng* 諸線皆由此而生). It is no surprise that the values of special arcs and the algorithms are geared toward finding the lengths of the sine lines.

To introduce new algorithms into the construction principles, *Numbers and Principles* first points at a flaw in *Grand Measure* as a segue: (certain) sine lengths for as small as 45 min can be obtained through the algorithms, yet for arcs less than 45 min, their trigonometric lengths need to be obtained through approximation (*bili fa* 比例法, the rule of three). To remedy this situation, *Numbers and Principles* then introduces two algorithms described as *yishi guichu* 益實歸除法 (the method of augmenting dividend and dividing numbers), to be described below. These algorithms prescribe procedures to solve problems equivalent to finding a positive root of a particular type of polynomials of third degree. Two important applications are to find the lengths of the sides of an inscribed regular tetradecagon (fourteen-sided) and octadecagon (eighteen-sided), and consequently, the lengths of the sides of inscribed regular nonagon and heptagon (seven-sided) can be found as well. Moreover, the one-third arc formula that finds the sine value of one-third of the arc given the sine value of the arc can also be derived with the same *yishi guichu* algorithm. In short, this particular algorithm can augment both the special sine values and the algorithms that generate more sine lengths.

Numbers and principles elaborates further that with the sine values from these additional inscribed regular polygons, one can find three hundred and sixty sine values of arcs in a quadrant with a 15-min increment. The value of $r \sin(5')$ can be calculated by applying the one-third arc formula to $r \sin(15')$; consequently, only the sine values of one through 4 min need to be obtained through the rule of three.

Numbers and Principles continues to use "six essentials, three important, and two simple methods" to label the construction principles even though there are now ten essentials and eight algorithms available. As the purpose of the discussion is to find sine values of special arcs, the texts do not include geometric construction of these inscribed regular polygons, in contrast to Mei's and Yang's practice in their treatise.

The two algorithms of *yishi guichu* method introduced in *Numbers and Principles* are in the form of two similar questions[76]:

1. Given four quantities of continued ratio (*lian biyi*, 連比例, i.e., $a{:}b = b{:}c = c{:}d$), suppose $a = 10{,}000$ and $a + d = 3b$, find b, c, and d.
2. A similar statement with the condition $a + d = 3b$ is replaced by $a + d = 2b + c$.

While the *yishi guichu* method resembles the traditional procedures of extracting cubic root with positive lower-degree terms (*kai daizong lifang*, 開帶縱立方, equiva-

[76] The two questions, the procedures to solve them, and the geometric explanation can be found in *Yüzhi shuli jingyun* (3:556–558, 3:560–573).

Table 5 Comparison of computational methods in *Numbers and Principles* and in *Grand Measure*

Given quantities	Desired quantities	Counterparts in *Grand Measure*
$r\sin(\alpha)$	$r\cos(\alpha)$	(3-1)
$r\sin(\alpha)$, $r\cos(\alpha)$	$r\sin(2\alpha)$, $r\cos(2\alpha)$	(3-2), (3-1)
$r\sin(\alpha)$, $r\cos(\alpha)$,	$r\sin\left(\frac{\alpha}{2}\right)$, $r\cos\left(\frac{\alpha}{2}\right)$	(3-3), (3-1)
$r\cos(\alpha)$	$r\cos(2\alpha)$, $r\cos\left(\frac{\alpha}{2}\right)$	
$r\sin(\alpha)$	$r\sin\left(\frac{\alpha}{3}\right)$	
$r\sin(\alpha)$, $r\cos(\alpha) r\sin(\beta)$, $r\cos(\beta)$	$r\sin(\alpha \pm \beta)$	(2-2)
$r\sin(60° + \alpha) r\sin(60° - \alpha)$	$r\sin(\alpha)$	(2-1)

lent to finding a root of certain polynomials of third degree),[77] it is in effect influenced by Western mathematics. The texts state without elaboration that in a logarithmic table of Western method, four quantities in the continued ratio were used to find inscribed heptagons and nonagons; the text goes on to say that the idea was expanded to devise the methods and find [the side lengths of] an inscribed regular tetradecagon and octadecagon.[78]

The algorithms discussed in the part of "three important" and "two simple methods" in *Numbers and Principles* are summarized in Table 5. Both the given and desired quantities in the algorithms are stated with arcs of concrete measures, and their trigonometric values instead of general terms: the explanation, construction, or "proof," though formulated with the same numerical values in question, can be applied directly to general arcs. Historians of Chinese mathematics in general agree that these explanatory texts constitute proofs for these algorithms.

The algorithm of finding the sine value of the one-third of the arc with a known sine length is demonstrated with an example: Given the sine value of 36°, 58,778.5252292, find the sine of 12°, one third of 36°.[79] The solution: Double sin 36° to get the chord of 72°. Then use it to multiple the square of the radius ten thousand as the dividend; triple the square of the radius ten thousand as the divider. Apply the [first] *yishi guichu* algorithm to get [the second quantity] 41,582.3381634, which is the chord of 24°. One-half of this value, 20,791.1690817, is the sine value of 12°.[80] The geometric

[77] For example, in the question where the condition is $a + d = 3b$, it can be deduced that in the modern language, the polynomial equation to solve should be $x^3 + (100{,}000)^3 = 3(100{,}000)^2 x$. See Guo and Li (2010, 663). In Chinese tradition, finding a solution to a polynomial of the second or third degree is considered part of theory of extracting roots, the process of which is considered similar to division and consequently shares certain terms in their operations. A discussion of Chinese theories of extracting roots would be long-winding and distracting. We refer interested readers to Martzloff (1997, 240–268).

[78] The texts read, "爾來, 西法對數表內有設連比例四率以求圜內容七邊九邊二法, 因推廣其理...增求圜內容十八邊十四邊形之法." See *Yüzhi shuli jingyun* (3:547).

[79] The decimal expression of the sine length of 36° is described in *Numbers and Principles* as 58,778 and, in smaller characters, "with a small extra" (*xiaoyu* 小餘) 5,252,292. See *Yüzhi shuli jingyun* (3:569). Other decimal values follow the same practice.

[80] See *Yüzhi shuli jingyun* (3:569–570).

explanation following the procedure also works for any acute angle. Therefore, the algorithm of finding the sine of the one-third of an arc is then established.

The *Numbers and Principles* does not articulate how a trigonometric table with a 1-min increment was amended to one with a 10-s increment. One can speculate that the first step should be finding the sine length of $10''$. One way to achieve this goal is to take the sine length of the smallest measure $1'$ from the table, apply the half-arc formula to get the sine value of $30''$, and then apply the algorithm to find the sine of one-third arc to get the length of $\sin 10''$. Once the sine length of the smallest measure is obtained, the sum and difference formula for sine and cosine can be applied repeatedly to get those sine values with $10''$ increments between sine values of arcs of consecutive minutes. It is evident from our analysis that the augmentation of the trigonometric table in *Numbers and Principles* is directly related to the one-third arc formula for sine lines. This is a departure from the previous treatises in which the tables provided were simply copied or simplified from existing ones, hence completely separated from the principles discussed.

7 A trigonometric table potentially constructed from scratch

An Qingqiao's *Yixian biao* 一線表 (*The Table of One Line*) might potentially be the only extant trigonometric table that was constructed afresh by an individual Chinese scholar based on the construction principles available. It can be found in three different treatises, *Juxian yuanben* 矩線原本 (Elements of Lines of the Carpenter Square) prefaced in 1818, *Yixian biao* yong 一線表用 (Applications of the Table of One Line) prefaced in 1817, and *Xuesuan cunlue* 學算存略 (Brief [Notes] of Learning Mathematics), with slight variations.[81] This table is extremely short with a total of two hundred and fifty entries. It has only values for the sine lines, hence the title, *Table of One Line*. Secondly, An's choice of the unit renders the measure of the entire circle to be one hundred *du* 度 (degree), each of which is equal to one hundred *fen* 分 (minute), each of which in turn is equal to one hundred *miao* 秒 (second).[82] Such a measuring system for arcs stands in stark contrast to the system, 360° for the full circle, used by the Jesuits and the majority of the scholars since late Ming, as well as to the traditional systems, roughly 365.25 *du*, used in various Chinese calendar systems prior to the Jesuit presence in China.[83] Considering that the shape of heaven is circular in Chinese

[81] The sine table in *Elements of the Lines of Carpenter Square* is titled *Geyuan yixian biao* 割圓一線表 (The Table of One Line Dividing the Circle), while the tables in the other two treatises are both titled *Yixian biao* 一線表 (The Table of One Line). In *Elements of the Lines of the Carpenter Square* and *Brief Notes of Mathematical Learning*, the table is one page long while in *Applications of the Table of One Line*, it is a little over three pages. They all nevertheless have 250 entries. See An (1818, 12:335; 1817, 12:339–341; 1997, 12:595) for the table in the respective treatise. In *Brief Notes*, An also has a partial tangent table at the end.

[82] An's circular unit was considered by Dai Zhen (1724–1777) in composing his *Gougu geyuan ji* 句股割圜記 (Records of Base-Altitude and Circle-Division) but eventually discarded. See Chen (2011) for the discussion on Dai's choices of measuring unit for the circle.

[83] Each of the Chinese calendric systems has its own constant for the *suishi* 歲實 (year numerator), which comes from the constant *suizhou* 歲周 (year cycle or year length), the number of days in a year. See Sivin

Fig. 7 *Table of One Line*

Classics, that the sun makes a full circle on the celestial sphere over a course of a day, and that one full day has one hundred *ke* 刻 (14.4 min), An's choice of the unit effectively establishes the correspondence between the one *du* of the circle and the distance travelled by the sun on the celestial sphere during the period of one *ke*. In short, An's choice of circular arcs results from the convergence of the temporal and spatial notions.

The Table of One Line has values for sine lines from 0 to 25 *du* (90°) with the increment of 10 *fen* (0.36°). The radius is 100,000.

Figure 7 is the complete *Table of One Line*.[84] The entries in the first column are 0 du (*chudu* 初度), 10 *fen*, 20 *fen* . . ., 90 *fen* and those in the second the sine values for the corresponding arcs in the first column. The right leaf contains the entries for arcs ranging from 0 du to 12 *du* 50 *fen* in the ascending order while the left 25 *du* to 12 *du*

Footnote 83 continued
(2009, 272–277) for the discussion on the length of the tropic year in various systems. These systems did not explicitly use their year length as the measure of the full circle. Chinese scholars in the seventeenth and eighteenth century, however, construed these numbers in the ancient systems as the measures of the circle used by the ancients. For example, Wang Xichan 王錫闡 (1628–1682) in his *Xiao'an xinfa* 曉庵新法 (New Methods by Xiao'an [Wang Xichan]) used three different units for measuring the circular arcs, the measures for the circle are 360°, 384 *yaoxian* 爻限, and 365 and some *ridu* 日度 (sun degree). See Wang (1983, 459). Dai Zhen in discussing his choice of circular unit explicitly stated 365 plus some *du* was used by the ancients as the measure for the circle. See Dai (2002, 129).

[84] See An (1818, 12:335).

Table 6 Computational algorithms in the *Table of One Line*

Given quantities	Desired quantities	
$r\sin(\alpha)$	$r\sin\left(\frac{\alpha}{5}\right)$	(1-1)
$2r\sin(\alpha)$	$2r\sin\left(\frac{\alpha}{2}\right)$	(1-2)
$r\sin(\alpha)$	$r\cos(\alpha)$	
$r\sin(\alpha), r\cos(\alpha)$	$r\sin(2\alpha), r\cos(2\alpha)$	
$r\sin(\alpha)$	$r\sin(3\alpha)$	(1-3)
$r\sin(\alpha)$	$r\sin(5\alpha)$	
$r\sin(\alpha), r\cos(\alpha)\ r\sin(\beta), r\cos(\beta)$	$r\sin(\alpha \pm \beta)$	(1-4)

50 *fen* in the descending order, reading from top to bottom and from right to left. The entries in the first column of the left leave are 25 *du*, 90 *fen*, 80 *fen*, ..., 10 *fen*. The measures 90 *fen*, 80 *fen*, and so on are to be understood as 24 *du* 90 *fen*, 24 *du* 80 *fen* ..., etc. Again, the next column consists of the corresponding sine values.

The construction principles accompanying the *Table of One Line* are described by An as taken from both the Chinese and Western methods. *Application of the Table of One Line* only lists two algorithms and directs readers to *Elements of Lines of the Carpenter Square* for details. The explanatory texts before the table discuss the construction principles as well. Although *Six Essentials*, *Three Important*, and *Two Simple Methods* are not mentioned, the construction more or less follows the similar pattern. We summarized An's algorithms for construction in Table 6.[85]

Algorithms (1-1) through (1-4) were provided with geometric figures and textual explanation, while the rest were simply stated with minimal explication. An describes Algorithm (1-1) of finding the sine length of the fifth of an arc as a method from *Jiegenfang* 借根方 (calculation by borrowed root and powers), the version of cossic algebra introduced to China by the Jesuits at the end of seventeenth century.[86] To establish the procedure of the algorithm, An first portrays the sine lines of an arc, α, and its one-fifth, $\frac{\alpha}{5}$, on a circle. An then describes in words with the aid of a simple geometric figure a process involving geometric reasoning as well as some form of "symbolic algebra" manipulation to obtain a relation, equivalent to a fifth-degree polynomial in which the root is the chord of the one-fifth of the given arc and can be found by applying a technique in "calculation by borrowed root and powers." Though deserving careful investigation and explication, the derivation of Algorithm (1-1) through geometry and symbolic algebra will prove long-winding and decidedly distracting. Therefore, it will be discussed separately in a different article.[87]

[85] The order of the algorithms in the table is the same as that of appearance in the treatise. See An (1818, 12:330–334).

[86] For a detailed discussion on cossic algebra and the "calculation by borrowed root and powers" in eighteenth-century China, see Han (2003) and Jami (2012, 201–210).

[87] Algorithm (1-1) was also discussed by Wang Lai 汪萊 (1768–1813) and Jiao Xun (1763–1820). Jiao basically elaborates on Wang's geometric construction, and their derivations of the fifth-degree polynomial relation are of purely geometric nature. See Wang (1798, 4:1505–1509). An's derivation in algorithm (1-1) involves the squares of expressions of fractions and multiplications of fractions in words not numbers with

Algorithms (1-1) and (1-2) are crucial in the construction of An's sine table due to his peculiar choice of 100 du as the arc measure for a full circle. To An, the main issue here is to find the sine value for the "smallest" arc, 1 *miao* (=0.00036° or 1.296 s). To do so, An applies Algorithm (1-1) to the sine of 25 *du*, the radius (1,000,000), to get sine of 5 *du*; continuing with this process, An obtains the sine values for 1 *du*, 20 *fen*, 4 *fen*, and 80 *miao* and 16 *miao*. Switching to (1-2), An then acquires the sine values for 8, 4, 2, 1, and then $\frac{1}{2}$ *miao*. In describing this process, An used the chord (*tongxian* 通弦) instead of the sine line, which is one-half of the chord of the half-arc. Moreover, An does not provide concrete numerical values for these sine lengths except for the chord of one *miao*, the smallest unit in his system, as 62.831853.[88] From this value, An concludes the ratio, equivalent to an approximation for the value of π, between the diameter and the circumference is: the diameter, twenty million, and the circumference, 62,831,853.

To construct a trigonometric table from scratch is no small feat, especially when no institutional support or mechanic computational aid is in place to expedite the process. So, did An construct his sine table manually on his own? According to our analysis below, it is very likely that the table was constructed independent of previously existing ones. Since most measures for the arcs in An's table, e.g., 10 fen = 21′36″, do not conform to the 360° system and therefore do not appear in any of other existing trigonometric tables, it can be said with certainty that An at least worked out the sine values for these measures. Is it possible that An took the sine values of the arc measures that do appear in early tables, e.g., 50 fen = 1°48′ to form some kind of grid and then computed the rest? By comparing these values in the *Table of One Line* with those in the *Table of Eight Lines* and in *Numbers and Principles Table*, we concluded that this is highly unlikely.

Fifty arc measures in An's *Table of One Line* also appear in the previously analyzed tables. These measures form an arithmetic sequence starting at 50 *fen* (= 1°48′), with an increment in the same measure. When compared to the sine values of those in the *Table of Eight Lines*, which uses the same radius, seven out of fifty have different sine lengths. Granted the discrepancy is in the last digit, a rate of 14 % seems to be very high, considering that the task was merely to copy the values. Had the sine values been taken from *Numbers and Principles*, where the radius of the circle is ten million, certain rounding should have been applied. Whatever rounding system was used, the rate of discrepancy is at least 12 %.[89] With these high discrepancy rates and the singular measuring system, we argue that it is highly unlikely that An took these values from the existing tables or any other source. It is rather difficult to comment on whether

Footnote 87 continued
other expressions. Existing evidences and current scholarships seem to suggest that the discussion of algorithm (1-1) originated from Ming Antu, to be discussed below.

88 An's description for this value is, six two with small extra (*xiaoyü* 小餘) eight three one eight five three, similar to the usage in *Shuli jingyun*. The discussion on the process of finding the chord of one *miao* and the rest can be seen in An (1818, 12:332).

89 It is not very clear how these values were rounded or if such a system was in place. The system that produces the fewest discrepancies is to round up when the rounding digit is 6 or more and to round down otherwise. The measures that do not conform to this system are 45°, 57°36′, 61°12′, 64°48′, 68°24′, and 77°24′.

An had any collaborator, foreign or Chinese, because little is known about An's life. Based on the evidence presented, we posit that the *Table of One Line* was probably constructed afresh by An Qingqiao independently, heavily relying on the algorithms of finding the sine values of one-fifth and one-half of an arc.

8 Reflection on construction principles and "accuracy" of the table

The system of construction principles codified in *Six Essentials, Three Important*, and *Two Simple Methods* consists of trigonometric values of certain special arcs and two kinds of algorithms. The first kind will generate more trigonometric values from the known values to populate the table, and the second kind helps obtain the trigonometric lengths of the "smallest" arc in the table. In particular, the algorithms that find the sine values of $\frac{\alpha}{2}$, $\frac{\alpha}{3}$, and $\frac{\alpha}{5}$, where the sine and cosine values of α have already been obtained, serve both capacities. The algorithm equivalent to $\sin(a+b)$, e.g., (2-2) in *Grand Measure*, is the main tool to populate a trigonometric table. To find the sine value of the "smallest" arc in the table, the rule of three was employed to obtain its approximation. Only after the algorithms equivalent to finding $\sin(\frac{\alpha}{5})$, $\sin(\frac{\alpha}{2})$, and $\sin(\frac{\alpha}{3})$ were developed, the sine value of the smallest arc in the table could be obtained independent of the application of the rule of three. Whether the system is sexagesimal or centesimal, one can obtain the values of $r\sin(1')$, $r\sin(1'')$ or the sine length of 1 *miao* by applying these three algorithms to the sine length of the special arc 60° (or 45°, 12.5 *du* in An's case) since only 2, 3, and 5 appear in the integer factorization of 60 and 100. In particular, the approximation by the rule of three is not needed.

To create a trigonometric table "better" than an existing one, one could in principle improve the precision of the values of the entries in the table in two different ways. The first one is to make the increment in the table smaller; or a greater radius of the circle is used, which amounts to providing more accurate significant digits. Both approaches are rather prohibitive for individuals without resources due to the tremendous amount of computation involved. In the direct applications of the table in aiding calculation, there should be another issue of how accurate the entries are in the table. The reality is, most Chinese authors treating trigonometric tables did not seem concerned with this issue, judging from the absence of discussion in their treatises.[90] Whether this was because the level of accuracy of entries in the table provided more than satisfied the need of computation in application or whether the authors did not consider the issue worthy requires further investigation. Just when one might naively conclude that the construction principles had reached the zenith in China when the algorithm of finding $\sin(\frac{\alpha}{5})$ was developed, and therefore, no enhancement of construction principles should be proposed or needed, a brand new approach to constructing trigonometric tables began to emerge in the early nineteenth century.

[90] As discussed earlier, the particular texts in *Numbers and Principles* are the only extant record that explicitly mentions the imprecision of the values of tangent and secant lines when the measures of the angles are close to 90°.

9 A Revolution of Construction Principles of Trigonometric Table

This revolutionary approach produces algorithms similar to calculating finite sums in the series expansion of trigonometric functions; consequently, many historians of Chinese mathematics characterize it as achievements of power series in China. Such a characterization is somewhat misleading because it imposes the observer's view over the actor's. It should be emphasized that these algorithms produced from this approach were for the most part considered by the Chinese scholars in the eighteenth and nineteenth century as an alternative and *quick* way to find the trigonometric lengths of given arcs; in other words, they are better tools in construction of trigonometric tables.

Before 1819, circulated among certain milieu of scholars well versed in mathematics were manuscripts of algorithms, including one that finds the sine length directly from the measure of the arc. Though other names were also associated with the manuscripts, historians of Chinese mathematics generally agree that these manuscripts were abridged copies of Minggantu's or Ming Antu's 明安圖 work.[91] The importance of these algorithms, above all, is that they are for general arcs, not merely for arcs of special measures. They amount to, in the modern mathematical language, calculating a finite sum of the power-series expansion of sine and inverse sine functions.[92] This revolutionary approach to finding trigonometric values of an arbitrary arc decidedly upends the traditional construction principles of trigonometric tables pioneered in China by the Jesuits, and later studied and perfected by generations of Chinese scholars. Though they emerged in the early nineteenth century, the story of these "power-series" algorithms started earlier in the eighteenth century.

The extant record of the earliest appearance of this type of algorithms is the treatise, *Chishui yizhen* 赤水遺珍 (The Pearls Recovered from the Red River) by Mei Juecheng 梅㲄成 (1681–1764), the oldest grandson of Mei Wending and one of editors and compilers of *Numbers and Principles* (*Yüzhi shuli jingyun*). We summarize all three algorithms here in Table 7.

In his treatise, Mei Juecheng compares these algorithms with the traditional circle division methods, emphasizes the surprising efficiency of these three algorithms, and consequently names them "quick method" (*jiefa* 捷法).[93] He also indicates that these

[91] The numbers of the manuscripts and versions are not clear. What was clear is that these manuscripts did not include diagrams or explanatory texts. As a result, initially the readers of these manuscripts did not know how these algorithms were established. Several scholars in the nineteenth century worked to derive and expand the results.

[92] These algorithms demonstrate examples of finite sums of the series; see accounts in Martzloff (1997, 353–361), Li and Du (1987, 234–240) and Guo and Li (2010, 730–735). In Ming Antu's treatise on this topic (discussed below), it was said that "if ... adding these terms ... indefinitely (*wuqiong*, 無窮), [one] can get a more accurate value (*mishu* 密數) [of the value of trigonometric function]." See Ming (1839, 6). The scholarship in Chinese on this topic is too numerous to be listed here. See for example Luo (1988, 1990), and Tegus (1996) to name a few. Jami offers a comprehensive study of this subject in French; see Jami (1990).

[93] The text reads, "割圓舊術...至精至密...但開數十位之方, 非旬日不能辨. 今以圓內六等邊別立乘除之數以求之, 得之頃刻, 與屢求句股者無異, 故稱捷法焉," (The old circle division methods [are]. . .most refined [and] most accurate. . .but extracting the square root of a number of tens of digits, it cannot be done in less than tens of days. Now [we] can set up multiplication and division numbers from a regular hexagon inscribed in a circle to find them. They can be obtained instantly and are not different from the values

Table 7 Three algorithms introduced by Pierre Jartoux

Given quantities	Desired quantities	
Diameter	Circumference	(D-1)
An arc α ($\leq 90°$)	$r\sin(\alpha)$	(D-2)
An arc α ($\leq 90°$)	$\mathrm{vers}(\alpha)$	(D-3)

quick methods of finding the [*accurate* values of] circumference, sine [length], and [that of the] sagitta are translation of old methods by a Jesuit missionary, Pierre Jartoux (1669–1720) or Du Demei 杜德美 his Chinese name.[94] The algorithms were presented using concrete examples, not stated as general procedures;[95] more importantly, no proof or explanation of their derivation was provided either by Mei Juecheng or by Jartoux. No existing evidence shows that Mei Juecheng did any further study on the quick methods of "power-series" algorithms.

The treatises treating this topic in the nineteenth century started with Ming Antu, a colleague of Mei Juecheng at the court.[96] Ming's treatise, *Geyuan milü jiefa* (Quick Methods for the Circle's Division and Precise Lü), formerly published in 1839, lists thirteen algorithms with the first three being those in Mei's treatise and the last four as corollaries of the first nine.[97] Chinese scholars in the nineteenth century believed that Ming learned of the first three from Jartoux and developed the rest on his own;

Footnote 93 continued
obtained from repeated uses of right triangles. Therefore, they are called quick methods; see Mei (1759, 25a). Brief as it is, the text suggests that these quick methods were derived based on an inscribed regular hexagon and therefore has its origin from the "circle division." It is difficult to determine with certainty that Mei had seen the derivation of the algorithms. Had Mei had seen it, Ming Antu, a colleague of Mei and the author of the major treatise of this subject, should have had seen it, too. But according to the preface in Ming's treatise, he did not know initially how these algorithms were derived. See discussions below.

94 In a commentary in smaller characters, Mei specifies, "*yi xishi Du Demei fa*, 譯西士杜德美法 (Translating the methods [by the] Western scholar, Du Demei [i.e., Pierre Jartoux])"; see Mei (1759, 23a).

95 The algorithm of finding the circumference is demonstrated with a circle of diameter two billion (*ershi yi* 二十億) and the others are applied to arcs of and $21°19'51''$ and $16°27'43''$, respectively. See Mei (1759, 24a, 25b, and 26a).

96 Both Mei and Ming were students at Kangxi Emperor's Office of Mathematics. For the establishment of the Office of Mathematics within the Studio of Cultivating the Young (Mengyang zhai 蒙養齋), see Han (1999) and also Jami (2012, 268). For the editorial project of *Yüzhi lüli yuanyuan* 御製律曆淵源 (Origins of Pitch Pipe and Calendar), which includes *Numbers and Principles* as one of the three parts, Mei and Ming both were listed among the personnel involved; see Han (1999). Mei was listed as "comprehensive collation (*huibian* 彙編)" and Ming as "Determination and Measure (*kaoce* 考測)"; see Jami (2012, 374). The personnel at the Office of Mathematics was first reported by Han Qi in an unpublished presentation in 2007, "1713: A Year of Significance," at REHSEIS, CNRS, Paris, January 9, 2007. Its file is accessible at https://www.academia.edu/8278554. This presentation is significant because it contains many original findings, chief among them the comparison of the Chinese Office of Mathematics with the French Académie Royale des Sciences. For the conflict between Jesuit Astronomers and the non-Christian literati in relation to the compilation project of *Origins of Pitch Pipe and Calendar*, see Han (2015).

97 Instead of providing procedures in the last four algorithms, the text explains that by utilizing the basic relations between trigonometric lines, the given quantities in the last four algorithms will result in certain quantities in some of the previous nine algorithms; therefore, the desired quantity can be found by applying one of the 9 previously described algorithms. See Ming (1839, 10–11).

Table 8 Other algorithms in *Quick Methods* for the *Circle's Division* and *Precise Lü*

Given quantities	Desired quantities	
2α	$2r\sin(\alpha)$	(D-4)
2α	$r\text{vers}(\alpha)$	(D-5)
$2r\sin(\alpha)$	2α	(D-6)
$r\sin(\alpha)$	α	(D-7)
$r\text{vers}(\alpha)$	α	(D-8)
$r\text{vers}(\alpha)$	2α	(D-9)

nevertheless, they referred to these algorithms as the nine algorithms of Du (*Dushi jiushu* 杜氏九術). We list the results of the additional algorithms developed by Ming in Table 8.[98] Though the precise sources for the Du's algorithms, (D-1)–(D-3), cannot be ascertained, it is generally agreed that these algorithms are of European origin.[99]

While these algorithms can produce results *quickly*, the absence of proper notation renders the statement of these algorithms cumbersome to the modern readers.[100] To illustrate the lengthy description of the algorithms, the complete procedures of (D-4), which is to find the chord of a given arc, are translated below[101]:

> The method: Takes the length of an arc as the first item. Use the radius as the first *lü* of *lianbili* 連比例 (continued proportion) and the [length of the] arc as the second *lü* of continued proportion to find the third *lü* of continued proportion. Next, multiply the first item with the third *lü* and divide [the product] by the first *lü* to get the fourth *lü*. Divide [the fourth *lü*] by 4, by 2, and then by 3 to get the second item, which is supposed to be subtracted. Write it separately. Next, take the second item, multiply [it] with the third *lü* and divide [the product] by the first *lü* to get the sixth *lü*. Divide [it] by 4, again by 4, and then by 5 to get

98 When these algorithms are described in modern notation, some of them seem redundant, e.g., algorithms (D-7) and (D-9). It is worth emphasizing that these quantities represent distinct line segments as defined in Fig. 1. Per Fig. 1, the sagitta (*shi* 矢) of the arc BDH $(=2BD)$ is the line segment AD, the versed sine (*zhengshi* 正矢) of the arc BD; the chord (*xian* 弦) of BDH is the line segment BAH $(=2AB = 2\sin(BD))$ twice of the sine (*zhengxian* 正弦) of BD. That is, $2\sin(\alpha)$ is the chord for the arc 2α and $\text{vers}(\alpha)$ is its sagitta.

99 Even though two manuscripts were mentioned as possible sources for the algorithms, no treatises or manuscripts could corroborate with the record. For the discussion on these manuscripts, see Martzloff (1997, 355), in particular, footnote 10. Jami argues that the Chinese titles of the two manuscripts are simply the description of the formula themselves and consequently implies that these manuscripts did not exist; see Jami (1990, 44). There is also speculation as whether all nine algorithms were actually given by Jartoux and Ming Antu simply copied them. Jami again argued against this possibility. See Jami (1990, 44–45). Moreover, Martzloff identifies in the library catalogue at the Pe'Tang Church, one of the Jesuit Residences in Beijing in the seventeenth and eighteenth century, six European treatises as related to infinite series. See Martzloff (1997, 356).

100 Almost all contemporary scholars treating this subject opt to use the modern mathematical notation in their discussion. Martzloff, however, translates the algorithm (D-1) into non-mathematical expressions. See Martzloff (1997, 355).

101 See Ming (1839, 7) for the Chinese text for the description. A series of numbers A, B, C, D, and $E \ldots$ are said to be in the relation of *lianbili* (continued proportion) if $A{:}B = B{:}C = C{:}D = D{:}E = \cdots$. Any number in the relation of continued proportion relation is a *lü*. See Jami (1990, 73–87) for the discussion on how Ming employed continued proportions in explicating his algorithms.

Table 9 Various terms that appear in Ming Antu's Algorithm D-4

	Lü	Item
First	r	α
Second	α	$\frac{\alpha^3}{4\times2\times3\times r^2}$
Third	$\frac{\alpha^2}{r}$	$\frac{\alpha^5}{4^2\times2\times3\times4\times5\times r^4}$
Fourth	$\frac{\alpha^3}{r^2}$	$\frac{\alpha^7}{4^3\times2\times3\times4\times5\times6\times7\times r^6}$
Sixth	$\frac{\alpha^5}{4\times2\times3\times r^4}$	
Eighth	$\frac{\alpha^7}{4^2\times2\times3\times4\times5\times r^6}$	

the third item, which should be written under the first item. Next, take the third item, multiply [it] with the third *lü*, and divide [the product] by the first *lü* to get the eighth *lü*. Divide [it] by 4, by 6, and then by 7 to get the fourth item, which is supposed to be subtracted. Write [the fourth item] under the second item. Add the first and third items; add the second and fourth items. Subtract one sum from the other to get the value of the chord [of the arc].

Following the above description, the *lü*'s and numbers in the text are displayed in Table 9 with modern mathematical notation:

Let α be the length of the arc in question and r the radius.

Therefore, $r\sin(\alpha)$ is [more of less] equal to the first item minus the second, plus the third, and then minus the fourth, or in the modern notation,

$$r\sin(\alpha) \approx \alpha - \frac{\alpha^3}{2\times3\times r^2} + \frac{\alpha^5}{2\times3\times4\times5\times r^4} - \frac{\alpha^7}{2\times3\times4\times5\times6\times7\times r^6}.$$

From the description or the above expression, it is clear that the computation of the sine length consists of a combination of four basic arithmetic operations; most importantly, it is free of time-consuming root extraction, as emphasized in Mei Juecheng's passage. With the length of the radius, r, usually being a power of ten, the divisions are relatively simple. Consequently, an accurate number of the value can be obtained *quickly*.

The derivation of these algorithms is long-winding and will lead us astray from the main discussion. A description of the basic idea behind the algorithms instead will be given below, using again the algorithm (D-4) as an example. Ming first considers and establishes the procedures in which the desired quantities are the chords of arcs whose lengths are one-half, one-third, and one-fifth $\left(\text{i.e., } \frac{\alpha}{2}, \frac{\alpha}{3} \text{ and } \frac{\alpha}{5}\right)$ of an arc α whose arc length and its chord are known.[102] These procedures are all derived through the interplay of geometry and algebra.[103]

[102] The geometric figure used by An Qingqiao on his algorithm (1-1) is very similar to the one in Ming's treatise regarding the algorithm that finds the chord of the one-fifth arc. It is not clear whether An had seen Ming's manuscript.

[103] Ming makes it clear that the chord of $\frac{\alpha}{2}$ can be found rather easily by applying the old method of right triangles, but this old approach cannot be generalized to other arcs. Ming in fact provides three different

First, Ming forms a sequence of similar isosceles triangles, in which the base of one triangle is the side of the next. For any two consecutive triangle, the proportionality relation holds:

$$S{:}B = s{:}b,$$

where S and B are the side and the base of one triangle in the pair, and s and b those of the next; more importantly, $B = s$. Furthermore, geometric construction makes it possible to compare the chord of α, the given quantity, and that of $\frac{\alpha}{k}$,where k is an integer. An expression involving both chords are then obtained. Once the three cases of $\frac{\alpha}{2}$, $\frac{\alpha}{3}$, and $\frac{\alpha}{5}$ are established, Ming utilizes the procedures of $\frac{\alpha}{2}$ and $\frac{\alpha}{5}$ to construct those for the length of the chord for $\frac{\alpha}{10}$, based on the fact that $2 \times 5 = 10$; afterward, the algorithms of finding the chords of one-hundredth, one-thousandth, and ten-thousandth of α arc can also be established similarly.

A pattern emerges in computing the chords of $\frac{\alpha}{k}$, and this pattern allows Ming to determine the numbers to be used as divisors in each step of finding the next item. For example, the numbers in the denominator of the second term in the three cases of $\frac{\alpha}{100}$, $\frac{\alpha}{1000}$, and $\frac{\alpha}{100{,}000}$ are 24.0024, 24.000024, and 24.00000024, respectively. Ming argues that those small extras beyond 24 come from the fact that in computation, the chord of the arc is used instead of the arc itself. As the discrepancy between the chord and the arc diminishes as the arc becomes smaller, he claims that the precise number 24 will be achieved when the chord is replaced by the actual length of the arc, with 24 being the product of 4 times 2 time 3 in the denominator of the second item in Table 9.[104] Ming thus goes through the steps and uses the pattern established from these cases to determine the numbers that should be employed to divide a *lü* to obtain an item.[105]

Similar to Ming, Dong Youcheng 董祐誠 (1791–1823) also discussed the sine and versed sine lines in his treatise. Dong, who in 1819, read an abridged, hand-copied version of Ming's algorithms without proofs or diagrams,[106] established algorithms similar to Ming's in his treatise, *Geyuan lianbili shu tujie* 割圜連比例術圖解 (The Explication with Diagrams of Continued proportions in the Circle Division). Published in 1830, almost a decade ahead of Ming's, Dong's treatise contributes to a wider circulation of these algorithms and connects these algorithms and their derivation in the continued proportions from similar isosceles triangles with another branch of Chinese mathematics, *duiduo shu* 堆垛術 or *duoji shu* 垛積術 (Methods of Piles),

Footnote 103 continued
ways to derive the chord of $\frac{\alpha}{2}$. The three methods in the texts in Ming's treatise can be seen, respectively, in Ming (1839, 21–26, 26–29, and 29–31); for Jami's comparison of the third method with Newton's approach; see Jami (1990, 125–140).

104 These calculated numbers are described respectively on 49b, 50a, and 50b of Ming's treatises; see Ming (1839, 44–45) and also see Jami (1990, 152–156) for more discussions.

105 See Ming (1839, 44–49).

106 In Dong's own preface dated 1819, it is described that Zhu Hong 朱鴻 (1802 civil examination *jinshi* 進士 degree) showed to Dong the nine algorithms [copied] by Zhang Zhiguan 張豸冠 (early nineteenth century). See Dong (1830, 435). The manuscript of the algorithms did not have any diagram or explanation with them. Independently of Ming's approach, Dong found ways to derive the algorithms. See Tegus (1996, 322–324).

which is similar to the topic of finding recursive relations in sequences of positive integers.[107]

Neither of Ming's and Dong's treatise discusses other trigonometric lines. The treatment of tangent and secant lines can be found in Xu Youren's 徐有壬 (1800–1860) *Ceyuan milü* 測圓密率 (Measuring Circle and Its Precise Lü) composed prior to 1840,[108] Li Shanlan's 李善蘭 (1811–1882) *Hushi qimi* 弧矢啟密 (Opening the Secrete of Arcs and Sagitta) composed in 1845,[109] Xiang Mingda's 項名達 (1789–1850) *Xiangshu yiyuan* 象數一原 (Images and numbers [are] of one origin), prefaced in 1843 and 1849,[110] and Dai Xu's 戴煦 (1806–1860) *Waiqie milü* 外切密率 (Tangent outside [the Circle] and the Precise lü) composed in 1852.[111] These treatises each have its own particular approach and topics other than "power-series" algorithms. To focus our discussion on construction of trigonometric tables, we only list related algorithms in these treatises in Table 10.[112]

From the table and the dates of composition for these treatises, it is clear that by early 1850s, Chinese scholars were able to find the lengths of the trigonometric lines of an arc and vice versa. In other words, the computational utility of a trigonometric table can be completely accomplished by these algorithms.[113] Surprisingly, no scholar at the time advocated the replacement of trigonometric tables with these algorithms.

[107] For Dong's treatise and contribution in these "power-series" algorithms, see Guo and Li (2010, 731–733). For the discussions on the Methods of Piles, see Martzloff (1997, 302–304, 341–351).

[108] Han Qi determines the date of composition of Xu's treatise to be prior to 1840; see Han (1993). For more discussion on Xu's treatise, see Li (1993) and Guo and Li (2010, 748–750). Note that Li's work contains incorrect information on the composition date of Xu's treatise, cf. Han (1993). Xu's treatise can be seen in Xu (1840).

[109] Li's treatise can be seen in Li (1845). Li Shanlan was an outstanding Chinese mathematician and educator in the nineteenth century. In addition to his own original mathematical results and treatises, his translation works include Books 7 through 15 of Euclid's *Element*, an algebra book, and the first calculus book in China. For brief discussion of Li Shanlan's contributions to Chinese mathematics in the nineteenth century, see Martzloff (1997, 173–176) and Guo and Li (2010, 738–748, 753–754).

[110] Xiang's treatise discusses numerous topics. Algorithms of finding the tangent and secant lines of an arc only appear in the sixth Chapter (*Juan* 卷), based on the known length of the sine or versed sine line. The treatise was not finished when Xiang died in 1850. On the request of Xiang's son, Dai Xu edited and supplemented some of the chapters; see Xiang (1849, 600). For more discussion on Xiang's treatise, see Guo and Li (2010, 733–735).

[111] For more discussion on Dai Xu's treatise, see Guo and Li (2010, 735–738). For Dai Xu's treatises, see Dai (1852).

[112] Two somewhat similar tables can be found in Li (1993) and Li (2000, 190–194). I thank Professor Han Qi for bringing these two sources to my attentions. The topics of "series expansion" of trigonometric and other functions are very rich and active in nineteenth-century China. For example, Dong's treatise has algorithms which find the chords and sagittae of multiples and fractions of the arc with the sine and the versed sine lines are given. Xu's includes algorithms involving the area of a circle, the volumes of a sphere, and an ellipse as well as finding trigonometric lines of an arc provided other trigonometric lines for an integer multiple or fraction of the arc are known. In some treatises, the expansion of tangent, secant, cotangent, and cosecant lines is in terms of sine and versed sine lines, not in terms of the length of the arc. Separated projects and discussions are needed to do these topics justice.

[113] With a trigonometric table, complex and tedious computations were carried out during the process of construction, and therefore, computational tasks can be made easier and expedited by checking the values in the table; with these "quick-method" algorithms, a practitioner still has to carry out the computation manually no matter how efficient and quick the algorithms make the computations appear.

Table 10 Algorithms in various 19th century treatises on "power-series" expansions of trigonometric functions

Given quantities	Desired quantities	Ming	Dong	Xu	Xiang	Li	Dai
Diameter	Circumference	√	√	√	√		
α	$r\sin(\alpha)$	√	√	√	√	√	
α	$r\mathrm{vers}(\alpha)$	√	√	√	√	√	
2α	$2r\sin(\alpha)$	√	√		√		
2α	$r\mathrm{vers}(\alpha)$	√	√		√		
$2r\sin(\alpha)$	2α	√	√		√		
$r\sin(\alpha)$	α	√	√	√	√	√	
$r\mathrm{vers}(\alpha)$	α	√	√	√	√	√	
$r\mathrm{vers}(\alpha)$	2α	√	√	√	√		
$r\tan(\alpha)$	α			√		√	√
α	$r\tan(\alpha)$			√		√	√
α	$r\cot(\alpha)$						√
$r\cot(\alpha)$	α						√
$r\sec(\alpha)$	α					√	√
α	$r\sec(\alpha)$					√	√
$r\csc(\alpha)$	α						√
α	$r\csc(\alpha)$						√

Instead, they emphasized that the trigonometric tables can be constructed quickly by applying these algorithms. That is to say, what these algorithms replaced was not the trigonometric table itself but its construction principles, the Six Essentials, Three Important, and Two Simple Methods.

Xu Youren and Dai Xu each have a treatise discussing the quick method of constructing tables. Xu's is called *Zao gebiao jianfa* 造各表簡法 (Simple Methods for Constructing Each Table), which discusses the methods of constructing tables of sine, versed sine, and tangent lines; it also gives the instruction as how a logarithmic table and a logarithmic trigonometric table can be made.[114] *Qiubiao jiefa* 求表捷術 (Quick Methods of Finding Tables), in which Dai's *Waiqie milü* mentioned earlier is included, prescribes methods of finding values of tangent and secant lines based on the measure of the arc under discussion. It also provides a "table" that has the lengths for arcs of one through 10 s, next every 10 s from twenty to fifty, after that every minute from one to ten, followed by every 10 min from twenty to fifty, then every degree from one to ten, after that every 10° from twenty to eighty, and lastly, the arc length of a quadrant, one-half of a circle, and the full circle.[115] Neither treatise actually constructs or provides a trigonometric table.

114 See Xu (1859).

115 For the table, the radius of the circle is 10 billion (*yibaiyi* 一百億). These values appear in the actual computation of trigonometric lines.

This "power-series" approach to constructing tables merited an introduction in the Protestant journal *Liuhe congtan* 六合叢談 (Shanghae Series), a monthly publication in Chinese disseminating new knowledge and introducing new fields in the Western science to the general public in addition to reporting certain news around the world. Edited by the missionary Alexander Wylie (1815–1887), the journal was in circulation from January 1857 to June 1858 before abruptly stopped due to poor finance.[116] In the June issue of 1857, the notices of new books briefly discuss Xu's and Dai's books on constructing tables.[117] Starting with the old construction principles, the notice quickly moves to describing the new approach initiated by Jartoux and pioneered in China by Ming Antu and his disciple Chen Jixin 陳際新 (late eighteenth century). It also mentioned all the treatises of the series approach discussed earlier, describing the scope of the algorithms in the books and at times commenting on their strengths. Besides trigonometric tables, logarithmic tables and their construction are also mentioned. This notice to new books demonstrates that the new mathematical developments fashioned by Chinese scholars were also accessible to and noticed by foreigners working in China.

10 Discoursing trigonometric table and the construction principles

As far as we know, no mathematical treatise post 1860 discusses the "power-series" approach to constructing trigonometric tables. Our story of trigonometric tables and the evolution and explication of their construction principles in China is drawing to an end. Let us recount the episodes briefly.

Trigonometric tables and the algorithms to construct them were introduced into China by the Jesuits as part of the effort to reform the Calendar circa 1630. While the computational power of the trigonometric table was acknowledged by all, the "absent" explanation of how these algorithms were established in the treatise remained an enduring criticism of the Jesuits by some Chinese scholars.[118] Generations of scholars strived to find how the construction principles were derived. Successful as they were in deriving the principles and in adding new algorithms to facilitate edification, few new trigonometric tables were actually constructed afresh. Some tables included in the treatises were outright plagiarized or simplified from the translated table by the Jesuits; some were re-worked based on existing tables. The only trigonometric table built anew consists entirely of sine lines utilizing a peculiar measure for arcs that no one else uses.

Starting from three early eighteenth-century, Jesuit-introduced algorithms resembling the modern-day series expansion for the circumference of a circle, sine lines, and versine lines, Chinese scholars in the late eighteenth century and the first half

116 For a discussion on this journal, see Elman (2005, 297–299).

117 Though the journal was published in Chinese, it did have a table of content in English. The title of this Notice is "Tae Heu's Logarithms, and Seu Yew-jin on the construction of Trigonometric and Logarithmic tables." Tae Heu and Seu Yew-jin are the Romanization of the names pronounced in a local dialect for Dai Xu and Xu Youren, respectively. See Shen (2006, 627–628).

118 To be fair, the explanation to the three important and two simple methods in *Grand Measure* amounts to proving their validity with the exception of (2-2), as discussed earlier.

of the nineteenth century again investigated how these algorithms were established. They succeeded spectacularly in deriving these algorithms from a series of similar isosceles triangles and the pattern they demonstrate; moreover, they greatly expanded the repertoire of algorithms of similar nature. With these powerful and efficient algorithms, the task of constructing a trigonometric table was made much easier, yet the general direction to construct trigonometric tables was disseminated without much fanfare for actual construction. This uneven treatment of the construction principles and trigonometric tables is worth our reflection.

Foremost, constructing a trigonometric table is a time-consuming business, commanding great resources. In our earlier discussion, the table included in *Numbers and Principles* was augmented from an existing table and completed with the backing of a dynasty's resources. It would have been almost impossible for an individual scholar to accomplish such a feat. Speculated to be constructed anew, An Qingqiao's table has merely 250 entries, built with new, powerful algorithms and an obscure measuring system, both of which facilitate and expedite computation. These examples point to the difficulties an individual encounters in constructing a trigonometric table alone. Therefore, it is understandable and expected that a treatise authored by an individual scholar not have a trigonometric table constructed from scratch by following the principles elucidated in the treatise.

The second interpretation of such an uneven treatment is more subtle and of intricate nature. It is in the context of classification of "scientific knowledge" in early modern China and the views on this issue held by the actors of the time. A little background discussion is in order before we delve into our specific case, situating mathematics in the general intellectual history at the time.

According to certain intellectual historians, Chinese scholars well versed in mathematics and calendric science had tried to reconcile natural philosophy with mathematical studies since the seventeenth century.[119] Mei Wending's works are a prime example. In his "*Discussions of Rectangular Array*" (*Fangcheng lun* 方程論), Mei explicitly describes his efforts to explain the reason behind the procedures as "illuminating the principle of mathematics" (*ming suanli* 明算理), *li* 理 being the pervasive notion in Neo-Confucianism in the state-sanctioned curriculum in the civil examinations in Ming and Qing.[120] Efforts such as Mei's culminate in the integration of mathematics in evidential scholarship (*kaozheng xue* 考證學) in the eighteenth century.[121] Consequently, mathematics along with astronomy and calendric science by then had simply become an integral part of classical studies, at least for many evidential scholars.

This perspective of mathematics allows us to place trigonometric tables and the explication of their construction principles into different genres of knowledge from

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119 For the discussions on the efforts by various Qing scholars, see Elman (2001, 218–221).

120 Neo-Confucianism is also described in Chinese as "Song-Ming *lixue* 宋明理學 (the learning if *li* during the Song (960–1278) and Ming Dynasties)." The meaning of *li* is rather fluid, depending on the context. A universal translation that fits all contexts is almost impossible. The usual translation of *li* is "moral principle," which does not fit our purpose here. As *li* is omnipresent in all things, its governing of mathematics is sometimes referred to as *suanli* or *li* of mathematics.

121 See Elman (2001, 218–221) for detailed discussions, cited from Jami (2007, 147).

the actor's viewpoint. On the one hand, the explication of principles, including the geometric figures, the derivation and establishment of algorithms, and the assertive explanation connecting the Western approach to classical texts, serves to illuminate the connection between the subject matter and the underlying governing principle, *li*, and then by extension to the Classic knowledge. As the discussions of construction principles constitute the main body of treatises, they fit perfectly with what Catherine Jami terms as "science as discourse." The actual construction of trigonometric tables, on the other hand, can be carried out by practitioners with limited training to produce a tool, which contributes in the service of statecraft to "enhance the control over the world—heaven, earth, and man."[122] Once a trigonometric table is constructed, it can continually be utilized without any need to comprehend how it was built by a large number of practitioners over a long time until the need arises to improve or replace it. That is, trigonometric tables fall under what Jami describes as "science as action." Literati were well aware that knowledge of this type is distinctively different from that contained in the discourse and explanation of the principles. For example, Qin Huitian 秦蕙田 (1702–1764), a minister of Justice at Qianlong Emperor's (reigned 1735–1795) court, describes in his preface to Dai Zhen's *Gougu geyuan ji* 句股割圜記 (The Records of Base-Altitude and Circle-Division) that "learning the [mathematical] algorithms without obtaining their principles, [this kind of learning] then [is] repetitive and fragmental, similar to [learning] skills."[123]

The traditional Chinese term for the knowledge of the action kind was *ji* 技 or *jineng* 技能 (skills).[124] Treatises as the main platform for discoursing knowledge and as the product of "science as discourse" are therefore partial to discussions of construction principles over the actual making of the table. The view that trigonometric tables are merely products of certain skills can also explain why scholars might consider it acceptable to include in their treatises an existing table or a simplified version of it. Moreover, such a practice would not create contention of priority.

Another interesting observation brings out a related but somewhat different issue: Two authors in our analysis boasted explicitly that their tables are shortened, be it with fewer trigonometric lines (functions) or adopting a coarser increment, and therefore *improvement* to the Western-made tables, replacing numerous entries with a few algorithms. Both Li Zijin and An Qingqiao expressed such a view. Li claims, "The old table has ninety some pages; [we] cannot see [the usefulness] of its multitude. This table has merely seven and one-half pages; [we] do not experience [the deficiency] of its meagerness." An's states, "Now [we] take from both the Chinese and Western methods… to make the table of one line. [This approach] increases the completeness of the principle, but the entries in the tables are rather simplistic. The table of eight lines in the Western method can be left to rest."[125]

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122 See Jami (2007, 148) for the quote.

123 The text by Qin reads, "習其術不得其理, 則繁碎而近於藝." See Dai (1961, 770).

124 The actual construction of the table should be considered a skill. The quote and Jami's description of science can be seen in Jami (2007, 146–150).

125 Li's texts read, "*jiubiao jiushiyu ye…er bujian qiduo, jinbiao buguo qiye you ban… er bujian qishao* 舊表九十餘葉, 而不見其多…今表不過七葉有半…而不見其少." See (Li Zijing 1763). An's read,

This peculiar practice runs directly counter to the core function of trigonometric tables—minimizing and expediting calculations. As the governing principles should be "simplistic" (*jian* 簡) and "refined" (*jing* 精), the practice of shortening trigonometric tables seems to suggest an effort by these scholars to repackage a product in the science-as-action genre to be in line with knowledge of possession of certain characteristics of the principles. The motivations for such adjustments to trigonometric tables can be manifold: the desire to show that Chinese works are superior to and therefore can replace the Western ones, foreign knowledge should be domesticated and conform to the Chinese standard, or a simple economic reason to reduce the cost of printing.

In 1877, the translation of a trigonometric treatise introduced a new, different kinds of trigonometric knowledge into China.[126] The difference commences from the very beginning: The trigonometric values of an (acute) angle were defined as the ratios of the lengths of sides in a right triangle, in contrast to the old definition as the line segments associated with an arc. This intrinsically different treatment of trigonometry should have undoubtedly enriched the knowledge developed and accumulated from before 1850. The transition, appropriation, and the conflicts of the old and new treatments of trigonometry in China should prove to be interesting as a sequel to our current investigation. And that will require further exploration and examination; for sure, it will be a story for another day.

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Appendix: Compendium abbreviations

DJTHM *Zhongguo kexue jishu dianji tonghui shuxue juan* 中國科學技術典籍通匯數學卷 (Comprehensive Collection of Treatises in Chinese Sciences and Technology, Mathematics). Zhengzhou: Henan Education Press, 1993.

DJTHA *Zhongguo kexue jishu dianji tonghui tianwen juan* 中國科學技術典籍通匯天文卷 (Comprehensive Collection of Treatises in Chinese Sciences and Technology, Astronomy). Zhengzhou: Henan Education Press, 1997.

Footnote 125 continued
"*jin canzhuo zhongxi, …, wei yixianbiao, yuli jiami, er biaoshu shenjian, xifazhi baxianbiao keyi buyong yi.* 今參酌中西…為一線表. 於理加密, 而表數則甚簡. 西法之八線表可以不用矣"; see (An 1819, 12:330).

[126] *A Treatise on Plane and Spherical Trigonometry*, 4th ed. by John Hymers was translated into Chinese in 1877 as *Sanjiao shuli* 三角數理 (The mathematical principles of trigonometry). The translation is based on the 1858 edition.

CZLS *Chongzhen Lishu* 崇禎曆書 (Astronomical Compendium of the Chongzhen Reign). Compiled and translated by Adam Schall von Bell et al. I use the copy collated by Pan Nai. Shanghai: Shanghai Guji Chubanshe, 2009.

GGZBCK *Gugong Zhengben congkan* 故宮珍本叢刊 (The Collection of Precious Treatises at the Palace Museum). Reprint of a collection of treatises housed in the Palace museum in Beijing. Haikou: Hainan Chubanshe, 2000–2001.

SKQS *Siku quanshu* 四庫全書 (The Complete Texts of the Four Repositories). Reprint, Taipei: Taiwan Shangwu Yingshuguan, 1983.

SKWSSJK *Siku weishou shujie kan* 四庫未收書籍刊 (The Collections of Treatises Not Included in the Complete Texts of the Four Repositories). Beijing: Beijing Chubanshe, 1997.

XXSKQS *Xuxiu siku quanshu* 續修四庫全書 (The Sequel of the Complete Texts of the Four Repositories). Reprint. Shanghai: Guji Chubanshe, 2002.

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