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"Local–Global": the first twenty years

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Abstract This paper investigates how and when pairs of terms such as "local–global" and "*im Kleinen–im Grossen*" began to be used by mathematicians as explicit reflexive categories. A first phase of automatic search led to the delineation of the relevant corpus, and to the identification of the period from 1898 to 1918 as that of emergence. The emergence appears to have been, from the very start, both transdisciplinary (function theory, calculus of variations, differential geometry) and international, although the AMS-Göttingen connection played a specific part. First used as an expository and didactic tool (e.g. by Osgood), it soon played a crucial part in the creation of new mathematical concepts (e.g. in Hahn's work), in the shaping of research agendas (e.g. Blaschke's global differential geometry), and in Weyl's axiomatic foundation of the manifold concept. We finally turn to France, where in the 1910s, in the wake of Poincaré's work, Hadamard began to promote a research agenda in terms of "*passage du local au general.*"

0 Introduction

> All mathematics is more or less "in the large" or "in the small." It is highly improbable that any definition of those terms could be given that would be satisfactory to all mathematicians. Nor does it seem necessary or even desirable that hard and fast definitions be given. The German terms *im Grossen* and *im Kleinen* have been used for some time with varying meanings. It will perhaps be

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> interesting and useful to the reader to approach the subject historically by way of examples. (Morse 1989, p. 259)

With these words, the old Master of differential topology and "calculus of variations in the large" introduced his talk at a 1967 conference on "global differential geometry." Given the social context, no definitions of "global" were needed: both Morse and the members of the audience already knew what "global" meant; *tacit* knowledge, subdisciplinary boundaries and professional identities were in play.

In this paper, our goal is to go beyond the *tacit* by means of a historical investigation. The meaning and role of the concepts of "local" and "global" in mathematics can be investigated from a great diversity of viewpoints. One could set out to identify a core-meaning, a conceptual invariant that would shed light on the various uses; although history could serve as a tool for such an endeavour, its pursuit is probably either mathematical, or philosophical (as part of an epistemological investigation into fundamental concepts of mathematics). From a more historical viewpoint, at least two investigative lines can be followed. Along the first line, one would study the emergence of global results, the accumulation of mathematical knowledge in emerging subdisciplines (e.g. "global differential geometry"), etc. For this study, the corpus of texts and the list of mathematicians to be considered would probably have to be delineated with the benefit of hindsight, relying on more contemporary understanding of what global and local results and theories are. Another line of investigation would focus on the concepts of "local" and "global" themselves, concepts which, as Morse's quote illustrates, are used by mathematicians to expound and discuss mathematics. Along this line, the "local–global" pair is seen as a reflexive category used (or not) by the actors. These two historical lines of investigations are clearly related. We feel, however, that they should be analytically distinguished, for at least two reasons. First, the lists of texts to consider need not be the same, since the criteria for corpus-building differ significantly. Second, distinguishing between the two questions seems to be a necessary prerequisite for the study—as a second phase—of the interplay between the rise of "global mathematics" and the spread of "local–global" as an actor's category.

The goal of this paper is to tackle the second question: we aim at identifying those mathematicians who first made explicit use of pairs of terms such as "local–global" or "*im Kleinen – im Grossen*," thus promoting a new conceptual distinction within mathematics. This focus on the *explicit* will lead us to include in this study several mathematicians who also contributed to the rise of global mathematics—such as Weyl, Blaschke and Hadamard; but to exclude from our core-corpus several mathematicians who, in spite of their proofs of numerous global results and their clear understanding of the local-global distinction, did not promote this distinction quite as actively. For instance, the cases of Poincaré and Hilbert will be discussed in the first section of this paper.

Hence, we will endeavour study the emergence of the "local–global" pair[1] within printed mathematical texts. In order to capture the range of uses and meanings which

[1] We will occasionally use the word "*articulation*" instead of the merely descriptive term "pair." In some semantic theories (Courtès 1991), an "articulation" is a pair of terms which define the two directions on a semantic axis. We will use this notion with this specific but rather neutral meaning, independently of Bruno Latour's use of Greimas' semiotics (Latour 1999).

Morse pointed to, we do not want to start from any explicit definition of these terms. It turns out that such a definition is not necessary to gather empirical data, on the basis of which these terms—and those to which qualitative sampling showed they were, to some extent, equivalent—can be studied in terms of both meaning (*semantic* aspect) and use (*pragmatic* aspect).

To put it in a nutshell, we shall show that "*im Kleinen–im Grossen*" was first used at the turn of the twentieth century, in several disciplinary contexts, with a specific and stable meaning, with a growing range of uses in the period from 1898 to 1918, and within a specifiable social context—that of the "special relationship" between the Göttingen school and the emerging American mathematical research community. Alongside these aspects, to which the bulk of this essay will be devoted, we will also come across occurrences of "*im Kleinen*" denoting the infinitely small (infinitesimal meaning); we will show that this use is marginal (at least in pure mathematics), and that one of the goals of the mathematicians who promoted the use of "*im Kleinen*"–"*im Grossen*" was to distinguish the local from the infinitesimal.

Finally, the use of the word "local" will be studied in the French context, the emphasis being laid on Hadamard's analysis of the current state of mathematics, in the wake of Poincaré's manifold work. In his own work and as a pillar of the French mathematical community, Hadamard would, in the 1910s, begin to promote a new research agenda in terms of "*passage du local au general.*"

To reach these conclusions, some methodological issues will have to be tackled. In the introductory quote, Morse stressed the fact that expressions such as "in the small—in the large" or "local–global" were poorly localised, both in disciplinary terms (they concern virtually all mathematics) and in time ("used for some time"). This sounds like a pretty elusive object for the historian to track down! Since we want to study how expressions like these can spread (or not) and stabilise (or not) through examples only, and in spite of the fact that no definitions are given, it sounds unwise to start from our own definition. But this raises the question of how to start without making a decision about either what to look for, or where to look.

It seemed to us that a reasonable way out of this predicament is to try to make use of the vast amount of retrodigitised mathematical literature that has become available in recent years. For historians of science working on the modern period in particular, the systematic use of word-search for corpus building (among other possible uses) has expanded investigative possibilities. However, this technique raises thorny methodological questions of its own; in particular, relying on it—even for a preliminary phase of corpus building—heavily influences the type of historical objects that can be captured.

This is why the first part of this essay is devoted to the deeply intertwined questions of the *nature* of the historical investigation and of the *modus operandi* for automatic corpus analysis. In particular, we will endeavour to spell out more precisely the difference between a history of "global mathematics"—global theorems, subdisciplines such as global differential geometry, etc.—and the history of "local–global" as expressions used in printed mathematical texts. As we shall see, this work is devoted to "local–global," not only because this is what word-searches help us capture, but also because the explicit use of new classifying terms in mathematical communities is a specific historical phenomenon that deserves a study of its own.

This work is part and parcel of a more general research program on the historical development of "global mathematics," and complements two former papers: one on the creation of the sheaf structure in the context of the theory of functions of several variables (Chorlay 2010) and one on the advent of global issues and techniques in Elie Cartan's work in the 1920s, both in differential geometry and in the theory of Lie groups (Chorlay 2009).

1 Setting the problem

1.1 "Local–global" or global mathematics?

Two quotations should help us define the scope of this study. The first one is taken from the introduction of Poincaré's first paper on curves defined by a differential equation:

> Looking for the properties of differential equations is a task of the highest interest. Along this line, a first step is taken by studying the given function *in the neighbourhood of one of the points of the plane*. Today we are to proceed further and study this function in the whole spread of the plane. In this investigation, our starting point will be what is known already about the function studied in a given area of the plane.
> The complete study consists in two parts:
> 1. The qualitative part (so to speak), or the geometric study of the curve defined by the function.
> 2. The quantitative part, or the numerical calculation of the values of the function.
>
> (Poincaré 1881, p. 3)[2]

In this series of papers on curves defined by first order differential equations with rational coefficients, Poincaré proved, among other global results, the formula linking the number singular points (assumed to be isolated and of a generic type) and the genus of the surface on which the differential equation is studied.

The second quotation is taken from Hilbert's 1901 paper on the embedding of surfaces of constant negative curvature into ordinary space:

[2] « Rechercher quelles sont les propriétés des équations différentielles est donc une question du plus haut intérêt. On a déjà fait un premier pas dans cette voie en étudiant la fonction proposée *dans le voisinage d'un des points du plan*. Il s'agit aujourd'hui d'aller plus loin et d'étudier cette fonction *dans toute l'étendue du plan*. Dans cette recherche, notre point de départ sera ce qu'on sait déjà de la fonction étudiée *dans une certaine région du plan*.
L'étude complète d'une fonction comprend deux parties:
1. Partie qualitative (pour ainsi dire), où étude géométrique de la courbe définie par la fonction;
2. Partie quantitative, ou calcul numérique des valeurs de la fonction.»
All quotations are freely translated by the author of this paper.

> Finally, for our investigation, it is necessary to realise that formulae (1) represent every point of the surface by a pair of values u, v; that is, that the mapping (1) of our surface on the uv-plane must be univocally invertible not only in sufficiently small domains, but taken as a whole.
>
> (Hilbert 1901b, p. 91).[3]

In this celebrated paper, Hilbert proved that the hyperbolic plane cannot be globally smoothly embedded into E^3, in spite of the fact that the embedding is locally possible. Both papers present significant mathematical results which Morse could have used as examples of "analysis in the large" without causing any puzzlement in a mathematical audience of 1967.[4] Actually, these two examples had been used for the very same purpose for quite some time, as is shown by Struik (1933) and Kasner (1905). Furthermore, these two quotations show that both mathematicians were completely explicit about their purpose to go global, the local results being mere starting points.

However, we will not include these two texts in our core corpus, for several reasons which need to be spelled out.

Were we to include these two texts in our corpus, two thorny questions would arise: Where to look? What criteria should be used to delineate the corpus? If early (say: nineteenth century) global results were to be identified and accounted for, we should certainly read all of Poincaré and Hilbert. Probably Riemann's work—and Klein's—should also be studied closely. But why Riemann and not Weierstrass? Why Poincaré rather than Picard? And if number theory is not to be left aside, probably Dedekind and Kronecker should be taken on board too. To prevent indefinite corpus expansion, it seems advisable to select the works (and the places in the works) where *we know* some global result is proven. However, this "we know" has a history and is the result of history: As we shall see later, whether or not Lie's theory of invariants of algebraic curves belongs to the corpus of nineteenth century global theories was a disputed issue at the turn of the twentieth century; likewise, if number theorists had been asked whether or not the concepts "local" and "global" were relevant in their research field, the answer in 1910 would probably have been very different from the answer in 1940.

Even if the issue of the historical process of selection of the relevant corpus could be left aside, writing the history of "local" and "global" as reflexive concepts in mathematics requires that some criterion be devised to distinguish between results that "we" (whoever that may be) consider to be of a global nature *in spite* of the fact that those who proved them never used anything even barely reminiscent of these concepts, from results that were—to some extent—originally described in terms of local and global. Yet, what "to some extent" might mean in the absence of terms such as "local," "*im kleinen*" or "in the small" is, again, a thorny question. The presence of these terms provides an easy criterion for saying that something is explicitly here. We do not know the right tools for drawing of a line between the still too implicit and the explicit enough,

[3] „Endlich ist es für unsere Untersuchung notwendig, einzusehen, daß die Formeln (1) jeden Punkt der Fläche nur durch ein Wertepaar u, v darstellen, d. h. daß die gefundene Abbildung (1) unserer Fläche auf die uv-Ebene nicht bloß für genügend kleine Gebiete, sondern im ganzen genommen eine umkehrbar-eindeutige sein muß."

[4] He actually used examples from Poincaré's work (Morse 1989).

and we doubt these could be used without assuming an a-historical definite notion of what "local" and global" means. In doing so, the very tools would destroy that which they are meant to capture from a historical viewpoint. Moreover, setting the problem in terms of an ever more explicit use of concepts rests on the dubious notion of an implicit concept, and might lead to a narrative centred on successive eye-opening, epoch-making papers written by mathematicians who eventually understood what it was all about (and had always been).

In this paper, we chose to focus on the use of *expressions* such as "*im kleinen*" in order to build the *core corpus*. Further, the *extended corpus* consists of the texts which are explicitly referred to in the core corpus. Accordingly, neither Poincaré's paper on curves defined by differential equations nor Hilbert's paper on the Euclidean embedding of the hyperbolic plane belong to the core corpus; but they do belong to the extended corpus, and this of course cannot be seen by reading Hilbert and Poincaré's papers themselves. This focus on explicit terms could be seen as a cheap way to bypass (and not tackle) the two problems, that of corpus building, and that of the criteria for assessing degrees of explicitness; pusillanimity wearing the mask of positivism, so to speak. However, we claim that not only does this focus help us avoid some methodological traps; it is also the background against which several important questions can be studied.

First, we chose to focus on the explicit use of a small list of terms because our study does not bear on local and global mathematics, but on the use of "local" and "global" in mathematics; in other words, we aim at studying how and when "we" came to "know" that, for instance, in these two papers, Poincaré and Hilbert proved global results on the basis of well-known local results. To say that the history of how "we" came to "be aware" of such a thing is deeply linked to the historical emergence of new proof methods, of new results and theories, and of specific research agendas, etc., is merely stating the obvious; to confuse the two is to miss our target.

Indeed, what is at stake here is a matter of *agency*. Leaving Poincaré's and Hilbert's papers out of the core corpus does not mean that we consider that they did not know what they were really *doing*; such a statement would be preposterous, and the quotations we gave clearly contradict it. Hibert's and Poincaré's papers are left out of the core corpus because, from a pragmatic viewpoint, proving a global theorem (say Poincaré's) and referring to this theorem to explain what mathematics in the large is (as Morse did) belong to quite different categories of "doing." As we shall study in detail in the next parts of this paper, "local" and "global" can be used to clarify the expository structure of a proof or a lecture, to warn against an all-too-common mistakes (beware, you're trying to draw global conclusions from local theorems), to sort out problems and point to the relevant tool-boxes, to devise completely new definitions for familiar objects, to mark disciplinary boundaries around an emerging research area, etc. All these will be exemplified in the 1898–1918 period to which this paper restricts its investigations. Reflexive terms such as "local" and "global" emerge somewhere and for some reason; they circulate in some networks and not in some others; they are given a definite meaning by some but another definite meaning (or some loose meaning) by others ... all these questions are specific to the historical study of "local" and "global," and distinguish it quite clearly from a history of global mathematical theories.

Finally, this investigation can be seen as a *part* of a more general study of the emergence of global theories and disciplines in mathematics; a preliminary part that looks tractable and should be instrumental in paving the way for the much larger task. This is not, however, the only potential context for a historical investigation into "local" and "global." As we shall see below, it could just as well be seen as a new tool to study the formation of an international mathematical community at the turn of the twentieth century or to study the role of the doctoral or post-doctoral journey to Germany in the making of the American mathematical community. To point to a third potential context, it could be part of a study of the emergence and *competition* of several reflexive categories, the case of "qualitative–quantitative" being exemplified by Poincaré's quotation. Of these three possible contexts, only the first two will be dealt with in this paper.

1.2 A 1930s sample

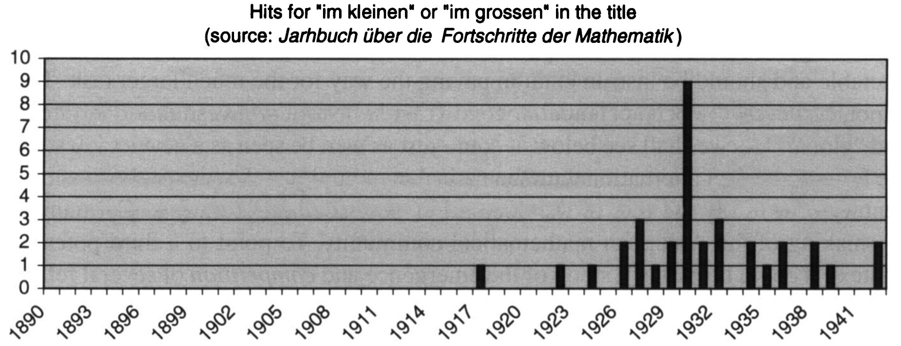
Given the question that we set out to address, it should seem clear that our raw material will be a set of occurrences of terms such as "local," "*im kleinen*" or "in the small" within printed mathematical texts; it is likely that the gathering of these occurrences can only be performed through automatic procedures, using digital documents. Hence, before the qualitative study can even begin, some more methodological problems need to be tackled regarding a variety of issues such as: the choice of terms, the digitised corpus to investigate, the reliability of the automatic investigation, the restriction to "printed" and "mathematical texts." These issues are very different in nature, ranging from the technical problem of character recognition to deep questions such as that of the elusiveness of oral transmission in small mathematical communities, or that of the historical variations of the thresholds between the oral and the written down, between written and printed.

To flesh out these arid questions, let us look at a simple-minded example; needless to say its role is merely heuristic. As is well known, the *Jahrbuch über die Fortschritte der Mathematik* (JFM) is available in digital form, making the titles and reviews of mathematical research papers and books published between 1868 and 1942 available for word-search.[5] What do we get if we search for either "*im kleinen*" or "*im grossen*" in the title of research papers? A total of 35 hits, starting in 1917.

This first sample is striking for the number and variety of the research fields involved. They include quite a lot of point-set topology, the titles mentioning technical notions built on the very same template: local connectivity (*Zusammenhang im Kleinen* in the German[6] titles, *connectedness im Kleinen* in English), local convexity (*Konvexheit im Kleinen*), local compactness (*im Kleinen kompakt topologischen Räume*). Some papers deal with topological aspects of problems which are not strictly topological, for instance, Schreier's *Die Verwandtschaft stetiger Gruppen im Grossen* (Schreier 1927), where the notion of a covering space is used to study the global continuation of

[5] Groups of words can be searcher as "expressions," the upper/lower case distinction is not taken into account, and the β was systematically turned into a double s.

[6] "German" and "English" refer to the language used, not to the nationality or location of the author(s).



local homomorphisms of topological groups. In this sample, number theoretic papers appear in 1930, with Schmidt's *Zur Klassenkörpertheorie im Kleinen* (Schmidt 1930) and Hasse's *Die Normenresttheorie relativ-Abelscher Zahlkörper als Klassenkörpertheorie im Kleinen* (Hasse 1930) and account for a significant proportion of the hits from then on. In this case, "*im kleinen*" is used to delineate a specific theory or field of research within number theory. Same for Rinow's use, but in differential geometry: *Über Zusammenhänge zwischen der Differentialgeometrie im Grossen und im Kleinen* (Rinow 1932). Analysis is not left out, with some qualitative theory of differential equations (*Sur la stabilité des intégrales des équations différentielles "im Grossen"* (Halikoff 1938); partial differential equations (*Über das Dirichletsche Problem im Grossen für nichtlineare elliptische Differentialgleichungen* (Schauder 1933)); the calculus of variations, with *Über minimalflächen im Grossen* (Wernick 1934), *Über eine neue Methode zur Behandlung einer Klasse isoperimetrischer Aufgaben im Grossen* (Schmidt 1942), and of course Seifert and Threlfall's book on *Variationsrechnung im Grossen* (*Morsesche Theorie*) (Seifert and Threlfall 1938); complex analysis is here too [*Zur Theorie der Funktionen mehrerer komplexer Veränderlichen. Konvexität in bezug auf analytische Ebenen im kleinen und grossen* (Behnke and Ernst 1935)].

Searching for "in the small" and "in the large" yields similar results in terms of dates and variety of research fields, with a lot of *calculus of variation in the large* (starting, of course, with (Morse 1925), differential topology, and a few papers on the theory of Lie groups, such as G. Birkhoff's (1936) *Lie groups isomorphic in the large with no linear groups* (Birkhoff's 1936). The only striking difference would be that "in the small" and "in the large" are used in the English language only (and mainly by American mathematicians), whereas "*im kleinen*" and "*im grossen*" were used in papers in German, English, and in one paper in French (by a Russian mathematician, in a Russian journal).

Again, this merely has a heuristic value. On the basis of this rather superficial search in the titles, and crude classification by research field, two things can be remarked.

First, the state of affair which Morse described in 1967 seems to apply to the 1930s: "in the small," "*im kleinen*," etc. seem to be used in "more or less" all mathematics. But this very fact makes it suspicious that these terms *started* being used in the late 1920s, in spite of what the word-search seems to show. The time period appearing here is probably the artifactual outcome of a clumsy search and not the period of emergence.

In this paper, we want to try to go as far back in time as our means permit, in order to study the emergence of these reflexive terms. To go back in time, it seems necessary to do a little more work, by using a larger corpus (even though, by its very nature, the *Jahrbuch* provides a nice overview), and by searching full texts and not mere titles. However, as we shall see, doing so creates its own difficulties.

Along with the variety of research fields, another feature of this first sample points to a bundle of interesting questions. In addition to the multiplicity of research fields, theories and disciplines, there is a multiplicity of terms, often but not always linked to a difference in the languages in which the papers are written. To what extent is "in the small" synonymous with "*im kleinen*" and "local" for mathematicians in the 1930s? At this point, we do not want to actually read the texts to try to decide whether meanings are identical or equivalent; in some cases, this endeavour would be highly problematic: who is to tell if "*im kleinen*" and "in the small" have the same meaning in "*Klassenkörpertheorie im Kleinen*" and "calculus of variations in the small?" Even if a reasonable answer could be given from an early twenty-first century viewpoint, it would leave undecided whether these terms were regarded and used as equivalent by mathematicians in the 1930s. For now, we shall use inter-translatability as criterion for equivalence.

A small but significant example seems to indicate that in this somewhat later phase of our story (leaving undecided for the moment the dates of our early phase), equivalence is not problematic. When Seifert and Threlfall wrote a book to present Morse's theory to the German public, they translated "*calculus of variations in the large*" by "*Variationsrechnung im Grossen*," and "*calculus of variation in the small*" by "*Variationsrechnung im Kleinen*"; the French text derived from a 1938 talk by Threlfall, and published in *L'Enseignement Mathématique*, reads "le calcul des variations global" (Threlfall 1939). For a second example: in 1932, when Georges de Rham translated into French a talk by Hopf on the relation between infinitesimal geometry and topology, he used "structure globale" (Hopf 1931, p. 233) and "géométrie globale" (Hopf 1931, p. 234); although the German text is not available, in the numerous papers by Hopf on this topic which are available in German, "*im grossen*" is systematically used. Many more examples could be given, and would be well worth studying, were we to focus on the 1930s. For now we just need to point out that this inter-translatability testifies to the fact that some degree of equivalence was tacitly agreed upon in the 1930s, in spite of the variety of disciplines and languages.

This leads to a bundle of questions for anyone who sets out to study the emergence of these terms. An obvious question is which came first. The mere multiplicity of terms seems to point to multiple sites of emergence. If it turned out that these terms began to be used in several poorly connected contexts, the later history of inter-translatability would have to be studied. (A detailed methodological discussion of the corpus-building phase of our work is given in the Appendix.)

Before engaging in qualitative analysis, it should be remarked that the bulk of this paper will be devoted to "*im kleinen*"–"*im grossen*." As far as our zero corpus shows, "in the small"–"in the large" were not in use before the 1920s. It is quite possible that Morse chose these expressions to avoid of "*im kleinen*"–"*im grossen*" (which were of common use in the US, as we shall see); German expressions may not have sounded as appealing to the WWI veteran as they had to Göttingen trained Osgood. The case of "local" will be dealt with in the last section of this paper.

2 William Fogg Osgood

Born in Boston in 1864, W.F. Osgood graduated from Harvard in 1887 and, taking the advice of Frank Nelson Cole, left for Germany (Göttingen 1887–1889, Erlangen 1889–1890). He got his Ph.D. in Erlangen, under the supervision of Max Noether, for a work on the theory of Abelian functions associated to the "*algebraische Gebilde*" $y^m = R(x)$ on which he had started to work in Göttingen, under the supervision of Klein. Back in the U.S., he started teaching in Harvard in 1890, was appointed as full professor in 1903 and worked there until he retired, in 1933.[7] There, he taught analysis to generations of American mathematicians.[8] He was a prominent member of the recently established American Mathematical Society, serving as president in 1904–1905 and as AMS Colloquium lecturer in 1898 and in 1913. These lectures, as well as his early papers, demonstrate his command of the latest trends in mathematical analysis, both in real analysis and the theory of *Punktmannigfaltigkeiten* (tricky convergence problems, Cantor sets, space-filling curves, abstract definition of the integral, existence theorems in the calculus of variations), and in the theory of complex functions (of one variable in 1898, of several variables in 1913). His 1898 lectures show that, in spite of his Göttingen training, he was well aware of the various approaches to complex function theory; his exposition is of a syncretic style, using and contrasting Riemannian and Weierstrassian definitions and proof-methods. His knowledge of recent developments in function theory was not limited to the German-speaking world, his presentations include works by Picard, Hadamard and Poincaré (in particular his 1883 general uniformisation theorem for analytic functions; Poincaré 1883). As we shall see, he published textbooks and research papers both in English and in German.

Although Osgood occasionally used "*im kleinen*" and "*im grossen*" in research papers (e.g. Osgood 1898), most of the texts in which the expressions appear are of a didactic—at least an expository—nature: the six lectures on *Selected Topics in the General Theory of Function* delivered at the AMS Cambridge Colloquium in 1898 (Osgood 1899); the general introductory article in the second *Band* of the Encyclopädie der mathematischenWissenschaften mit Einschluss ihrer Anwendungen (Encyclopädie), entitled *Analysis der komplexen Größen. Allgemeine Theorie der analytischen Functionen (a) einer und (b) mehrerer komplexen Größen* (Osgood 1901a); his German textbook, *Lehrbuch der Funktionentheorie* (first edition in 1906, second edition in 1912; Osgood 1906, 1912); finally, the lectures on *Topics in the Theory of Functions of Several Complex Variables*, delivered at the AMS Madison Colloquium in 1913 (Osgood 1914).

[7] He also taught in Beijing in 1934–1936. This move may not have been motivated by academic reasons only. In 1932, Osgood married Celeste Phelpes, who had recently divorced Marston Morse; this apparently created quite a stir. For biographical elements on Osgood, see, for instance, (Archibald 1938, pp. 153–158), (Koopman 1944), (Parshall and Rowe 1994), (Walsh 2002).

[8] As G. Birkhoff recalled, Osgood's " (...) course on functions of a complex variable remained the key course for Harvard graduate students until World War II" (quoted in (Parshall and Rowe 1989, p. 13)).

A selection of quotes will give us some clue, both to the *semantic* and *pragmatic* aspects: what does Osgood mean by "*im kleinen*" and "*im grossen*"? How and for what purpose do those terms appear in the texts? If, as the automatic search seems to indicate, "*im kleinen*" and "*im grossen*" were not widely used at that time—and that's an understatement –, a close look at the texts is necessary to capture this intriguing emergence phenomenon.

2.1 *Meta* use: contrasting theorems

The first of the six 1898 lectures is devoted to *Picard's theorem, and the application of Riemann's geometric methods in the general theory of functions*. Osgood presented and proved two "forms" of the theorem, a "restricted form" and a "more general form." First comes the restricted form:

> Any function $G(z)$ which is single valued and analytic for all finite values of z takes on in general for at least one value of z any arbitrarily assigned value C. There may be one value, a, which the function does not take on. But is there is a second such value, b, the function reduces to a constant. (Osgood 1899, p. 59)

After its proof:

> We now turn to the more general form of Picard's theorem:
> If $F(z)$ is any analytic function of z which in the neighborhood of a point A is single valued and has in this region no other singularities than poles, and if A is an essential singular point of $F(z)$, then there are at most two values which $F(z)$ does not take on in every neighborhood of the point A. (Osgood 1899, p. 63)

The statement of the general form of the general theorem is immediately followed by a comment:

> This theorem, it will be noticed, is concerned with the behavior of a function *im Kleinen*, *i.e.* throughout a certain arbitrarily small region; while the earlier theorem was one *im Grossen*, the domain of the independent variable being there the whole finite region of the plane. (Osgood 1899, p. 63)[9]

No such comment was made by Picard, who, however, remarked that the second theorem was an easy consequence of a more general form of the first theorem, the more general form being the polynomial form of the first (global) theorem.[10] Consequently, the emphasis in Picard was by no means on anything like local and global (Picard 1880).

It should be noticed that Osgood used these terms in German, in the English text of a talk delivered before an American audience (of twenty-six[11]). He did not pause and make an aside to explain what he meant by these unusual (and foreign) terms,

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9 Italics in the original text.

10 What Picard called the general theorem says that if, for an entire function G, the equation $G(z) = a$ (a finite) has a finite number of roots for more than one a, then G is a polynomial.

11 (Osgood 1899, p. 58).

and it is not entirely clear whether the short descriptions (i.e. throughout a certain arbitrarily small region) are here to help understand what "*im kleinen*" means, or to point to what makes the second theorem a theorem on the behaviour of functions "*im kleinen*." They actually serve both purposes. This has to do with the type of discourse to which these terms belong in Osgood's writings. As is clear in this example, "*im kleinen*" and "*im grossen*" are used to say something *about* a pair of theorems; something that is optional; something that does not have to do with proof and does not affect the mathematical validity of the statement. It simply points to some aspects of what has just been said or proved, thus performing a *meta* function in the mathematical discourse.[12] Stepping back from the specific case studied in this paper, many other examples can be mentioned, all of which could be well worth investigating from a historical viewpoint. In mathematical texts, many words or expressions perform similar *meta* functions: emphasising logical relations (e.g. "theorem B is the converse of theorem A"), pointing to disciplinary boundaries (as in "proof A is more algebraic than proof B"), classifying theorems by type (e.g. existence theorem, uniqueness theorem, convergence theorem, ...), expressing epistemic values (e.g. rigour, purity, generality, ability to convey understanding, etc.). We actually encountered a few other terms performing a *meta* function in the few quotes given so far: general/particular in Osgood and Picard, qualitative/quantitative in Poincaré. Needless to say, the fact that these words or expressions play *a meta* function doesn't mean that they play the *same* function in the text. In this paper, our goal is not to classify the variety of such functions, but to study the case of "local"–"global" from close.

A case in which Osgood repeatedly used "*im kleinen*" and "*im grossen*" is that of inversion. As in the case of Picard's theorems, these terms are used to compare and contrasts two theorems. Both in the *Encyclopädie* and in the *Lehrbuch*, these expressions appear in paragraphs titles, testifying to its importance for the whole architecture of function theory. In the *Encyclopädie*, for instance, the title of §5 reads "*Die konforme Abbildung im Kleinen*"; in §18, entitled "*Die Umkehrfunktion und die konforme Abbildung im Großen*," Osgood pointed to the difference between local and global inversions and mentioned a global theorem:

> 18. The inverse function and conformal mapping im Grossen
> An analytic function $w = f(z)$ in a domain T defines a one-to-one mapping of a domain T_1' on a domain $\mathfrak{T}_1'$ of a Riemann surface spread over the w-plane. If $f'(z)$ vanishes nowhere in T_1', then the mapping of the neighbourhood of an arbitrary point z_0 of T_1' on the neighbourhood of the corresponding point w_0 will be conformal. However, this fact is not sufficient to conclude that $\mathfrak{T}_1'$ does not overlap itself, i.e. that the inverse function $z(w)$ is one to one for the values of w being considered. A sufficient condition to this effect is given by this proposition: Let $w = f(z)$ be a function of z which is continuous on a

[12] By "meta," we certainly do not mean here anything like metamathematics nor any mathematical study of formal languages. Neither do we mean anything like what Caroline Dunmore (after Michael Crowe, or, in a more philosophical context, Philip Kitcher; Kitcher 1984) called the "meta-level" in her macro-historical reflections on "revolutions" in mathematics, a "meta-level" where values and beliefs of mathematical communities are to be found (Dunmore 1992). Our approach is more local (no pun intended) and proceeds through textual analysis.

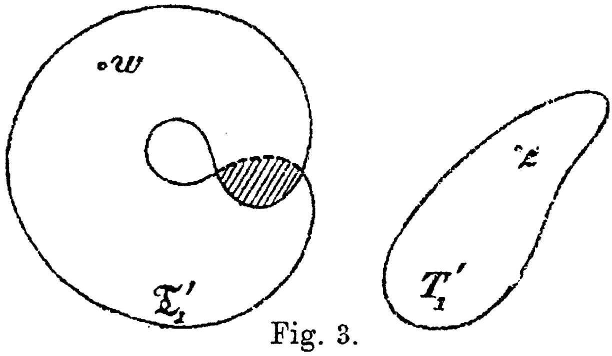

Fig. 3.

(Osgood 1901a, p. 52)

domain B_1(No.1), analytic in the interior of B_1, and which never takes on the same value at two different points of the boundary C of B_1, then C goes into a closed and non-intersecting Jordan curve Γ of the w-plane; the simple domain of the w-plane which Γ bounds will be correlated in a continuous, one-to-one and onto manner to B_1, so that, moreover, its interior is conformally correlated to the interior of B_1. (Osgood 1901a, p. 52)[13]

In Osgood's *Encyclopädie* chapter, the letter T denotes domains (i.e. connected open parts of the complex plane, usually bounded), B denotes the closure of a T-domain (the boundary of which is usually assumed to consist in a finite number of analytic arcs), T' denotes a domain of type B, chosen at will within a given T-domain.

The fact that domain $\mathfrak{T}_1'$ might be self-overlapping, even if domain T_1' is simply connected and the derivative $f'(z)$ never vanishes, is illustrated by the following figure:

The proof of Osgood's global inversion theorem is given in his *Lehrbuch*: injectivity on the boundary of the simply connected domain B_1 implies that its image is a Jordan-curve (the Jordan curve theorem was proved as the *Fundamentalsatz* of *Mengenlehre* in the fifth chapter of the *Lehrbuch*); the fact that the mapping is one-to-one and onto between the bounded Jordan-domains is proved with a standard argument in complex analysis: the number of solutions to an analytic equations $f(z) = 0$ which lie within a domain bounded by closed curve V is expressed by the line-integral of the

[13] „18. Die Umkehrfunktion und die konforme Abbildung im Grossen.
Eine in einem Bereich T analytische Funktion $w = f(z)$ definiert eine ein-eindeutige Abbildung eines Bereiches T_1' auf einen Bereich $\mathfrak{T}_1'$ einer über die w-Ebene ausgebreiteten Riemann'schen Fläche. Verschwindet $f'(z)$ in T_1' nirgends, so wird die Abbildung der Umgebung eines beliebigen Punktes z_0 von T_1' auf die Umgebung des entsprechenden Punktes w_0 konform sein. Dieser Umstand reicht jedoch nicht zum Schlusse aus, das $\mathfrak{T}_1'$ nicht über sich selbst greift, m.a.W. dass die Umkehrfunktion $z(w)$ für die in Betracht kommenden Werte von w eindeutig ist. Eine dazu hinreichende Bedingung gibt der Satz: Ist $w = f(z)$ eine in einem Bereich B_1 (Nr.1) stetige und innerhalb B_1 analytischer Funktion von z, die auf der Begrenzung C von B_1 ein und denselben Wert in zwei verschiedenen Punkten niemals annimmt, so geht C in eine geschlossene sich selbst nicht schneidende Jordan'sche Kurve Γ der w-Ebene über; der von Γ abgegrenzte schlichte Bereich der w-Ebene wird ein-eindeutig und stetig auf B_1, das Innere dieses Bereiches ausserdem noch konform auf das Innere von B_1 bezogen."

logarithmic derivative, hence by the number of loops of the image of V around any given point (Osgood 1912, p. 378).

In this case, "*im kleinen*" and "*im grossen*" are not used only to show that there is an interesting connection between the theorems that are more than a hundred pages apart in the textbook. They help draw the attention to the fact that, in spite of their similarities (as inversion theorems), these theorems are of a different nature and must not be confused one for the other. Applying the "*im kleinen*" theorem to draw "*im grossen*" conclusions is an *all too common mistake*, as Osgood points out in a footnote in §18 of the *Encyclopädie*. The footnote mentions Briot and Bouquet's faulty exposition of the inversion of elliptic integrals, and Klein's criticism of Fuchs' uniformisation results for special classes of ordinary differential equations of order two with algebraic coefficients. It should be noted that Klein's criticism was by no means worded in terms of local and global; there are several ways to describe that mistake. Klein wrote that Fuchs mistook non-ramified functions (which the inverse of f indeed is, if f' does not vanish) for single-valued functions (Klein 1883, p. 214). The same confusion had been pointed out to Fuchs by Poincaré in their exchange of letters of 1880 (Poincaré 1951–1956, vol. 11, pp. 14–17); Poincaré's explanation is of a more geometric flavour than Klein's, and deals with self-overlapping domains, but again, the criticism was not articulated in terms of something like local and global.

2.2 *Meta* use: exhibiting proof patterns and theorem patterns

Leaving the sorting of theorems for a while, we can also find "*im kleinen*" and "*im grossen*" in the wording of theorems and proofs.

As far as proofs are concerned, one example will suffice. Chapter 13 of the *Lehrbuch* deals with logarithmic potentials. Its 6th paragraph presents Schwartz's symmetry principle and a series of applications; the straightening of analytic curves plays a key part here: "4th Proposition. Let C be an analytic curve. Then the neighbourhood of C can be correlated in a one-to-one, onto and continuous manner with the neighbourhood of a segment Γ, so that curve C goes into segment Γ." (Osgood 1912, p. 669).[14] After proving that the straightening is possible in the neighbourhood of any point of the curve, Osgood paused and remarked: "Here we have reached im Kleinen, for the neighbourhood of point (x_0, y_0), what the proposition requires im Grossen for the whole curve C" (Osgood 1912, p. 670).[15] He thus inserted a *meta* comment, pointing to the two phases of the proof of a global theorem, while expounding this proof. It exemplifies the general strategy: when proving a global theorem, first state and prove a local version of it, then try to go global (in this particular case, using compactness and *reductio ad absurdum*).

Another case is more intriguing, that has to do with the wording of theorems and the exemplifying of general proof strategies. Although the *Lehrbuch* deals mainly

[14] „4. Satz. Sei C eine analytische Kurve. Dann läßt sich die Umgebung von C auf die Umgebung einer geraden Strecke Γ ein-eindeutig und konform beziehen, dergestalt daß die Kurve C in die Strecke Γ übergeht."

[15] "Hiermit ist im Kleinen für die Umgebung des Punktes (x_0, y_0) das erreicht, was der Satz im Großen für die ganze Kurve C verlangt."

with complex function theory, its first five chapters (which means about 200 pages) consist in a state-of-the-art exposition of real analysis, Weierstrass style and including a good share of point-set topology. The first chapter presents the elementary notions of function of a real variable, limits, continuity, derivatives, etc.; none of these came as a surprise for the 1906 reader (at least in Germany). The 10th paragraph of this first chapter, however, contains more unexpected material. It deals with multi-valued functions, in the real context, and since it is the first time in the textbook that multi-valued functions are considered, Osgood presented a standard strategy for dealing with these unfriendly functions: "To deal with a multi-valued function, it is often advisable to strive to represent it by means of single-valued functions" (Osgood 1912, p. 44).[16] Then comes the first of three theorems on the separation of branches for multi-valued *real* functions:

> 1st Proposition. Let it be required that, for every point x_0 of the domain of definition of a multi-valued function, the following be given: (a) a given neighbourhood $|x - x_0| < \delta$, (b) a series of single-valued functions $y_1 = f_1(x), y_2 = f_2(x), \ldots$ defined without exceptions in these very neighbourhoods $|x - x_0| < \delta$, so that in the given neighbourhood a one-to-one and onto relation holds between these functional values and those of the given multi-valued function. Then, a similar aggregation of the values of the multi-valued function is also possible im Grossen, which yields the stock of values of the multi-valued function once and only once. (Osgood 1912, p. 44)[17]

Again, the proof relies on *reductio ad absurdum*: working on a bounded closed interval, if the conclusion did not hold for the interval, then it would not hold for at least one of its half-intervals ... a contradiction would arise at the point of intersection of the nested intervals. The next proposition is a corollary:

> 2nd Proposition: Let us add the two following hypotheses to those of the first proposition: (c) at all points of T the values of the multi-valued function all differ one from the other; (d) the functions $f_k(x)$ have been chosen so that they are continuous. Then the single-valued functions into which, according to the first proposition, the values of the multi-valued function can be split, can be determined so as to be continuous on the whole interval (and, actually, in only one way, disregarding the order of the series). (Osgood 1912, p. 45)[18]

[16] "Zur Behandlung einer mehrdeutigen Funktion empfiehlt es sich meist, eine Darstellung derselbe mittelst eindeutiger Funktionen anzustreben."

[17] "1. Satz. Jeder Stelle x_0 des Definitionsbereiches T einer mehrdeutigen Funktion sollen sich (a) eine bestimmte Umgebung $|x - x_0| < \delta$ (b) eine Reihe je in derselben ausnahmlos definierter eindeutiger Funktionen $y_1 = f_1(x), y_2 = f_2(x), \ldots, |x - x_0| < \delta$, so zuordnen lassen, daβ zwischen diesen Funktionswerten und den Werten der vorgelegten mehrdeutigen Funktion in der gennanten Umgebung eine ein-eindeutig Beziehung statt hat. Dann wird eine ähnliche Zusammenfassung der Werte des mehrdeutigen Funktion auch im Groβen möglich sein, deren Werte geradezu den Wertvorrat der mehrdeutigen Funktion einmal, aber auch nur einmal, liefern.„

[18] "2. Satz. Zu den Voraussetzungen des 1. Satzes füge man noch die beiden weiteren hinzu: (c) in jedem Punkte von T sollen die Werte der mehrdeutigen Funktion sämtlich voneinander verschieden sein; (d) die Funktionen $f_k(x)$ sollen so gewählt werden können, daβ sie stetig sind. Dann lassen sich die eindeutigen Funktionen, auf die sich nach dem 1. Satze die Werte der mehrdeutigen Funktion verteilen, so bestimmen

These two propositions clearly do not belong to the standard list of elementary propositions of real analysis,[19] as a quick look at any other late nineteenth or early twentieth century analysis textbook would confirm. Moreover, these theorems are not inserted in the first chapter because of their deductive value; actually, they are not used as lemmas in later parts of the book where the separation of branches of functions of a complex variable is investigated. Does that mean that these propositions are of no use, save maybe for showing yet another (rather convoluted) application of the principle of nested intervals? In the 8th chapter, which is the first chapter dealing with multi-valued functions of a complex variable, a central theorem reads:

> Proposition. Let several values $f(z)$ be associated to every inner point of a simply connected domain T of the extended plane. Were the number of values to be infinite, it should however be countable. Let these values be such that to every inner point of T there corresponds a given neighbourhood on which the whole stock of values of the function can be aggregated into a series of one-valued analytic functions. Then, the given values can also be aggregated im Grossen, that is in the whole of domain T, into one-valued functions $f_1(z)$, $f_2(z)$, ..., all of which are analytic in T, and which, taken as a whole, exhaust the values $f(z)$ exactly once. (Osgood 1912, p. 396)[20]

In Osgood's textbook, this theorem on the separation of branches stands in the more standard monodromy theorem for the analytic continuation of a function element in a simply connected domain. This theorem was already mentioned in the *Encyclopädie*, where a footnote remarked that analyticity was not the heart of the matter: continuity "*im kleinen*" was enough (Osgood 1901a, p. 29). Actually, Osgood borrowed[21] this investigation of the separation of *continuous* branches of functions (on a simply connected domain) from the second volume of O. Stolz's textbook *Grundzüge der Differential- und Integralrechnung* (Stolz 1893). In the paragraph *Über eindeutige und stetige Zweige vieldeutige Functionen einer complexen Veränderlichen* (Stolz 1893, pp. 15–24), Stolz dealt with functions of a complex variable, but first studied them on a curve in the domain; dividing that into a real analysis part and a complex analysis part, as Osgood did, was merely a matter of presentation. Something that did not come from Stolz, however, is the wording of both the real and complex theorems in

Footnote 18 continued
(und zwar, von der Reihenfolge abgesehen, nur auf eine einzige Weise), daβ auch sie im ganzen Intervalle stetig ist."

19 By the first decade of the twentieth century, the list of theorems which belong to the general exposition of "modern" analysis was already quite standardised.

20 "Satz. Jedem inneren Punkt eines einfach zusammenhängenden Bereichs T der erweiterten Ebene mögen mehrere Werte $f(z)$ zugeordnet werden. Im Falle die Anzahl der Werte nicht endlich ist, soll sie jedoch abzählbar sein. Diese Werte sollen so beschaffen sein, daβ jedem inneren Punkte von T eine bestimmte Umgebung entpricht, in welcher sich die ganze Vorrat der Funktionswerte zu einer Reihe eindeutiger analytischer Funktionen zusammenfassen läβt. Dann können besagte Werte auch im Groβen, also im ganzen Bereiche T, zu eindeutigen Funktionen $f_1(z)$, $f_2(z)$, ... zusamenfaβt werden, deren jede sich in T analytisch verhält und deren Gesammtheit die Werte $f(z)$ gerade erschöpft."

21 A hint is given by Osgood's footnote 44 (Osgood 1901a, p.29).

terms of "*im kleinen*" and "*im grossen.*" Thanks to this rewriting, the central analytic theorem now clearly displays a general form, a general *theorem-pattern*, namely: in some well-behaved domains (here: simply connected), if a property holds locally at every point, then it also holds globally. The relationship between real the propositions of the first chapter and the complex theorem of the 8th chapter is by no means a deductive relationship. Considered in terms of their role in the deductive structure of the book, the real theorems of Chap. 1–§10 are null and void; they could be called *dummy theorems*. The absence of a deductive role does not imply, however, that they are useless. They have a purely *meta* role, displaying the same theorem-pattern for the reader to memorise and, later, identify.

2.3 Neumann made rigorous

Both the theorem on the separation of branches and the global inversion theorem deal with the behaviour of analytic functions on simply connected domains. They play a significant role in Osgood's exposition of the theory of functions of a complex variable, which, when it comes to algebraic functions, is a variant of Carl Neumann's "cut-and-paste" approach to Riemann surfaces. The procedure is well known: once the ramification points have been identified, choose a system of cuts that links them and chop the domain into simply connected sub-domains over which the function is unramified (but multi-valued); consider that these simply connected components consist of several independent layers (one for each branch of the specific function you want to study), then glue the layers along the borders according to the permutations rules derived from the study of the ramification points. As Osgood pointed out in the *Encyclopädie* (Osgood 1901a, p. 29), the standard presentation starts with the study of the ramification points, then proceeds through analytic continuation. Both in the *Encyclopädie* and in the *Lehrbuch*, he changed the order of presentation, starting with the two general theorems on analytic functions on simply connected domains and then presenting the standard cut-and-paste procedure. These theorems provide a rigorous justification for the "paste the simply connected sub-domains along the borders to get a well-behaved function"-step of the Neumann procedure; before Osgood, this step was usually dealt with in a purely intuitive, hand-waving, diagram-drawing manner.

Not only these two global theorems, but the very notions if "*im kleinen*" and "*im grossen*" are central for Osgood's revamping of the Neumann-style introduction to Riemann's theory of algebraic functions of one complex variable. As in Neumann, Osgood presented both general theorems and a series of examples showing how the surface is to be constructed in ever more complicated cases; the theorems play an ancillary role, establishing either that some construction step is possible, or that it actually yields what is expected. In spite of its didactical value, this presentation lacks generality, in two senses. First, there is no general description of what a Riemann surface is; a Riemann surface is what you get when you apply a given procedure (whose steps can be accounted for) to particular cases. Second, when applying the procedure, arbitrary choices have to be made (the cuts), which lead to theoretical problems, for instance, when questions of isomorphism (or worse, describing parameter spaces for Riemann surfaces[22]) are to be tackled. Osgood was well aware of this problem, and, after presenting the construction of the Riemann surface for w (defined by $w^3 - 3w = z$) he took a step back and paused for a *meta* comment:

> Let us again clear up what is essential and what is adventitious in the Riemann surface. The following facts are essential, (a) that three simple leaves run over the neighbourhood of any point $z_0 \neq -2, 2, \infty$ that serve as bearers for three functions which are analytic and one-valued in this neighbourhood; (b) that one simple leaf runs in the neighbourhood of the points $z = -2, 2$, while two others are connected in a cycle; similarly, all three leaves are connected in the point $z = \infty$. So much for the im Kleinen part; to this one should add (c) that the leaves join together as the course of the several determinations of the function im Grossen requires. (Osgood 1912, p. 374)[23]

This classification of the essential pieces of information in terms of "*im kleinen*" and "*im grossen*" clearly echoes the wording of the theorem on the separation of branches: (a) is its local version, and the two global theorems warrant step (c).

The goal of the construction of the Riemann surface is to replace a multi-valued function defined on a domain of the complex sphere, by a single-valued function defined on a new domain which is a surface over the complex sphere. If a single-valued parametric representation for the Riemann surface can be found (using a function of an ordinary complex variable), then complex uniformisation of the original function has been achieved. The general uniformisation theorems, either for algebraic functions or for general analytic functions, were beyond the scope of the *Lehrbuch*, but they were central to Osgood's 1898 AMS talk on the latest issues in function theory. Osgood actually played a part in that story as a researcher: in his 1898 talk, he pointed to one of the flaws in Poincaré's 1883 proof of the general uniformisation theorem (Osgood 1898, pp. 69–74); he would later give a proof of the missing fundamental lemma, as Poincaré would acknowledge in his revised proof (Poincaré 1908). It comes as no surprise (to us) that Osgood introduced the general problem of uniformisation to his audience in terms of "*im kleinen*" and "*im grossen*." To quote one of several examples[24]:

22 These issues are explicitly dealt with by Klein and Poincaré in their attempts at proving general uniformisation theorems for algebraic functions (in terms of automorphic functions).

23 „Machen wir uns noch klar, was an der Riemannschen Fläche wesentlich und was nur zufällig ist. Wesentlich ist, (a) daβ über der Umgebung eines jeden der Punktes $z_0 \neq -2, 2, \infty$ drei Blätter schlicht verlaufen, welche als Träger dreier in dieser Umgebung eindeutiger, sich analytisch verhaltender Funktionen dienen; (b) daβ in der Umgebung der Punkte $z = -2, 2$ ein Blatt schlicht verläuft, während zwei andere dort im Zyklus zusammenhängen; sowie daβ im Punkte $z = \infty$ alle drei Blätter zusammenhängen. So viel im Kleinen; dazu kommt noch, (c) daβ die Blätter so miteinander verbunden werden, wie es der Verlauf der verschiedenen Bestimmungen der Funktion im Groβen verlangt."

24 Also, in the 1898 talk: "3. Next may be mentioned the representation of the coordinates of an algebraic curve by the elliptic functions when $p = 1$, and, generally, by automorphic functions. Here the relation between (x, y) and z continues to be one-to-one im Kleinen, but is one-to-infinity im Grossen." (Osgood 1899, p. 70)

> When dealing with multi-valued functions, it is often advisable to represent them by single-valued functions. Thus, for instance, the Riemann surface first of all serves this purpose of providing a domain on which a given multi-valued function becomes one-valued. Another case is dealt with in Chap. 8, §14, where an analytic function $w = f(z)$ has a ramification point of finite order in $z = a$. Here, we are to express the components of the pair of values (w, z) belonging to the function by means of two single-valued functions of a parameter $t : z = a + t^m, w = \varphi(t)$. But this holds only im Kleinen, that is, for a restricted part of the domain of definition of the function. In contrast, we already know from integral calculus certain classes of functions by means of which it is possible to represent the function in its entire course through one-valued functions—to *uniformise* it, as is customarily said. (Osgood 1912, p. 710)[25]

Checking against Poincaré's or Koebe's introduction to uniformisation theory shows that this description in terms of local and global was by no means standard in the first decade of the twentieth century.[26]

2.4 Defining "*im Grossen*" and "*im Kleinen*": the syntactic view

After this review[27] of the ways and contexts in which Osgood used "*im kleinen*" and "*im grossen*," the question of meaning somehow remains. A semantic analysis could very well be carried out, which would show connections between "*im kleinen*"–"*im grossen*" and pairs such as small–large, whole–part, neighbourhood–domain, etc. None of this would come as a surprise for the early twenty-first century reader, nor would the fact that the meaning is conveyed *in context*, in the title of chapters or paragraphs, in the names of theorems, in the wording of theorems, in the laying out of a proof, in the introductory setting of a problem that a specific theorem solves, etc. This is ascribable to the fact that these terms perform a *meta* function in the mathematical discourse: they appear along that material which they describe or a feature of which they help underline; they are instrumental in the organising and shaping of some more primitive contents. A philosophically minded historian could describe this in Wittgensteinian terms, with meaning conveyed through use only, in some language game that can be

[25] „Wenn wir es mit einer mehrdeutigen Funktion zu thun haben, empfiehlt es sich meist, dieselbe durch eindeutige Funktionen darzustellen. So dient beispeilweise die Riemannsche Fläche vor allem der Zweck, einen Bereich zu schaffen, in welchem eine vorgelegte vieldeutige Funktion eindeutig wird. Ein anderer Fall ist der Kap.8, §14 behandelte, wo eine analytische Funktion: $w = f(z)$ einen Verzweigungspunkt endlicher Ordnung in $z = a$ hat. Hier gelang es uns, die Bestandteile eines der Funktions zugehörigen Wertepaares (w,z) vermöge zweier eindeutiger Funktionen eines Parameter t auszudrücken: $z = a + t^m, w = \varphi(t)$. Doch galt diese Darstellung nur im Kleinen, also für einen beschränkten Teil des Definitionsbereiches der Funktion. Dagegen sind schon von der Integralrechnung her gewisse Klassen von Funktionen bekannt, wobei es möglich ist, die Funktion in ihrem Gesamtverlaufe durch eindeutige Funktionen zur Darstellung zu bringen,- zu uniformisieren, wie man sich wohl auszudrücken pflegt.„

[26] (Poincaré 1883); (Poincaré 1921); (Poincaré 1908); (Koebe 1909).

[27] We left out a few interesting instances, in particular in the theory of functions of several complex variables. Osgood called Weierstrass' preparation theorem the local divisibility (*Teilbarkeit im Kleinen*) theorem (Osgood 1901a, p. 105), (Osgood 1914, p. 83). The Cousin problems are mentioned in the *Encyclopädie* in a paragraph entitled "*Einige Sätze über das Verhalten im Großen.*" (Osgood 1914, p. 111).

located within history (first decade of the twentieth century), within mathematics (at the meeting of Weierstrass' rigorous analysis based on point-set topology, and Neumann style presentation of Riemann surfaces) and in terms of textual genres (didactic and expository texts). However, we feel that in the case of terms which perform a *meta* function, that type of description probably always fits. Moreover, it so happens that it is not the case that meaning was conveyed through use *only*. Osgood did *once* give a definition—at least a precise explanation, abstracted from the immediate context of use—of what he meant by "*im kleinen*" and "*im grossen*."

In the second paragraph of the *Encyclopädie* article, Osgood discussed the very notion of an analytic function:

> The concept of analytic continuation is part and parcel of the complete definition of the analytic function (No. 13). It could be said that the definition used up to this point bears on the behaviour of the function *im Kleinen* (indeed, nothing more was known before Weierstrass); a stipulation regarding the behaviour *im Grossen* is still missing (footnote 8). (Osgood 1901a, p. 12)[28]

For their first appearance in the text, the terms are italicised, and footnote 8 reads:

> The concept of behaviour of a function *im Kleinen* and *im Grossen* plays an important role in Analysis, and concerns all part of mathematics (in particular Geometry as well) where a continuous set of elements form the substrate for the configuration to be studied. In the theory of functions, the behaviour of a function *im Kleinen* resp. *im Grossen* means its behaviour in the neighbourhood of a given point a, $(a_1, a_2, \ldots, a_n)$, or a point-set P (N° 40) [for the sake of brevity, one incorrectly speaks of the behaviour *in* a, $(a_1, a_2, \ldots, a_n)$, or *in* point-set P], resp. in a domain T, T', $\mathfrak{T}$, $\mathfrak{T}'$, etc., the extent of which is set from the start [*von vornherein feststeht*] and not determined afterwards to meet the requirements of the given problem. In many cases, in domains T', $\mathfrak{T}'$ the corresponding *uniform* (N° 6) behaviour im Grossen stems from the behaviour im Kleinen. (Osgood 1901a, p. 12)[29]

It turns out that "*im kleinen*" and "*im grossen*" are not, in Osgood's use, metaphorical terms (even though they may convey intuitive grasp *as well*). We call this the *syntactic view* of "*im kleinen*" and "*im grossen*." Being local or global is a property of

28 „Zur vollständigen Definition der analytischen Funktion gehört noch der Begriff der analytischen Fortsetzung (No. 13). Man darf wohl sagen, die bisherige Definition bezieht sich auf des Verhaltens der Funktion im Kleinen (weiter war man ja vor Weierstrass nicht gekommen); es fehlt noch eine Festsetzung bezügl. des Verhaltens der Funktion im Grossen."

29 „Der Begriff des Verhaltens einer Funktion im Kleinen und im Grossen spielt in der Analysis einer wichtige Rolle und erstreckt sich auf alle Gebiete der Mathematik (namentlich auch auf die Geometrie), wo eine stetige Menge von Elementen das Substrat für die in Betracht zu ziehenden Gebilde bildet. In der Funktionentheorie versteht man unter dem Verhalten einer Funktion im Kleinen resp. im Grossen ihr Verhalten in der Umgebung eines festen Punktes a, $(a_1, a_2, \ldots, a_n)$ oder einer Punktmenge P (No. 40) [der Kürze halber spricht man dann schlechtweg von ihrem Verhalten im a, $(a_1, a_2, \ldots, a_n)$ oder in der Punktmenge P] resp. in einem Bereich T, T', $\mathfrak{T}$, $\mathfrak{T}'$ u.s.w., dessen Ausdehnung von vornherein feststeht und nicht erst hinterher den Bedürfnissen des vorgelegten Problems entsprechend bestimmt wird. In vielen Fällen folgt aus einem gegebenen Verhalten im Kleinen in jedem Punkt eines Bereiches T', $\mathfrak{T}'$ das entsprechende gleichmässig (No. 6) Verhalten im Grossen."

mathematical *statements* (be they definitions or propositions) which refer to domains. The key distinction is between "set from the start" (*von vornherein feststeht*) and "afterwards" or "in retrospect" (*interher*): if the domain over which the conclusion holds is the very domain that was referred to in the hypotheses, then the proposition is "*im grossen*"; if the domain which is referred to in the conclusion is not the one mentioned in the hypotheses, but some new domain which has to be determined so as to meet some extra requirements, then the position is "*im kleinen*." The case of local versus global inversion theorems is a perfect example; so is the case of the basic theorem on the convergence of power series, the convergence being locally uniform (on any compact subsets of the convergence disc). What matters here is the strength of the coupling between domains referred to in different parts of a complex mathematical assertion. To some extent, it expresses in terms of domains what formal, Weierstrass style, propositions or definitions in analysis express completely syntactically in terms of quantifiers, be it through the introduction of a new quantified variable (e.g. in the definition of a local maximum), or by the order of quantifiers (as in everywhere pointwise versus uniform continuity).

The fact is, Osgood spelled that out once and just once, as far as our core corpus shows.[30] However, his use is consistent with this quite specific meaning, and that is specific enough to allow for comparison with the use of the same terms by other mathematicians.

It should be noted that Osgood's precise and context-free explanation of what he meant by "*im kleinen*" and "*im grossen*" is unique in our core corpus.

3 "*im Kleinen*"–"*im Grossen*" in the U.S.

Besides, and *after*, Osgood, we find "*im kleinen*" and "*im grossen*" used by members of the American Mathematical Society, in texts written in English, with a meaning and a use which are similar to Osgood's.

3.1 Kasner's problems of "Geometry *im Grossen*"

The terms are used in Edward Kasner's address delivered before the section of Geometry at the International Congress of Arts and Science, in Saint Louis, in 1904. As mentioned earlier, several distinguished foreign mathematicians attended that Congress, among them Picard, Darboux and Poincaré. Kasner presented a general overview of the "present problems in Geometry"; he aimed at presenting "a survey of the leading problems or groups of problems in certain selected (but hopefully representative) fields

30 Some passages are very reminiscent of this explanation, in spite of the fact that they are "in context." For instance, in the *Lehrbuch*, the theorem on global analytic inversion is introduced as follows: „Bisher haben wir uns im allegemein Falle bloβ mit der konformen Abbildung im Kleinen beschäftigt, indem wir zeigten, daβ unter gewissen Bedingungen die Umgebung eines Punktes z_0, deren Ausdehnung also von vornherein nicht feststand, ein-eindeutig und konform auf eine Umgebung eines Punktes w_0 bezogen wird. Jetzt wollen wir ein Kriterium kennen lernen, wonach ein vorgelegter Bereich inkl. Der Berandung ein-eindeutig und stetig, und im Innern konform auf einen zweiten vorgegebenen Bereich abgebildet werden kann." (Osgood 1912, p. 377).

of contemporary investigation" (Kasner 1905, p. 87)), from foundational issues to the birational geometry of algebraic surfaces and the geometry of transformation groups. The seventh of the nine groups of problems he called "Geometry im Grossen":

> The questions we have just been considering, in common with almost all the developments of general or infinitesimal geometry, deal with the properties of the figures studied im kleinen, that is, in the sufficiently small neighborhood of a given point. Algebraic geometry, however, on the other hand, deals with curves and surfaces in their entirety. This distinction, however, is not inherent to the subject matter, but is rather a subjective one due to the limitations of our analysis: our results being obtained by the use of power series are valid only in the region of convergence. (...) Only the merest traces of such a transcendental geometry im Grossen are in existence, but the interest of many investigators is undoubtedly tending in this direction. (Kasner 1905, p. 304)

Three families of problems were identified as "*im grossen*". First the study of geodesics on a given surface, in particular closed geodesics and those which are asymptotic to these; Kasner referred his audience to Hadamard's papers on the geodesics on a surface of negative curvature (Hadamard 1898). At the very same Congress, Poincaré gave a talk on the geodesics on a surface of positive curvature; a topic on which he published a paper in the *Transactions of the AMS* in 1905 (Poincaré 1905). The second family of "*im grossen*" problems deal with "the determination of applicability criteria valid for entire surfaces"; he mentioned the case of surfaces of constant positive curvature (for which applicability on a sphere is locally trivial) and, more generally, results of rigidity for convex surfaces (Lagrange, Minding, Jellet and Liebmann). The third problem is that of the models for non-Euclidean geometry; of course, the only result "*im grossen*" is Hilbert's proof to the effect that the whole hyperbolic plane cannot be represented by an analytic surface of ordinary Euclidean space. The Hilbert quote we gave in the first part of this paper was from this proof. Kasner finally mentioned some problems which he did not choose to list under the heading "problems of geometry *im grossen*" but which are strongly related:

> Other theories belonging essentially to geometry im Grossen are the questions of analysis situs or topology to which reference has been made on several occasions, and the properties of the very general convex surfaces introduced by Minkowski in connection with his *Geometrie der Zahlen*. (Kasner 1905, p. 306).

We could remark in passing that the connection described here between "*im grossen*" problems and *analysis situs* is not specific to Kasner. In our zero corpus, several hits appear in texts of point-set topology, but *none* in texts that bear directly on *analysis situs*, at a time when it is gaining autonomy as a research field. Pioneers of *analysis situs* such as Poincaré (1895) or Dehn and Heegaard (1907) did *not* describe this discipline as that which is necessary to pass from local to global[31]; but, as we shall see, several among those who use "*im kleinen*" and "*im grossen*" or similar

[31] A survey of the literature would show general descriptions of *analysis situs* in terms of groups of transformation (usually "general point transformations," i.e. diffeomorphisms), or as qualitative hypergeometry (for instance, in Poincaré), etc.

terms in other theoretical contexts stress the fact that *analysis situs* is of fundamental importance when striving for global results.

3.2 "*im Kleinen*"–"*im Grossen*" in the calculus of variations and the theory of PDEs

Still on American ground, we find "*im kleinen*" and "*im grossen*" in another theoretical context, namely the calculus of variations. There is one and only one hit for "*im kleinen*" in Oskar Bolza's *Lectures on the Calculus of Variations* (Bolza 1904); the 27th paragraph is devoted to the study of several necessary or sufficient conditions involving the second variation, the last in the list being:

> (e) *Existence of a minimum "im Kleinen"*: We add here an important theorem which has been used, without proof, by several authors in various investigations of the Calculus of Variations, viz., the theorem that under certain conditions two points can always be joined by a minimizing extremal, provided only that the two points are sufficiently near to each other. (Bolza 1904, p. 146)

This use of "*im kleinen*" in the labelling of theorems in a textbook that has many of these is exactly the same as Osgood's. The latter, however, used "*im kleinen*" and "*im grossen*" is a much more systematic way, and it played a central part in his take on function theory as a unitary whole. When we said that Bolza used "*im kleinen*" just once in the text, we meant the body of the text; if we take the index into account, something new comes up. Of course, the theorem presented on p. 146 is mentioned in the final index (as existence theorem for a minimum "*im kleinen*," in appears twice in the index, under "existence theorems" and "minimum); but in that index, another theorem is referred to as 'existence theorem for a minimum "*im grossen*"' (Bolza 1904, pp. 269–270). No theorems, however, were described as "*im grossen*" in the body of the text. The index does not actually refer to a specific theorem, but to the final chapter of the book, whose introduction reads:

> Chapter VII. Hilbert's existence theorem
> §43 Introductory remarks
> If a function $f(x)$ is defined for an interval (*ab*), it has in this interval a lower (upper) limit, finite or infinite, which may or may not be reached. If, however, the function is continuous in (*ab*), then the lower (upper) limit is always finite and is always reached at some point of the interval: the function has a minimum (maximum in the interval).
> Similarly, if the integral
>
> $$J = \int_{t_0}^{t_1} F(x, y, x', y')\mathrm{d}t$$
>
> is defined for a certain manifoldness **M** of curves, we can, in general, say *a priori* whether the values of the integral have a minimum or a maximum. But the question arises whether it is not possible to impose such restrictions either upon

the function F or upon the manifoldness **M** (or upon both), that the existence of an extremum can be ascertained *a priori*. (Bolza 1904, p. 245)

The content of this chapter is based on Hilbert's recent works in the calculus of variations,[32] in which he provided new and general methods for tackling problems of *absolute* maxima or minima, problems which were usually beyond the reach of the ordinary tools. The proof of Dirichlet's principle was, of course, a prominent example. Remaining closer to classical calculus of variations, Bolza only gave the details of Hilbert's proof for the existence of a line for which a well-behaved integral reaches its absolute minimum, the "manifoldness" being that of lines with fixed end-points. In the introductory passage quoted above, Bolza presented this family of existence theorems with the same terms that Hilbert used, that is, in terms of whether or not some upper bound is actually attained (which was the gist of Weierstrass' criticism of Dirichlet and Riemann's proof method in potential theory); which means, neither Hilbert nor Bolza described it as a family of problems "*im grossen*." It is possible that this feature struck Bolza a little later, when writing the index: Hilbert's theorem was listed under "existence theorems," just after the "existence theorem '*im kleinen*'," mentioned above; it probably seemed natural, at that point, to label Hilbert's theorem "existence theorem '*im grossen*'," since the two end-points are set from the start.

In 1909, Bolza published the German edition of his American textbook. In this much more comprehensive work, "*im kleinen*" and "*im grossen*" are used more systematically. When "*im kleinen*" appears for the first time in the book, it appears in quotation marks, indicating that the expression was being introduced as a neologism or a metaphor in the German (mathematical) language (Bolza 1909, p. 270); a few pages down, the quotation marks are gone (Bolza 1909, p. 294). New results appear, for which Bolza considered that "*im kleinen*"–"*im grossen*" provides relevant descriptive means. In §34, Bolza presented theorems proved by Osgood (Osgood 1901b, p. 173); he called the first one "*Der Fall eines Extremums 'im Großen*'" (Bolza 1909, p. 280), and the other one "*Der Fall des Extremums 'im Kleinen*'" (Bolza 1909, p. 283). Osgood, Bolza wrote, discovered that, in the calculus of variation, a property holds which is similar to the following property of elementary analysis: if a function f of a real variable has a relative (i.e. local) minimum for $x = a$ (value $f(a)$ being reached in a only), and is continuous in $[a-k, a+k]$, then for any l such that $0 < l < k$ there is a positive ε_l such that $f(x) - f(a) \geq \varepsilon_l$ on $[a-k, a-l]$ and $[a+l, a+k]$. What Bolza called the "*im grossen*" theorem in the calculus of variation says that (skipping several hypotheses) if curve L is a simple extremal[33] joining two points P_1, P_2, then there is a neighbourhood[34] S of L such that, for any neighbourhood U of L that is strictly contained in S, there is a positive ε_U such that for any curve L' which joins P_1 and P_2 in S but does not lie entirely in U, $J'_L - J_L \geq \varepsilon_U$ (Bolza 1909, p. 281). Upon first reading it, it is not striking what makes this an "*im grossen*" theorem. The fact

32 See, for instance, Hilbert 1900, 1901a.

33 Following Kneser, Bolza called extremals the curves for which the first variation vanishes. Thus, they need not yield either a maximum or a minimum of the integral, even a relative one.

34 In Bolza's textbook, the neighbourhood of a curve is just an ordinary neighbourhood in the plane, no conditions on the derivative are imposed.

that the theorems deals with extremals, which may be local mimina or maxima, is not what Bolza takes into account to label it "*im grossen*": as far as maxima and minima are concerned, Bolza consistently used the adjectives "relative" and "absolute," never "*im kleinen*" and "*im grossen.*" What makes it an "*im grossen*" theorem is that it deals with extremals which have fixed end-points; by contrast, the "*Fall des Extremums 'im Kleinen*'" says that a similar conclusion holds for those extremals which are known to exist by an "*im kleinen*" existence theorem, i.e. extremals whose end-points, wherever they may be, are not too far apart (Bolza 1909, p. 283).

The very same year, Gilbert Bliss served as AMS Colloquium lecturer in Princeton (Bliss 1913), the other lecturer being Kasner. Bliss' talks dealt with "Existence theorems," for implicit function problems and then for differential equations. The phrase "*im kleinen*" is used just once, in the introduction, to summarise the general goal of the last part of the exposition (§17 and fol.):

> One of the principal purposes of the paragraphs which follow, however, is to free the existence theorems as far as possible from the often inconvenient restriction which is implied by the words "in a neighborhood of," or which is so aptly expressed in German by the phrase "im Kleinen". (Bliss 1913, p. 3).

We can find several other American hits for "*im kleinen*" or "*im grossen*" in our zero corpus, although slightly later and in research papers or dissertations rather than survey talks and textbooks. For instance, Earl Gordon Bill defended at Yale in 1908 a dissertation on *An A Priori Existence Theorem for Three Dimensions in the Calculus of Variations*. At the AMS annual meeting of 1908, he gave two talks, the first one was entitled *Existence "im Kleinen" of a curve which minimizes a definite integral*; the summary of the second talk reads: "In the second paper, Dr. Bill proves the existence "*im grossen*", by a method whereby the minimising curve is obtained by applying the existence theorem "*im kleinen*" to a "finite" number of points which are defined as limiting points of the points lying on a sequence of approximating curves" (Cole 1909, p. 285). The expressions were used by Bill himself, and not only by the secretary of the AMS who wrote the report (Frank Cole), as Bill's paper shows (Bill 1912). The terms are used exactly as in Bolza's textbook, and Bill refers to several papers of Osgood and Bliss.

Finally, we should mention the case of Wallie Hurwitz's book's *Randwertaufgaben bei Systemen von Linearen Partiellen Differentialgleichungen erster Ordnung* (Hurwitz 1910). The book is actually his dissertation, which this American, Harvard trained mathematician defended in Göttingen under the supervision of Hilbert.[35] The terms "*im kleinen*" and "*im grossen*" are used in the introduction only, where he presents the various types of PDE and classifies the existence theorems and proof methods:

> In the general study of linear partial differential equations, two types of problems [*Problemstellungen*] are of particular import. In the initial value problem or Cauchy problem, one seeks to determine a solution by giving its values, and that of some of its derivatives, along a curve; there, all the functions that occur are supposed to be analytic in a small neighbourhood of the curve, and so are

[35] A biographical note is included at the end of the book.

> the curve itself and the given values; and the solution sought will be analytic in a possibly even smaller neighbourhood: it is therefore perfectly described as an analytic problem im kleinen. In contrast, the boundary condition or Dirichlet problem requires of the given and sought for functions only continuity, and existence and continuity of a limited number of derivatives; it prescribes the values on a complete curve given from the start, and seeks the solution in the whole given domain; the problem is a non-analytic problem im grossen. (Hurwitz 1910, p. 7)[36]

The theoretical context is quite far from that of Osgood's general function theory, but the meaning is of the "*im kleinen*"–"*im grossen*" distinction is the same. In spite of the variety of contexts, a *network* of problems can clearly be identified which links most of the texts read so far: the Dirichlet problem is fundamental both for the theory of PDEs (as a paradigm for the theory of elliptic PDEs) and the general theory of functions (for Riemann-style theory of functions of a complex variable, and later, for Poincaré's uniformisation theorems); the calculus of variation depends on existence theorems for ODEs and PDEs; Hilbert's new methods for variational problems deal with both line-integral problems (as in standard calculus of variations) and surface-integral problems (as in Dirichlet's problems); the latter context is that of Hurwitz's dissertation:

> In recent years, the investigations into equations of the second order, in the several cases which necessarily occur, have taken a unified shape through the methods of integral equations. (Hurwitz 1910, p. 8)[37]

Hurwitz then wrote that he endeavoured to apply the new methods of integral equations to the theory of first order PDEs, following the suggestion of his dissertation supervisor, namely Hilbert. Only Kasner's collection of problems "*im grossen*" in differential geometry is not so clearly related to this network.

3.3 The Göttingen connection

Of course, the case of Hurwitz shows the extent to which the use of "*im kleinen*" and "*im grossen*" is more an AMS-Göttingen phenomenon than a strictly American

[36] „Bei Untersuchungen allgemeinen Charakters über lineare partielle Differentialgleichungen sind zwei Problemstellungen von besonderer Wichtigkeit. Die Anfangswertaufgabe oder das Cauchysche Problem versucht, eine Lösung durch Angabe ihrer Werte und der Werte gewisser Ableitungen auf einer Kurve zu bestimmen; dabei werden alle vorkommenden Funktionen in einer kleinen Nachbarschaft der Kurve, sowie die Kurve selbst und die vorgeschriebenen Werte in einer kleinen Nachbarschaft eines Punktes als analytisch vorausgesetzt; und die Lösung wird als analytische Funktion in einer eventuell noch kleinern Nachbarschaft gesucht: das Problem ist hervorragend als analytisches Problem im kleinen zu bezeichnen. Dagegen fordert die Randwertaufgabe oder das Dirichletsche Problem von den gegebenen und gesuchten Funktionen nur Stetigkeit und die Existenz und Stetigkeit einer geringen Anzahl von Ableitungen, schreibt die Werte auf einem ganzen vorgegebenen Kurvenstück vor, und sucht die Lösung in einem ganzen vorgegebenen Gebiet; das Problem ist ein nicht-analytisches Problem im großen."

[37] „In den letzten Jahren haben die Betrachtungen für Gleichungen zweiter Ordnung in den verschiedenen Fällen, welche notwendig vorkommen, durch die Methode der Integralgleichungen eine einheitliche Gestalt angenommen."

phenomenon. As mentioned earlier, Osgood had studied in Göttingen and Erlangen and remained in close contact with the Göttingen people, as his involvement in the *Encyclopädie* shows. As for German Oskar Bolza, he trained in Berlin (attending Weierstrass' lectures on the calculus of variations in 1879) but did his dissertation under Klein's supervision; after 22 years in the states (1888–1910, in Chicago as from 1892), he would return to Germany in 1910. Kasner spent a postdoctoral year in Göttingen in 1899–1900. After a dissertation under Bolza's supervision, Bliss spent his post-doctoral year in Göttingen in 1902–1903. Wallie Hurwitz studied in Missouri then in Harvard; in the *Lebenslauf* at the end of his dissertation, he thanked Harvard Professors Osgood and Bôcher for sending him to Göttingen for his doctoral work. The role of this German tour in the making of the first two generations of American mathematicians—in particular to Klein's and then Hilbert's Göttingen—is well documented by qualitative (Parshall and Rowe 1989, 1994) and quantitative (Fenster and Parshall 1994) reference works.

When studying the use of "*im kleinen*" and "*im grossen*," the roles of the AMS and Göttingen are not symmetrical, however. Our zero corpus included the *Mathematische Annalen* as well as Klein's and Hilbert's collected papers (and several other works, see Appendix A); we found no hits for these expressions in the collected papers, and none in the *Mathematische Annalen* before 1904. In the theory of PDEs, W. Hurwitz sorted the problems in terms of "*im kleinen*"–"*im grossen*"; Hilbert did not. In differential geometry, Kasner gathered several problems under the heading "differential geometry im grossen," problems which neither Hilbert nor Hadamard had described in these *meta* terms (nor any of similar meaning). Hilbert's theorems in the calculus of variations were labelled "*im grossen*" by Bolza, not by Hilbert. Within the still small American Mathematical Society (think of the 26 participants mentioned in the report of the 1898 meeting), Osgood's systematic use of these expressions had a strong impact. All the more since, being foreign terms, they could not go unnoticed by the American audience; as foreign expressions they immediately rang like technical terms rather than metaphors, and were used by other mathematicians with *no variations*, hence with a greater stability than in the German-speaking context. In contrast, a native German speaker could find many ways to express the same thing.[38] At least for early members, the AMS such as E.B. Van Vleck, "*im kleinen*" and "*im grossen*" were clearly identified as Osgood's pet expressions. In his presidential address to the society, in 1915, entitled *The Rôle of Point-Set Theory in Geometry and Dynamics*, we can read:

> I was much interested to find that one of my well-informed colleagues had thought of point-set theory as a tool especially adapted for use "im Kleinen"—to use a significant term of Osgood. To me, on the other hand, it had appealed because of its power "im Grossen." (...) I doubt not that much of the characteristic strength of the point-set theory lies precisely in the union of consideration im Kleinen and im Grossen. (Van Vleck 1915, p. 330)

[38] To give a chronologically much later example: in a letter to Weyl, Hasse spoke of "Schluβ vom Kleinen aufs Groβe" (Schwermer 2007, p. 171).

It will be noticed that the quotation marks disappear after the first occurrence of the terms.

It could be argued that Osgood started using *in print* expressions which were used only in informal talks in the Klein and Hilbert circles. What our word-search based inquiry recorded could just be a minute shift of the threshold between what can be said *and* printed, and what can be said but not printed. The problem of documenting local oral traditions, or diagram-drawing practices, is well known in the history of science.[39]

However, several pieces of information make it not so likely that what we captured is a mere shift of threshold. The fact that Göttingen-trained Van Vleck identified these expressions as Osgood's and not Klein's is an indication already (Parshall and Rowe 1994, p. 213). In Göttingen, Osgood attended Klein's lecture and participated in Klein's seminar along with fellow American doctoral students H.W. Tyler, H.S. White, M.W. Haskell and H.D. Thompson; they even took turn writing down Klein's lectures, under the master's close supervision (Parshall and Rowe 1994, p. 209). As far as our core corpus shows, none but Osgood used "*im kleinen*"–"*im grossen*."

Another indication comes from a memory of Grace Chisholm Young,[40] in a 1926 review for *L'Enseignement Mathématique*. Discussing the validity of non-Euclidean geometry, she mentioned in passing: "(…) comme s'exprimait Klein, *im Kleinen ist jede Geometrie Euklidisch*" (Young 1926, p. 326). It seems to be the only evidence of an oral use of "*im kleinen*" around Klein, but it is all the more interesting since the meaning differs significantly from that of Osgood's "*im kleinen*". It is not likely that Klein thought all (Riemannian) geometries were *locally* Euclidean; he more probably expressed the standard idea that they were *infinitesimally* Euclidean. This infinitesimal meaning of "*im kleinen*" corresponds to a few (actually two) hits in the core corpus, which are both connected to Klein's views on geometry. In a 1911 paper *On the Analytical Basis of Non-Euclidean Geometry*, William Henry Young wrote in the introduction: "Other writers have made the assumption that Euclidean Geometry holds in the smallest parts (*im kleinen*)" (Young 1911, p. 250). In Gino Fano's *Encyclopädie* 1907 chapter on *Kontinuirliche Geometrische Gruppen*, there is one hit. In the list of the most usual groups, Fano mentioned the group of conformal transformations of the real plane; these transformations are defined as those which preserve angles; Fano then reformulated: "These transformations therefore behave im Kleinen, in the neighbourhood of a regular point, like conformal transformations" (Fano 1907, p. 343).[41] These traditional descriptions in terms of "smallest parts" or the behaviour in the infinitely small (*im unendlich Kleinen*) are exactly what Osgood would *not* use in his modern, Weierstrass-style take on elementary analysis. Ironically, a few lines after his description of conformal plane transformations as similitudes *im Kleinen*, Fano referred his reader to Osgood's *Encyclopädie* chapter for the interpretation in terms of complex functions. In the passage the *Lehrbuch* where he discussed these

39 For the Göttingen case, see Rowe (2004).

40 Grace Chisholm got her PhD in Göttingen in 1895, under Klein's supervision. The Youngs (Grace married William Henry in 1896) visited Göttingen several times, and lived there from 1899 to 1908.

41 „Diese Transformationen verhalten sich also im Kleinen, in der Umgebung regulärer Stellen, wie Ähnlichkeitstransformationen."

aspects, Osgood avoided all talk of "smallest parts" or "infinitesimal elements"; when discussing the differential properties of functions of two real variables he clearly distinguished between the plane transformation associated to the function and the linear transformation (of the whole plane) associated to a point; their actions on curves (not infinitesimal lines) are compared, and tangency is the key notion (Osgood 1912, pp. 70 & fol.). In these few texts in our core corpus where "*im kleinen*" was used merely as a new way to express the old, pre-Weiersrassian (and now seen as merely metaphorical) infinitesimal meaning, "*im grossen*" is not used; the opposite of infinitesimal is finite (which is usually only local).

All the other hits have a clearly non-infinitesimal meaning and use "*im kleinen*" with the same non-metaphorical meaning as in Osgood's texts. For instance, the first systematic use of both "*im kleinen*" and "*im grossen*" in a text written in the German[42] context is Ludwig Schlesinger's 1905 paper in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (JDM-V) entitled *Über eine Darstellung des Systems der absoluten Geometrie* (Schlesinger 1905). The paper contains a short exposition of the intrinsic Riemannian geometry of surfaces; after introducing $\mathrm{d}s^2 = E\mathrm{d}p^2 + 2F\mathrm{d}p\mathrm{d}q + G\mathrm{d}q^2$ to measure the length line-elements, he remarked:

> As long as E, F, G are seen as defined only in a given neighbourhood of a point determined by the nature of the study, we have the geometry of M_2 im kleinen. In contrast, if E, F, G are given for all values, or even for a domain of values of p and q which is limited by *a priori* given inequalities, then we have the geometry of M_2 im grossen. To the latter belong in particular the extension relations and domains relations (Analysis situs). (Schlesinger 1905, pp. 562–563)[43]

The distinction is nearly word for word that of Osgood's definitional footnote.

Another element must be mentioned here, which is of Göttingen origin. A distinction which clearly rings like "*im kleinen*"–"*im grossen*" was repeatedly used by Felix Klein in his lectures on geometry. For instance, in his 1892–1893 introduction to higher geometry, the introductory part dealt with the notion of function. After presenting the two fundamental notions of (analytic) function, that of function element and that of *Gesamtfunction* derived from a function element by maximal analytic continuation, he presented the main division in the geometrical sciences:

> 2. Main division of Geometry.
> Along these lines, we can also divide geometry itself in two different parts, namely: 1) Geometry in a limited portion of space, corresponding to the use of function elements only. 2) Geometry in the whole space [*Gesamtraum*], corresponding to the use of whole functions [*Gesamtfunktionen*]. Nearly all the applications of diff. and integral calculus to Geometry belong in the first part.

[42] Hungarian/German, actually.

[43] „Solange die E,F,G nur als in einer durch die Natur der Untersuchung bestimmten Umgebung einer Stelle definiert angesehen werden, haben wir Geometrie der M_2 im kleinen, sind dagegen die E,F,G für alle Werte oder doch für Wertgebiete der p,q, die durch a priori gegebene Ungleichheitsbedingungen beschränkt sind, gegeben, so haben wir Geometrie der M_2 im groβen. Der letzteren gehören namentlich die Ausdehnungs- und Gebietsverhältnisse (Analysis situs) an."

> (…) On the other hand, the theory of algebraic curves and surfaces belongs for its greater part in the second part. (Klein 1893, pp. 6–7)[44]

This distinction would remain unchanged through the various reprints [1907, 1926 (with Blaschke)]. In terms of meaning, the connection between this distinction and Osgood's articulation is clear, in spite of the fact that Klein's distinction gets its precise meaning only in the realm of analytic functions, whereas Osgood's syntactic distinction does not depend on analyticity. This closeness of meanings may account for the fact that, in his definitional footnote, Osgood noted that the "*im kleinen*"–"*im grossen*" distinction was relevant for analysis *and* geometry, in a context where no geometry would be touched upon.

4 "*im Kleinen*"–"*im Grossen*" in Germany

With Schlesinger, we have already left the American Mathematical Society. The AMS hits for "*im kleinen*"–"*im grossen*" showed stability in meaning through several disciplinary contexts (general function theory, calculus of variations, differential geometry); they also showed a similar use of these expressions, a use which we called *meta* to stress the fact that it stands in direct connection with some specific primary content: labelling theorems, associating theorems in pairs, grouping apparently unrelated theorems under one heading, spelling out the steps of a long proof, emphasising the scope of a theorem (and warning the reader against a common mistake). The German hits for "*im kleinen*" or "*im grossen*" in the core corpus come after the AMS hits, the meaning is the same as in Osgood, Bolza and Kasner (save for the few infinitesimal cases mentioned above), but, from a pragmatic viewpoint, they show a greater diversity of uses. We shall discuss three of these uses: introducing a new technical property by localisation; setting disciplinary boundaries; selecting axioms in the reshaping of a well-known theory.

4.1 Definition by localisation

In 1913, Hans Hahn gave a talk at the *Versammlung deutscher Naturforscher und Ärzte zu Wien*, which was published as a paper in the 1914 volume of the *JDM-V* (Hahn 1914). Dealing with point-set topology, it tackled the already well-known problem of characterisation of the subsets of the plane which are the image of a segment by a continuous function. Hahn started by mentioning three simple necessary conditions: if a *Punktmenge* M is a such image, then it is bounded (*geschränkt*), closed (*abgeschlossen*) and connected [*zusammenhängend*, i.e. the set is not the disjoint union of two

[44] „2. Haupteinteilung der Geometrie.
Entsprechend der entwickelten Auffassung können wir auch die Geometrie selbst, in zwei verschiedene Teile spalten, nämlich: 1) Geometrie im begrenzten Raumstück, entsprechend der Verwendung allein von Funktionselementen. 2) Geometrie im Gesamtraum, entsprechend der Verwendung von Gesamtfunktionen. Zu dem ersten Teile gehört fast die ganze Anwendung der Diff.- und Integralrechnung auf Geometrie. (…) Andererseits gehört die Theorie der algebraischen Kurven und Flächen grösstenteils zu dem zweiten Teile"

non-empty closed subsets (Hahn 1914, p. 318)]. He then proved that M also enjoys a fourth property, which he defined as follows:

> Let P be a point of M; to every positive number ε corresponds a positive number η such that, for any point P' of M in the η-neighbourhood of P there is a closed and connected part of M containing both points P and P', and lying entirely within the ε-neighbourhood of P.
> A set in which every point has this property, we shall call *connected im kleinen*. (Hahn 1914, p. 319)[45]

He then proved that these four conditions are also sufficient for any subset M of the plane to be the continuous image of a segment. This technical notion of local connectivity was soon taken up and used (under the names "*Zusammenhang im kleinen*" and "connectedness *im kleinen*") in point set topology, wherever these questions of Jordan continua were studied: by Mazurkiewicz, Sierpinski and Kuratovski in Poland; in the U.S. by R.L. Moore, J.L. Kline in the US, and, later, W.A. Wilson and G.T. Whyburn; by Tietze (in Vienna, then Erlangen). A citation network would show that these form a strongly connected network in a well-identified research field. Local connectivity would account for the greater part of the hits, where we investigate the use of "*im kleinen*" the 1914–1925 period. Since local connectivity is technically defined, and "*Zusammenhang im kleinen*" is a closed syntagm, we do not consider using this expression as way of performing a *meta* role in the mathematical text.

Before introducing the notion of local connectivity, Hahn had used "*im kleinen*" several times in a context that is already familiar to us, that of the calculus of variations. He had written several works in that field, starting with his dissertation on the second variation of simple integrals (1902, supervised in Vienna by G. Ritter von Escherich). His use of "*im kleinen*" and "*im grossen*" followed Bolza's model, for instance, in his review for the JFM of a paper by Hadamard on the isoperimetric problem (1913); after his summary of Hadamard's proof: "Once the possibility of the Weierstrass construction im Kleinen has been demonstrated, the existence of the absolute extremum im Grossen can be established by a well-known method." (Hahn 1913).[46] It should be noted that his definition of "*Zusammenhang im Kleinen*" is not a simple localisation of the notion of connectivity (as in: every neighbourhood of point P contains a connected neighbourhood of P), but follows the two-neighbourhood pattern of Bolza's wording of Osgood's theorems in the calculus of variations. The many notions that would then be introduced in point-set topology would follow a more straightforward localisation

[45] „Sei P ein Punkt von M; zu jeder positiven Zahl ε gehört dann eine positive Zahl η derart, dass es zu jedem in der Umgebung η von P liegenden Punkt P' von M einen die beiden Punkte P und P' enthaltenden abgeschlossenen und zusammenhängenden Teil von M gibt, der ganz in der Umgebung ε von P liegt.
Eine Menge, die in jedem ihrer Punkte diese Eigenschaft hat, wollen wir zusammenhängend im kleinen nennen."

[46] "Nachdem so die Möglichkeit der Weierstraβschen Konstruktion im Kleinen dargetan ist, kann einer bekannten Methode die Existenz des absoluten Extrems im Groβen dargetan werden." Same in (Hahn 1912). The fact that for the "general isomeperimetric problem" (to minimise or maximise an integral, while another remains constant), the theorem "im Kleinen" may fail to hold had already been pointed out by Hadamard a few years earlier (1907); this was described using "im Kleinen" in Haussner's review for the *JFM*. (Haussner 1907).

pattern, as for "*Kompaktheit im Kleinen*" (Alexandroff 1924) or "*Konvexheit im Kleinen*" (Tietze 1928).

4.2 "*im Kleinen*"–"*im Grossen*" in the defence or creation of disciplinary boundaries

A quite different use of "*im kleinen*"–"*im grossen*" can be found a little earlier and in a completely different theoretical context. In 1908 Eduard Study published his *Kritische Betrachtungen über Lies Invariantentheorie der endlichen kontinuirlichen Gruppen* in the JDM-V (Study 1908). Starting from an erroneous statement that he spotted in Lie and Scheffers' *Vorlesungen über Continuirmichen Gruppen mit geometrischen und anderen Anwendungen*, Study presented a strongly worded but detailed analysis of what he considered to be a fundamental and systematic flaw in Lie's reasoning. For instance, Study argued, Lie systematically relies on the counting of constants (*Konstantenabzählung*); whenever he is faced with a system of parameters or a system of equations, he relies on dimensional arguments (counting arguments) to eliminate "dependent" parameters or equations[47] (Study 1908 p. 130). His quest for a general treatment of all groups leads to a fundamental error (*Grundirrtum*):

> At this point, one should remember that in his general theory of finite continuous groups, Lie wanted to deduce the properties that are common to all of them. But, by the very nature of things, that could only be achieved by renouncing to grasp the whole space studied in each case, and, as a rule, by renouncing to take into account the totality of the transformations of a group. (Study 1908, p. 137)[48]

As to the formation of invariants, when geometric objects are to be classified under the action of a given Lie group, Lie's theory may lead us astray. For instance, Study wrote, they cannot help us distinguish between the action of a continuous (i.e. connected) group such as that of *direct* isometries, and the action of a mixed group (i.e. with multiple connected components) such as the general group of isometries (Study 1908, p. 133). Likewise, when faced with multi-valued invariants, Lie usually arbitrarily chooses one branch and, for instance, passes from $r^2 = r'^2$ to $r = r'$; hence, Study went on, he mistakes sufficient equivalence conditions for necessary and sufficient conditions (Study 1908, p. 135). About halfway through his paper, Study summarised his manifold criticisms by saying that Lie's theory is a wonderful *local* theory, but that it is not likely that it is a tractable starting point when global results are sought for:

[47] For instance, the group of fractional linear transformations of the complex projective line (i.e. homographies $z' = (az + b)/(cz + d)$) has three essential parameters, since (a, b, c, d) is determined up to an arbitrary (non null) factor. Hence, Lie chose to write these as $z' = (z + b)/(cz + d)$. This looks like mere tidying up, and reflects the fact that the group is 3-dimensional indeed, but it implies (Study argues) that only a neighbourhood of the identity is taken into account.

[48] „Hier dürfte nun zunächst daran zu erinnern sein, daβ Lie in seiner allgemeinen Theorie der endlichen kontinuirlichen Gruppen die Eigenschaften entwickeln wollte, die allen diesen gemeinsam sind. Das aber konnte der Natur der Sache nach durchaus nur dann erzielt werden, wenn gleichzeitig darauf Verzicht geleistet wurde, den ganzen jedesmal in Betracht kommenden Raum zu umfassen, und in der Regel auch darauf, die Gesamtheit der Transformationen einer Gruppe mitzunehmen."

> (...) wherever a complete theory of invariants for a specific group is to be developed, Lie's invariants provide an essential component.
> But it is a quite different question, whether or not it is a tractable path to necessarily refer back to Lie's general theory; whether or not in concrete cases one will be able to proceed by starting from a theory of invariants "im kleinen," and then extend it to a theory of invariants "im grossen." (Study 1908, p. 138)[49]

Lie's complete systems of local invariants may not be analytically continuable; even if they were building invariants through analytic continuation would probably lead to invariants that are multi-valued, hence nearly useless (Study 1908, p. 140). The paper ends on even harsher notes. Not only is Lie's theory valuable in its limited scope only, but, as a mathematician, Lie probably was not aware of these limitations:

> As a foreign necessity imposed from the outside, the indubitable theoretical insight he had that his concepts and theorems were valid only in limited domains never rooted itself properly in Lie's creative and intuitive mind. It was merely felt to be like tiresome fetters, to be dropped at the first occasion. As we mentioned, in the theory of invariants of the whole space, Lie never spoke of multi-valued invariants any differently than if they had been single-valued; and every dropping of redundant equations—an operation which, with due caution, is possible im kleinen—led, in completely different circumstances, to illicit applications. (Study 1908, p. 141)[50]

The (excessive) drive for generality is not all there is to it. Study implied that Lie did not think correctly; if he did, still he wrote in the loose and unrigorous style of old times. Therefore, his writings cannot be trusted, and the whole theory is built on sand (Study 1908, p. 132). A crafty and experienced polemicist, Study played with quotation marks: the first time "*im kleinen*" and "*im grossen*" appear, they appear in quotation marks; these disappear afterwards. On the contrary, terms that are central to Lie's reasoning, such as "independent" (*unabhängig*; Study 1908, p. 131), "essential" (*wesentlich*; Study 1908, p. 134), or "invariants" (Study 1908, p. 134) are used with no quotation marks in the beginning of the text, then get some as Study explains why they should be viewed with suspicion.

The meaning of "*im kleinen*" and "*im grossen*" in this text is not new to us, even if Study's take on "*im grossen*" is probably less specific than Osgood's syntactic

49 „(...) wo immer eine vollständige Invariantentheorie einer speziellen Gruppe entwickelt werden wird, Lies Invarianten einen wesentlichen Bestandteil von ihr ausmachen werden.
Eine ganz andere Frage aber ist es nun, ob der Weg gangbar ist, auf den Lies allgemeine Theorie notwendig verweist, ob es im konkreten Falle durchführbar sein wird, mit einer Invariantentheorie „im kleinen" zu beginnen, und diese dann zu einer Invariantentheorie „im groβen" zu erweitern."

50 „Die unzweifelhaft bei ihm vorhandene theoretische Einsicht, daβ seine Begriffe und Theoreme nur in beschränkten Bereichen Geltung haben, hat als eine fremdartige von auβen her aufgedrängte Forderung in Lies schaffensfrohem intuitivem Geiste wohl nie recht Wurzel gefaβt. Sie wurde wohl kaum anders denn als eine lästige Fessel empfunden, die bei erster Gelegenheit abgeschüttelt werden durfte. So hat, wie wir gesehen haben, Lie auch in der Theorie des Gesamtraumes von mehrwertigen Invarianten nicht anders geredet, als ob sie einwertig wären; und jenes Weglassen überzähliger Gleichungen besteht ebenfalls darin, daβ eine im kleinen bei gehöriger Vorsicht mögliche Operation unter ganz anders gearteteten Verhältnissen eine nunmehr unerlaubte Anwendung findet."

characterisation and is more closely associated to the process of analytic continuation. Also, Osgood had already warned his readers against faulty reasoning where one seeks to establish global facts by using local theorems only. However, what we have here is a general analysis of traditional modes of reasoning (and writing) in which systematic mistakes are made, either in the name of generality (generic reasoning) or by appealing to a form of reasoning that would be specifically geometric (as opposed to rigorous analysis (Study 1908, pp. 131–132) and painstaking writing[51]).

This particular text can be—and has been—studied in several contexts, be it the life-long difficult relationship between Study and his Master Klein (Hartwich 2005) or the long-term history of Lie groups theory (Hawkins 2000, Chorlay forthcoming, Chap. 6). But Study chose to criticise Lie's theory of invariants not only because of his general dissatisfaction with a style of thinking and writing that was by no means specific to Klein and Lie, but because Lie and himself ploughed similar fields (classification of geometric objects) with similar tools (groups and invariants). The matters of rigour are only one side of the coin, the other being a question of disciplinary pre-eminence: Lie's attempts at developing a theory of invariants for algebraic curves and surfaces is doomed, Study argued; the right tools come from the algebraic theory of invariants, of which Study was a Master and which he would defend until his death (Hartwich 2005, Chap. 7). Now, is Study (1908) paper a turning-point in this history of two disciplines fighting for the same patch of mathematical territory? Did this emphasis on the purely "*im kleinen*" nature of Lie's theory lead its proponents to take up the challenge and articulate a new and "*im grossen*" research agenda?

On the whole, we believe the answer to both questions is no. The fact that, due to the very tools on which it rests, Lie's theory is of local scope had been pointed out by Klein himself for quite some time; for instance, in he Evanston Colloquium talk (1893), Klein's second talk on Lie's theory started with:

> The distinction between analytic and algebraic functions, so important in pure analysis, also enters into the treatment of geometry.
> Analytic functions are those that can be represented by a power series, convergent within a certain region bounded by the so-called circle of convergence. Outside of this region the analytic function is not regarded as given a priori; its continuation into wider regions remains a matter of special investigation and may give very different results, according to the particular case considered. On the other hand, an algebraic function, $w = \text{Alg.}\ (z)$, is supposed to be known for the whole complex plane, having a finite number of values for every value of z. Similarly, in geometry, we may confine our attention to a limited portion of an analytic curve or surface, as, for instance, in constructing the tangent, investigating the curvature, etc.; or we may have to consider the whole extent of algebraic curves and surfaces in space. Almost the whole of the applications of the differential and integral calculus to geometry belongs to the former branch of geometry. (Klein 1894, p. 18)

[51] See Study's the letter to Engel quoted in (Hartwich 2005, p. 123).

A similar statement can be found in Fano's *Encyclopädie* chapter on *Kontinuirliche geometrische Gruppen. Die Gruppentheorie als geometrisches Einteilungsprinzip*:

> The group-theoretic view of geometry also shed light on the true nature of "differential geometry," and showed that the latter is not the opposite of "projective geometry" or "algebraic geometry," but only that of the "geometry of the whole space" (IIIA B 4a, Fano No. 35); and that within both, one can identify an elementary (or metric) viewpoint, a projective viewpoint, a birational viewpoint, etc. (Fano 1907, p. 297)[52]

Likewise, Study's criticism was explicitly mentioned a few years later, in Cartan's version of Fano's article (for the French edition of the *Encyclopädie*):

> Lie's theory has the advantage of great generality; but, in addition to the drawback of requiring integrations, it has an even more serious one, of solving invariant problems solely from the viewpoint of analytic functions. Its results generally pertain to some domain about a point, and cannot, due to the very generality of the theory, be extended to the whole space. In particular, Lie's theory cannot replace the *algebraic* theory of invariants. (Cartan 1915, p. 1845)[53]

It seems that between 1893 and 1915, the frontline did not move by an inch. Study's criticism did not lead the proponents of Lie's theory to engage in a new and global theory of Lie groups; that would happen in the 1920s, in a quite different context (Hawkins 2000; Chorlay 2009). For now, Klein and his allies opted for a truce, at least for the public eye: Lie's theory and the theory of algebraic invariants both have their merits; Lie's theory is local all right, let everyone do their thing; as to the charge about rigour, Engel acknowledged that, indeed, they should be more careful (Engel 1908).

It should be noted, however, that neither Klein, Engel, Fano nor Cartan used "*im kleinen*"–"*im grossen*" or any similar expressions. Their description was in terms of "analytic" versus "algebraic," which was echoed in a distinction in terms of "part" versus "whole" (e.g. *Gesamtraum*), the "part" being the convergence disc of an analytic function. The fact that "*im kleinen*"–"*im grossen*" became part of Study's writing tools may not have had a direct impact on the history of Lie groups or on the frontline between the competing theories of differential and algebraic invariants; however, a quick look at another discipline—differential geometry—gives a different picture.

Apart from the 1908 paper on Lie's invariants theory, Study occasionally used "*im kleinen*" and "*im grossen*" the *meta* way. For instance, in his 1905 paper entitled *Kürzeste Wege im komplexen Gebiet*, he endeavoured to lay the foundation of the

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52 "Die gruppentheoretische Auffassung der Geometrie hat auch das eigentliche Wesen der "Differential-geometrie" ans Licht gestellt und gezeigt, daβ letztere kein Gegensatz zur "projektiven Geometrie" oder zur "algebraische Geometrie" ist, sondern nur zur "Geometrie des Gesamtraumes" (III A B 4a, Fano No. 35), und daβ man innerhalb beider eine elemenrare (oder metrische), eine projektive, eine birationale usw. Auffassung unterscheiden kann."

53 «La théorie de S. Lie a l'avantage d'une très grande généralité; mais, outre l'inconvénient d'exiger des intégrations, elle en a un autre plus grave, c'est de ne résoudre les problèmes relatifs aux invariants que du point de vue des fonctions analytiques. Ses résultats ne se rapportent en général qu'à un certain domaine autour d'un point et ne peuvent pas, à cause de la généralité même de la théorie, être étendus à tout l'espace. En particulier, la théorie de S. Lie ne peut remplacer la théorie *algébrique* des invariants.»

geometric theory of Hermitian forms (thanks to which a notion of distance between to points can be defined even in the complex context, hence the title). Among the many topics dealt with in the 60 page paper, he studied the groups of the properly discontinuous subgroups of the group of those linear transformations which preserve the Hermitian form. After describing the fundamental domain for a class of such groups, he pointed that it could be seen from two viewpoints: either as a bounded domain in n-dimensional complex projective space or when border points which are equivalent under the action of the group are identified, as an ideally closed figure: "It yields a new type of—as we might call it—quasi-Hermitian space, which "im kleinen" has the same properties as that of the space we considered, but shows very different connectivity properties (...)" (Study 1905, p. 367).[54] Studying the applicability of those Hermitian spaces which generalise the ordinary non-Euclidean elliptic and hyperbolic spaces, he proved that: "As from $n > 2$, the elliptic and the hyperbolic $(2n-2)$-dimensional hermitian spaces cannot be, even im kleinen, either geodesically or conformally mapped one onto the other, or onto a manifold of constant curvature" (Study 1905, p. 371).[55]

This occasional *meta* use would become a systematic use for one of Study's students, Wilhelm Blaschke. After a dissertation in Vienna, under the supervision of Wirtinger (1908), Blaschke went on a tour of his own, through Bonn (to work with Study), Pisa (for Bianchi) and Göttingen[56]; he became a *Privatdozent* at Study's side in Bonn, in 1910, but quickly left for Greifswald (where he could work with Engel; see Reichardt 1967). In the period from 1910 to 1916, Study and Blaschke often cited one another's work[57]; the second volume of Study's *Vorlesungen über ausgewählte Gegenstände der Geometrie* (part II: *Konforme Abbildungen und einfach zusammenhängenden Bereiche*; Study 1913) was written with Blaschke, who wrote the part on Koebe's proof.

To say that "*im kleinen*"–"*im grossen*" had not been used very often differential geometry before 1913 is an understatement. That seems to be all the more surprising since (as far as our core corpus shows) the first use of these expressions in that field—namely Kasner's—was very public. The only hits for these expressions in our core corpus, after Kasner and before Blaschke, are in Schlesinger's paper on the applicability of surfaces (quoted above) and in Mangoldt's *Encyclopädie* chapter entitled *Die Begriffe "Linie" und Fläche"* (Mangoldt 1907).[58] In this 1906 text, Mangoldt mentioned in passing that the representation of a space curve by a pair of implicit equations quite often holds globally: "A similar representation as that im kleinen is often possible im grossen" (Mangoldt 1907, p. 138)[59]; Mangoldt also used "*im*

54 „Es liefert dann eine weitere Art von—wie wir etwa sagen mögen, Quasi-Hermitschem—Raume, der "im kleinen" dieselben Eigenschaften hat, wie der von uns betrachtete, aber eine andere Art des Zusammenhangs aufweist (...)"

55 „Der elliptische und der hyperbolische $(2n-2)$-dimensionale Hermitsche Raum können, auch im kleinen, weder geodätisch noch konform aufeinander oder auf eine Mannigfaltigkeit konstanten Krümmungsmaβes abgebildet werden, sobald $n > 2$ ist"

56 (Blaschke 1961, p. 148). In Göttingen, he worked with Koebe.

57 For instance, in Study's *Vorlesungen über ausgewählte Gegenstände der Geometrie* (Study 1913, pp. 121–122). In the preface, Study also thanked Blaschke for his critical reading and for doing the drawings.

58 It could be argued, of course, that Study's use of "im Kleinen" to study the local applicability of his hermitian hyperbolic and elliptic spaces should be counted as a hit in the field of differential geometry.

59 "Eine ähnliche Darstellung wie im kleinen ist häufig auch im grossen zulässig"

ganzen" to denote this "*im grossen*" analytic representation.[60] On the whole, it seems that Kasner's use of "*im grossen*" to gather under one heading problems of seemingly unrelated natures did not have a direct impact on the way differential geometers viewed and described their field,[61] nor did it serve as a research agenda. In contrast, Blaschke made a systematic use of "*im kleinen*"–"*im grossen*," turning Kasner's list of problems into a new-subfield of differential geometry. The first line of his 1913 paper in the *JDM-V Über isometrische Flächenpaare* reads: "The following research is meant as preliminary work on the treatment of questions pertaining to the deformation of surfaces im Groβen" (Blaschke 1913, p. 155).[62] Many such examples could be taken from Blaschke's work in this period.[63] There is a swift shift from "*Fragen im Grossen*" (type of question) to "*Differentialgeometrie im Grossen*" (subdiscipline), for instance, in his 1916 book *Kreis und Kugel* (Blaschke 1916, p. vi)[64], in which both applicability and isoperimetric problems are tackled. The general description of these problems which constitute that research field comes as no surprise: "While most of the theorems of differential geometry bear on a sufficiently restricted neighbourhood of an element of the geometric figure under study [footnote: in the following, we will assume knowledge of the basic notions of this differential geometry "im kleinen"], the propositions we will establish deal with the boundary curves and surfaces of convex domains and bodies in their whole extent." (Blaschke 1916, p. 113).[65] Blaschke also stressed the novelty of this subfield, by remarking that Bianchi's textbook on differential geometry (which he prized) was almost entirely devoted to local differential geometry (Blaschke 1916, p. 114). These views will be echoed in the 1920s in his textbooks on differential geometry, in the prestigious *Grundleheren der mathematischen Wissenschaften* series. For instance, the first two chapters of the second volume are entitled *Ebene Kurven im Kleinen* (Chap. I), and *Ebene Kurven im Groβen* (Chap. II) (Blaschke 1923).

60 It should be noted that, in this chapter, Mangoldt refers his reader Osgood's work quite often; usually for Osgood's papers in point-set topology (space-filling curves, proofs of the Jordan curve theorem, etc.), but also for his presentation of Weierstrass' *analytisches Gebilde* in his *Encyclopädie* chapter (Mangoldt 1907, p. 132).

61 This statement has but a heuristic value, since it is grounded on our word-search in the core corpus, and not in a study of differential geometry as a discipline in the 1900s, in terms of body of knowledge and image of knowledge (to use Leo Corry's terms), or in terms of network of practitioners, etc (Corry 1996).

62 „Die vorliegende Untersuchung ist als eine Vorarbeit gedacht zur Behandlung von Fragen nach der Verbiegbarkeit der Flächen im Groβen"

63 For instance, in his 1915 *JDM-V* paper on *Kreis und Kugel* (Blaschke 1915, p. 203). *Fragen der Differentialgeometrie im Grossen* is the title of a talk he gave in December 1915 at the *Berliner Mathematische Gesellschaft*.

64 In his review of this book for the *JDM-V*, Heinrich Liebmann wrote „(...) das reiche Schatzkästlein der in schüchternen und glänzenden Anfängen schon längst vorhandenen, aber abseits stehenden und wohl erst im Jahre 1905 von Kasner unter dem Namen "Geometrie im Großen" in das mathematische Standesamtsregister eingetragenen Sparte dem freudig überraschten Leser aufzuschließen." (Liebmann 1917, p. 123)

65 „Whärend nämlich die meiste Lehrsatz der Differentialgeometrie sich auf eine genügend enge Nachbarschaft eines Elements des betrachteten geometrischen Gebildes beziehen [footnote: Wir werden im folgenden die Anfangsgründe dieser Differentialgeometrie "im kleinen" als bekannt voraussetzen], handeln die Sätze, die hier aufgestellt werden sollen, von den Begrenzungskurven und Begrenzungsflächen konvexer Bereiche und konvexer Körper in ihrer ganzen Ausdehnung"

4.3 *"im Kleinen"–"im Grossen"* in the design of an axiomatic theory

In 1912, H. Weyl wrote a critical review of N. Nielsen's *Elemente der Funktionentheorie* for the *JDM-V*. This author, Weyl wrote, derived everything from power series:

> This is naturally and innerly consistent, if one leaves out of the presentation Riemann's ideas, and even Weierstrass' principle of analytic continuation (which is objectively so closely connected to these ideas), and limits oneself to the behaviour of functions "im kleinen" (that is, in the domain of validity of the underlying series, or the like.) (Weyl 1912, p. 97)[66]

Unfortunately, Weyl deplored, Nielsen did not safely remain within the limits of the *"im kleinen"* part of the theory and touched on topics such as Cauchy's integral theorem,[67] and the theory of algebraic functions:

> But unfortunately, the parts of the book where the two-dimensional domain as a whole come into play can hardly be called a success; the vagueness of its concept-formation [*Begriffsbildung*] and proof-devising [*Beweisführung*] (...) is in sharp contrast with the painstaking accuracy which prevails in nearly all the rest. (Weyl 1912, p. 97)[68]

When he wrote this review, Weyl himself was busy lecturing on *Die Idee der Riemannschen Fläche*.

Weyl's 1913 book is a landmark of mathematics and has been studied in several historical contexts: the history of complex function theory, the history of topology (general and algebraic), the history of the manifold concept,[69] etc. Its striking features are well known: the choice of an *axiomatic* definition of the two structures involved (two-dimensional topological surface, complex-analytic structure on the latter), instead of a standardised construction procedure; its endorsement of the Kleinian 1882 Copernican revolution, according to which surfaces per se come first, and functions on these come second[70]; its vocal claim that the geometric-topological setting which Riemann had chosen for the theory of algebraic functions (and their integrals)

66 „Es ist natürlich und innerlich konsequent, wenn dabei die Riemannschen Ideenbildungen, ja selbst das mit ihnen in engem sachlichen Zusammenhang stehende Weierstraβsche Princip der analytischen Fortsetzung von der Darstellung ausgeschlossen bleiben und die Betrachtungen sich auf das Verhalten der Funktionen "im Kleinen" (d.h. im jeweiligen Gültigkeitsbereich der zugrunde gelegten Reihenentwicklungen oder dgl.) beschränken."

67 Which states that, in a simply connected domain of the complex plane, a meromorphic differential with no residues is the differential of a meromorphic function. (Weyl 1919, p. 56)

68 „Leider aber sind gerade diese Abschnitte des Buches, wo das zweidimensionale Gebiet als Ganzes in Frage kommt, kaum als gelungen zu bezeichnen, und stechen in der Verschwommenheit ihrer Begriffsbildung und Beweisführung (...) merkwürdig ab gegen die im übrigen fast ausnahmslos herrschende peinliche Genauigkeit."

69 See Scholz (1980, 1999).

70 In the Neumann presentation of Riemann's theory, first came (multi-valued) functions on the complex plane, then came surfaces *over* the plane *associated* to functions, eventually functions (and integrals) on these surfaces.

is not doomed to a intuitive-but-hopelessly unrigorous hell, but can be made just as rigorous as its numerous competitors.[71]

Weyl certainly did all these things—define, endorse, claim, etc.—but the questions of *how* he reached the goals he so vocally set for himself in the famous preface has to be accounted for. We feel it is necessary but not sufficient to say that he relied on recent works which showed how key parts of theory could be develop in a thoroughly rigorous manner: Brouwer's work on topology, Hilbert's work on the Dirichlet principle, Koebe's work on uniformisation. These served as prime material indeed, but, again, knowing what Weyl selected as the right building blocks leaves the *how* question, to a large extent, unanswered. On the contrary, its makes it all the more challenging to describe what Weyl did that was new, since the novelty lies more in the general design of the book rather than in some new theorem, or in a new proof for a formerly loosely founded proposition. The architectural metaphor sounds right, for once: the building-blocks may have been borrowed from the Masters, still, the new layout or design has to be accounted for.

What we called "design," Weyl more specifically described as *Begriffsbildung* and *Beweisführung* in his critical review of Nielsen. Weyl certainly aimed for rigour, but, as that review shows, his specific goal was to find the proper fundamental concepts (*Begriffe*) and proof schemes for the "*im grossen*" part of the theory; for the "*im kleinen*" part, power series are perfectly adequate. As we shall see, the "*im kleinen*"–"*im grossen*" articulation is not only useful to make *specific* sense of the too general epistemic value of rigour; it also sheds light on how the new layout was designed.

Begriffsbildung and *Beweisführung* are indeed what sets Weyl's exposition of the geometric theory of algebraic functions apart from the standard cut-and-paste exposition. But quoting Weyl's axiomatic definitions of topological surfaces, complex analytic curves and covering spaces fails to show why these definitions were crafted, what made them right for a rigorous treatment of the global part of the theory. Actually, Weyl is perfectly explicit on this issue. After spelling out the axioms for a topological surface, he endeavoured to show that they meet the requirements he considered relevant: "Let us now briefly explain how, on the basis of the concept of neighbourhood, all notions pertaining to continuity can be transferred [*übertragen*] from the ordinary plane to any two-dimensional manifold" (Weyl 1919, p. 18).[72] The notion that is implicitly defined by the axiom is that of *Umgebung*, and this local basis is enough for the standard notions of point set topology to be defined (condensation point, interior point, continuous curve, continuous map, etc.).[73] Similar reasons are laid out in the paragraph that comes *before* the definition of the analytic structure:

> To claim that z and u are analytic functions on surface $\mathfrak{G}$, it is essential that $\mathfrak{G}$ be given *not only as a surface in the sense of Analysis situs*. For, on a surface of

[71] Among which: the *funktiontheoretisch* theory of analytic configurations (Weierstrass style); the algebraic-geometry-with-invariants theory of Gordan, Clebsch and Noether; the theory of function fields of transcendence degree one over some primary field (Dedekind-Weber, with no theory of Abelian integrals in that context, of course), etc. (Brill and Noether 1894).

[72] „Wir legen jetzt kurz dar, wie auf Grund des Begriffes der Umgebung alle Kontinuitätsbegriffe von der gewöhnliche Ebene auf beliebige zweidimensionale Mannigfaltigkeiten übertragen werden können."

[73] It is well known that this emphasis on local axioms only led Weyl to fail to include a separation axiom.

which only Analysis-situs properties are taken into account, one can indeed talk about continuous functions, but not about "continuously differentiable," "analytic" (or even "entire rational") functions, or the like. In order to deal with analytic functions on a surface $\mathfrak{F}$ just as in the plane, we must moreover (in addition to the definition of the surface) provide a stipulation [*Erklärung*] which will set the meaning of the expression "analytic function on the surface," in such as way that all propositions about analytic functions in the plane which are valid "im Kleinen" transfer to this more general concept. Here, propositions valid "im Kleinen" are those whose validity is claimed only for some neighbourhood of a point; some neighbourhood about the size of which no information is given by the proposition. With such a stipulation for the expression "analytic function on $\mathfrak{F}$," the surface $\mathfrak{F}$ becomes a **Riemann surface**. (Weyl 1919, p. 35, emphasis in original)[74].

In both the topological and analytic cases, the notion of *Übertragung* (transfer) is fundamental. The distinction between "*im kleinen*" and "*im grossen*" statements (definitions, properties, theorems) in the ordinary theory of complex functions is the basis on which the requirements that the general definition of Riemann surfaces has to meet can be spelled out: the axiomatic definition has to be such that (1) some statement concerning the newly defined objects can be said to be "*im kleinen*" (here, the topological structure is adequate), thus forming a core sub-class of all the syntactically correct statements within this axiomatic context, (2) the class of "*im kleinen*" *valid* statements[75] has to be the same for all Riemann surfaces, i.e. it has to be the same[76] as the well-known class of "*im kleinen*" statements for functions defined in the complex plane. Weyl put it again in the clearest of ways after the spelling out of the axioms[77]:

74 „Für die Behauptung, daβ z und u analytischen Funktionen auf der Fläche $\mathfrak{G}$ sind, ist es wesentlich, das $\mathfrak{G}$ *nicht bloβ als eine Fläche im Sinne der Analysis situs gegeben ist*. Denn auf einer Flächen, von der allein Analysis-situs Eigenschaften in Betracht gezogen werden, kann man wohl von stetigen Funktionen sprechen, nicht aber von „stetige differentiierbaren", „analytischen" (oder gar „ganz rationalen") Funktionen oder dergl. Um auf einer Fläche $\mathfrak{F}$ analytische Funktionentheorie in analoger Weise wie in der komplexen Ebene treiben zu können, muβ vielmehr (auβer der Definition der Fläche) eine Erklärung abgegeben sein, durch welche der Sinn des Ausdrucks „analytische Funktion auf der Fläche" so festgelegt wird, daβ alle Sätze über analytischen Funktionen in der Ebene, die„im Kleinen" gültig sind, auf diesen allgemeineren Begriff übertragen. „Im Kleinen" gültige Sätze sind dabei solche, deren Richtigkeit immer nür für eine gewisse Umgebung eines Punktes, über deren Gröβe der Satz selbst keine Auskunft gibt, behauptet wird. Durch eine solche Erklärung des Ausdrucks „analytische Funktion auf $\mathfrak{F}$" wird die Fläche $\mathfrak{F}$ zur **Riemannschen Fläche**."

75 "Valid" meaning "well defined" for definitions, and "true" for theorems.

76 Of course, "sameness" here is not defined only syntactically, but in terms of set theoretic admissible local maps.

77 "**Allgemeine Definition des Begriffs der Riemannschen Fläche**. Liegt eine Fläche $\mathfrak{F}$ vor und ist auβerdem für jeden Punkt $\mathfrak{p}_0$ von $\mathfrak{F}$ und jede in irgend einer Umgebung von $\mathfrak{p}_0$ vorhandene Funktion $f(\mathfrak{p})$ auf $\mathfrak{F}$

Footnote 77 continued

erklärt, wann $f(\mathfrak{p})$ um Punkte $\mathfrak{p}_0$ regülar-analytische heiβen soll, so ist damit eine **Riemannsche Fläche** $\mathfrak{R}_{\mathfrak{F}}$ gegeben, als deren Punkte die Punkte von $\mathfrak{F}$ betrachtet werden. Jene Erklärung aber muβ den foldenden Bedingungen genügen: 1. Ist $\mathfrak{p}_0$ irgend ein Punkt von $\mathfrak{F}$, so gibt es eine Funktion $t(\mathfrak{p})$, die nicht nur im $\mathfrak{p}_0$ (woselbst sie den Wert 0 besitzt) sondern auch in allen Punkten $\mathfrak{p}$ einer gewissen Umgebung von $\mathfrak{p}_0$ auf $\mathfrak{F}$ regulär-analytisch und von dieser Umgebung ein umkehrbar-eindeutiges,-gebietsstetiges Bild in der komplexen t-Ebene entwirft; eine solche Funktion heiβt eine Ortsuniformisierende zu $\mathfrak{p}_0$. 2. Ist $f(\mathfrak{p})$ irgend

> Let us close this paragraph with a few general remarks on the *idea of the Riemann surface*. The fundamental idea which grounds its introduction is by no means restricted to complex function theory. A function of two real variables x, y is *a function on the plane*; but it is indeed just as legitimate to study functions on the sphere, on the torus or on a surface, as to study them on the plane. So long as one only cares about the behaviour of functions "im Kleinen"—and most of the propositions of analysis bear on this – the notion of function of two real variables is generally adequate, since the neighbourhood of any point on a two-dimensional manifold can be represented by x, y(or $x + iy$). But as soon as one proceeds further and studies the behaviour of functions "im Grossen," the functions on the plane become an important but *special* case *among many equally legitimate cases*; Riemann and Klein have taught us not to limit ourselves to this special case. (Weyl 1910, p. 42)[78]

Following these guiding principles or *Grundgedanke*, it is clear that the axioms should not impose any global constraints. Hence all the objects that are to be defined on the basis of these axioms should be defined by a syntactically local definition, even though they are defined on the whole surface: harmonic functions (Weyl 1919, p. 38), differentials (Weyl 1919, p. 55), covering space of a given surface (Weyl 1919, p.47).[79]

The definitions of "integral functions" (*Integralfunktionen*) show an interesting variant. Weyl started from the notion of a "linear curve function" (*lineare Kurvenfunktion*) which, to each smooth curve γ on the surface, associate a number $F(\gamma)$ so that if the end point of γ is the starting point of γ', $F(\gamma+\gamma') = F(\gamma) + F(\gamma')$. A such function is said to be homologous to 0 (denoted $F \sim 0$) if it vanishes on closed curves. But the important notion is defined through localisation: "We shall consider only these linear line functions which are everywhere "im Kleinen" ~ 0; these shall be

Footnote 77 continued
eine im Punkte $\mathfrak{p}_0$ regulär-analytische Funktion und $t(\mathfrak{p})$ eine zu $\mathfrak{p}_0$ gehörige Ortsuniformisierende, so gibt es stets eine Umgebung U_0 von $\mathfrak{p}_0$, in welcher $f(\mathfrak{p})$ sich als eine reguläre Potenzreihe in $t(\mathfrak{p})$

$$f(\mathfrak{p}) = a_0 + a_1 t(\mathfrak{p}) + a_2(t(\mathfrak{p}))^2 + \dots$$

darstellen läβt." (Weyl 1919, p. 36).

78 „Wir schlieβen diesen Paragraphen mit einige allgemeinen Bemerkungen über die Idee der Riemannschen Fläche. Der Grundgedanke, der ihrer Einführung zugrunde liegt, ist keineswegs auf die komplexe Funktionentheorie beschränkt. Eine Funktion von zwei reellen Veränderlichen x, y ist eine Funktion in der Ebene; aber es ist gewiβ ebenso berechtigt, Funktionen auf der Kugel, auf dem Torus oder überhaupt auf einer Fläche zu untersuchen als gerade in der Ebene. Solange man sich freilich nur um das Verhalten der Funktionen « im Kleinen » kümmert – und darauf beziehen sich die meisten Betrachtungen der Analysis –, ist der Begriff der Funktion von zwei reellen Veränderlichen allgemein genug, da sich die Umgebung eines jeden Punktes einer zweidimensionalen Mannigfaltigkeit durch x,y (oder $x + iy$) zur Darstellung bringen läβt. Sobald man aber zur Untersuchung des Verhaltens von Funktionen « im Groβen » fortschreitet, bilden die Funktionen in der Ebene einen wichtigen, aber speziellen Fall unter unendlichvielen andern gleichberechtigten; Riemann und Klein haben uns gelehrt, bei diesem speziellen Fall nicht stehen zu bleiben."

79 We will use "covering space" for what Weyl called unlimited (*unbegrenzt*), and unramified (*unverzweigt*) covering space (*Überlagerungsfläche*). Of course there is one non-local element in the definition, the *unbegrenztheit* (i.e. paths-lifting property), but it is definable in that context.

called "integral functions"" (Weyl 1919, p. 68).[80] One of the goals of the theory is to study the representation of these abstract "integral functions" by actual line integrals of harmonic differentials.

One of the advantages of these axiomatic definitions is that only local properties are to be checked in order to prove that a newly defined object is a Riemann surface. This is how, nearly by virtue of their definitions alone, the universal covering (Weyl 1919, p. 51) and the *Überlagerung der Integralfunktionen*[81] (Weyl 1918, p. 74) are proved to exist. The *Begriffsbildung* leads to a smooth and elegant *Beweisführung*.

5 Jacques Hadamard's "*passage du local au général*"

The number of relevant hits for "*im kleinen*" and "*im grossen*" in the period from 1890 to 1918 was both small enough to allow for qualitative treatment (in terms of meaning and use), and large enough to ground the hypothesis that something *specific* happened there and then; something that was neither mere background noise, nor a misleading artefact emerging from clumsy sampling; something that could be studied historically.

This is not so with "local" or "*lokal.*" With less than 10 relevant hits for these in our zero corpus, we cannot claim to be clearly above noise level, and we cannot claim that any collective dynamics can be mapped out with any degree of confidence. This is why we will opt only for qualitative analysis, centring on the case of Jacques Hadamard: he is the only mathematician whom we could identify (using automatic and non-automatic means of inquiry) who began to use "local" in a coherent and systematic way that matured from 1898 to 1918. In this context, the hull of hits for "local" and "*lokal*" can be used heuristically to suggest possible dynamics.

Studying Hadamard's take on "local" will also help show the extent to which what we described in the previous paragraphs is specific: "*im kleinen*" and "*im grossen*" may have been used in several disciplinary contexts right from the start, and with a pretty stable meaning[82]; however, it was used by very few authors, and even fewer used these expressions more than once or twice, or in passing. Hadamard's case will show that, at the same period of time and within the same mathematical context, other mathematicians could come up with something quite different. As we will endeavour to show, Hadamard's notion of "*passage du local au général*" differs significantly from Osgood's "*im kleinen*"–"*im grossen*," even at the semantic level.

In 1906, Hadamard used "local" in a context that is already familiar to us, namely that of inversion theorems (Hadamard 1906). Before the proof of the main theorem, this research paper contains a long introduction in which the very nature of the problem

[80] "Wir betrachten nur solche lineare Kurvenfunktionen, welche "im Kleinen" überall ~ 0 sind; diese mögen "Integralfunktionen" heiβen".

[81] In the third edition (1955), Weyl chose to call it the class covering, to make the analogy with class field theory more visible.

[82] Save for a residual but persistent use of "im Kleinen" with an infinitesimal meaning (in mathematics, and without "im Grossen" as its counterpart). If we had included research papers in physics, we might have found this use to be prevailing.

is discussed in some detail. The questions seem pretty straightforward: given a system of equations (in the real domain)

$$(I)\begin{cases} X_1 = f_1(x_1, x_2, \ldots, x_n) \\ X_2 = f_2(x_1, x_2, \ldots, x_n) \\ \quad\ldots \\ X_n = f_n(x_1, x_2, \ldots, x_n) \end{cases}$$

can it be solved for any values $(X_1, \ldots, X_n)$ (question 1)? And, if so, does it have more than one solution (question 2)? Hadamard first reminded his reader of the well-known necessary condition: if the functions involved are continuously differentiable, the Jacobian determinant must not vanish.

> Taken in the strict sense, this condition happens to be *locally* sufficient, which means that, a $(a_1, a_2, \ldots, a_n)$ being a point of e_n where determinant (2) is non-null, and A $(\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n)$ its image, a small enough sphere can be drawn around the latter point, so that any point inside it is the image of one and only one point neighbouring a.
> This condition has sometimes been assumed to be generally sufficient (...). (Hadamard 1906, p. 72)[83]

After showing that this is downright false (for $n > 1$), Hadamard proved that the answer to both questions is yes, provided an extra condition is added to the effect that no infinite line drawn in the first Euclidean space e_n has a bounded image in E_n (i.e. the map is proper). No notion of covering is introduced; Hadamard just remarked at the end of the paper that the theorem does not hold as such for multiply connected surfaces such as a torus or an infinite cylinder (Hadamard 1906, p. 83). He also mentioned the fact that calculus was not of the essence here, the key notion being that of local invertibility, which is perfectly adequate to state the problem (Hadamard 1906, p. 84).

However familiar *to us* this context might be, it was not standard to use "*localement*" in this context and with this meaning in 1906. The word-search showed many hits that we counted as not relevant, but which must be mentioned here. As stressed earlier in the essay, "*local*" was of common use in scientific texts, for instance, in mathematical physics (*refroidissement local*, *ébranlement local*) or in geodesy. Of course, physical problems may be studied from a purely mathematical viewpoint or suggest classes of problems for the mathematician to tackle. This builds up a trading zone where words or expressions associated with the physical problem can pass on to purely mathematical contexts. An example is given by Burkhardt's chapter of the *Encyclopädie*, which deals with trigonometric series and integrals. When studying the asymptotic integral representation of the waves caused by an initial localised perturbation, he used

[83] «Cette condition prise au sens étroit est d'ailleurs suffisante *localement*, c'est-à-dire que, a $(a_1, a_2, \ldots, a_n)$ étant un point de e_n où le déterminant (2) n'est pas nul et $\mathrm{A}(\mathrm{A}_1, \mathrm{A}_2 \ldots, \mathrm{A}_n)$ son image, on peut, autour de ce dernier point, décrire une sphère assez petite pour qu'un point quelconque pris à son intérieur soit l'image d'un point et d'un seul voisin de a.
On a quelquefois admis que la condition était suffisante d'une manière générale (...).»

the expression "*lokalisierte Anfangstörung in einem elastischen Medium*" (Burkhardt 1914, p. 1286).

As we saw earlier, the issue of local versus global invertibility was also one of the key examples for Osgood. The relationship between Hadamard's "*local*" and the American "*im Kleinen*" is not easy to ascertain, however. Hadamard's work on the geodesics on a surface of negative curvature was one of the main instances of "*Geometry im Grossen*" according to Kasner.[84] His results on the asymptotic behaviour of the function of a complex variable defined by a power series were dealt with by Osgood in his 1898 AMS Colloquium talks, and included in the 1901 *Encyclopädie* chapter. In the calculus of variations, his numerous contributions were widely known (and sometimes reviewed in terms of "*im kleinen*" and "*im grossen*" (Hahn 1913)). When lecturing on the calculus of variations, he referred his reader to what he considered to be the two standard textbooks in that field, namely Kneser's and Bolza's (Hadamard 1910, p. viii); these lectures, as well as his research papers show that he was well aware of Osgood's theorems in that field. It seems unlikely that Hadamard was not aware of "*im kleinen*," "*im grossen*" being used by some (few but visible) mathematicians active in the many research fields to which he contributed prominently. He probably could have imported these expressions into the French mathematical community (probably in a translated form) ... but he did not. A detailed analysis of his 1910–1912 writings should show that he had a pretty different view of local and global.

This view matured over time. In the 1898 paper on the geodesics of surfaces of negative curvature, the reflexive categories as well as the technical notions and research agenda are those that can be found in Poincaré's qualitative study of differential equations (either in the early 1880s papers on curves defined by a differential equation or in the *Méthodes nouvelles de la mécanique céleste*). In the final paragraph, Hadamard passed a short comment on the importance of *Analysis situs*:

> (...) once more, our conclusions highlight the fundamental role which *Analysis situs* plays in these questions. That it is absurd to study the integral curves drawn in a given domain without taking into account the very shape of this domain, that is a truth on which it seems unnecessary to dwell at length. However, this truth remained unsuspected before Mr. Poincaré's works. (Hadamard 1898, p. 775[85])[86]

Much more was said in 1909, in his inaugural lecture for the chair of analytical and celestial mechanics at the *Collège de France*. This lecture is entitled *La géométrie de situation et son rôle en mathématiques*, which testifies to the impact of Poincaré's

[84] We can also mention the fact that the *Lectures on the Calculus of Variations* he gave at Columbia University in the fall of 1911 were proof-read by Kasner (Hadamard 1915).

[85] The page number refers to the Complete Works (Hadamard 1968).

[86] «(...) nos conclusions mettent en évidence, une fois de plus, le rôle fondamental que joue dans ces questions l'*Analysis situs*. Qu'il soit absurde d'étudier des courbes intégrales tracées dans un domaine déterminé sans faire entrer en ligne de compte la forme même de ce domaine, c'est une vérité sur laquelle il peut sembler inutile d'insister longuement. Cette vérité est cependant restée insoupçonnée jusqu'aux travaux de M. Poincaré.»

work on what constituted, at least for Hadamard, the contemporary agenda in celestial mechanics. In this inaugural lecture, Hadamard stressed the fact that, until recently, most mathematicians had been unaware of the fundamental role of *Analysis situs* (or "*géométrie de situation*"), whether in the theory of polyhedra, in inversion problems or in the theory of algebraic functions of one complex variable. Hadamard described this as a form of blindness,[87] which either hindered the advancement of mathematics or led to false results. So far, this is basically what could be read in the 1898 paper on surfaces of negative curvature or in the 1906 inversion paper, and the emphasis on the eye-opening role of the works of Riemann and Poincaré comes as no surprise. To these classical elements, Hadamard added an analysis of the *causes* of this specific form of blindness. Hadamard argued that, in the history of mathematics, after the advent of Algebra in the sixteenth century, two new sciences prompted major development: analytic geometry and calculus ("*le calcul intinitésimal*"; Hadamard 1909, p. 830). The wealth of result was so impressive that too little attention was paid to the fact that these tools, by their very nature, have powers of limited scope. In analytic geometry, coordinates usually cannot represent the whole space that is being investigated, as the case of the sphere shows (Hadamard 1909, p. 821). As to calculus, its power lies in the fact that, in spite of the variety of objects (curves, for instance), their properties in the infinitely small are both simple and common to all of them (Hadamard 1909, p. 820). What is more: "(...) in many cases, one can derive from these infinitesimal properties those that the functions or curves at stake show in the finite domain" (Hadamard 1909, p. 820).[88] However, "(...) the powerful tools which had prompted the great development of Analysis and Geometry since Descartes, Newton and Leibniz had, however, wonderful they may have been, a common weakness. The representation of figures by numbers and their decomposition into infinitely small elements indeed shed a bright light on the problems which we may ask ourselves about these figures; they nevertheless hide an essential property" (Hadamard 1909, p. 822).[89] In most cases, the conclusion at the finite level which calculus endeavours to reach only bear on small regions; not infinitely small, but small enough. To express this, Hadamard repeatedly used the metaphor of the geographical atlas, seen as a collection of maps/sheets:

> Now, we saw it a moment ago, problems which behave similarly when taken in a sufficiently small region of a domain may differ totally on the whole. That is to say, it does no suffice to study the different sheets of the map, it has to be complemented by that of the combination scheme, and there are several

[87] In a much later text (and in the different context), Hadamard would use the Greek word *ablepsia* to describe such a failure or inability to see (Hadamard 1954).

[88] «(...) on peut, dans beaucoup de cas, remonter de ces propriétés infinitésimales à celles que présentent les fonctions ou les courbes en question dans le domaine fini. »

[89] « Ainsi les puissants outils qui avaient donné à l'Analyse et à la Géométrie le grand développement qu'elles ont pris depuis Descartes, Newton et Leibniz présentaient, si merveilleux qu'ils fussent, une même faiblesse. La représentation des figures par les nombres, leur décomposition en éléments infiniment petits éclairent assurément d'une vive lumière les problèmes que nous pouvons nous poser sur ces figures; elles en masquent néanmoins une propriété essentielle.»

> such combinations which differ essentially one from the other. (Hadamard 1909, p. 820)[90]

The uniform treatment of all spaces at the infinitesimal level, or at the level of sufficiently small domains (for smooth manifold), blinds us to the fact that domains taken as a whole differ significantly. Partial domains—whether coordinated domains or domains over which calculus theorems prove some conclusion holds—have to be fitted together; functions have to be extended. The seeming uniformity vanishes, as in the case of the theory of curves defined by differential equation; as Poincaré's work showed us, there exists

> (...) a theory of curves defined by a differential equations on a surface of zero genus; another one—of which it is only a slight exaggeration to say it bears no relation to the first—on a surface of genus one; and so on. (Hadamard 1909, p. 826)[91]

This 1909 overview of the role of *analysis situs* in contemporary mathematics would be the basis for a much more detailed work, namely the analysis of Poincaré's mathematical work, which Hadamard wrote as a tribute on the occasion of Poincaré's death, in 1912. The tribute issue of *Acta Mathematica* was ready for publication in 1915, but was actually published after the war (in 1921).

The main theme or *leitmotiv* is introduced right from the start. Hadamard quoted Poincaré's comment to the effect that "There are no longer problems which have been solved and others which haven't, there are only problems which have been *more or less* solved" (Hadamard 1921, p. 204)[92]:

> In this regard, one may say that a first solution is reached in most cases—and this achievement, sketched out by Newton, is mainly ascribable to Cauchy and Weierstrass: from the relationship between *infinitely* neighbouring states we can infer something significantly different, namely, knowledge of all the states which are *sufficiently neighbouring* to a given state. (...) At any rate, these first results, even if we do not have to content ourselves with them, serve as compulsory intermediate steps on the way to better ones, so that, almost everywhere, the walk of contemporary mathematical sciences consists of two steps:

90 « Or, nous l'avons vu tout à l'heure, des problèmes qui se comportent de la même façon si on les prend dans une région suffisamment petite d'un domaine, peuvent différer totalement dans l'ensemble. Autrement dit, il ne suffit pas d'avoir fait l'étude des différentes feuilles de la carte, mais il faut la compléter par celle du tableau d'assemblage, et il y a plusieurs sortes de tableaux d'assemblages, essentiellement différents entre eux. »

91 « (...) une théorie des courbes définies par une équation différentielle sur une surface de genre zéro; une autre—dont il est à peine exagéré de dire qu'elle est sans rapport avec la première—sur une surface de genre un; et ainsi de suite. »

92 « Il n'y a plus des problèmes résolus et d'autres qui ne le sont pas, il y a seulement des problèmes *plus ou moins* résolus. »

> The local solution of problems;
> The passage from this solution to a whole solution [*solution d'ensemble*], if this kind of synthesis is possible.
> (Hadamard 1921, p. 205)[93]

The central element is spelled out slightly more clearly than in 1909 and consists in what we will call a *two-stage scheme*: to pass from the infinitesimal to the local (the right tool being calculus), then, to fit together local solutions to reach a conclusion about the whole space (this is where *analysis situs* enters the stage); this two-stage dynamic scheme relies on the identification of *three levels* at which mathematical objects can be studied: infinitesimal, local (partial but finite) and whole (for lack of a better word in Hadamard).

A quick comparison with "*im kleinen*"–"*im grossen*" already points to do differences. First, there is no stable word to say what "*im grossen*" says in German (and in English!). In sharp contrast to "local," which is perfectly stable in Hadamard's text, various counterparts are used: we saw "*solution d'ensemble*" and "*synthèse*" (the classical counterpart of analysis); "*propriété [des fonctions] dans tout leur domaine d'existence*" (Hadamard 1921, p. 230), and "*passage du local au général*" (Hadamard 1921, p. 247) can also be found. Further, "*général*" was the standard counterpart of "*local*" in the medical sciences, as a quick search in the *Comptes Rendus de l'Académie des Sciences de Paris* suggests: "*accident local ou général*" (in 1847), "*moyen anesthésique, soit générale, soit locale*" (in 1868), "*traitement général* (...) *traitement local* (1876), "*développement général ou local des muscles*" (1909), etc. As mentioned earlier, there was still no standard and stable counterpart for "local" in the French mathematical vocabulary in the 1930s.

A second element of comparison is even more striking. For Osgood, the property of being "*im kleinen*" or "*im grossen*" was a definable, syntactic property of mathematical statements (definitions or propositions). In 1912, Hadamard did not come up with syntactic criteria to classify statements; rather, he discussed the nature of problems and tactics for solving them. These may not be amenable to syntactic study; writing down exactly what the problem or the strategy is constitutes, more often than not, a problem of its own! Moreover, the true nature of a problem or the real scope of a technique is not something can be read off a statement of that problem or tactic: it has to be discovered through painstaking work; hints as to this true nature are gathered in time; new insights may lead to a complete change of view. For instance, commenting on Poincaré's study on the divergent developments of irregular solutions of linear differential equations, Hadamard wrote:

> In one respect,—the investigation of the limit of the logarithmic derivative of the solution—the method used is closely related to those which we will meet again

[93] « On peut dire à ce point de vue qu'une première solution est acquise dans la plupart des cas, – et cette conquête, ébauchée dès Newton, est surtout l'œuvre de Cauchy et de Weierstrass: – des relations entre états *infiniment* voisins, on sait déduire, ce qui est fort différent, la connaissance de tous les états *suffisamment voisins* d'un état donné. (...) Quoi qu'il en soit, ces premiers résultats, même si l'on est pas réduit à s'en contenter servent tout au moins d'intermédiaires obligés pour en obtenir de meilleurs, de sorte que, presque partout, la marche de la science mathématique actuelle comporte deux étapes: La solution locale des problèmes; Le passage de celle-ci à une solution d'ensemble, si cette sorte de synthèse est possible. »

> later, not in the local but in the general study of the problem of differential equations; and the true reason for the great difficulty of this question—which would deserve so much more research work—lies in the fact that it is only seemingly "local." (Hadamard 1921, p. 237)[94]

Another striking example can be taken from Hadamard's comments on Poincaré's qualitative study of curves defined by differential equations, more precisely on the study of curves in the neighbourhood of a closed (periodic) solution:

> One can immediately see that such a question straddles the two viewpoints around which the whole theory of differential equations revolves; and combines the advantages of both. Well within the scope of the same procedures which apply in the local domain, it however lies from the outset beyond this domain, since the new trajectories run by no means in the neighbourhood of a unique point, and are studied on courses just as extended as the primitive periodic solution itself. (Hadamard 1921, p. 249)[95]

Neither strategies that straddle two viewpoints nor problems that are only seemingly of a local nature fit in Osgood's "*im kleinen*"–"*im grossen*" syntactic framework.

Even though the two-stage scheme applies to nearly all mathematics, Hadamard argued many other types of problems and many other tactics also play a significant part. For instance, as far as ordinary differential equations are concerned, Hadamard listed five classical (i.e. non-qualitative) contemporary research lines (Hadamard 1921, p. 236). What is more surprising is that many problems or results that we, and Osgood, would call "*im grossen*" are not described by Hadamard in terms of "*général*" or "*de synthèse*." We need to quote in full the introductory passage to the third and last part of Hadamard's text, devoted to Poincaré's work and problems of partial differential equations:

> The difficulties which they display can, depending on cases, be of very different natures.
> It can be that they resemble, with a difference of degree, the difficulties displayed by differential equations, so that, from a theoretical viewpoint, the solution can be seen as locally given by Cauchy's method, even if, in a second stage of the work, the various elements of solution thus obtained have to be synthesized.
> This is what happens—assuming the equation has been introduced for the study of some physical phenomenon—when the latter freely evolves in unlimited space

94 « Sur un point,—la recherche de la limite vers laquelle tend la dérivée logarithmique de la solution—la méthode employée se rapproche beaucoup de celles que nous retrouverons plus loin à propos de l'étude, non plus locale, mais générale du problème des équations différentielles; et dans le fait que la question dont nous parlons en ce moment n'est « locale » qu'en apparence réside sans doute la véritable raison des grandes difficultés de cette question qui mériterait encore tant de nouvelles recherches. »

95 « On voit immédiatement qu'une telle question est à cheval sur les deux points de vue entre lesquels pivote toute la théorie des équations différentielles; et cela en combinant les avantages de toutes deux. Accessible aux mêmes procédés qui s'appliquent au domaine local, elle est d'emblée cependant en dehors de ce domaine, puisque les nouvelles trajectoires obtenues n'évoluent nullement au voisinage d'un point unique et sont étudiées sur des parcours aussi étendus que la solution périodique primitive elle-même. »

> and where, consequently, to define its evolution, it is sufficient to give *initial conditions*, that is, its state at a given moment in time.
> But if the phenomenon is set in some enclosed area—so that, to finish defining it, one has to write down a *system of boundary conditions* expressing the part played by the wall—an altogether new kind of difficulty occurs.
> It remains true that, in the neighbourhood of any point, the solution can most of the time be represented by series development of the same type as in the previous problems. But in this case, none of these *elements* of solution—not even the first one, as happens for ordinary differential equations—can be determined in isolation: knowledge of any one of them is inseparable from that of *all* of them. This is a reversal of the principle which, in all other circumstances, guides the walk of integral calculus: split the problem into a local problem and a synthesis problem. Here, such a division is radically impossible. (Hadamard 1921, pp. 278–279)[96]

In his endeavour to shed light on this motley of problems and partial solutions that mathematics is, Hadamard chose to draw a clear-cut line; not one between ODEs and PDEs, but one with ODEs and non-elliptic PDEs on the one side, and elliptic PDEs on the other side. Both ODEs and non-elliptic PDEs fit the two-stage scheme, since global solutions can be reached by patching together local solutions (hopefully); on the other hand, elliptic problems—such as the Dirichlet problem—are of a completely different nature, and call for a completely different proof strategy. As far as these are concerned, Hadamard was not only commenting on Poincaré's work. Of course, he presented Poincaré's "*méthode de balayage*," but added:

> But, whereas the sweeping-out method itself relates to former works devoted to the Dirichlet problem, this theory was soon afterwards to enter a completely new phase, and undergo a deep revolution whose usefulness is also highlighted by the preceding remarks.
> Its principle consists in replacing the *partial differential* equation, as well as the other conditions which the unknown function should meet, by an *integral*

[96] « Les difficultés que ceux-ci présentent peuvent être, suivant les cas, de nature très différente.
Il peut arriver qu'elles ressemblent, avec des différences de degré, à ce qu'elles sont pour les équations différentielles, de sorte que la solution puisse être considérée, au point de vue théorique, comme fournie localement par les méthodes de Cauchy, quitte, dans une seconde partie du travail, à faire la synthèse des différents éléments de solution ainsi obtenus. C'est ce qui se passe—l'équation étant supposée introduite par l'étude d'un phénomène physique—lorsque celui-ci se déroule librement dans l'espace illimité et où, par conséquent, et pour définir son évolution, il suffit de se donner les *conditions initiales*, c'est-à-dire son état à un instant déterminé.
Mais si le phénomène a pour théâtre une enceinte limitée par des parois—de sorte que pour achever de le définir, il faut écrire un système de *conditions aux limites*, exprimant le rôle joué par les parois en question,—une difficulté d'un tout autre ordre apparaît. Il est encore vrai que, au voisinage d'un point quelconque, la solution est le plus souvent représentable par des développements en série du même type que dans les problèmes précédents. Mais cette fois, aucun de ces *éléments* de solution,—non pas même le premier, comme il arrivait pour les équations différentielles ordinaires—ne peut être déterminé isolément: la connaissance de chacun d'eux est inséparable de celle de *tous* les autres.
C'est ce renversement du principe même qui, en toutes les autres circonstances, guide la marche du calcul intégral: la division de la difficulté en une difficulté locale et une difficulté de synthèse. Une telle division est ici radicalement impossible."

equation. Instead of writing the unknown under the derivative sign, it is made to appear under the integral sign. (Hadamard 1921, p. 279)[97]

This new phase in the development of the theory of elliptic PDEs is closer to Hadamard's own research agenda (on integral equations and the emerging functional analysis) than to Poincaré's work.[98]

Hence, mathematical problems that would lead to typical "*im grossen*" statements—where Osgood's syntactic criterion to be used—fall into two distinct categories according to Hadamard: problems for which "*passage du local au général*" is the right method and problems for which it is altogether irrelevant. Hadamard coined no specific words or expressions for the latter; he occasionally referred to them as synthesis problems as well (Hadamard 1921, p. 279).

This may account for the fact that many of Poincaré's results that were already considered as standard "*im grossen*" results are not described by Hadamard in terms of "*passage du local au général.*" For instance, the issue of uniformisation of analytic functions was described in terms of "*im grossen*" and "*im kleinen*" by Osgood as from 1898; the description was the same, nearly word for word, in Weyl's *Idee der Riemannschen Fläche*, and his notion of universal covering owed much to Poincaré's work. When Hadamard presented this part of Poincaré's work, neither the two stage-scheme nor *analysis situs* were even mentioned; he simply emphasised the connection with potential theory.

The idea that, in 1912, the passage from local solutions to "*solutions d'ensembles*" has become a pressing task in almost all branches of mathematics (Hadamard 1921, p. 205) was not widely expressed; that is an understatement. Nor was the idea that Poincaré was the Master of this type of synthesis, and that the two-stage scheme is the key (or one of the two keys) to a deep and unified understanding of his many mathematical achievements. Actually, this had not struck Poincaré himself! In his own analysis of his mathematical work—written in 1901 but published in the tribute issue of *Acta Mathematica*—no such idea is expressed (Poincaré 1921). This theme of the passage from the local to the global (or the general, or regarding space as a whole, etc.) is not mentioned either in Poincaré's 1908 survey talk at the International Congress of Mathematicians, bearing on the future of mathematics (Poincaré 1909). To the best of our knowledge, Poincaré used "local" once in a technical context, that of the classification of the conformal classes of bounded domains in $\mathbf{C}^2$; he split the problem into three subproblems: a "*problème local*" (local conformal mapping of a

[97] « Mais alors que la méthode du balayage elle-même se rattache aux autres travaux antérieurs consacrés à la théorie du problème de Dirichlet, cette théorie devait peu après entrer dans une phase toute nouvelle et subir une révolution profonde dont l'utilité ressort, elle aussi, des remarques précédentes.
Son principe consiste à remplacer l'équation *aux dérivées partielles*, ainsi que les autres conditions auxquelles doit satisfaire la fonction inconnue, par une équation *intégrale*. Au lieu de faire figurer l'inconnue sous des signes de dérivation, on la fait apparaître sous un signe d'intégration. »

[98] Consequently, Hadamard had to justify his excursus on the blooming theory of integral equations by arguing that (1) "les manifestations les plus importantes, les plus inattendues de l'esprit humain, sont le produit non seulement du cerveau de leur auteur, mais de toute l'époque qui les a vu naître," (2) "notre époque, au point de vue mathématique, c'est avant tout Poincaré," therefore (3) "son œuvre a été une condition indispensable à la naissance de la nouvelle méthode"(Hadamard 1921, p. 280).

piece of the boundary), a "*problème étendu*" (Poincaré 1907, p. 244) and a "mixed problem [*problème mixte*], because it stands halfway, so to speak, between the local problem and the extended problem"[99] (conformal mapping in a neighbourhood of the whole boundary; Poincaré 1907, p. 264).[100]

Among the very, very few relevant hits for "local" in our zero corpus, two fall within the milieu of young French analysts who published in the Borel series. In the introduction to his *Leçons sur les fonctions définies par les équations différentielles du premier ordre, professées au Collège de France* (Boutroux 1908), Pierre Boutroux drew a parallel between the historical development of the general theory of (analytic) functions and that of the theory of ordinary differential equations of the first degree. According to him, until recently, both theories only dealt with "*études locales*" (Boutroux 1908, p. 2) and dared not go beyond the "*point de vue local*" (Boutroux 1908, p. 4). Fortunately, Boutroux argued, recent developments have shown us how to go beyond; in particular: "M. Painlevé has abandoned Cauchy's local viewpoint" (Boutroux 1908, p. 5).[101] Similar views are expressed in Ludovic Zoretti's *Leçons sur le prolongement analytique professées au Collège de France*, taught in 1908–1909, and published in 1911 in the Borel series (Zoretti 1911). Zoretti distinguished between "*le point de vue de Cauchy, ou local*" and "*le point de vue de Weierstrass ou général*" (Zoretti 1911, p. 96), and took up Weierstrass' notion of analytic configuration. There was no hit for "local" in Borel's texts included in our zero corpus (five monographs and numerous research papers) nor in Painlevé's texts (two monographs and several research papers). The use of "*général*"as the counterpart of "*local*," and the fact that both lectures were given at the Collège de France (where Hadamard taught) suggest possible connections with Hadamard.

Hadamard actively promoted the "*passage du local au général*" as a research program, as is exemplified by the question for the 1916 *Grand Prix de Mathématiques* of the French Academy of Sciences:

> The iteration of a substitution in one or several variables, that is, the construction of a system of successive points $P_1, P_2, \ldots, P_n, \ldots$, every one of which is deduced from the previous one by a same given operation:
>
> $$P_n = \varphi(P_{n-1}) \quad (n = 1, 2, \ldots, \infty)$$
>
> (φ depending rationally, for instance, of point P_{n-1}), and whose first term P_0 is also given, plays a part in several classical theories, and in some of the most celebrated memoirs of Poincaré.
>
> Until now, the well-known works dedicated to this study deal with the "local" viewpoint.

[99] «problème mixte, parce qu'il tient pour ainsi dire le milieu entre le problème local et le problème étendu.»

[100] As mentioned earlier, a similar splitting of the problem could be found, in the much more elementary context of functions of one complex variable, in Osgood's *Lehrbuch* (Osgood 1912, p. 669). The first part of the problem would be taken up by Kasner, for instance, in his paper *Conformal Classification of Analytic Arcs or Elements: Poincaré's Local Problem of Conformal Geometry* (Kasner 1915).

[101] «M. Painlevé abandonne le point de vue local de Cauchy.»

> The Academy believes it would be valuable to move on from this to the examination of the whole domain of values which the variables may take on. (CRAS vol. 163, 1916, pp. 911–912)[102]

The prize went to Gaston Julia (with a second prize for Samuel Lattès). There is no definite proof that the question was suggested by Hadamard, who had been elected to the Academy after Poincaré's demise. However, the direct connections between this question and both his and Poincaré's works on dynamical systems (Poincaré's section), and the fact that he was a member of the commission for that prize[103] back up this hypothesis.[104]

In spite of the patchy data, it seems that, in the 1900s, some degree of dissatisfaction with more traditional works in analysis (that were viewed as merely local) was voiced by several—but very few—French mathematicians using the word "local." In this context, Jacques Hadamard clearly stands out, both for his personal research which features several global results in several branches of mathematics, and for his highly articulate analysis of the state of contemporary mathematics, written on the occasion of Poincaré's death. As a leading member of the *Société Mathématique de France*, then as a member of the Academy of Sciences, he seems to have promoted the "*passage du local au général*" as a research agenda, though in very specific theoretical contexts.

In spite of Hadamard's involvement within the international mathematical community—in the general theory of analytic functions, in functional analysis, in the theory of integral equations—this development seems to have been quite *autonomous* from what we studied earlier with "*im kleinen*"–"*im grossen*." From a semantic viewpoint, we saw that the conceptual framework expounded by Hadamard in 1912 differs significantly from Osgood's. Another indication is given by the translation into French of Osgood's *Encyclopädie* article.[105] In the French version of Osgood's definitional footnote,[106]

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102 « L'itération d'une substitution à une ou plusieurs variables, c'est-à-dire la construction d'un système de points successifs $P_1, P_2, \ldots, P_n, \ldots$, dont chacun se déduit du précédent par une même opération donnée: $P_n = \varphi(P_{n-1})$ $(n = 1, 2, \ldots, \infty)$ (φ dépendant rationnellement, par exemple, du point P_{n-1}), et dont le premier terme P_0 est également donné, intervient dans plusieurs théories classiques et dans quelques uns des plus célèbres mémoires de Poincaré. Jusqu'ici les travaux bien connus consacrés à cette étude concernent le point de vue "local."
L'Académie estime qu'il y aurait intérêt à passer de là à l'examen du domaine entier des valeurs que peuvent prendre les variables. »

103 Along with Jordan, Appell, Painlevé, Boussinesq, Lecornu, Picard and Humbert.

104 Michèle Audin reached similar conclusions, in her recent book on this prize question and subsequent developments (Audin 2009). The very wording is the same as Hadamard's: « L'Académie avait mis au concours l'étude de l'itération d'une substitution, en rappelant que le point de vue local avait seul été considéré jusqu'alors et invitant les concurrents à se placer au point de vue général » (CRAS vol. 176, 1918, p. 811).

105 See Gispert (2001).

106 « [note 22] Les notions d'*allure locale* d'une fonction et d'allure d'une fonction *dans tout un domaine*, qui s'opposent, jouent en analyse un rôle important, et s'étendent à tous les domaines des mathématiques dans lesquels les éléments considérés appartiennent à un ensemble parfait, notamment en géométrie et en mécanique. Dans la théorie des fonctions, on entend par *allure locale* d'une fonction l'allure de cette fonction au voisinage d'un point fixe a de coordonnées $(a_1, a_2, \ldots, a_n)$ ou d'un ensemble de points P [no. 40], et pour abréger, on dit simplement l'allure de la fonction *au* point a ou *au* point $(a_1, a_2, \ldots, a_n)$, ou l'allure

as expected, "*im kleinen*" was translated by "*local*," and no specific terms were used to translate "*im grossen*." But something more surprising comes next. The translators (or adapters), Boutroux and Chazy, actually *skipped* the passage in Osgood's footnote which we saw as definitional: "(...) in a domain T, T', $\mathfrak{T}$, $\mathfrak{T}'$, etc., the extent of which is set from the start [*von vornherein feststeht*] and not determined afterwards to meet the requirements of the given problem." (Osgood 1901a, p. 12). In 1911, for Boutroux and Chazy, "*local*" made perfect sense; but, apparently, the syntactic distinction between statements that bear on a domain that was set from the start, and statements which bear on some new *ad hoc* domain, just did not ring a bell. So much so that it could be skipped altogether.

6 Conclusion

Let us summarise some of the main facts gathered from the empirical study, then conclude with more general comments on the type of historical phenomenon we endeavoured to capture.

From a *pragmatic* viewpoint, the introduction pairs of terms such as "*im Kleinen –im Grossen*" first served expository purposes: expressing the scope of a theorem, warning against common mistakes, stressing the conceptual connection in a series of theorems or among the sections of a treatise. By providing new classificatory concepts to describe mathematical knowledge, it soon began to be used in the assessments of current research mathematics and in the formulation of new research agendas. In particular, Hilbert's work in the calculus of variations (including his existence proof for the Dirichlet problem) led to a distinction between ODEs and non-elliptic PDEs, on the one hand (for which patching up local solutions is relevant) and elliptic PDEs, on the other hand, for which integral equations are seen as a new and fundamental tool; the cases of Wallie Hurwitz and Jacques Hadamard document this shift of agenda in pure analysis. Another quite distinct research program, that of *Differentialgeometrie im Grossen*, would be launched at the end of our period of study by Whilhelm Blaschke, only to bloom in the 1920s (Chorlay 2009).[107]

The "*im kleinen–im grossen*" distinction also led to direct conceptual innovation. In point-set topology, a series of terms for local-connectedness, local-compactness, etc. began to appear in the 1910s, following a similar definitional template which we called localisation. More importantly, we claim that the "*im kleinen–im grossen*" pair played a key role in Weyl's axiomatic rewriting of the theory of Riemann surfaces.

Footnote 106 continued
de la fonction *sur* l'ensemble P. Dans beaucoup de cas, un domaine T étant donné, l'allure locale d'une fonction en tout point d'un domaine T' est *uniforme* [no. 6] et l'allure de cette fonction dans tout le domaine T en résulte immédiatement. » (Chazy and Boutroux 1912, p.105)

[107] A list of *negative* results should be mentioned: (1) The introduction of "*im kleinen–im grossen*" in polemics about the theory of Lie groups did not prompt its proponents to engage in an "*im grossen*" theory of Lie groups. (2) It seems that "local–global" was not used in the theory of numbers before Hasse, even though his "local–global principle" has its roots in Hensel's turn of the century work (Schwermer 2007). (3) The founders of *Analysis situs* as an autonomous research field (Dehn and Heegaard, or Brouwer) did not use the "*im kleinen–im grossen*" to define its scope and word their agenda.

These *meta*-level descriptive concepts provided him with the tools to express what the axioms should do, Riemann surfaces being the objects of a theory in which a sub-class of statements could be identified as "*im kleinen*" (hence the key role of the notion of neighbourhood) which were to be valid if and only if they were for a standard model (the z-plane). Needless to say the investigation of similar guiding principles for the design of axiomatic theories should be a fruitful research topic for those historians who work on the contemporary period.

From a *syntactic* viewpoint, samplings from the 1920s and 1930s showed that, at that time, to some extent, "local–global," "*im kleinen–im grossen*" and "in the small–in the large" could be used one for the other. Going as far back in time as our means allowed, we showed that this intertranslatibility between several pairs of terms belonged to a second phase of development, after a first phase, in the period from 1898 to 1918, where "*im kleinen–im grossen*" prevailed, both the English- and German-speaking worlds. This pair seems to have first been used in a very consistent and systematic way by Göttingen-trained American mathematician William Fogg Osgood, in numerous high-level didactic publications, both in the English and German languages. It was quickly adopted within the still small American mathematical community, in such diverse fields as function theory (Osgood), the calculus of variations (Bolza) and differential geometry (Kasner).

Finally, from a *semantic* viewpoint, it turns out that the meaning of "*im kleinen–im grossen*" was not transmitted through use *only*, since Osgood once endeavoured to define these terms in his *Encylclopädie* chapter on the general theory of functions of one or several variables. We called his take on "*im kleinen–im grossen*" the *syntactic view*, since it refers to a syntactic property of mathematical statements and can be rephrased in terms of the order of quantifiers. The fact that, on one occasion, a definition was put forward does not imply that every one used this pair with the very same meaning, or even that those who did were aware of the definition (Bolza, Schlesinger). In some cases, it can be argued that "*im kleinen–im grossen*" was only added to an already well-identified series of pairs such as "*im begrenzten Raumstück–im Gesamtraum*" and "*analytic–algebraic*"; these pairs, however, were used in a more specific disciplinary context—that of Klein's conception of geometry as a theory of invariants—and with a much more restricted meaning—the *Raumstück* referring to the disc of convergence of some power series, in a context where only analytic functions were considered. Another use of "*im kleinen*" clearly stands apart from the semantic viewpoint, since it clearly refers to the infinitesimal (*im unendlich kleinen*) and not to the local. It appears marginally within our corpus and is never used in pair with "*im grossen*." Moreover, close reading of the texts shows that both Osgood and Hadamard clearly meant to delineate to realm of "*im kleinen*" or "local" statements and properties from that of infinitesimal statements and properties. The coining of these terms served a purpose of conceptual differentiation.

From a more general perspective, the "local–global" pair presents itself as a challenging object for the historian to study, since it stands *in between* better identified categories. It is more than a mere metaphor (although with metaphorical undertones), but needs not be explicitly defined (although definitions can be given, in specific contexts). It is used to help the readers find their way about mathematics and make sense of the motley of notions, theorems and techniques. Performing this *meta* function, it stands in between the body of knowledge and the image of knowledge and *actively links both*. In some cases, it can be internalised within the body of knowledge, for instance, when a new pattern of concept definition is introduced (e.g. definition by localisation) or when it provides the guidelines to select the right axioms for the revamping of a theory. In other cases, it can be used at higher levels of discourse which are more typically analysed in terms of image of knowledge: discussing the scope and strength of a theory (as in Study's criticism of Lie's theory), articulating research agendas (as for Blaschke's *Differentialgeometrie im Grossen* or Hadamard's theory of integral equations). It helps connect notions, theorems or theories that belong to different disciplines; these are *weak* links, if strength is to be assessed in terms of logical dependence, but they are long-distance and cross-disciplinary links which help present mathematics as an organised whole at times of speedy growth.

For these reasons, it comes as no surprise that, as Morse did, mathematicians see this articulation as something that has been used "for some time with varying meanings" and concerns "more or less" all mathematics; something that is part of the mathematical landscape; part and parcel of the toolkit that mathematicians think and write with. What may come as a surprise is that the emergence of such a conceptual tool can be studied historically, and that this study gives quite definite results. These results depend heavily on methodological choices that we endeavoured to spell out and ground.

We chose to focus on the *explicit* use of a series of terms, but we did not tell a story of the passage from the implicit to the explicit. This is not a tale of growing awareness, but one of *collective making*.[108] We feel this is the background against which two other historical issues can be tackled. First, that of the historical development of theories and disciplines such as, for instance, global differential geometry or local algebra. Second, that of the history of other (possibly competing) reflexive categories or articulations, such as "qualitative–quantitative."

This principled focus on the explicit led us to resort to use a combination of *not-reading* (here: computer-aided word-search) and *close-reading*.[109] As far as the emergence of "*im kleinen–im grossen*" is concerned, this approach showed that in spite of their apparent elusiveness, both meaning and social context could be delineated with some confidence. Empirical data show that "*im kleinen–im grossen*" was not used just anywhere and with no discernible core-meaning. Quite the contrary: the fact that other meanings (for instance, an infinitesimal meaning for "*im kleinen*") and other social contexts (for instance, that of young French analysts working along Borel and Hadamard) can be distinguished from that of the AMS–Göttingen "*im kleinen–im grossen*" testifies to the specificity of the latter. One of the advantages of this combination of not- and close-reading is that it captures the conceptual and the social at the same

108 For a very different but not unrelated work on the historical emergence of a semantic category that we cannot even imagine was not "always there," see Michel Pastoureau's fascinating work on the history of the colour "blue" (Pastoureau 2006). For a more philosophical background, see (Goodman 1992) and (Hacking 1999).

109 These awkward terms have somehow become standard in the field of "digital humanities." See Crane (2006) and Kirschenbaum (2007). For an overview of the French scene and several stimulating examples (very few dealing with mathematics and the sciences, however), see http://www.revue-texto.net/.

time, thus shedding light on the *dual* dynamics of conceptual innovation. Another great advantage, though double-edged, is that it makes our conclusions falsifiable.

Appendix: Corpus building

In Sect. 1, we strove to identify the specific of question that we set out to tackle, the kind of tools which we want to use in this first phase of the work and some potential biases to beware of; so it seems we are ready to let the computer "not read" for us. For now, the period of emergence is still undetermined; so is the "site," be it a research domain or a linguistic area. Hence, the building of the core corpus has to proceed through full-text search for the words or expressions: "*im kleinen*," "*im grossen*," "local," "*lokal*," "global," "in the small" and "in the large"; starting from a corpus which is as large as possible in terms of topics and languages. However, only printed and published mathematical texts will be selected for this *zero corpus*. This restriction is not only an obvious consequence of the choice of tools; it also reflects the fact that we want to study the public and collective use of these reflexive terms.

This led us to include in the zero corpus the *Jahrbuch*, of course, as well as the *Proceedings of the International Congress of Mathematicians*. Other highly visible publications were included, be they periodical such as the *Jahresbericht der Deutschen Mathematiker-Vereinigung* and the *Comptes Rendus de l'Académie des Sciences de Paris*; or non-periodical, such as the *Encylopädie der mathematischen Wissenschaften* (the 13 volumes from Band I to Band III). Several journals were included, such as the *Mathematische Annalen*, the *Bulletin de la société mathématique de France* and the *AMS Journals*.[110] We also included something over which we have less control, namely the corpus of retrodigitised mathematical texts available in the *Cornell Library of Historical Monographs* and the *Michigan Historical Maths Collection*. We feel this loss of control is well worth it, especially when the diversity of *genres* and textual forms is taken into account: these online libraries bring monographs and textbooks to a corpus which otherwise consists primarily of proceedings and research papers. We also included the collected works of Klein and Hilbert.

Zero corpus:

AMS Colloquium publications (1903, 1906, 1909, 1913, 1916).
Annales Scientifiques de l'ENS (1864–)
Bulletin de la Société Mathématique de France (1872–)
Comptes Rendus de l'Académie des Sciences de Paris (vol. 130, 134, 139, 140, 141, 144, 149, 153, 154, 155, 159[111])

110 The *Journal für die reine und angewandte Mathematik* (JRAM) was not part of the zero corpus. However, we OCRed it and ran the word-search after completion of this essay, from volume 106 (1890) to volume 148 (1918). The results are thoroughly in keeping with the conclusions drawn from the study of the core corpus: one hit for "lokal" in vol. 131. The first hits for "*im Kleinen*" or "*im Grossen*" appear after 1914, in volumes 145 (in the calculus of variations) and 146; the first systematic use is to be found in Robert König's paper in volume 148 (1918) (König 1918).

111 We only used those for which full-text search was available.

Encyclopädie der mathematischen Wissenschaften I.1, I.2, II.1.1, II.1.2, II.2, II.3.1, II.3.2, III.1.1, III.1.2, III.2.1, III.2.2A, III.2.2B, III.3
Jahrbuch über die Fortschritte der Mathematik (1868–1942)
Jahresbericht der Deutschen Mathematiker-Vereinigung [from vol. 1 (1890/1891) to vol. 27 (1918)]
Journal, Bulletin, Proceedings, and Transactions of the American Mathematical Society
Journal de l'Ecole Polytechnique (1881 (cahier 49), 1882 (51, 52), 1884 (54), 1886 (56), 1890 (60), 1891 (61), 1894 (64), 1895 (1), 1897 (2), 1902 (7), 1903 (8), 1904 (9), 1906 (11), 1907 (12), 1910 (14), 1912 (16), 1921 (21), 1923 (23), 1924 (24), 1927 (26), 1929 (27), 1931 (29), 1932 (30), 1933 (31), 1935 (34)[112]).
L'enseignement mathématique (1899–)
Mathematische Annalen [from vol. 35 (1890) to vol. 79 (1919)]
Proceedings of the International Congress of Mathematicians (1897, 1904, 1908, 1912, 1920)
The University of Michigan Digital Maths Collection, http://quod.lib.umich.edu/u/umhistmath/ (accessed 7 Nov. 2009) which includes Klein's collected papers.
Cornell University Library. Historical Maths Monographs, http://digital.library.cornell.edu/m/math/ (accessed 7 Nov. 2009)
Hilbert's collected papers (Hilbert 1932), retrieved from GDZ and OCRed.

Now, the *core corpus* is given by the list of relevant hits for "*im kleinen*" or "*im grossen*" is the following, ordered by year of publication,[113] until 1918:

(1) (Osgood 1898)
(2) (Osgood 1899)
(3) (Osgood 1901a)
(4) (Bolza 1904)
(5) (Kasner 1905), (6) (Schlesinger 1905), (7) (Study 1905), (8) (Young 1905)
(9) (Osgood 1906)
(10) (Haussner 1907), (11) (Mangoldt 1907), (12) (Hartogs 1907), (13) (Bernstein 1907), (14) (Fano 1907)
(15) (Study 1908), (16) (Riesz 1908)
(17) (Jung 1909), (18) (Bliss 1913 [1909]), (19) (Bolza 1909), (20) (Cole 1909),
(21) (Carathéodory 1909)
(22) (Hurwitz 1910), (23) (Stäckel 1910)
(24) (Study 1911), (25) (Young 1911)
(26) (Weyl 1912), (27) (Hahn 1912), (28) (Bill 1912)
(29) (Osgood 1914 [1913]), (30) (Rosenblatt 1913), (31) (Hahn 1913), (32) (Voss 1913), (33) (Blaschke 1913),
(34) (Hahn 1914), (35) (Van Vleck 1915 [1914]), (36) (Jacobstahl 1914), (37) (Koebe 1914)
(38) (Blaschke 1915)

112 Depending on availability of full-text search.

113 For the *Jahrbuch*, we used the year of publication of the reviewed paper or book. For talks, we gave the date of publication first, and mentioned the date of delivery between square brackets.

(39) (Osgood 1916)
(40) (Osgood 1917), (41) (Moore 1917), (42) (König 1917), (43) (Liebmann 1917), (44) (Carathéodory and Rademacher 1917)
(45) (Blaschke 1918), (46) (Williams 1918)

It should be emphasised that this collection should not be seen as raw material. Indeed, it is the result of a processing that involves more than selecting the documents and, more often than not, running an OCR software.[114] For reasons which pertain to the kind of words or expressions for which we searched, we actually had to check every hit to decide if it would appear in our collection. Hence, we will work on a collection of *relevant* hits ... so much for *not* reading.

Actually we were faced with two rather different problems.

Since the words and expressions which we targeted are not strictly technical, we got much more "noise" than if we had searched for occurrences of, say, "non-Euclidean" or "Fuchsian function." Many hits are not remotely mathematical: "*im kleinen* + [noun]" or "in the small + [noun]" (such as "*im kleinen Dorf*," "in the small village") account for a large proportion of the hits. "*im grossen*" appears in the German phrase "*im grossen und ganzen*," which means "on the whole," "by and large." Further, "local" can be used to thank the "local organising committee" for the great job they did. The German "lokal" and the French "local" can be used to refer to the room where some meeting was held; needless to say you get quite a few of these in conference proceedings. This large proportion of non-mathematical hits makes the purely automatic not-reading of the corpus highly inadequate for further study, even if one were to engage in a purely quantitative study in terms of date and place of publication, nationality of the author(s), research field (identified by key-words or classification in the *Jahrbuch*), references to published papers (for quotation networks), etc. Of course, more sophisticated not-reading techniques could be devised to partially solve this problem, although probably at the cost of new biases. In the case at hand, it turned out that the parting of clearly non-mathematical hits from mathematical hits was tractable by old fashioned reading.

A completely different problem arises when one wants to select relevant hits among hits that have a somewhat technical meaning in a scientific text. For instance, in English, "local" can be used to describe problems or theorems bearing on loci, as in Samuel Roberts' 1880 paper *On an Immediate Generalization of Local Theorems in which the Generating Point Divides a Variable Linear Segment in a Constant Ratio* (Roberts 1880). It is occasionally used with the same meaning in French (e.g. "*théorème local*"; Breton 1853, p. 1011). A related meaning can be found in the theory of "local probability" or "geometric probability," as in Morgan Crofton (1868) paper *On the Theory of Local Probability, applied to Straight Lines drawn at random in a plane* (Crofton 1868). We did not include these in our list of relevant hits. The word "local" was (and is) also quite common in physics and some fields of applied mathematics, with a meaning that is clear in the context. Since our zero corpus in French features scientific

[114] And hoping that it will work well enough, not reading "im kleinen" as "im dleinen," for instance. In fact, many hits for "local" are mistakes, the actual text reading "focal".... The quality of the scanning process, and the rate of failure of the OCR software are things over which we have no control. The fact that this rate is non-null implies that specific statements such as " "im kleinen" are *never* used by X in book B" should be taken *cum grano salis*.

papers beyond mathematics (in the *Comptes Rendus* or in the *Annales scientifiques de l'Ecole Normal Supérieure*, for instance), this is where we get the largest variety of hits. For instance, we come across "*déformation locale*" (in a paper on the deformation of copper wires), "*cause locale*" (in a paper on meteorology, dealing with the causes of winds), "*traitement local du cancer*" (one of many, many hits in the medical field), "*abaissement local de température*" (in a paper on sun spots), "*courant local*" (in electrodynamics), "*attractions locales*" (in geodesy), etc. In all cases, "local" draws the attention to the fact that something happens at a specific place, as opposed to everywhere in the medium (be it the earth, a metal layer or the human body). We did not include these in the list of relevant hits, even though they may help understand how, at some point in time, some mathematicians chose to import terms which were quite commonly used in the sciences into mathematics. The case of "temps local" is borderline, since it started its career both in the outer world (for travellers, merchants and soldiers) and in the sciences (in geodesy, meteorology and astronomical observation), then, after 1905, became a central notion in a highly mathematical theory, namely relativity theory (special, then general). However, interesting this case may be, we didn't include the technical use of "local" or "*lokal*" in relativity theory in our list of relevant hits.

Some final remarks have to be made regarding these empirical data. Although these remarks are part of methodological nature, this is not methodological discussion for the sake of methodological discussion. The choice of methods of analysis depends on the type of empirical data available to us, and the type of historical object that we endeavour to capture.

It should first be noted that our list of occurrences is heterogeneous in several different senses. It is heterogeneous in terms of genres, including textbooks (e.g. (4), (9)), research papers (e.g. (33)), texts of talks given at mathematical meetings (e.g. (2), (5)), short reviews (e.g. (10), (23)), etc. In terms of size, these texts range from a 10 line review to a thick textbook (e.g. 766 pages in the second edition of (9), in 1912). The number of occurrences of "*im kleinen*" or "*im grossen*" also varies greatly, even in texts of similar natures and sizes. For instance, there is *one* sentence in Mangoldt's *Encyclopädie* article on the concepts of curve and surface in which "*im kleinen*" and "*im grossen*" are used (11); in contrast, both terms are used a significant number of times in Osgood's *Encyclopädie* article on the general theory of functions of one or several complex variables (3); what's more, Osgood used these terms in the titles of paragraphs, which probably means that it should be considered more significant—in some way—than use in a mere sentence. One could think of automatic means to account for this textual importance; for instance, a weight could be assigned to the various hits, a footnote being of less weight than a sentence in the core of the text, a paragraph title being of less weight than a chapter title, etc. However, as we shall see in the next section, a footnote *can* be of fundamental significance. Finally, this list of texts shows *mirror effects*. For instance, (19) is the German edition of American textbook (4); (13) is a review of (9). We could have removed (13) from our list, but we think we would have lost significant information: whether "*im kleinen*" and "*im grossen*" plays a lesser or a greater role in a new and improved edition of a textbook seems to us to be significant; as does whether or not a reviewer chooses to explain to the German audience the meaning of an expression used by an American mathematician.[115]

Of course, it should also be noted that this list is extremely short, given the quite extended time period and zero corpus. For instance, there are no relevant hits in the *Proceedings of the ICM* (from 1897 to 1920); nor any in the collected works of either Klein or Hilbert.

For reasons of both heterogeneity and scarcity, we opted for a qualitative analysis rather than a quantitative one (or a combination of qualitative and quantitative methods). A third type of reasons drove us away from further quantitative treatment, either of the core corpus itself or starting from the core corpus. Let us take a slightly closer look at some of the first texts in the "*im kleinen*"–"*im grossen*" list, from 1898 to 1906. There are several texts by William Fogg Osgood, including three of great national (US) and international visibility: a series of six lectures delivered at the Cambridge Colloquium of the AMS in 1898, on general function theory (2); the first article of the *Encyclopädie* volume on complex function theory (3); and a high-level textbook on complex analysis, based on state of the art point set topology and real analysis, and published in German (9). Number (5) is a lecture delivered by Edward Kasner on *The Present Problems of Geometry* at the *International Congress of Arts and Sciences* in Saint Louis in 1904, the seventh part of which is entitled "*Geometry im Grossen.*"[116] Number (4) is Oskar Bolza's textbook on the calculus of variations, which was subsequently be translated into German by its German born author (19). Surprisingly enough, these first occurrences already cover very different research fields; maybe not yet "more or less all mathematics"—for instance, no number theory appears in our core corpus—but a significant share indeed. Since these texts survey a vast literature, listing who they quote might not bring very interesting information and probably not much more information that can be got by actually reading the texts. Because of their nature and high visibility, listing who quote them might prove incredibly tedious; in addition to looking rather intractable, the prospect of getting significant information from such a citation network is not great: textbooks and encyclopaedia chapters are not quoted the way research papers are; the former convey an image of a discipline, gather tools into tool-boxes, tell more or less mythical histories, articulate research agendas, etc. All of this may leave a deep imprint on their (very numerous) readers, yet referring to them is not compulsory the way referring to a research paper in which some new technical definition or theorem appears is. Because of the fact that several large mathematical domains are covered, and very few texts involved, we doubt that

[115] We didn't include in our list of relevant hits those cases in which the expressions appear in a review as a mere quotation of the original text, or the cases in which the review is written by the author of the original paper/book.

[116] In the biographical Memoir written by Jesse Douglas, one reads: "The speech, published in the *Bulletin of the American Mathematical Society*, also aroused wide interest abroad; an indication of this was the publication of a translation in a Polish mathematical journal. Incidentally, one of Kasner's auditors at the St. Louis Congress was the great French mathematician Henri Poincaré, himself one of the principal invited speakers" (Douglas 1958, p. 13). To name a few of the participants in that Congress: H.S. White, G.A. Bliss, H. Maschke, M. Bôcher, J. Pierpont, E. Moore; among the non-American participants: E. Picard, G. Darboux and Poincaré. (White 1905).

we could get much more information using other quantitative methods[117] such as, for instance, co-word graphs used to map "dynamics of problematisation" (Callon et al. 1986).

To sum up, for reasons of scarcity, heterogeneity, visibility and interdisciplinarity which are all specific to our empirical data, we have engaged in qualitative analysis only. The fact that many of the methodological questions which we were faced with are usually associated with quantitative approaches results from the fact we built our core corpus by using automatic (i.e. blind) methods to survey a quantitatively large zero corpus.

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117 On quantitative methods as heuristic tools for the history of mathematics, see, for instance, Goldstein (1999b).

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