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Poncelet's porism: a long story of renewed discoveries, I

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Abstract In 1813, J.-V. Poncelet discovered that if there exists a polygon of n-sides, which is inscribed in a given conic and circumscribed about another conic, then infinitely many such polygons exist. This theorem became known as Poncelet's porism, and the related polygons were called Poncelet's polygons. In this article, we trace the history of the research about the existence of such polygons, from the "prehistorical" work of W. Chapple, of the middle of the eighteenth century, to the modern approach of P. Griffiths in the late 1970s, and beyond. For reasons of space, the article has been divided into two parts, the second of which will appear in the next issue of this journal.

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To all my teachers, friends and colleagues.

Communicated by: Jeremy Gray.

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General introduction

In 1822, Jean-Victor Poncelet published the *Traité sur les propriétés projectives des figures*. In this fundamental work, he gave a synthetic geometric proof of the following theorem, which became known as *Poncelet's closure theorem*,[1] or *Poncelet's porism*[2]: let two smooth (real) conics be given in the plane, if there exists a polygon of n sides, which is inscribed in one conic and circumscribed about the other, then there are infinitely many such polygons, and every point of the first conic is vertex of one of them (Poncelet 1822, sections 565–567) (Fig. 1a, b).

This theorem, which undoubtedly is one of the most important and beautiful theorems of projective geometry, was discovered by Poncelet in 1813 during his captivity in Russia as a prisoner of war.

Any polygon inscribed in C and circumscribed about D is called *Poncelet's polygon* (related to C and D), and sometimes *inter-scribed polygon*, or even *in-and-circumscribed polygon* (to C and D).

Poncelet proved the theorem as a corollary of a more general one, to which we refer as *Poncelet's general theorem* and that can be stated as follows: let $C, D_1, D_2, \ldots, D_n$ be smooth conics from a pencil, if there exists a polygon of n sides whose vertices lie on C, and each side is tangent to one of the others $D_1, D_2, \ldots, D_n$, then infinitely many such polygons exist.

The proofs that Poncelet gave of his theorems were heavily based on the unproved "principle of continuity," and for this he was criticized by some contemporaries, and especially by Cauchy.

Six years after the publication of the *Traité*, Jacobi recognized the existence of a relation between these theorems and the elliptic function "amplitude." In his elegant paper (Jacobi 1828), he gave a new proof of the theorem, in the "particular" case of two circles lying within each other, by applying some recursion formulae that arise in the iterated addition of a constant to an elliptic integral of the first kind.

Numerous mathematicians have been inspired to further studies in order to extend Jacobi's method to conics and to find the conditions on two conics C and D allowing the existence of a polygon, of a given number of sides, inter-scribed to them. At the same time, since the problem was originally an algebraic one (indeed, conics are algebraic curves, and the conditions of intersection and tangency are algebraic conditions too), some geometers considered Jacobi's transcendental solution somehow unsatisfactory and looked for a purely algebraic–geometric approach to the problem.

[1] From the German word *Schliessungstheorem* see Hurwitz (1879), also Dilgeldey (1903, p. 46).

[2] From the Greek word $\pi o\rho\iota\sigma\mu o\nu$: a proposition affirming the possibility of finding the conditions under which a certain problem becomes indeterminate or capable of infinite solutions. See for instance (Cayley 1853b).

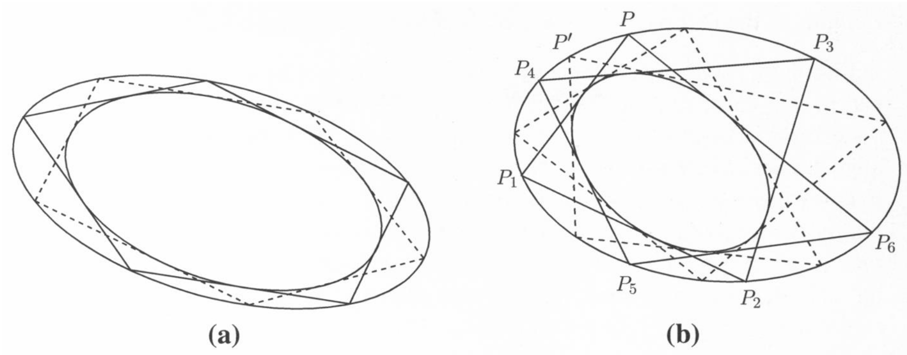

Fig. 1 Poncelet's closure theorem, **a** for $n = 5$, and **b** for $n = 7$

These themes, especially until the last decade of the nineteenth century, produced a huge literature concerning variants, alternative proofs and generalizations of the Poncelet theorem. The major works of this period are due to Nicola Trudi, Arthur Cayley, George Salmon, Adolf Hurwitz, Gaston Darboux and George H. Halphen.

Around 1920, two new important contributions appeared. In 1919, Fran-cesco Gerbaldi applied what he called Halphen's continued fractions, in order to get the bidegree of the covariant whose vanishing guarantees the existence of an inter-scribed n-gon (Gerbaldi 1919), and 2 years later, Henry Lebesgue gave an elegant geometrical proof of the general Poncelet theorem and a re-interpretation of Cayley's result (Lebesgue 1921).

After the 1920s, Poncelet's theorem seemed less appealing, and it almost fell into oblivion. Only a few isolated papers on the subject were published in the following 50 years, and, with the exception of Todd (1948), they were either not very relevant or considered Poncelet's problem only for circles. It was only in the late 1970s that Phillip Griffiths and Joseph Harris, with their papers (1977, 1978a), renewed the interest of mathematicians (and physicists) in Poncelet polygons and related questions.[3] Then in the nineties, W. Barth and J. Michel presented in a modern algebraic–geometric setting some result of Halphen and Gerbaldi (Barth and Michel 1993).

Over the years, several historical contributions on the subject have been published. We mention here only those we believe to be of historical relevance—others will be quoted in the text. (Loria 1889a), a detailed report on the papers that were published on this subject after (Jacobi 1828) until the last decade of the nineteenth century; (Bos et al. 1987), which extensively discusses the works of Poncelet, Jacobi and Griffiths–Harris; (Flatto 2009), the first monograph devoted to Poncelet's theorem; (Dragović 2011), which presents, in a modern setting, some of the major works on the subject, and explains the mechanical significance of the Poncelet closure theorem.[4]

[3] The trajectory of a free particle [called *billiard*, see Birkhoff (1927)], which moves along a straight line inside an ellipse and reflects at the boundary according to the law for a light ray, is tangent to a confocal conic (Sinai 1976).

[4] The book offers an excellent insight into the applications of Poncelet's closure theorem and its generalizations, to the theory of integrable system, billiard dynamics, PDEs and statistical mechanics, with an extended bibliography. Moreover, chapter four is a very good summary of such geometrical topics as pencils of conics, polarity, invariants of pairs of conics, duality, etc.

Leibniz, in the first page of *Historia et origo calculi differentialis*, wrote:

> Utilissimum est cognosci veras inventionum memorabilium origines, presertim earum non casu, sed meditandi innotuere. Id enim non eo tantum prodest, ut historia literaria suum cuique tribuat et alii ad pares laudes invitentur, sed etiam ut augeatur ars inveniendi, cognita methodo illustribus exemplis [It is most valuable to know the true origins of memorable inventions, particularly of those revealed not by chance, but through the force of reasoning. Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its method known through illustrious examples.[5]] (Leibniz 1846).

This work aims at giving a thorough historical account of these studies, spanning two centuries before the true nature and all facets of the problem were unveiled, on the occasion of the bicentennial jubilee of Poncelet's famous theorem.[6]

For reasons of space, the article, which has sixteen sections, has been divided into two parts. Part I, sections 1–10, deals with the history of the research developed on the subject until the end of the nineteenth century; Part II, sections 1–6, takes into account those developed in the twentieth century. For the convenience of the reader, each part has been equipped with its own list of references.

Part II will be published in the next issue of this journal.

Introduction to part I

The first section of Part I is devoted to the "prehistory," i.e., to the studies relating to the existence of triangles, and other polygons of a small number of sides, inter-scribed to two circles, carried out from the middle of the seventeenth century until about 1822. We count contributions by William Chapple, John Landen, Leonhard Euler, Nicolas Fuss, Simon A.J. Lhuilier and Jakob Steiner.

In the second section, we present the theorems of Poncelet and the proofs he gave. His methods were those proper of the school of Monge, but that Poncelet perfected, by formalizing the method of central projection and introducing some new concepts and tools, such as the controversial "principle of continuity." This, roughly speaking, can be stated as follows: A projective property that holds for a particular position of the figures involved holds true for any position of the figures, even if some of them disappear becoming imaginary. This principle allowed him to prove the theorems for circles and then pass to conics by a central projection.

Section three is devoted to illustrating the proof of the closure theorem for two circles, one lying inside the other, that Jacobi gave by means of the elliptic function "amplitude" (Jacobi 1828), and gives a glimpse of the "elliptic nature" of the problem. Jacobi ended his paper by saying that it would be of great interest to make similar considerations directly for a system of two conics, so avoiding the use of the principle of continuity, but he never returned to this subject.

[5] For the last sentence we have adopted the translation due to André Weil (Weil 1980, p. 226).

[6] In the same occasion (Dragović and Radnović 2014) offers the current state of the art on billiard dynamics.

His program was carried out by Trudi and, independently, by Cayley 25 years later.

The work of Trudi is presented in the fourth section. In his main memoir (1853), he used the algebraicity of the complete integral of Euler's differential equation, and the addition theorem for elliptic integrals of the first kind, to prove the closure theorem. Through his method, the role that symmetric (2, 2)-correspondences will play in closure problems can be seen to emerge. Unfortunately, Trudi's paper remained almost unknown outside the Kingdom of Naples.

Section five is mainly devoted to the explicit conditions that allow the existence of an inter-scribed n-gon to two conics, first found by Cayley in (1853b), and further detailed in (1861), by using Abel's addition theorem. His method revealed, in the form of the "Cayley cubic" $y^2 - \square(x) = 0$ ($\square(x)$ is the discriminant of the pencil generated by the two conics), the existence of an elliptic curve closely connected with the problem.

Section six is dedicated to Salmon's algebraic approach, developed in his work (1857), that uses the projective invariants of a pair of conics.

In the third quarter of the nineteenth century, many papers were published concerning new proofs of the Poncelet theorem and its generalizations, to curves of higher degree or to quadrics surfaces in space. We illustrate this literature in section seven, remarking that some of these works, such as (Weyr 1870) and (Darboux 1870a, b, 1873a, b), strongly inspired mathematicians and physicists many decades later.

Around the 1870, the deep connection between Poncelet's closure theorem and symmetric (2, 2)-correspondences emerged clearly. This aspect of the story, which reached its height in Hurwitz (1879), is discussed in section eight.

In section nine, we present, in some detail, the work of Darboux, who devoted a large part of his mathematical studies to questions connected with Poncelet polygons. The new system of plane coordinates that Darboux introduced, now called "Darboux coordinates," and the symmetric (2, 2)-correspondences were the main tools that he used for developing the theory.

In 1888, the second volume of Halphen's treatise on elliptic functions was published. In it, he applied the theory developed in the first volume in terms of the Weierstrass $\wp$ and σ functions, to several questions of geometry, mechanics and geodesy. Chapter ten of the book was expressly devoted to the Poncelet polygons, but other results on the same subject were inserted in chapters nine and fourteen. In section ten of the present article, we illustrate the content of these three chapters and discuss, at some length, what he called the "elliptic representations of point of the plane," and his use of the development in continued fractions of $\sqrt{X}$, X a polynomial of degree 3 or 4, to provide a new proof of Poncelet's theorem.

1 The prehistory: from Chapple to Steiner

Properties of triangles, which are inscribed in, or circumscribed about, a given circle, have been known since the Hellenistic period. For instance, it was known how to express the area of the inscribed triangle in terms of its sides and the radius and that among all the inscribed triangles the equilateral has maximal area. It was also known that given two concentric circles, of radii r, R with $r < R$, a triangle inscribed in the larger and circumscribed about the smaller exists, only if $r = R/2$, and that, in this case, all the (infinitely many) triangles that can be inter-scribed to the two circles are equilateral. Similar results were also known for other regular polygons.

In this section, we present some results on triangles, and other polygons, inscribed in one circle and circumscribed about another eccentric to the first, that were discovered from about the middle of the eighteenth century until few years after the publication of Poncelet's treatise.

1.1 Chapple

It seems Chapple was the first to study the problem of the existence of inter-scribed polygons, specifically triangles, to two non-concentric circles.[7] His essay (1746) starts as follows:

> The following enquiry into the properties of triangles inscrib'd in, and circumscrib'd about given circles, has let me to the discovery of some things relating to them, which I presume have not been hitherto taken notice of, having not met with them in any author; though an ingenious correspondent of mine, in the isle of *Scilly*, to whom I communicated some of the propositions herein after demonstrated, informs me that he had begun to consider it some years ago, but did not go thro' with it; however I must acknowledge that a query of his to me, relating thereto, gave me the first hint, and induc'd to pursue the subject with more attention, than perhaps otherwise I should have done.

In his paper, which probably remained unknown to professional mathematicians, Chapple stated that if there exists a triangle which is inscribed in one circle C of radius R and is circumscribed about another circle c of radius r (lying inside the first), then the distance a between the centers of the circles must satisfy the equation

$$a^2 = R^2 - 2rR. \tag{1.1}$$

This formula is sometimes called "Chapple's formula."

J.S. Mackay called attention on Chapple's essay only in 1887, but in his historical note (Mackay 1887) he gave just a partial transcription of it without commenting.[8]

To illustrate the work of Chapple, whose arguments are often confused and whose logic is very poor, even for the standard of his time, is not easy especially when trying to keep as faithful as possible to his thought.

[7] From the obituary published in the Exeter Flying Post we learn that William Chapple (1718–1781), who served for 40 years as Secretary of the Devon and Exeter Hospital, was an enthusiastic amateur of mathematics who studied John Ward's, *The Young Mathematician's Guide:/Being a Plain and Easy/Introduction/to the/Mathematicks/in five Parts*, whose fourth edition appeared in London in 1724. Chapple was capable of using *fluxions* and contributed several articles to the English periodicals the *Gentlemen's Magazine, Miscellanea Curiosa Mathematica* and *Ladies' Diary*.

[8] Chapple was quoted in Chapple (1901, pp. 552–553) and in Dingeldey (1903, p. 47) but not in Kötter (1901), whose sections 8–11 of chap. XVIII were devoted to the history of the Poncelet closure theorem. Let us remark that Chapple's paper also escaped to Gino Loria (see Loria 1889a, b, 1896). To our knowledge, the work of Chapple has been discussed in some depth only in Bos et al. (1987).

Chapple considered two (real) circles one lying inside the other. He first noticed (see his propositions I and II) that, if there exists an inter-scribed triangle having area A and whose sides have length x, y, z, one must have

$$A = \frac{r(x+y+z)}{2} = \frac{xyz}{4R}, \tag{1.2}$$

from which he deduced

$$2rR = \frac{xyz}{x+y+z}. \tag{1.3}$$

Chapple recalled (see his proposition III) that if the two given circles are concentric, then an inter-scribed triangle exists only if $2r = R$, and, in this case, it is always equilateral because—he observed—the circumcenter and incenter must coincide. He also remarked that the equilateral triangles have largest area among those inscribed in the exterior circle. Moreover, Chapple asserted (see his proposition IV, but the proof is very muddled) that, if the two circles are not concentric, then they may admit an inter-scribed triangle only if $2r \leq R$.

Next Chapple stated two propositions:

(V) (If there exists an inter-scribed triangle, then) "*An infinite number of triangles may be drawn, which shall inscribe and circumscribe the same two circles, provided their diameters, with respect to each other, be limited, as in the two last propositions*;"

(VI) "*The nearest distance of the peripheries of the two given circles, or, which amount the same, the distance of their centers, in order to render it possible to inscribe and circumscribe triangles, is fixed, and will be always the same*."

To prove (V), Chapple argued as follows. From (1.3), it is plain that, if x, y and z are required to be found from the given r, R, the question is capable of innumerable solutions. In fact, he remarked one side, at least, of the inter-scribed triangle can be chosen at pleasure, provided that it does not exceed the longest segment that can be drawn within the larger circle and tangent to the smaller circle inside. Hence, he continued, if the mutual position of the two circles is fixed so that a triangle may be inter-scribed to them, innumerable triangles may be inscribed and circumscribed to the same two circles (1746, p. 120). Let us notice that Chapple missed one more relation which the data R, r, x, y, z must satisfy when an inter-scribed triangle exists, the one derived from Heron's formula $A^2 = p(p-x)(p-y)(p-z)$, being p the half perimeter of the triangle.

To show (VI), Chapple proceeded in a very complicated way, but his reasoning was essentially as follows. He considered the two circles positioned so that an inter-scribed triangle exists and supposed that a is not fixed, i.e., that the inner circle can move freely while still allowing an inter-scribed triangle. He called AB the chord of the outside circle parallel to the direction of the motion and tangent to the inner circle, drew an inter-scribed triangle with this chord as one of its sides (so Chapple assumed the closure theorem, and accordingly his argument was circular), and denoted by E the third vertex of this triangle (Fig. 2a). He observed that moving the inner circle, from the initial position along the fixed direction AB, the vertex E also moves and leaves the outside circle, because the altitude of E decreases moving from the periphery of

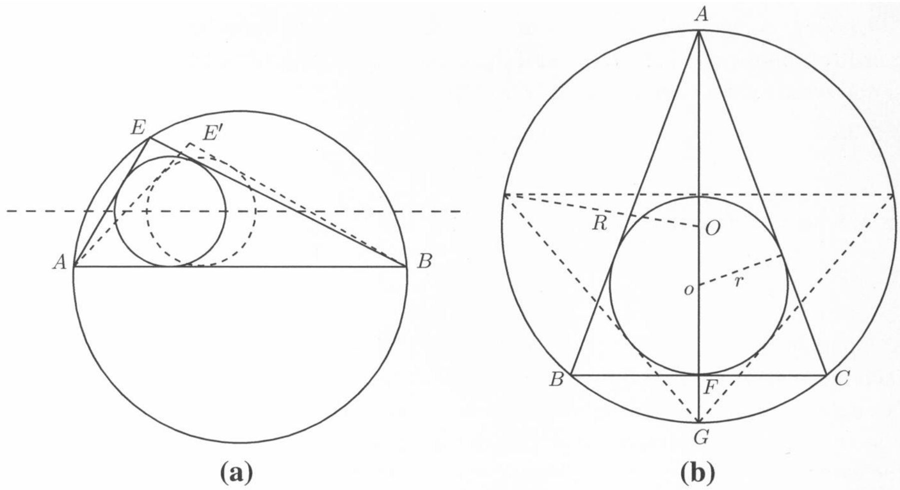

Fig. 2 **a** Let AEB be an inter-scribed triangle to two circles one inside the other. Chapple observed that moving the inner circle along the chord AB the vertex E leaves the outside circle. **b** Chapple's use of isosceles triangles to prove formula (1.1)

the chord toward the middle. So, concluded Chapple, the inner circle cannot move while still allowing an inter-scribed triangle, unless a remains the same.

Chapple used propositions (V) and (VI) (i.e., his unproved closure theorem) to derive the formula (1.1). For, he first stated the following proposition

(VII) "*Of the innumerable triangles that may be inscrib'd and circumscrib'd in and about two given (eccentric) circles, to must of course isoscelar, the vertexes of which will be in the common diameter of those circles, which will cut their bases at right angles; now the content of that isoscelar triangles which hath the least base, and the greatest altitude, will be the greatest, and that of the other the least of all the triangles that can be inscribed and circumscrib'd in the given circles.*"

He proved the proposition through a number of geometrical lemmas that we will not consider here. At the end of his proof, he observed:"the proposition is every way demonstrated; and tho' the method herein taken seems a little tedious and intricate, it is perhaps more concise and less troublesome than any which Fluxions would have afforded us." Finally (see pp. 123–124), by using the existence of the isosceles triangles he proved formula (1.1). Chapple argued as follows. He put (see Fig. 2b) $AF = x$, $FC = y$, hence $GF = 2R - x$ and $y = \sqrt{2R - x}$; then, since $\sin \angle CAF = \frac{r}{x-r} = \frac{y}{\sqrt{x^2+y^2}}$, with a simple computation he found $x = R + r + \sqrt{R^2 - 2Rr}$, and this, being $a = x - R - r$, is equivalent to formula (1.1).

There is no doubt that, despite the many failures in his proofs and logic, Chapple grasped some fundamental aspects of the problem.

In order to draw attention to his paper, Chapple proposed to prove formula (1.1) as "prize question" in the *Ladies' Diary* for the year 1746.

Robert Heath, editor of the journal, answered somewhat unsatisfactorily the year following (see The mathematical Questions 1817, pp. 393–395). In his answer, Heath

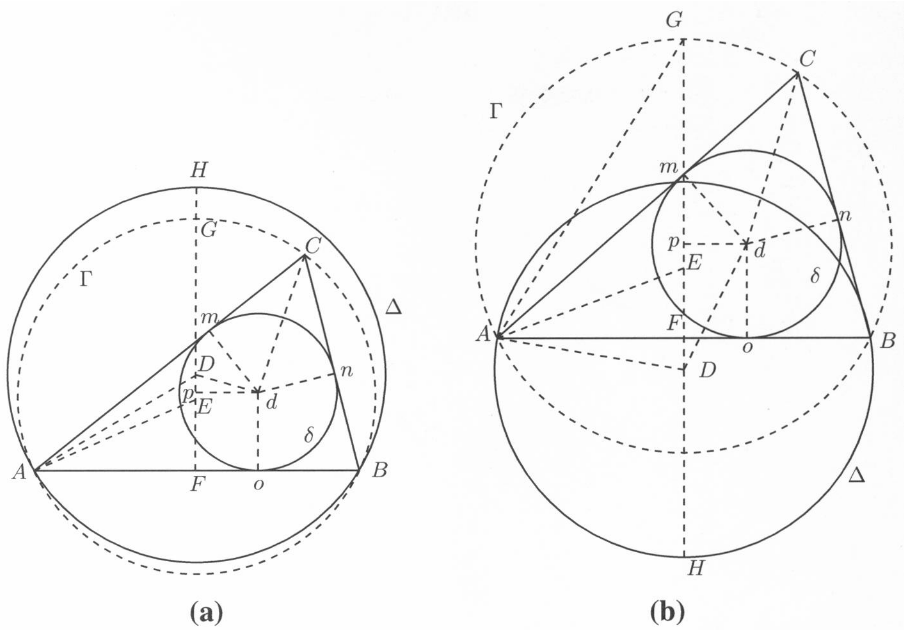

Fig. 3 The first case of Landen's question: the circle δ falls within ABC. **a** The circle δ is inside the circle Δ, and **b** the circle δ in outside the circle Δ

also mentioned the solution he had received from J. Landen, a non-professional mathematician, that a few years later would be tackling the problem from a new point of view.

1.2 Landen

Landen[9] devoted Part I of his *Mathematical Lucubrations* to the following question (Landen 1755, p. 1)[10]:

The two circles ABH and mno whose centers are D and d, respectively, being given in magnitude and position; let any given chord AB in the circle ABH touch the circle mno at o; and, from the extremities of that chord, let two other tangents be drawn to the circle mno, touching it at m and n, and intersecting each other at C: It is proposed to find the radius AE of the circle ABG circumscribing the triangle ABC (see Figs. 3a, b, 4a, b).

In the following, it is convenient to denote the circles *mno*, *ABH* and *ABG*, respectively, δ, Δ and Γ.

[9] Although considered a mathematician of high rank, John Landen (1719–1790) was never a professional one. He began contributing to the *Miscellanea Curiosa* and to the mathematical problem section of the *Ladies' Diary* from 1744. In 1754, he published the first of his eight papers in the *Philosophical Transactions*, and the following year he published the *Mathematical Lucubrations*. His name is often associated with an important transformation giving a relation between certain Eulerian integrals.

[10] This work of Landen, which was summarized in Mackay (1887), is also quoted in Dingeldey (1903, p. 47) but not in Loria (1889a, b, 1896), Cantor (1901), Kötter (1901) and Bos et al. (1987).

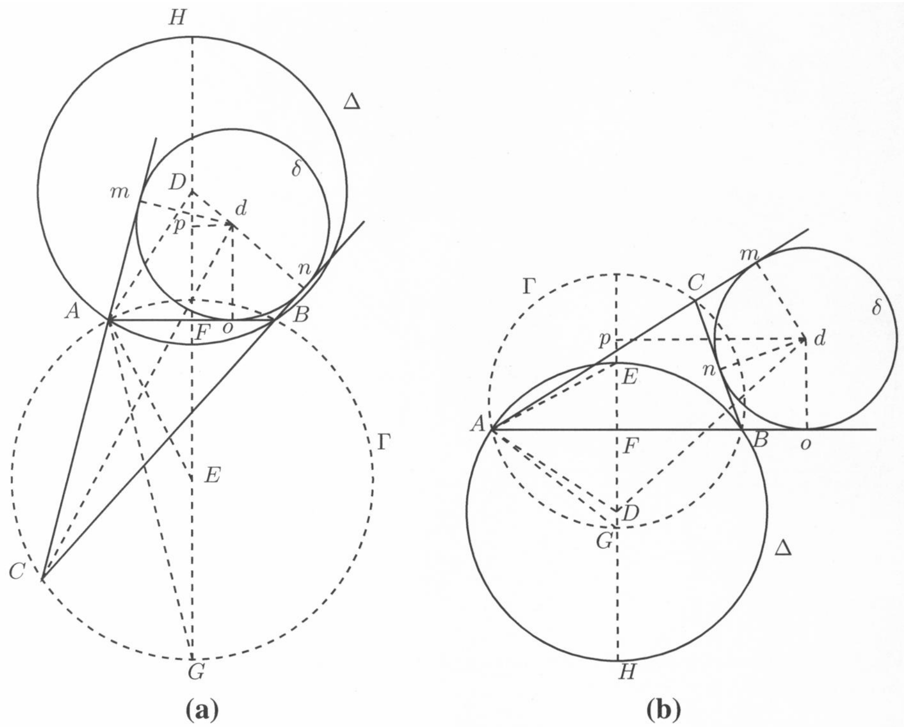

Fig. 4 **a** The second case of Landen's question: the circle δ falls without ABC and o falls between A and B. **b** The third case: the circle δ falls without ABC but o does not fall between A and B

Landen denoted by R and r the radii of Δ and δ, respectively, and called d the distance between their centers. He considered the axis l of the given chord and called F the intersection of AB with l. Moreover, he drew the parallel to AB passing through d and denoted p its intersection with l. He also denoted E the center of the circle Γ circumscribing the triangle ABC and put $b = AF$, $x = AE$. Always considering δ as standing upon AB (see the figures), he found the quotes of D above F, and above p, to be, respectively, $\pm\sqrt{R^2 - b^2}$, and $\pm\sqrt{R^2 - b^2} - r^2$. Then, he got

$$dp = \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

$$Ao = b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}},$$

$$Bo = b - \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}}.$$

At this point, Landen wrote: "To proceed with perspicuity, it will convenient to consider distinctly the different cases of the circle mno falling within or without the triangle (ABC)," namely

1. δ falls within the triangle ABC (Fig. 3a, b),
2. δ falls without ABC but o falls between A and B (Fig. 4a),
3. δ falls without ABC and o does not fall between A and B (Fig. 4b).

Case 1. Landen drew the segment AG, G being the point above AB where the line FH intersects Δ, and noticed that the triangles AFG and dmC are similar, in fact they are both right and $\widehat{AGF} = \widehat{mCd}$. Therefore, $\pm\sqrt{x^2 - b^2}$ and $x \pm \sqrt{x^2 - b^2}$ being the quotes of E and G above F, respectively, the proportion

$$b : x \pm \sqrt{x^2 - b^2} = r : Cm,$$

holds true, and so $Cm = Cn = \frac{rx \pm r\sqrt{x^2-b^2}}{b}$. Then,

$$AC = b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2} + \frac{rx \pm r\sqrt{x^2 - b^2}}{b}},$$
$$BC = b - \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2} + \frac{rx \pm r\sqrt{x^2 - b^2}}{b}},$$

b in this case being always greater than oF.[11]

The perimeter of the triangle ABC is $4b + \frac{2rx \pm 2r\sqrt{x^2-b^2}}{b}$; hence, its area $\mathcal{A}$ is equal to $2br + \frac{r^2x \pm r^2\sqrt{x^2-b^2}}{b}$. From the known formula

$$\mathcal{A} = \frac{1}{2}(AC \times BC)\sin\widehat{ACB}$$

it follows that

$$AC \times BC = 4rx + \frac{2r^2x^2 \pm 2r^2x\sqrt{x^2 - b^2}}{b},$$

then, by equating this value with that obtaining multiplying the values of AC and BC above, and solving with respect to x, Landen got

$$x = \frac{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}{4r} + \frac{b^2 r}{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}.$$

From this expression is clear that, if $d^2 = R^2 - 2Rr$, whatever b may be, one has $x = R$.

In the other two cases, that here for brevity we omit to discuss, Landen proceeded similarly and he found that

$$x = \frac{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}{4r} + \frac{b^2 r}{\pm 2r\sqrt{R^2 - b^2} + d^2 - R^2}.$$

Again he observed that if $d^2 = R^2 + 2rR$, then, whatever b may be, is $x = R$.

[11] In all cases, the proper sign of $\sqrt{R^2 - b^2}$ is $+$ or $-$, according if the center D is on the same, or on the contrary, side of AB with the center d; and the proper sign of $\sqrt{x^2 - b^2}$ is $+$ or $-$, according if the center E is on the same, or on the contrary, side of AB with the center d (Landen 1755, p. 5). See Figs. 3, and 4.

Finally, as a corollary, he stated (Landen 1755, p. 5):

It follows from what has been said that, d being equal to $\sqrt{R^2 - 2rR}$ *or* $\sqrt{R^2 + 2rR}$, *whatever b may be, E will fall in D, and the circle circumscribing the triangle always coincides with the given circle ABH; a thing very remarable!*

It seems to us that all this amounts to the following: given the two circles Δ and δ, if the distance d between their centers is given by $R^2 - 2rR$ (or by $R^2 + 2rR$ in the second and third case), then there exists a triangle ABC which is at the same time inscribed in Δ and circumscribed about δ, and moreover, the triangle can be constructed starting from any point A on Δ, the condition being independent of $2b$, i.e., the length of the chord AB.

This means that, not only condition (1.1) is sufficient for the existence of an interscribed triangle to the two circles, but also that, in this case, the closure theorem is proved since the chord AB can be arbitrarily chosen.

Landen was aware of this, although he did not explicitly state it at this point. On the other hand, as we have noticed at the end of the previous subsection, Landen had been aware since 1747 that (1.1) represents a necessary condition, and he had certainly read Chapple's (1746), so he knew that the existence of an inter-scribed triangle implies the existence of infinitely many others. This is well shown by the sequel of his memoir.

After having stated the corollary, Landen wrote:

> For this to happen and the circle *mno* fall within the triangle, it is obvious R must be no less than $2r$, for, if it be, $\sqrt{R^2 - 2rR}$, the quantity to which d ought to be equal, will be imaginary.
>
> But, that the circle falling without the triangle, the same thing may happen though R be less than $2r$, so that R be greater than $r/4$. The reason why R, in this case, must be greater than $r/4$ appears from this consideration. The distance of the center d from that point in the periphery ABH which is the farthest from the center is $d + R = \sqrt{R^2 - 2rR} + R$, whose distance must be greater than r; otherwise, the circle ABH will fall entirely within the circle mno, and no chord in that can be tangent to this. Therefore, since $\sqrt{R^2 - 2rR} + R$ must be greater than r, $\sqrt{R^2 - 2rR}$ must be greater than $r - R$, $R^2 + 2rR$ greater than $r^2 - 2rR + R^2$, and R greater than $r/4$.
>
> Consequently, since r must be less than $4R$, d must be less than $\sqrt{R^2 - 2rR}$, or its equal $3R$.

Here, Landen observed that, by a orthographic projection, i.e., a affine parallel projection, two circles—in a same plane—and their tangents can be mapped into two similar ellipses, i.e., having the same eccentricity, and their tangents (Landen 1755, p. 6). Then, he stated the following (see Fig. 5a, b):

If within or without any ellipsis whose transverse axis is T,[12] *a second concentric similar ellipsis be described with its transverse axis t, in the same direction with T, and a third ellipsis be described, similar to the other two, with its center any where in the periphery of the second ellipsis, and having its transverse axis equal to* $[\tau =]\frac{T^2 - t^2}{2T}$,[13]

[12] For "transverse axis" he meant the "half of the major axis".

[13] This condition translates formula (1.1) in the present case, where t stands for a, T for R, and τ for r.

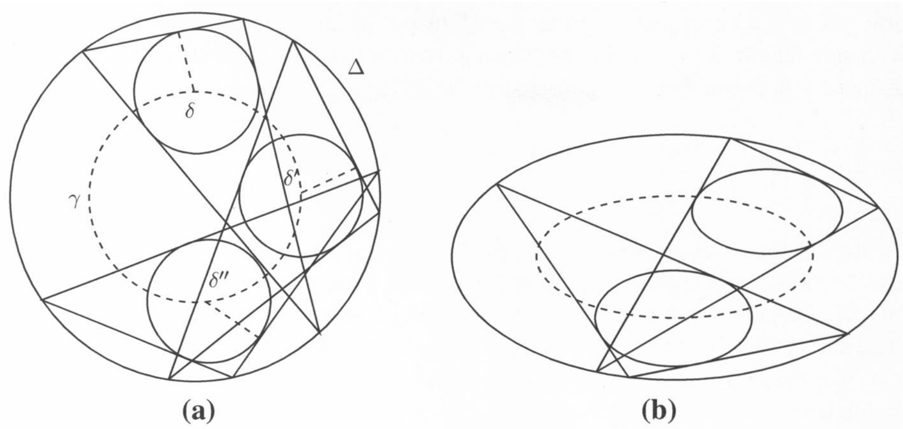

Fig. 5 **a** Landen observed that if there exists an inter-scribed triangle to the circles Δ and δ, then there exist such triangles for all circles δ', δ'',... whose centers have distance from the center of Δ equal to the distance of the center of δ. **b** By using orthographic projection, Landen extended the property illustrated in (**a**) to the case of concentric similar ellipsis

and parallel to the transverse axes of the other ellipses; any tangent being drawn to this third ellipsis and continued both ways till it intersects the periphery of the first ellipsis in two points, and two other tangents being drawn to the same third ellipsis from those points of intersection, the locus where these last tangents continued to intersect each other will always be in the periphery of the first ellipsis.

The drawing of the tangents in that manner will be impossible unless t be less than 3T.

It is worthy underlining that Landen wrote: "any tangent being drawn to this third ellipsis and continued both ways till it intersects the periphery of the first ellipsis in two points, and two other tangents being drawn to the same third ellipsis from those points of intersection, the locus where these last tangents continued to intersect each other will always be in the periphery of the first ellipsis," clearly this is the closure theorem for triangles and circles, or ellipses, in the previous configurations.

Landen concluded the first part of his essay by saying: "Other conclusions of a like nature may be drawn from what is done above and a consideration of other projections, but I have no inclination to pursue the speculation farther." The projective methods of the school of Monge were still far in the future; nevertheless, these words show how Landen had perception of the projective nature of the question.

We can safely state that Landen's work, although unfortunately it remained unknown outside England, contained in germ some ideas that Poncelet was to develop 60 years later.

1.3 Euler

Some nineteenth-century authors, first of all Steiner (see below), attributed formula (1.1) to Leonhard Euler. In his paper (1765), which can be considered a milestone in triangle geometry, he studied the positions and mutual distances of barycenter, orthocenter, incenter and circumcenter.[14] In particular, he found that if a triangle, with side lengths x, y, z, is inter-scribed to two circles, one inside the other, then the distance a between their centers satisfies the following equation:

$$a^2 = \frac{(xyz)^2}{16A^2} - \frac{xyz}{x+y+z}. \tag{1.4}$$

Surprisingly enough in his article, Euler did not investigate the relation between a and the radii r, R of the two circles, and in fact this formula does not express a in terms of r, R. We notice that one can get formula (1.1) from (1.4), by taking into account (1.2) and (1.3), which were known to Euler. Nevertheless, we should add that N. Fuss, who was certainly well acquainted with Euler's work, did not attribute formula (1.1) to him (see below). Moreover, as stressed in Bos et al. (1987, p. 295), formula (1.4) is not a formula by which a closure theorem could be detected, since a is a function of the sides of the triangle, which depend on it, and not of the radii, which are fixed.

1.4 Fuss

Nicolaus Fuss, in his paper (Fuss 1797), studied first several problems concerning quadrangles which are inscribed in, or circumscribed about, a given circle. For instance, he determined the radius R of the circumscribed circle, and the radius r of the inscribed circle, as functions of the length of the sides a, b, c, d of the quadrangle, for which he found, respectively:

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{abcd}},$$
$$r = \frac{\sqrt{abcd}}{a+c}.$$

Only at the end of his paper did he consider the following problem (Fuss 1797, section 30):

Datis radiis circolorum quadrilatero ABCD inscripti et circumscripti, invenire distantia centrorum [Find the distance between the centers of the circumscribed circle and of the inscribed circle to the quadrangle $ABCD$].

To solve the problem Fuss proceeded as follows (see Fig. 6a). He denoted $2\alpha, 2\beta, 2\gamma, 2\delta$ the four angles at the vertexes A, B, C, D of the quadrangle. From the above formulae for R and r, he got

$$R = \frac{r\sqrt{1 + 2\sin 2\alpha \sin 2\beta}}{\sin 2\alpha \sin 2\beta},$$

[14] In this paper, Euler discovered the *nine point circle* (i.e., the circle on which lie the three midpoints of the sides, the three feet of the altitudes, and the three midpoints of the line segments from each vertex to the orthocenter), and the line now referred to as *Euler line*, to which belong barycenter, orthocenter and circumcenter.

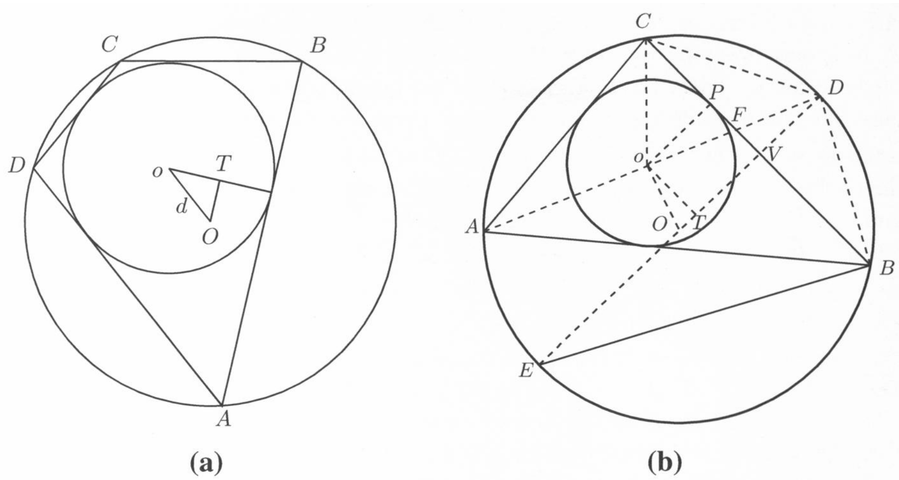

Fig. 6 **a** Fuss' construction for the quadrangle. **b** Illustration of how Fuss proceeded in order to get a direct proof of formula (1.1)

and, putting $m = \cos(\alpha + \beta)$, and $n = \cos(\alpha - \beta)$, he wrote R in the form

$$R = \frac{r\sqrt{1 + n^2 - m^2}}{n^2 - m^2}.$$

He called d the distance between the centers O and o and denoted by T the intersection between the line through o orthogonal to AB and the line through O orthogonal to the previous one. He found that

$$\overline{TO} = \frac{r\sqrt{1 - n^2}}{n - m}, \quad \overline{To} = \frac{r(n^2 + m^2 - 1)}{n^2 - m^2}, \quad d = \frac{r\sqrt{1 + m^2 - n^2}}{n^2 - m^2}.$$

Taking into account the last formula for R and computing $R^2 - d^2$, $R^2 + d^2$, he obtained

$$(R^2 - d^2)^2 = 2r^2(R^2 - d^2)$$

and so

$$d^2 = R^2 + r^2 \pm r\sqrt{4R^2 + r^2},$$

which clearly generalizes formula (1.1) to the case of quadrangles.

In the subsequent, final, section, Fuss gave a direct proof of formula (1.1), and for this he argued as follows (see Fig. 6b).

He considered a triangle ABC inscribed in the circle Δ of center O and circumscribed about the circle δ of center o, lying inside the first. He observed that the line through A and o meets Δ in a new point D and CB in the point F; moreover,

Ao and Co bisect $\widehat{BAC}$ and $\widehat{ACB}$, respectively; the line through D and O meets Δ in a new point E; the diameter DE meets perpendicularly in V the side CB of the triangle. Then, he let T and P be the points of intersection with the diameter ED and the side CB of the respective perpendicular through o. He found that $d^2 = R^2 - 2R \cdot \overline{DT} + \overline{Do}^2$. Moreover, $\widehat{CVD} = \widehat{DBV} = \pi/2$, and $\widehat{BED} = \widehat{DCV}$ since they belong to the same chord. It follows that the triangles CVD and EBD are similar, and so $\overline{DV}/\overline{CD} = \overline{BD}/\overline{DE}$; therefore, $\overline{CD} \times \overline{BD} = \overline{DV} \times \overline{DE} = 2R \cdot \overline{DV}$. Clearly, $\widehat{CoD} = \widehat{DAC} + \widehat{oCA}$ and $\widehat{oCD} = \widehat{oCF} + \widehat{DCF}$.[15] Since $\widehat{oCF} = \widehat{oCA}$ and $\widehat{DCF} = \widehat{DCB} = \widehat{DEB} = \widehat{DAB} = \widehat{oAC}$, is $\widehat{CoD} = \widehat{oCD}$. It follows that $\overline{Do} = \overline{CD} = \overline{BD}$ and $\overline{Do}^2 = \overline{CD} \times \overline{BD} = 2R \cdot \overline{DV}$, substituting this last value in the above formula for d^2, he found $d^2 = R^2 - 2R(\overline{DT} - \overline{DV})$, and, since $\overline{DT} - \overline{DV} = \overline{TV} = \overline{oP} = r$, Fuss finally got $d^2 = R^2 - 2Rr$.

Let us remark that this was the first occurrence of formula (1.1) in a widely circulated journal known to an international mathematical public.

In a subsequent paper (Fuss 1802), Fuss considered the same question for polygons having more than four sides. However, the great difficulties he encountered in dealing with the general problem forced him to limit his analysis to *polygona symmetrice irregularia* [symmetrically irregular polygons], i.e., irregular inter-scribed polygons that are divided into two equal parts by the line of centers of the circles.

He let R, r, a be as above, and put $p := R + a$, $q := R - a$ and $s := pq/r$. Then, using trigonometric formulae with great skill and a large amount of work, he got the following conditions for $n = 5, 6, 7, 8$, respectively:

$$\frac{p^2q^2 - r^2(p^2+q^2)}{p^2q^2 + r^2(p^2-q^2)} = \pm\sqrt{\frac{q-r}{p+q}},$$

$$3p^4q^4 - 2p^2q^2r^2(p^2+q^2) = r^4(p^2-q^2),$$

$$\pm(s^2 - s(p-q) - 2pq)\frac{\sqrt{p(p+q)(s-q)}}{2}$$

$$\pm (s^2 - p^2 - q^2)\frac{\sqrt{q(p+q)(s-p)}}{2} = \pm\left(\frac{s-p+q}{2}\right)(s^2+p^2-q^2),$$

$$p^2r\sqrt{q^2-r^2} + q^2r\sqrt{p^2-r^2} = p^2r^2 - \sqrt{(p^2-r^2)(q^2-r^2)}.$$

Since Fuss was unaware of any closure theorem, he did not realize that his results were true in general, i.e., without restriction to symmetrical irregular polygons.

1.5 Lhuilier

Although the results of Fuss were published in a prestigious journal, many mathematicians failed to notice them. In fact, in the first volume of Gergonne's *Annales*

[15] In (Fuss 1797, p. 124), it is erroneously written $\widehat{oCD} = \widehat{oCE} + \widehat{DCE}$, and then (in the line below) $\widehat{oCE} = \widehat{oCA}$, $\widehat{DCE} = \widehat{oAC}$ which is clearly false. Since our figure is the same as (Fuss 1797, Fig. 4) we may argue that the letter E and F were interchanged when printing the paper.

de mathématiques pures et appliqués for the years 1810–1811, the following two questions were proposed:

I) Un cercle étant donné et un point étant donné arbitrairement sur son plan et dans son intérieur, il y a toujours une longueur, et une seule longueur, laquelle étant prise pour rayon d'une nouveau cercle ayant pour centre le point donné, il arrivera qu'un même triangle pourra être à la fois inscrit au premier des deux cercle, et circonscrit au second;

II) Un cercle étant donné et un point étant donné arbitrairement sur son plan, il y a toujours une longueur, et une seule longueur, laquelle étant prise pour rayon d'une nouveau cercle ayant pour centre le point donné, il arrivera qu'un même triangle pourra être à la fois circonscrit au premier des deux cercle, et inscrit au second (Gergonne 1810, pp. 62–64).

The two questions can be reformulated as follows:

I) Given a circle C and a point P inside it, there is exactly one length r such that if D is the circle of center P and radius r, there exists a triangle which is inscribed in C and circumscribed about D;

II) Given a circle D and a point P, there is exactly one length R such that if C is the circle of center P and radius R, there exists a triangle which is circumscribed about D and inscribed in C.

It seems that the proposer knew the formula (1.1) (in fact it was so), and had noticed that, fixed a, there is only one R for any given r, and viceversa.

These questions were soon solved by Lhuilier[16] in his memoir (Lhuilier 1810).[17] From a footnote in the first page of Lhuilier's paper, the editors explain that they had known the theorem since 1807. It was communicated to them by *Monsieur* Mahieu, professor of mathematics in Alais, who learned it from *Monsieur* Maisonneuve, a mines engineer.

In the same footnote, the editors presented the proof given by Maisonneuve and that we think worthy enough to be reproduced here below.

If a denote the distance between the centers and x, y, z are the length of the sides of the inter-scribed triangle, one has:[18]

$$a^2 = \frac{(xyz)^2}{(x+y+z)(y+z-x)(x+z-y)(x+y-z)} - \frac{xyz}{x+y+z},$$

on the other side, as well known, the following relations hold true

$$\begin{gathered} 16A^2 = (x+y+z)(y+z-x)(x+z-y)(x+y-z), \\ 2A = r(x+y+z), \\ 4RA = xyz. \end{gathered}$$

[16] Simon Antoine Jean Lhuilier, sometime written L'Huilier (1750–1840), Swiss mathematician, professor at the University of Geneva and member of several European Academies. He is mainly known for his studies in Analysis and for having generalized the Euler formula to planar graphs.

[17] The questions posed by Gergonne, and the solution given by Lhuilier, completely escaped Loria but were cited in Lhuilier (1901, p. 151).

[18] Maisonneuve probably deduced this formula from (1.4) and Heron's formula.

From the first two relations, it follows that

$$\frac{xyz}{x+y+z} = 2rR,$$

and also, by equating the square of the third with the first, it follows that

$$\frac{(xyz)^2}{(x+y+z)(y+z-x)(x+z-y)(x+y-z)} = R^2.$$

Then, taking into account the above expression of a^2, from the last two formulae immediately follows $a^2 = R^2 - 2rR$.

Lhuilier expressed formula (1.1) in the form $\overline{Zz}^2 = R(R-2r)$, being Z and z the centers of the circles. The proof he submitted was quite complicated, but it can be seen as a trigonometric translation of Maisonnneve's proof. We will not present it here, preferring to insist on Lhuilier's clear perception of the closure theorem.

After having proved formula (1.1), he wrote:

> La relation entre la distance des centres de deux cercles et les rayons R et r de ces cercles, étant telle qu'ill vient d'être dit; si on circonscrit au cercle dont le rayon est r un triangle dont un des côtés soit une corde de l'autre cercle, ce triangle sera inscrit à ce dernier cercle; et reciproquement, si l'on inscrit au cercle dont le rayon est R un triangle dont un des côtés soit tangent à l'autre cercle, ce triangle sera circonscrit à ce dernier cercle. Il y a donc un nombre illimité de triangles qui peuvent être à la fois inscrits à un cercle et circonscrits à une autre cercle. Lorsque les rayons de ce cercles et la disatnce de leurs centres sont liés par l'équation $\overline{Zz}^2 = R(R-2r)$ [If the relation of the distance between the centers of the circles and the radii R and r of these circles is as we have just said, then if we circumscribe about the circle of radius r a triangle having a side which is a chord of the other circle, this triangle will be inscribed in the last circle and vice versa. If we inscribe in the circle of radius R a triangle having a side which is tangent to the other circle, this triangle will be circumscribed about this last circle. Hence, there are infinitely many triangles which are inscribed in a given circle and circumscribed about another if the distance between their centers satisfies the equation $\overline{Zz}^2 = R(R-2r)$].

So, similarly to Chapple, it was the degree of freedom implicit in the equation (1.1) that led Lhuilier to the clear enunciation of the closure theorem for triangles interscribed to two circles, one inside the other. Nevertheless, his proof, based exclusively on the above argument, was not complete.

1.6 Steiner and the attribution to Euler of formula (1.1)

In the second volume of Crelle's journal, in the section *Aufgaben und Lehrsät-ze*, Jakob Steiner asked the reader to solve the following problem (Steiner 1827, p. 96):

3. Aufgabe. Wenn ein gegebenrs (irreguläres)[19] Vieleck (n Eck) so beschaffen ist, dafs sowohl in als um dasselbe ein Kreis beschrieben werden kann, so soll man zwischen des Radien (r, R) der beiden Kreise und dem Abstande (a) ihrer Mittelpuncte von einander eine Gleichung finden. (Für das Dreieck ist diese zuerst von Euler gefundene Gleichung bekanntlich $a^2 = R^2 - 2rR$). [If a (irregular) polygon (n-gon) is given such that a circle can be drawn in and around it, it is required to find the equation relating the radii (r, R) of the two circles and the distance (a) between their centers. For triangles this equation, that was found for the first time by Euler, is $a^2 = R^2 - 2rR$)].[20]

Following Steiner, other authors attributed formula (1.1) to Euler; see for instance (Jacobi 1828; Loria 1889a, 1896; White 1916).[21] It seems that Steiner was not aware of Fuss (1797).

In a subsequent note, Steiner gave, without proofs, the equation that R, r, a must satisfy for the existence of an inter-scribed n-gon when $n = 4, 5, 6, 8$, respectively (Steiner 1827, p. 289):

$$(R^2 - a^2)^2 = 2r^2(R^2 + a^2),$$
$$r(R - a) = (R + a)\left[\sqrt{(R - r + a)(R - r - a)} + \sqrt{(R - r - a)2R}\right],$$
$$3(R^2 - a^2)^4 = 4r^2(R^2 + a^2)(R^2 - a^2) + 16r^4a^2R^2,$$

$$\left\{(R^2 + a^2)\left[(R^2 - a^2)^4 + 4r^4a^2R^2\right] - 8r^2a^2R^2(R^2 - a^2)\right\}$$
$$\times 8r^2\left[(R^2 - a^2)^2 - r^2(R^2 + a^2)\right] = \left[(R^2 - a^2)^4 - 4r^4a^2R^2\right]^2.$$

We do not know how Steiner obtained them, but, since he had Poncelet's treatise handy, we may argue that he proved the formulae for symmetrically irregular polygons and extended their validity to irregular polygons via Poncelet's theorem.

The lack of proofs was deplored by Jacobi (1828, p. 376). In this paper, that will be discussed in section 3, Jacobi felt compelled to challenge the paternity on behalf of Fuss, who had recently died, the results claimed by Steiner. He also affirmed that the case $n = 7$, that omitted by Steiner, was the most difficult to solve, and comparing the formulae of the two authors he established their equivalence in all cases except for $n = 8$.[22]

[19] It is quite probable that Steiner was referring to irregular polygons as "completely irregular polygons," and not to "symmetrically irregular polygons" in the sense of Fuss.

[20] Steiner did not quote any paper by Euler.

[21] No reference to Euler's papers were provided in Jacobi (1828) and White (1916). Loria quoted *Novi Comm. Ac. Sci. Imp. Petropolitanae*, vol. 2 (1749), published in 1750, but, although this volume contains two memoirs by Euler, none of them presents results which may lead to formula (1.1).

[22] In fact, Steiner's formula for $n = 8$ is not correct [see for instance (Dingeldey 1903, p. 47)].

2 The theorems and methods of Poncelet

In 1822, Jean-Victor Poncelet published his *Traité sur les propriétés projectives des figures* (Poncelet 1822).[23]

In this treatise, Poncelet adopted a highly synthetical approach and introduced two concepts that would be crucial in the setup of his entire book: the *ideal chord* and the *principle of continuity* (see below). Although Cauchy and Gergonne criticized the use of the principle of continuity,[24] the treatise was well received by contemporaries, and the tools developed therein were adopted for decades in the nineteenth century [see (Chasles 1837; Kötter 1901) and also (Kline 1972, pp. 163–165), (Gray 2007, chap. 4)].

In his extremely rich book, largely conceived between March 1813 and June 1814 during his captivity as a prisoner of war in Saratov[25] Poncelet formulated the following theorem that became known as the *Poncelet closure theorem* (PCT for short) or *Poncelet's porism* (Fig. 7a):

Theorem PCT *Let C and D be two smooth conics in the projective plane, if there exists a polygon of n sides which is inscribed in C and circumscribed about D, then for every point $P \in C$ there is one such polygon having P as one of its vertices.*

Poncelet obtained this theorem as a corollary of a more general result, which we will refer to as *Poncelet general theorem* (PGT for short), that can be formulated as follows (Fig. 7b):

Theorem PGT *Let $C, D_1, D_2, \dots, D_{n-1}$ be conics from a pencil $\mathfrak{F}$. Consider a n-gon $P, P_1, \dots, P_{n-1}$ inscribed in C and having the side PP_1 tangent to D_1, the side P_1P_2 tangent to D_2 and so on until $P_{n-2}P_{n-1}$ tangent to D_{n-1}. Then, if P move along C in such a way the sides PP_1, P_1P_2 etc. remain tangent, respectively, to D_1, D_2 etc., the nth side $P_{n-1}P$ envelops a conic belonging to $\mathfrak{F}$.*

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[23] A second enlarged edition of this work appeared in two volumes more than 40 years later (Poncelet 1865–1866).

[24] See Cauchy's report on the paper that Poncelet presented to the Academy in 1820 (Cauchy 1820), also reproduced in Poncelet (1822).

[25] Poncelet took part, as Lieutenant of Engineers, in Napoleon's Russian campaign in 1812. After the retreat of the French army, following the defeat at Borodino (September 1812), Poncelet was left for dead on the battlefield of Krasnoi (November 1812), here he was found by the enemy soldiers. As a prisoner of war, he was forced to march for almost five months until he was imprisoned in Saratov, on the banks of the river Volga (Didion 1870). During his captivity, Poncelet wrote seven notebooks, called by him "Cahiers de Saratoff," where on the basis of what he had learnt from Carnot and Brianchon at the École polytechnique, he developed the projective theory of conic sections. Poncelet wrote the seventh notebook with the intent of presenting it to the Academy of Sciences of St. Petersbourg, with the hope, if accepted, of being called to Moscow until a peace agreement was reached between France and Russia. The events of 1814 interrupted this project. Poncelet based the development of his treatise (Poncelet 1822) on the *Cahiers*, of which the seventh might be regarded as a first attempt at a redaction. After his return to France, Poncelet published some of the results he had obtained in Saratov. In particular, in 1820, he presented to the Paris Academy the paper *Mémoire relatif aux propriétés projective des sections coniques* in which, for the first time, he presented the "principle of continuity" as a tool for solving difficult problems concerning conics. But it was only in 1862 that he published the seven notebooks in their entirety, as a part of the first volume of his new treatise *Applications d'analyse et de géométrie* (Poncelet 1862, vol. 1). For a detailed analysis of the content of the *Cahiers* (see Belhoste 1998).

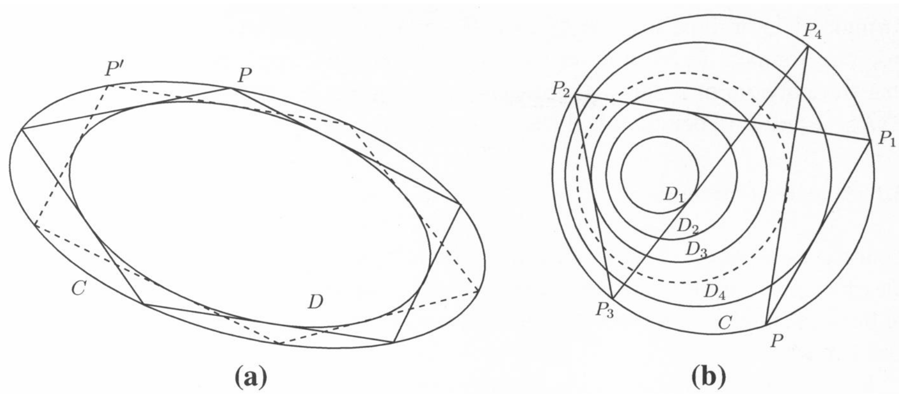

Fig. 7 **a** Poncelet's closure theorem for $n = 5$. **b** Poncelet's procedure in order to prove the general theorem for circles belonging to a same pencil

The proof of PGT presented in the *Traité* was a slight variation of one that he had written during his imprisonment, but which was published much later (Poncelet 1862, sixth *Cahier*). In both variants, Poncelet first proved the theorem for circles, proceeding by induction on n, and then he extended its validity to conics using the method of projection (see below). The two proofs differ only in the first step of the induction, i.e., in the case of three circles. In fact, in Saratov, Poncelet had proven it by an enormous straightforward computation, while in the *Traité* he proceeded by the synthetical method, developing several preliminary geometrical lemmas, and applying an ad hoc reasoning, heavily based on the principle of continuity. We will discuss these proofs later on.

In an article published in 1817, Poncelet announced new methods in geometry. The problems that he said were solvable by these methods included the following (Poncelet 1817, p. 154):

> Deux section coniques ètant tracées su un même plane, construire un polygone de tant de côtés qu'on voudra qui soit, à la fois, inscrit à l'une d'elles et circonscrit à l'autre, en ne faisant usage que de la règle seulement [given two conics in traced in the same a plane, it is asked to construct a polygon, inscribed in one of them and circumscribed about the other, having whatever number of sides making use only of the ruler].

The formulation of the problem does not allude at all to the existence of a closure theorem, but rather suggests the existence of such a polygon for any pair of given conics. This fact led G. Loria to doubt that Poncelet knew the PCT much before the publication of the *Traité* (Loria 1889a, pp. 9–10).[26] For this reason, many historians were unaware that Poncelet had achieved the proof of the theorem in 1813.

It is worthwhile stressing that, although in the *Traité* Poncelet did not explicitly express the link between PCT and the existence of conditional equations, such as

[26] This is curious because, in footnote n.4 Loria quoted the first volume of *Applications d'analyse et de géométrie*, where the general theorem is stated and proved, and on its title page is written: "Sept cahiers manuscrits rédigés à Saratoff dans les prisons de Russie (1813–1814)."

formula (1.1) or those given by Fuss and Steiner, he had been clear about this since his imprisonment in Saratov. In fact, in Poncelet (1862, pp. 357–358) he affirmed that if a polygon of n sides is inter-scribed to two circles of radii R, r, an equation $f(R, r, a; n) = 0$, being a the distance between the centers, must necessarily hold.

2.1 Poncelet's methods

Poncelet's geometry always concerned the real plane and real space extended with the elements at infinity. In this setting, he aimed at deducing properties for systems of lines and conics by projectively generalizing properties proved for systems of lines and circles.

With this goal in mind, during his captivity Poncelet developed systematically the method of central projection, introduced by Brianchon (1810) and that he based on five fundamental principles.

The first three, which he gave without proof, being already largely accepted by geometers, affirm the projective equivalence of a circle and a conic, and the projective equivalence of a pencil of parallel lines and a pencil of intersecting lines.

The last two, which are more delicate, state, respectively: the projective equivalence of a system of a conic and a line at finite distance with a system of one circle and the line at infinity, and the projective equivalence of a system of two conics with the system of two circles [(see Poncelet 1862, vol. 1, p. 122; pp. 287–307; pp. 380–388) also (Poncelet 1822, Art.s 109–111; 121)]. Precisely:

Theorem (4th principle) *Let C be a conic and l be a line in the real plane at finite distance. Then, C and l are projective images of a circle and of the line at infinity.*

Theorem (5th principle). *Any pair of conics is the projective image of a pair of circles.*

As already remarked, Poncelet considered only real objects in real planes and spaces. For this reason, he was able to prove the fourth and fifth principle only for certain positions of the figures involved. So, in order to extend their validity for all positions of the elements, even when some of them disappear becoming "imaginary," he invoked the principle of continuity.

Before we discuss Poncelet's proofs of the last two principles, it is useful to recall his concept of "ideal chord."

The concept of "ideal chord"

In his early studies at the *École Polytechnique*, Poncelet had already met imaginary elements in connection with the intersection of a conic and a line, or the intersection of two conics. To take into account these elements, without actually extending the real plane with complex points,[27] he introduced the concept of "ideal chord."

Let C be a (real) conic, and let l be a (real) line which intersects C in two points P_1 and P_2 (Fig. 8a). The segment P_1P_2 is the chord of C corresponding to l. Let d be

[27] Complex projective geometry entered into the mathematical scene only around the middle of the nineteenth century.

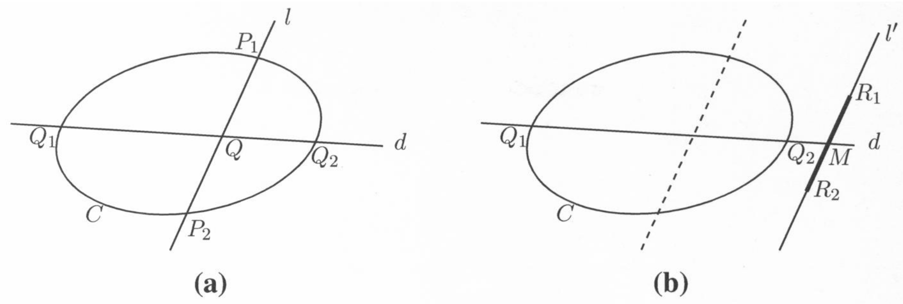

Fig. 8 **a** The real chord P_1 P_2 of the conic C. **b** The ideal chord R_1 R_2 of the conic C

the diameter of C conjugated to l, let Q_1, Q_2 be the points of intersection of C with d, and say Q the intersection $l \cap d$. Then, the one has

$$(QP_1)^2 = c(Q_1Q)(Q_2Q),$$

where c is a real number that does not change for any lines parallel to l intersecting C.

When l' is a line parallel to l, but "exterior" to C (Fig. 8b), Poncelet, unaware of the complex elements, associated with l' and C not the pair of conjugate points that constitute the intersection $l' \cap C$ in the complex projective plane, but a segment in the real plane. He did this by simply extending the above relation to the present case, precisely: he let $M = d \cap l'$ and let R_1 and R_2 be the two points on l' such that

$$(MR_1)^2 = c(Q_1M)(Q_2M),$$

and $MR_1 = MR_2$. Then, Poncelet defined the segment R_1R_2 to be the *ideal chord* intercepted by (the *ideal secant*) l' on C.

In Poncelet's mind, the ideal chord was a "real" justification for the "imaginary." His thinking is well expressed in (Poncelet 1822, Art.s 50, 54):

> En supposant qu'on ne veuille pas créer des termes nouveaux pour désigner la droite mn [l] et ce qui lui appartient, et qu'on persiste à la regarder comme une secante de la courbe quand elle cesse de la rencontrer, nous dirons, a fin de conserver l'analogie entre les idées et le langage, que ses points d'intersections avec la courbe, et par conséquent la corde correspondente, sont *imaginaires*, qu'elle est elle-même sécante idéale de cette courbe... et on pourra regarder $M'N'$ [R_1R_2] comme représentant, d'une manière fictive, la corde imaginaire qui correspond à la droit $m'n'$ [l'] considérée comme secante de la courbe [Let us suppose that we do not want to create new terms for the line l and for what pertains to it and that we continuing to look at this line as a secant of the curve even when the intersection no longer exists; we will say, with the objective of preserving the analogy btween the ideas and the language, that the points of intersection with the curve, and consequently the corresponding chord, are *imaginary*, it is itself an ideal secant of that curve ...and we can look at the segment R_1R_2 as

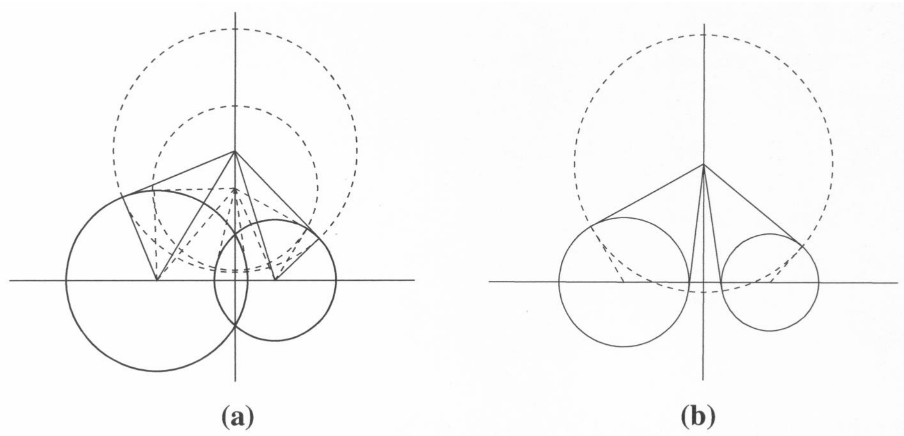

Fig. 9 Poncelet defined the radical axis of a pencil of circles as the locus of points from which tangent segments drawn to all circles of the pencil have equal length. **a** The construction for the case of intersecting circles and **b** the construction in case of not intersecting circles

a fictional representative of the imaginary chord corresponding to l' considered as a secant of the curve].

Let us observe that if $T_1 = (a + ib, c + id)$, $T_2 = (a - ib, c - id)$ are the two complex conjugate intersection points of l' and C in the complex plane, then we can write $T_k = S \pm iD$ where $S = (a, c)$ and $D = (b, d)$ represented real points in the plane. It is easy to see that $R_1 = S + D$ and $R_2 = S - D$ and S is the middle point of the segment $R_1 R_2$. Vice versa it is also clear that from R_1 and R_2 one can find T_1, T_2.

So, by the concept of ideal chord, Poncelet was able to develop many features that nowadays are introduced by embedding the real plane into the complex one. In particular, he showed that two (real) conics always have two common chords (real or ideal), i.e., two conics always intersect in four points (real or imaginary) (Poncelet 1822, Art.s 58–59).

Let us to stress that Poncelet was aware that endpoints and chords may lie on the line at infinity. In particular, he noticed the existence of the two points at infinity through which all circles pass, i.e., of those points that later were called *circular points* (Poncelet 1822, Art. 94).

Poncelet introduced the concept of a "system of conics having two common chords" (real or ideal), which corresponds to the modern concept of a pencil of conics. In particular, a pencil of circles was defined by Poncelet as the system of circles having a real or ideal common chord. He also defined the *radical axis* of a pencil of circles as the locus of points from which tangent segments drawn to all circles of the pencil have equal length (Poncelet 1822, Art.s 71–77).[28] This allowed Poncelet to extend the concept of radical axis to the case of non-intersecting circles (Fig. 9a, b); in fact, this

[28] This definition, as Poncelet remarked, had already been introduced in Gaultier (1813). J. Steiner called it *line of equal power* (Steiner 1826). Defining *power of a point P with respect to a circle* the quantity $h = d_P^2 - r^2$, where d_P is the distance of P from the center of the circle of radius r, the radical axis of a pencil of circles is the locus of points P whose power is the same with respect to all circles of the pencil.

line coincides with the real line on which lies the real or ideal common chord of all circles of the pencil.

Poncelet's "principle of continuity"

In its essence, the principle of continuity may be stated as follows: if for a figure in the plane certain properties have been deduced from the given data and theorems, these properties remain valid after a continuous deformation of the figure, even if during the deformation some aspects of the figure disappear becoming imaginary.

This principle, that "was admitted without saying by many geometers" as Poncelet recalled,[29] enabled him to avoid the use of the "imaginary," and to deduce results as if he was working into the complex projective space.

He formulated the principle of continuity in various ways and discussed it at length in several places of his treatise (see (Poncelet 1822, pp. xiii–xiv; Art.s 135–140)). When in Saratov, Poncelet expressed this principle as follows (Poncelet 1862, p. 379):

> Quand on se proposera de découvrir quelque propriété générale de position d'une figure, on pourra imaginer que cette figure soit projetée sur un nouveau plane (d'après la manière indiquée ci-après), de telle sorte qu'ne ou plusieurs parties de cette figure soient réduites à des circonstances plus simples; on aura ainsi une nouvelle figure qui pourra remplacer la première, sinon pour toutes les dispositions possibles au moins en général; on raisonnera sur cette figure comme tenant lieu de la première d'où l'on est parti, et les propriétés, les conséquences géérales qu'on en déduira seront également applicables à cette figure, quoiqu'il arrive des cas où la projection soit imaginaire [When it is proposed to discover certain general position properties of a figure, we can imagine that figure projected on a new plane (in the way specified here below), so that one or several parts of it are reduced to more simple situations; thus, we will have a new figure that can replace the first, if not in all cases at least in general; we will argue on this figure as on the first from which it came, and the properties, the general consequences that can be deduced will be equally applicable to this figure, even if obtained by an imaginary projection].

We clarify Poncelet's use of the principle of continuity by shortly discussing his proofs of the fourth and fifth fundamental principles mentioned above.

To prove the fourth principle, Poncelet proceeded as follows.

Let C and l be, respectively, a conic and a line in a plane π, which is thought of as immersed in real space. If l does not intersect C, one can find a point $S \notin \pi$ and a plane π' such that, under the projection from S in π', the conic C is mapped onto a circle and l onto the line at infinity. In Saratov, Poncelet gave an analytical proof of this fact, while in the *Traité* he provided a synthetic proof (Poncelet 1822, Art.s 109–111).

If l intersects C, such S and π' cannot be found in the real space, then Poncelet claimed, on the basis of the principle of continuity, the validity of the fourth principle also in this case.

[29] See Poncelet (1862, p. 124). The principle was used only occasionally until G. Monge revived it establishing certain theorems of descriptive geometry (Kline 1972, pp. 163–165). A similar principle was also used by Carnot in his *Géométrie de position* (1802).

Poncelet proceeded to prove the fifth principle similarly.

If C_1 and C_2 have at most two real points in common, then they have an ideal common chord along a certain line l not intersecting them. Hence, according to the fourth principle, C_1 and l are the projective image of a circle C and of the line at infinity l_∞. Then, the projective image of C_2 passes through $C \cap l_\infty$ (i.e., the circular points) and so is a circle.

If the two conics have more than two real points in common, this program is not realizable by a real projection, in fact two circles with three points in common coincide. Then, Poncelet extended the validity of the fifth principle by invoking the principle of continuity.

Poncelet enounced and proved also the following (Poncelet 1822, n. 131)

Theorem B *Two conics C_1, C_2 which are tangent to each other in two different points, are the projective image of two concentric circles.*

His reasoning was as follows. By the fifth principle, the two conics are the projective images of two circles and, at the same time, the two points of contact are the images of the circular points. Because the two circles are tangent to each other at the circular points, the line at infinity has the same pole with respect to them, and so they are concentric.

Let us remark that the fourth and fifth principle and theorem B are actually correct in the extended complex plane, so, ultimately the principle of continuity led Poncelet to correct results.

It can be observed (see Bos et al. 1987, p. 303) that these three results are successively more counterintuitive, in the context of the real geometry. The first two can still be seen to be correct in certain cases, but the third does not apply at all in the real case: two real circles which are tangent one each other in two points coincide. This shows how daring Poncelet's use of ideal chords and of the principle of continuity really was.

2.2 The analytical proof PGT

Poncelet first considered the case of two circles (C) and (C'), of centers c and c' and radii r and R, respectively (the first lying inside the second). He denoted by a the distance between the two centers and wrote the equations of the circles as:

$$(C): \quad x^2 + y^2 = r^2, \quad (C'): \quad (x-a)^2 + y^2 = R^2. \tag{2.1}$$

From a point $\alpha := (\alpha, \beta)$ on (C'), so that

$$\beta^2 + (\alpha - a)^2 = R^2, \tag{2.2}$$

he drew two chords of (C') tangent to (C), say αx_1 and $\alpha x''$ (Fig. 10a), and showed that, if α moves along (C'), then the chord $x_1 x''$ varies and envelopes a circle (C'') from the pencil determined by (C) and (C'). Poncelet proved this result by a direct laborious computation. He first found the equations of αx_1 and $\alpha x''$, then he obtained the equation for $x_1 x''$:

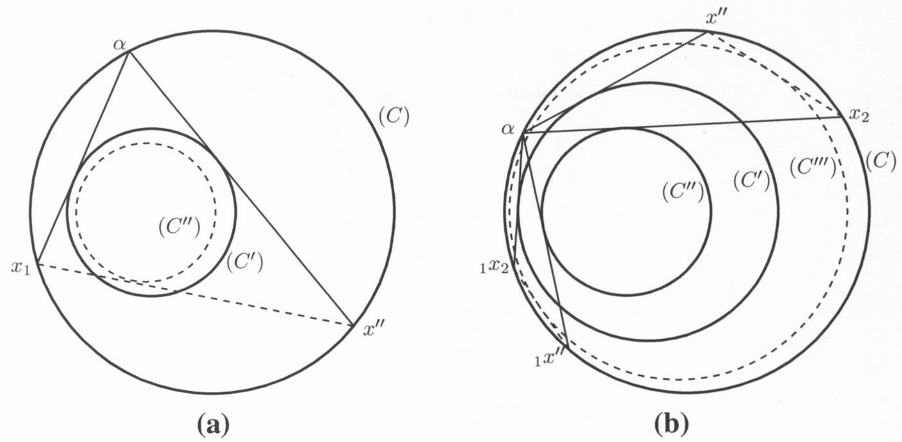

Fig. 10 Illustration of how Poncelet proceeded in order to prove his general theorem in case of two circles (**a**) and of three circles (**b**)

$$\beta(R^2-a^2)y+[a(R^2-a^2)+\alpha(R^2+a^2)]x+(R^2-a^2)-2R^2r^2+a(R^2-a^2)\alpha=0.$$

Taking into account the relation (2.2), he was able to eliminate α and β from this equation, and for (C'') he obtained the following equation:

$$\begin{aligned} &x^2(R^2-a^2)^2+y^2(R^2-a^2)^2-2a\left[(R^2-a^2)^2-4R^2r^2\right]x \\ &\quad = R^2(R^2-a^2-2r^2)^2-a^2(R^2-a^2)^2 \end{aligned} \tag{2.3}$$

(Poncelet 1862, pp. 314–319). Then, Poncelet proved that (C'') belongs to the pencil determined by (C) and (C') by showing that the ideal common chord of (C) and (C'') coincides with the ideal common chord of (C') and (C'') (Poncelet 1862, pp. 319–320).

Let us observe that the equation (2.3) reduces to that of (C) when one puts $a^2 = R^2 - 2rR$.

After having examined certain special cases in which (C'') may degenerate, Poncelet proceeded to the case of three circles (C), (C') and (C'') from a same pencil (Poncelet 1862, pp. 323–339).

He represented these circles by the following equations

$$(C): \ x^2+y^2=R^2, \quad (C'): \ (x-a)^2+y^2=r^2, \quad (C''): \ (x-a')^2+y^2=r'^2.$$

From a point $\alpha := (\alpha, \beta)$ on (C) he drew four chords of (C), say $\alpha x''$, αx_2, $\alpha_1 x''$ and $\alpha_1 x_2$, the first and the third tangent to (C'), the second and fourth tangent to (C'') (Fig. 10b).

Through extremely long and involved calculations, Poncelet was able to find the equation of the chord $x''x_2$,[30] but it proved too complicated to admit calculating the envelope by eliminating the parameters α, β.

[30] See for instance pp. 327, 330, 336 where the formulae to be written down require sheets folding out to the width of some quarto pages.

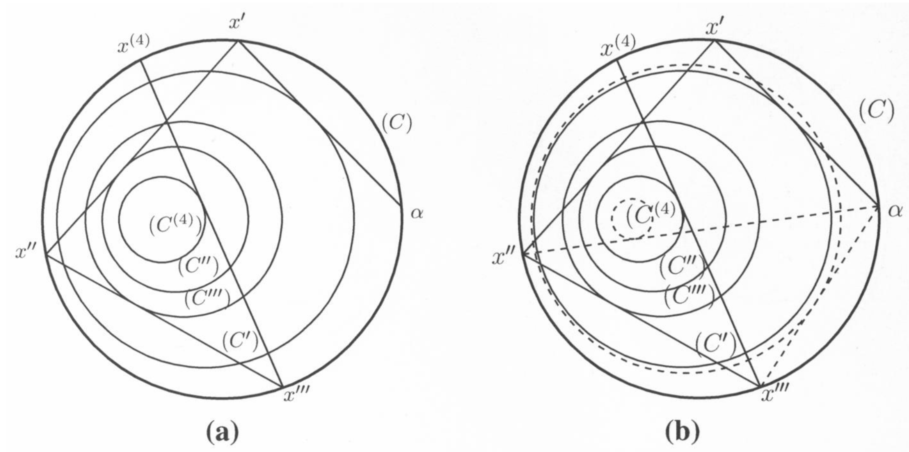

Fig. 11 Illustrations of how Poncelet applied the *main lemma* in order to prove the general theorem for $n > 3$

Then, he proved the existence of a point on the line of centers, whose distance from the chord $x''x_2$ is the same whatever is the point α on (C), and therefore, that the enveloped curve of this chord is a circle (C'''), for which he got the following equation

$$\left(x - \frac{R^2(a-a')\left(r' - r\sqrt{\frac{a'}{a}}\right)}{(R^2 - aa')\left(r' + r\sqrt{\frac{a'}{a}}\right)} \right)^2 + y^2 = \frac{R^2\left[(R^2 - a'^2) + r'(R^2 - a^2)\right]^2}{(R^2 - aa')^2\left(r' + r\sqrt{\frac{a'}{a}}\right)^2}.$$

Finally, arguing as in the previous case, he proved that (C''') is from the same pencil that (C), (C') and (C'') belong to. He also showed that the chord ${}_1x_1''x_1$ envelops the same circle (C''').

We will call this result *main lemma*.

At this point, Poncelet generalized these results to the case of $n > 3$ circles.

He considered n circles $(C), (C'), (C''), \ldots, (C^{(n-1)})$ from a same pencil $\mathfrak{F}$, and a transversal $\alpha x'x''x''' \ldots$ inscribed in (C), and whose sides are tangent, in some order, to the inner circles $(C'), (C''), \ldots, (C^{(n-1)})$. Clearly, without loss of generality, one can suppose that $\alpha x'$, $x'x''$, $\ldots$, $x^{(n-2)}x^{(n-1)}$ are, respectively, tangent to $(C'$, (C''), $\ldots$, $(C^{(n-1)})$ (see Fig. 11a). By the previous result, the chord $\alpha x''$ envelops a circle from $\mathfrak{F}$. Similarly, since the chord $\alpha x''$ and the chord $x''x'''$ are tangent to two circles from $\mathfrak{F}$, the chord $\alpha x'''$ envelops a circle from $\mathfrak{F}$, and so on (see Fig. 11b). Clearly, $x^{(n-1)}\alpha$ envelops a circle from the pencil $\mathfrak{F}$.

Poncelet used projection and the principle of continuity to extend these results to conics, but we will discuss this generalization later on, when we will comment the proof of PCT he gave in Poncelet (1822).

After having proved PGT, Poncelet enounced the closure theorem as follows (Poncelet 1862, p. 355):

Il est impossible, généralement parlant, d'inscrire à une courbe donnée du deuxiè-me degré un polygone qui soit en même temps circonscrit à une autre courbe de ce dedré, et quand la disposition particulière de ces courbes sera telle que l'inscription et la circonscription simultanées soient possibles pour un seul polygone essayé à volonté, il y aura, par la même, une infinité jouissant de cette propriété à l'égard des coniques données. Pour démontrer ce théorème directement, soient deux ligne quelconque du second degré; d'après nos principes, ces ligne pourront, en général, être projectées suivant deux circonférences de cercle, bie que, dans des cas particuliers cela puisse devenir illusoire... [In general it is impossible to inscribe in a conic a polygon which is at the same time circumscribed to another conic, but when, for the particular disposition of the two conics, it can be proved that this is possible for a particular polygon, then there will exist infinitely many polygons having the same property with respect to the given conics. For a direct proof of this theorem, let two curves of second degree be given, from our principles these curves can, in general, be projected onto two circles, although in some cases this may become illusory...].

Poncelet considered two circles, (C) and (C') (the second lying entirely in the interior of the first), and a polygon $\alpha, x, x', x'', \ldots, \alpha'$ inscribed in (C), whose sides are all tangent to (C') except $\alpha'\alpha$, which will be tangent only for particular positions of the two given circles. Then, in order to prove PCT, he reasoned as follows.

Suppose that one can deform the polygon, maintaining the same number of sides, in such a way that it assumes all possible positions around (C') while remaining inscribed in (C). If, by chance, there exists a polygon of the same number of sides which is inter-scribed to (C) and (C'), it is evident that there is a position of $\alpha, x, x', x'', \ldots, \alpha'$ in which $\alpha'\alpha$ is tangent to (C'). This is impossible, unless $\alpha'\alpha$ is tangent to (C') for any position of the polygon. In fact, $\alpha'\alpha$ envelops a circle (C'') (from the same pencil of (C) and (C')) and if, for a certain position of the polygon, $\alpha'\alpha$ is tangent to (C'), then (C'') necessarily coincides with (C') and all polygons $\alpha, x, x', x'', \ldots, \alpha'$ will be inter-scribed to the two circles (C) and (C').

Hence, on the basis of the fundamental principles, Poncelet concluded (Poncelet 1862, p. 357):

on peut conclure des principes posés au commencement du IIIe Cahier, que la double proposition d'abord énoncée est vrai quelle que soit la situation relative des deux cercles donné (C) e (C') [from the principles given at the beginning of the third Notebook, it follows that the proposition stated above holds true for every mutual position of the two given circles (C) e (C')]

and finally, he extended the validity of the theorem to the case of conics by projection (Poncelet 1862, p. 364):

Le deux cercles (C) et (C') peuvent être considérés comme la projection de deux courbes quelconques du second degré, au moins en général; car, pour des positions particulières de ces courbes, la projection peut devenir imaginaire, impossible geométriquement [the two circles (C) et (C') can be considered as the projection of two arbitrary curves of second degree, at least in general,

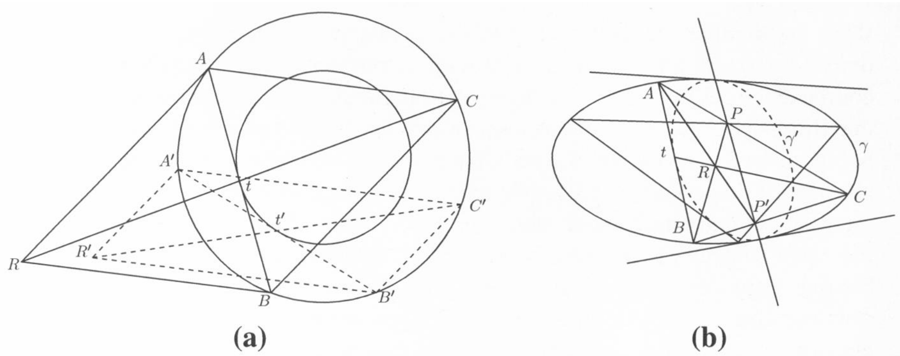

Fig. 12 Poncelet first proved Proposition 1 for two concentric circles (**a**), then, by applying the principle of continuity, he extended the proposition to the case of two bitangent conics (**b**)

because, for particular positions of these curves, the projection may become imaginary, geometrically impossible].

2.3 The synthetical proof of PGT

Poncelet presented the proof of PCT in Art. 534 of his treatise, as a corollary of the PGT. Much of the proof of this theorem rests on the following proposition that he inserted in Art. 531[31]:

Proposition (Main lemma) *Let* (c), (c'), (c'') *be three circles having a common chord* mn, *real or ideal (i.e., from the same pencil* $\mathfrak{F}$. *Let* ABC *be a triangle inscribed in* (c'') *whose sides* AB *and* AC *are tangent, respectively, to* (c') *and* (c). *If we move* A *on* (c'') *in such a way the sides* AB *and* AC *remain tangent, respectively, to* (c') *and* (c), *then the third side* BC *of the triangle will envelop a circle* (c''') *having the same common chord with the given circles (i.e., from the pencil* $\mathfrak{F}$*)*

As we have seen, Poncelet had already produced an analytical proof of the first part of this proposition in the notebooks of Saratov. In the *Traité*, in accordance with the spirit of the whole book, he wanted to give a synthetical proof.

Before we proceed to discuss this proof, it is useful to briefly review what Poncelet had shown in the previous Art.s 431–439 of his treatise.

In Art. 431, he stated the following:

Proposition 1 *Let* ABC *be an inscribed triangle to a given conic* γ. *If* C *moves along* γ *in such a way that the triangle remains inscribed in* γ *and the sides* CA *and* CB *rotate, respectively, around two fixed points* P *and* P', *arbitrarily chosen on them, then the side* AB *will envelop a conic* γ' *which is bitangent to* γ *at the two points (real or ideal) where the line* P, P' *meets* γ. *Moreover, if* t *is the point where* AB *touches* γ', *then* PB *and* $P'A$ *met in a point* R *on* Ct *(see Fig. 12b).*

[31] Here and in the following we maintain Poncelet's notation.

To prove it, Poncelet first considered (Art. 433) the simpler case of concentric circles. Let (C) and (C') be concentric circles with center in O, the second lying inside the first (Fig. 12a). Suppose ABC is a triangle inscribed in (C) with the side AB touching (C') at t. If C moves along (C), in such a way that the sides CA and CB remain parallel to the original direction, then the side AB rotates around (C'). This is evident, claimed Poncelet, because all the chords AB have the same length and so have the same distance from O. Moreover, considering the parallelogram $ABCR$ with $AR//BC$ and $BR//AC$, it follows that the line Ct passes through R.

To complete the proof Poncelet argued as follows. By theorem B, the conic γ and the line through P, P' are the projective image of a circle and of the line at infinity. In this way, the present configuration (Fig. 12b) is the projective image of the previous one (Fig. 12a), and then, the claim follows from what has been proved above for two concentric circles.

Unfortunately, the assumption that P, P' are real points is not correct, in fact they are the images of the circular points, and, as known, under a projective map, either all the real points of a line have real images, or at most two of them have real images. So in general P and P' will not be real (see Bos et al. 1987, p. 308).[32]

In Art. 434, Poncelet stated the reciprocal of the proposition above[33]:

Proposition 2 *Let ABC be a triangle inscribed in a conic γ. If C moves along γ so that the triangle remains inscribed in γ, the side AB moves remaining tangent to a same conic γ' having a double contact with γ along the direction TT', and the side AC rotates around a fixed point P placed on TT', then the third side will rotate around another point P' placed on TT'.*

Then, he observed (Art. 437) that the conic γ, enveloped by the side AB, reduces to the point O, pole of the line PP', when the points P and P' are placed so that the polar line of one of them passes through the other.

By applying propositions 1 and 2, in Art. 439 Poncelet proved the following generalization:

Proposition 3 *If a triangle inscribed in a conic moves, remaining inscribed in it, so that a first side passes through a fixed point, and a second side envelops a conic having a double contact with the first, then the third side will envelope another conic having double contact with the first.*

We will return on these propositions later on, when we will discuss certain generalizations due to A. Cayley.

Now let see how Poncelet proved the *main lemma*.

First of all he found the point A' at which CB touches the enveloped curve. He supposed the given circles have a real common chord MN on the line mn (see Fig. 13).

[32] According to Bos et al. (1987, p. 308), it is possible to prove this part of the theorem along the lines implicit in Poncelet's approach, by using cross-ratios to generalize the concept of parallelism and angles from the real to the complex case. This procedure, however, being rather laborious, these authors offered, in section 8.5 and specifically lemma 8.5 of their work, a modern alternative proof by means of closed conditions on Zariski-dense sets.

[33] This proposition will be useful in section five.

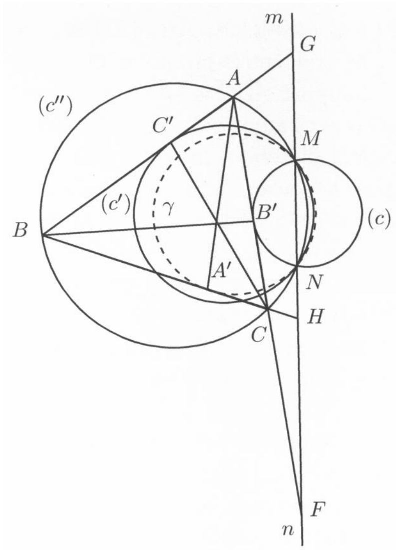

Fig. 13 Illustration of how Poncelet proceeded in order to prove the *main lemma*

He observed that an infinitesimal displacement of the triangle ABC causes an infinitesimal displacement of its sides, which, he affirmed, may be considered the same as the displacement that would occur if the points of contact B' and C' of the chords AB and AC with (c) and (c') were fixed and the chords rotate around them.[34] Then, by what he had observed above, it follows that the chord BC envelopes a conic γ having a double contact with (c''). Hence, the point A', where BC touches γ, is the same point where it touches the curve unknown (c'''). Therefore, he concluded, if D is the point of intersection of BB' and CC', the line AD will intersect AB in the point A'.

At this point, Poncelet considered the intersection points F, G, H of the line mn, respectively, with the lines AC, AB and BC. From well-known properties of the circle, he deduced that:

$$\overline{FB'}^2 = FM \cdot FN = FA \cdot FC,$$
$$\overline{GC'}^2 = GM \cdot GN = GA \cdot GB.$$

From these relations, by applying what he had already shown in Art.s 162–163, Poncelet deduced that

$$\overline{HA'}^2 = HB \cdot HC = HM \cdot HN$$

holds true. Finally, he remarked that this relation characterizes the circles passing through M and N. Thus, he had proved the proposition in case of circles having real intersections.[35]

[34] Poncelet gave no further argument. His reasoning can be made rigorous by using the modern theory of deformation (see Bos et al. 1987, section 8).

[35] For a deeper analysis of the arguments exposed in this subsection, we refer to Bos et al. (1987 sections. 4, 8).

For extending the proof to circles in any position, Poncelet used the principle of continuity as follows:

> Ce raisonnement suppose, il est vair, que les cercles (c), (c') et (c'') aient deux points communs réels; mais, en vertu du principe de continuité, on peut l'étendre directement à celui où ces poins deviennent imaginaires, et où par conséquent la droit *mn* est une sécante idéale commune aux cercles proposés. Ainsi notre théorème est général et comprend tous les cas... [This reasoning supposes, of course, that the circles (c), (c') and (c'') have a real common chord; but, by the principle of continuity, it can be directly extended to the case in which the points of intersections are all imaginary, and the line *mn* is an ideal secant of the given circles. So our theorem is general and includes all cases...]

Poncelet proved the PGT, in Art. 534, and the PCT, in Art.s 565–567, in the same way he had done in the *Cahiers de Saratoff.*

The work of Poncelet was appreciated by Dupin, Hachette, Malus and others former students at École Polytechnique, and partly also by Gergonne, Cauchy and Chasles. But it was in Germany that Poncelet had major followers: Plücker, Steiner and Von Staudt.

In 1834, Poncelet was elected to the *Académie des sciences*. In the following years, he virtually abandoned geometry in favor of more experimental studies, in particular mechanics, a discipline in which he pointed out the central role of geometry.

The year after his death the *Poncelet Prize* was established. It was to be awarded by the French Academy of Sciences for the advancement of the science. Darboux and Halphen were among the first recipient of the prize.

3 Jacobi and the use of the elliptic functions

In the extended *Note historique* in Poncelet (1862, pp. 480–498), Poncelet traced the history of the research on polygons inter-scribed to two conics in the 40 years since 1822. He recalled that Carl Gustav Jacob Jacobi went to Paris in 1829, shortly after the publication of Jacobi (1828) and that he had several meetings with him, during which they exchanged ideas on that subject. At page 485, we read:

> M. Jacobi... m'apprit qu'au début de ses études sur ce suject, il avait aussi imaginé de faire varier un tel polygone, d'une quantité infiniment petite, de manière que les arcs élémentaires décrits respectivement par les extrémités de l'un quelconque de ses côtes, divisés par la longueur des segments correspondantes formés sur sa direction, à partir du point de contact avec le cercle auquel il est tangent, représentaient autant de différentielles elliptiques de la première espèce, et fournissaient ainsi, par leur comparaison relative à chacun des côtés, autant d'équations distinctes, les mêmes qu'Euler Lagrange et Legendre avaient primitivement intégrées sous une forme rationelle, dans leurs mirables recherches sur la matière [Mr. Jacobi told me that at the beginning of his studies on this subject, he had also thought to move such a polygon by an infinitesimal displacement, in such a way that the elementary arches described by the extremities of each of

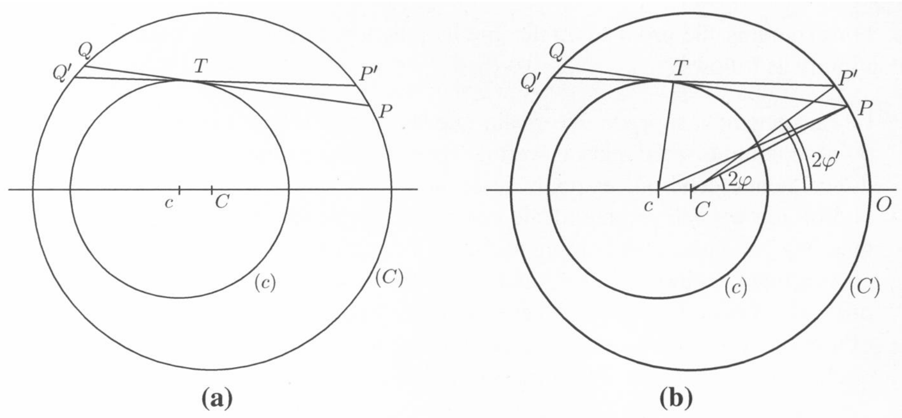

Fig. 14 Two steps of Jacobi's thought on the use elliptic functions in the proof of Poncelet's theorem in case of two circles, one inside the other. An infinitesimal displacement of the chord QP gives rise to an elliptic differential of the first kind. This idea was confirmed in the letter he wrote to Hermite on the 6th of August 1845

> its sides, divided by the length of the corresponding segments along its direction, represent as many elliptic differentials of first kind, which, by their relative comparison with each sides, give as many distinct equations, equal to those that Euler, Lagrange, and Legendre had previously integrated in rational form...].

All this seems to be confirmed by what Jacobi wrote to Hermite in a letter dated 6 of August 1845 (Jacobi 1846, pp. 178–179) (see Fig. 14a):

> Je suis aussi parvenu à étendre au théoréme d'Abel ma construction de l'addition des fonctions elliptiques. Dans cette dernière, la corde PQ d'un cercle touche constamment un autre cercle. Soit T le point d'intersection de deux positions consécutives de la droite; les deux angles $Q'QT$ at $PP'T$ étant égaux d'après une propriété du cercle, on aura $PP'/PT = QQ'/Q'T$ ce qui est l'équation differentielle, dont par la construction de la droite inscrite à l'un et circonscrite à l'autre cercle on trouve l'intégtrale complète et algébrique, la même qui à été donnée par Euler [I have been also able to extend my construction for the addition of elliptic functions to the theorem of Abel. In this latter case, the chord PQ of a circle constantly touch another circle. Let T be the point of intersection of two consecutive positions of the line; the two angles $Q'QT$ and $PP'T$ being equal by a property of the circle, one has $PP'/PT = QQ'/Q'T$ which is the differential equation, for which, by the construction of the line inscribed in one and circumscribed about the other circle, one finds the complete integral to be algebraic, the same that was given by Euler].

It is likely that Jacobi argued as follows (see Fig. 14b). The triangles PTP' and QTQ' are similar. So setting $\delta := PT$, $\mathrm{d}s := PP'$ and $\delta' = P'T$, $\mathrm{d}s' = QQ'$, it follows that:

$$\frac{\mathrm{d}s}{\delta} = \frac{\mathrm{d}s'}{\delta'} = \frac{2Rd\varphi}{\delta} = \frac{2Rd\varphi'}{\delta'},$$

where $2\varphi = \angle OCP$ and $2\varphi' = \angle OCP'$. If a denotes the distance between the centers C and c, one has $\delta^2 = \overline{Pc}^2 - r^2$. Therefore, since $\overline{Pc}^2 = R^2 + a^2 + 2Ra\cos 2\varphi$, one gets

$$\delta^2 = (R+a)^2 - r^2 - 4Ra\sin^2\varphi,$$

and hence

$$\int_0^s \frac{\mathrm{d}s}{\delta} = \int_0^\varphi \frac{2Rd\varphi}{\sqrt{(R+a)^2 - r^2 - 4Ra\sin^2\varphi}}.$$

Putting $k^2 = 4Ra((R+a)^2 - r^2)^{-\frac{1}{2}}$, this integral can be written

$$\frac{2R}{\sqrt{(R+a)^2 - r^2}} \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}$$

which is readily seen to be an elliptic integral of the first kind, of *modulus* k (see Legendre 1825).

Summing up: Jacobi was lead to consider the elliptic function amplitude by looking at "the rolling" of a side of the polygon on the inner circle.

From the historical note of Poncelet, we also learn that Steiner suggested to Jacobi the use of elliptic functions, in fact at page 481 we read:

> D'après ce qu'a bien voulu me faire savoir plus tard M. Steiner,... ce serait par ses encouragements propres, ses avis éclairés que Jacobi, ayant pris connaissance du *Traité des propriété projectives des figures*, aurait été conduit à appliquer la théorie des fonctions elliptiques à démonstration des théorèmes (p. 322 et suiv. de cet ouvrage) sur les polygones simultanément inscrits et circonscrits à plusieurs cercles [Later Mr. Steiner kindly informed me that it was through his enlightened advice and encouragement that Jacobi, when he learned of the *Traité des propriété projectives des figures*, was led to apply the theory of elliptic functions in order to prove theorems (p. 322 and following of this work) on polygons simultaneously inscribed and circumscribed about several circles].[36]

3.1 Jacobi's proof of PGT for circles

Let us see, in some detail, how Jacobi proved the general theorem of Poncelet. He considered two circles (C), (c), the second within the first, respectively, of centers C, c and radii R, r, and a polygonal line (or *transversal*) $AA'A''A'''\ldots$ inter-scribed to them (see Fig. 15). He put $\angle PCA = 2\varphi$, $\angle PCA' = 2\varphi'$, $\angle PCA'' = 2\varphi''$, etc. and denoted by a the distance between the two centers.

[36] From November 1822 to August 1824, Steiner attended courses at the University of Berlin. Jacobi, 8 years younger than Steiner, was at that time also a student in Berlin and soon they became friends.

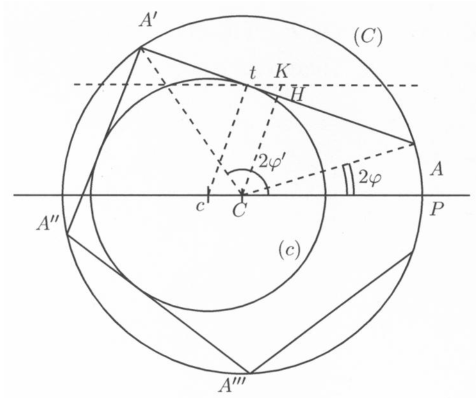

Fig. 15 Jacobi's procedure in order to prove PGT for circles

Then, one has the following equations

$$\begin{cases} R\cos(\varphi' - \varphi) + a\cos(\varphi' - \varphi) = r \\ R\cos(\varphi'' - \varphi') + a\cos(\varphi'' - \varphi') = r, \\ R\cos(\varphi''' - \varphi'') + a\cos(\varphi''' - \varphi'') = r, \\ \quad\vdots \end{cases}$$

which can be put in the form

$$\begin{cases} (R+a)\cos\varphi'\cos\varphi + (R-a)\sin\varphi'\sin\varphi = r, \\ (R+a)\cos\varphi''\cos\varphi' + (R-a)\sin\varphi''\sin\varphi' = r, \\ (R+a)\cos\varphi'''\cos\varphi'' + (R-a)\sin\varphi'''\sin\varphi'' = r, \\ \quad\vdots \end{cases} \tag{3.1}$$

By subtracting each of these equations from the following one, since $\frac{\cos x - \cos y}{\sin y - \sin x} = \tan\left(\frac{x+y}{2}\right)$, he got the following system of equations

$$\begin{cases} \tan\left(\frac{\varphi''+\varphi}{2}\right) = \frac{R-a}{R+a}\tan\varphi', \\ \tan\left(\frac{\varphi'''+\varphi'}{2}\right) = \frac{R-a}{R+a}\tan\varphi'', \\ \quad\vdots \end{cases}$$

about which, at page 35 of his paper, Jacobi wrote:

> In dieser Form der Gleichungen springt es sogleich in die Augen, dass sie mit denjenigen übersinkmmen, welche sur Vervielfachung der elliptischen Transcendenten aufgestellt werden [In this form, it is plain to see that these equations are the same as those for the multiplication of elliptic transcendentals].

With this in mind, he considered the elliptic integral of first kind

$$u = F(\varphi) = \int_0^{\varphi} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}, \tag{3.2}$$

with its inverse function (*amplitude*)

$$\varphi = \mathrm{am}(u).$$

Then, chosen any angle $\alpha := \mathrm{am}(t)$, he put $\varphi' = \mathrm{am}(u + t)$, $\varphi'' = \mathrm{am}(u + 2t)$, and from the basic results on elliptic functions [see for instance (Legendre 1825, pp. 19–25)] he deduced

$$\tan\left(\frac{\varphi'' + \varphi}{2}\right) = \Delta\mathrm{am}(t) \tan \varphi', \tag{3.3}$$

where $\Delta\mathrm{am}(t) := \sqrt{1 - k^2 \sin^2 \alpha}$.

This correlation suggested to Jacobi the possibility of determining k in (3.2) in such a way that the successive values of $\varphi, \varphi', \varphi'' \ldots$ correspond to the vertices $A, A', A'', \ldots$ of the polygonal line. He proceeded as follows.

From another basic relation in the theory of elliptic integrals (Legendre 1825, p. 19), for $\varphi = \mathrm{am}(u)$, $\varphi' = \mathrm{am}(u + t)$ and $\alpha = \mathrm{am}(t)$, one has

$$\cos \varphi \cos \varphi' + \sin \varphi \sin \varphi' \Delta\mathrm{am}(t) = \cos \alpha,$$

and since equation (3.1) can be put in the form

$$\cos \varphi \cos \varphi' + \sin \varphi \sin \varphi' \frac{R - a}{R + a} = \frac{r}{R + a},$$

by comparing the last two equations he obtained

$$\cos \alpha = \frac{r}{R + a}, \quad \text{and} \quad 1 - k^2 \sin^2 \alpha = \frac{(R - a)^2}{(R + a)^2},$$

which yield

$$k^2 = \frac{4Ra}{(R + a)^2 - r^2},$$

and consequently

$$R = \frac{r(1 + \Delta\mathrm{am}(t))}{2 \cos \alpha}, \quad a = \frac{r(1 - \Delta\mathrm{am}(t))}{1 + \Delta\mathrm{am}(t)}, \quad r = \frac{2R \cos \alpha}{1 + \Delta\mathrm{am}(t)}.$$

Jacobi observed that the quantities k and α do not depend on φ and u. This is very important, as Jacobi remarked, because in this way, starting from any point A on (C) and $\angle ACP/2 = \varphi = \mathrm{am}(u)$, $\angle A'CP/2 = \varphi' = \mathrm{am}(u + t)$, the line AA' is tangent to the circle determined by a and r as above. Moreover, the line AA'' is tangent to the circle determined by

$$a = R\frac{1-\Delta^{(2)}}{1+\Delta^{(2)}}, \quad r = \frac{2R\cos\alpha^{(2)}}{1+\Delta^{(2)}},$$

where

$$\alpha^{(2)} = \mathrm{am}(2t), \quad \Delta^{(2)} = \sqrt{1-k^2\sin^2\alpha^{(2)}},$$

and in general the line $AA^{(n)}$, which closes the polygonal line, will be tangent to a circle determined by

$$a = R\frac{1-\Delta^{(n)}}{1+\Delta^{(n)}}, \quad r = \frac{2R\cos\alpha^{(n)}}{1+\Delta^{(n)}},$$

where

$$\alpha^{(n)} = \mathrm{am}(nt), \quad \Delta^{(n)} = \sqrt{1-k^2\sin^2\alpha^{(n)}}.$$

Clearly, all these circles have centers on the line CP.

Here, Jacobi made the crucial remark: all the circles belong to the same pencil determined by (C) and (c), and all lead to the same modulus k:

> Wir wollen jetzt beweisen, dass diese Kreise ein System bilden, welche dieselbe Linie zum Orte der gleichen Tangenten haben, welche zweckmässige Benennung Herr Steiner in einen geometrischen Arbeiten in diesem Journal eingführt hat [Let us prove that all these circles form a system of circles having the same line as locus of *equal tangents*, a suitable name introduced by Mr. Steiner in a geometrical paper in this journal].[37]

To prove this, he reasoned in the following way (see Fig. 16).

For a point Q on the line CP let $d = QC$, so that its distance from c is $d - a$. Then, the tangential distances of Q from the two circles (C) and (c), are, respectively, $\sqrt{d^2 - r^2}$ and $\sqrt{(d-a)^2 - r^2}$). Comparing these, he deduced

$$d = \frac{(R+a)^2 - r^2}{2a} - R,$$

and, since

$$\frac{(R+a)^2 - r^2}{2a} = \frac{2R}{k^2},$$

he finally found

$$d = \frac{2R}{k^2} - R.$$

[37] See Steiner's paper *Einige geometrische Betrachtungen* (J. für die reihe und Ang. Math., 1 (1826), 161–184), p. 165. This line is the radical axis of the pencil. As already seen, for a system of circles to have the same line of equal distances or to have a common (real or imaginary) chord is the same they must belong to a same pencil. See the note at p. 386 in Jacobi (1828).

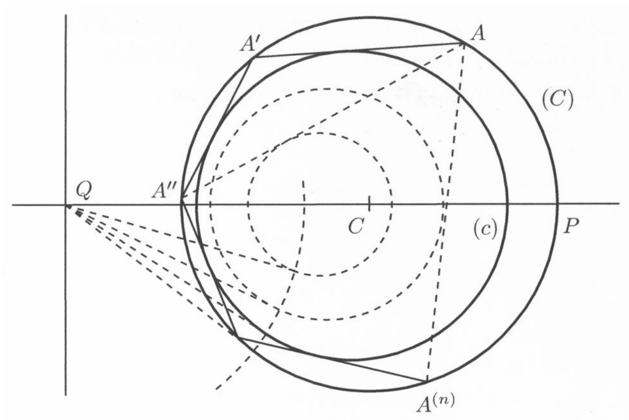

Fig. 16 In his construction, Jacobi found a series of circles that he proved belonging to a same pencil. To prove this he showed, following Steiner, that all these circles have the same line as locus of *equal tangents*

Then, Jacobi wrote:

> Wir sehen dass α in dem Ausdruck für d gar nicht vorkommt, sondern dass es bloss von k abhängt. Für alle jene Kreise aber ist dieses k dasselbe, und nur im α unterscheiden sie sich. Hätten wir daher für C [(C)] und irgend einen anderen Kreis den Ort ihrer gleichen Tangenten gesucht, se hätten wir denselben Ausdruck für D [d] gefunden, so dass also alle jene Kreise einen gemeinschaftlichen Ort der gleichen Tangenten haben [We see that α does not appear in the expression for d, which depends only on k. For all these circles, k is the same, and it changes only for α. Thus, if we had searched the locus of equal tangents for (C) and another circle, we will have found the same expression for d; hence, all these circles have the same locus of equal tangents].[38]

The perpendicular to CP passing through the point Q, whose distance from C is as above, is the so-called line of "equal tangents."

Jacobi was now in the position to prove the PGT.

He considered a sequence $(c), (c^{(1)}), \ldots, (c^{(n-1)})$ of circles, with centers $c, c^{(1)}, \ldots, c^{(n-1)}$ and radii $r, r^{(1)}, \ldots, r^{(n-1)}$, all lying within (C) and all belonging to the same pencil together with (C). He put $a^{(i)}$, $i = 1, \ldots, n-1$, the distance of $c^{(i)}$ from C, and determined the angles α's by

$$\cos\alpha = \frac{r}{R+a}, \quad \cos\alpha_i = \frac{r^{(i)}}{R+a^{(i)}}, i = 1, \ldots, n-1.$$

Jacobi remarked that k^2 is the same for all inner circles, and so that the same function "am" occurs for each of them. Then, he found $t, t^{(1)}, t^{(2)}, \ldots$ by putting $\alpha = \mathrm{am}(t)$,

[38] One may argue as follows. It can be supposed (C), D_1, D_2 be given, respectively, by $x^2 + y^2 = R^2$, $(x - a_1)^2 + y^2 = r_1^2$ and $(x - a_2)^2 + y^2 = r_2^2$. Then, the abscissas of the points of intersection are $x_i = (r_i^2 - R^2 - a_i^2)/2a_i$, and for circles belonging to the same pencil we must have $x_1 = x_2 = x$. From $k^2 = \frac{2R}{R - \frac{r^2 - R^2 - a^2}{2a}}$, it follows that $k^2 = \frac{2R}{R-x}$ which proves the claim.

$\alpha_i = \mathrm{am}(t^{(i)})$ for $i = 1, \ldots, n-1$. From a point A on (C), he drew the tangent $AA^{(1)}$ to the circle (c) (proceeding counterclockwise), then he drew the tangent $A^{(1)}A^{(2)}$ to $(c^{(1)})$, and proceeded similarly until he had drawn the tangent $A^{(n-1)}A^{(n)}$ to $(c^{(n-1)})$. Putting

$$\angle ACP = 2\varphi, \quad \angle A^{(i)}CP = 2\varphi^{(i)}, i = 1, \ldots, n,$$

he obtained

$$\varphi = \mathrm{am}(u), \quad \varphi^{(i)} = \mathrm{am}(u + t + \cdots + t^{(i-1)}), i = 1, \ldots, n.$$

Then, Jacobi observed that, with $s := t + \cdots + t^{(n-1)}$, the line $A^{(n)}A$ (which closes the transversal) envelops a circle, which is determined by

$$r^{(n)} = \frac{2R \cos \mathrm{am}(s)}{1 + \Delta \mathrm{am}(s)}, \quad a^{(n)} = \frac{R(1 - \Delta \mathrm{am}(s))}{1 + \Delta \mathrm{am}(s)},$$

being $r^{(n)}$ its radius and $a^{(n)}$ the distance of its center from C. This circle, being associated with the same modulus k, necessarily belongs to the same pencil that(C), (c), $(c^{(1)}), \ldots, (c^{(n-1)})$ belong to.

3.2 Jacobi's condition allowing inter-scribed n-gons to two circles

By using his approach, Jacobi was able not only to prove PCT but also to find a necessary and sufficient condition that allows the existence of a polygon, of any number n of sides, inter-scribed to two given circles(C) and (c), the second lying within the first. To do this, he proceeded as follows.

Suppose that starting from A_0 on (C) the transversal $A_0, A_0^{(1)}, \ldots, A_0^{(n)}$, constructed as above, closes, i.e., that $A_0 = A_0^{(n)}$, after having turned i times around c. Let $\angle PCA_0 = 2\varphi_0$ and $\angle PCA_0^{(n-1)} = 2\psi$, then one has

$$\psi = \mathrm{am}(u + nc) = \varphi_0 + i\pi = \mathrm{am}(u) + i\pi.$$

Defined

$$K := \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - k^2 \sin^2 x}},$$

one gets $\mathrm{am}(K) = \pi/2$, $\mathrm{am}(u + 2K) = \mathrm{am}(u) + \pi$, and $\mathrm{am}(u + 2iK) = i\pi + \mathrm{am}(u)$ for any integer i. From above, it follows that

$$\mathrm{am}(u + nt) = \mathrm{am}(u + 2iK),$$

hence

$$u + nt = u + 2iK,$$

and then

$$t = \frac{2iK}{n}. \tag{3.4}$$

This equation is clearly a necessary and sufficient condition on (C) and (c) for the existence of a n-gon inter-scribed to them. Since this condition does not depend on φ_0, it follows that, if satisfied, the inter-scribed polygonal line $AA^{(1)}A^{(2)}\dots$ always closes after n steps (and i turns), whatever is the point A from which one starts.

We stress that Jacobi proved PCT for a pair of circles in a particular position (one lying within the other), not for *any* pair of circles. Although he was aware of Poncelet's projection methods and principle of continuity, he did not apply them to generalize his results to conics; possibly because he was not totally convinced of the validity of that principle.

Jacobi's paper immediately attracted attention. Shortly after its publication, Legendre inserted Jacobi's result as a section of the third volume of his *Traité des fonctions elliptiques* (Legendre 1828, pp. 174–180). Two years later Jacobi's pupil F.J. Richelot,[39] by applying Legendre's duplication law for elliptic integrals, gave a recursive formula for determining the relation among the radii and the distance between the centers of two circles, allowing the existence of an inter-scribed $2n$-gon, knowing the analogous relation for a n-gon. Defined

$$p := \frac{R+a}{r}, \quad q := \frac{R-a}{r}$$

and denoted by R', r', a', p', q' the analogous quantities for a $2n$-gon, he found (Richelot 1830, p. 27):

$$p = \frac{p'^2 + p'^2q'^2 - q'^2}{p'^2 - p'^2q'^2 + q'^2}, \quad q = \frac{q'^2 + q'^2p'^2 - p'^2}{q'^2 - q'^2p'^2 + p'^2}.$$

At the end of his paper, feeling his work was not fully completed, Jacobi wrote:

> Es dürfte nict ohne Interesse für die Theoria der elliptischen Functionen sein, ähnliche Betrachtungen unmittelbar für das System zweier Kegelschnitte anzustellen. Das Integral dürste dann in einer complicirteren Form erscheinen, die sichjedoch auf jene einfachere reduciren lassen muss. Vielleicht nehme ich später Gelagenheit, hieraus wieder zurückzukommen [It would be not without interest for the theory of elliptic functions, to make similar considerations directly for a system of two conics. The integral may appear in a more complicated form, which must, however, be reduced to the simplest form found above. I will return on this subject on another occasion]

but, to judge by what appears in his published works, he did not do so. Jacobi's program was carried out by Nicola Trudi and, independently, by Arthur Cayley, 25 years later.

[39] Friedrich Julius Richelot (1808–1875), was a student of Jacobi at Könisberg. He graduated in 1831 with a thesis on the subdivision of the circle in 257 equal parts. In 1844, he succeeded to Jacobi at the University of Könisberg.

4 Trudi: the forgotten work

In 1839, Nicola Trudi,[40] stimulated by his teacher Vincenzo Flauti, became interested in questions related to the existence of polygons inscribed in, or circumscribed about, a conic and satisfying given conditions. Two years later he gathered the results of his studies in his extended memoir (Trudi 1841). The last part of the memoir was devoted to finding the algebraic relation, among the coefficients of two conics, which guarantees the existence of an in-and-circumscribed triangle to them. This work, published at Flauti's expense, remained unknown outside the borders of the Kingdom of the Two Sicilies.[41]

In the spring of 1843, Trudi read a paper on this subject at the Royal Academy of Naples, but only a four-page summary, written in third person, was published in the *Rendiconti* of the Academy (Trudi 1843). This summary clearly indicates that: (1) he solved the question for the triangle; (2) he trusted his method in order to find the conditional relation for the existence of an inter-scribed polygon of any number of sides and that only the difficulties encountered with a problem of elimination led him to temporarily abandon the general question; (3) he had been induced to deal with that question as a continuation of the first of three Flauti's research proposals; (4) he had been hitherto unaware of the interest raised by this topic, because of his late discovery of Jacobi's paper.[42]

We extract from (Trudi 1843, p. 93) what follows:

> In ultimo ei ritorna al caso generale delle sezioni coniche per mostrare come questo metodo si applichi alla ricerca delle relazioni pei poligoni di qualsivoglia numero di lati iscrittibili tra esse: metodo che vincendo tutta la difficoltà che circonda la quistione, di cui trattasi, finisce per non recare altra pena, che quella di scrivere le formule corrispondenti. Intanto, mosso il Trudi dalle savie indicazioni dell'illustre Jacobi,[43] promette di ritornare su questo argomento, per guardare la questione sott'altro punto di vista, e propriamente in rapporto all'utile, che può trarsene nella teorica delle funzioni ellittiche [Finally he returns to the general case of conic sections, to show how this method is applicable in order to find the relations for polygons of any number of sides to be inter-scribed to them: the method, which overcomes all the difficulty which surround the question that we are concerned with, gives without further effort the corresponding formulae. Trudi, following the wise guidance of Jacobi,[44] promises to return to this

[40] Nicola Trudi (1811–1884), was born in Campobasso and studied in the University of Naples. In 1851, he became professor of infinitesimal calculus in that University and then member of the local Royal Academy of Sciences. He contributed to the theory of elliptic functions, and to the theory of determinants with the publication of the treatise *Teoria dei determinanti e loro applicazioni*, Napoli 1862. For a biographical note and information on his scientific production (see Amodeo 1924, part two, pp. 190–213).

[41] This memoir of Trudi, with three others by him, appeared at the end of Part II of (Flauti 1840–1844).

[42] This is to say that Trudi read (Jacobi 1828) with great delate.

[43] At page 90, Trudi explicitly refer to the phrase " Es dürfte nicht onhe Interesse..." that we have transcribed from (Jacobi 1828) at the end of the previous section.

[44] Idem.

argument, looking at the question from another point of view, and properly in relation to the advantage which may result to the theory of elliptic functions.]

Years later, having overcome all these difficulties, Trudi published the paper (Trudi 1853) answering to the general question, and completing Jacobi's program. Unfortunately, this work also remained almost unknown outside the Neapolitan milieu.

4.1 Trudi's first approach of 1841

In section 59 of his paper, Trudi raised the question of the construction of a triangle which is inscribed in a conic C and circumscribed about another conic C'. He observed that, if one takes as axis x a diameter of C, and for axis y the tangent to C at one of its intersection points with the chosen diameter, then C can be represented by the equation

$$y^2 = m^2x^2 + 2nx.$$

Moreover, he put

$$Ay^2 + 2Bxy + 2Cy + Dx^2 + 2Ex + F = 0,$$

the equation of C'. For (z, v) and (z', v') (general) points on C, he set $r := v/z$ and $r' := v'/z'$. With respect to these parameters, the chord of C joining the two points has equation

$$y(r + r') - x(rr' + m) = 2n.$$

He also observed that the line $y = ax + b$ is tangent to C', if and only if,

$$A' + a^2B' + 2bC + 2abD' + 2aE' + b^2F' = 0 \tag{4.1}$$

where

$$\begin{aligned} A' &= E^2 - DF, \quad B' = C^2 - AF, \quad C' = BE - CD \\ D' &= AE - BC, \quad E' = CE - BF, \quad F' = B^2 - AD. \end{aligned}$$

If $RR'R''$ is a triangle inscribed in C and circumscribed about C', then the three sides RR', $R'R''$ and $R''R$ have, respectively, equation

$$\begin{aligned} y(r + r') - x(rr' + m) &= 2n, \\ y(r' + r'') - x(r'r'' + m) &= 2n, \\ y(r'' + r) - x(r''r + m) &= 2n. \end{aligned}$$

So, taking into account (4.1), Trudi was led to the following equations, which link the parameters r, r', r'' and the coefficients of C and C':

$$\begin{aligned}&A'(r+r')^2+B'(rr'+m)^2+4mC'(r+r')+4nD'(rr'+m)\\&\quad+2E'(r+r')(rr'+m)+4n^2F'=0,\\&A'(r'+r'')^2+B'(r'r''+m)^2+4mC'(r'+r')+4nD'(r'r''+m)\\&\quad+2E'(r'+r'')(r'r''+m)+4n^2F'=0,\\&A'(r''+r)^2+B'(r''r+m)^2+4mC'(r''+r)+4nD'(r''r+m)\\&\quad+2E'(r''+r)(rr'+m)+4n^2F'=0.\end{aligned}$$

By eliminating r, r' and r'' among them, he got the equation

$$\begin{cases}A'^2+2mB'A'+4nA'D'-8nC'E'-4mE'^2+4mnD'B'\\+4n^2B'F'+m^2B'^2=0\end{cases}\tag{4.2}$$

which expresses a necessary condition for the existence of an inter-scribed triangle to C and C'.

Trudi remarked that, as this equation does not depend on the parameter of the initial point, if such a triangle exists, then infinitely many others exist. In case C and C' are circles, he also observed that his result leads to formula (1.1), that he attributed to Lhuilier.

At this point, Trudi tried to apply his method to the case of an inter-scribed quadrangle, but the exceedingly long computation forced him to limit himself to the case of circles, that he represented by the equations

$$y^2+x^2-2nx=0,\quad y^2+x^2-2ax+a^2-n'^2=0,$$

the first with center in $(n,0)$ and radius n, the second with center in $(a,0)$ and radius n'. In this case, he found that for the existence of an inter-scribed quadrangle to the two given circles the equation

$$n'^4=(n'^2-a^2)[n'^2-(2n-a)^2]$$

must hold. Trudi unaware of Fuss (1797) did not compare his formula with that of Fuss. Nevertheless, an easy computation shows that the two formulae are in fact identical.

From a footnote on page 97, we learn that Trudi tried to apply his method to polygons with $n\geq 3$ sides, but that he was unable, using standard procedures, to eliminate the n parameters among the n equations that can be deduced in this case. He expressed the desire to return on this subject as being of great importance.

Trudi kept his promise. In the spring of 1843 at the Royal Academy of Naples, he read the new memoir *Delle relazioni fra i determinanti di due sezioni coniche l'una iscritta l'altra circoscritta ad un poligono irregolare* [On the relations between the determinants (coefficients) of two conic sections, one inscribed and the other circumscribed to an irregular polygon]. As we have already said, a four-page summary of it was published in the reports of the Academy (Trudi 1843). Here, besides relation 4.2, the conditional equations for the existence of inter-scribed polygons to two circles, of 3, 6, 12, 24 sides and of 4, 8, 16, 32 sides are published. Moreover, since the volume of Crelle's *Journal* for the year 1828, containing Jacobi's famous paper, had recently arrived in Naples, Trudi was able to compare his formulae with those of Fuss (Trudi

1843, p. 90).[45] Finally, he returned to the general case of two conics and showed how his method could be applied to any polygons.

As we will see below, it was through the relation 4.2 that Trudi perceived a link between the existence of inter-scribed polygons and the complete integral of Euler's differential equation

$$\frac{\mathrm{d}x}{\sqrt{\alpha x^4+\beta x^3+\gamma x^2+\delta x+\varepsilon}}=\frac{\mathrm{d}y}{\sqrt{\alpha y^4+\beta y^3+\gamma y^2+\delta y+\varepsilon}}.$$

Later in 1843, Jacobi went to Italy. He joined Steiner in Rome, and in April of 1844 they both visited Naples. Trudi had the opportunity to meet them and to talk with Jacobi about his studies on Poncelet's theorem.[46] According to Trudi, Jacobi manifested real interest in his results and encouraged him to pursue research in this field (Trudi 1863a, p. 4).[47]

In 1845, the seventh Congress of the Italian Scientists was held in Naples. On this occasion, Trudi read a paper titled *Sull'eliminazione fra le equazioni algebriche eseguita per mezzo della differenziazione e della integrazione* [On the elimination among algebraic equations by means of differentiation and integration].[48] In particular, Trudi announced to have easily deduced many theorems of Poncelet and got the relations among the coefficients of the two conics as requested by Jacobi. Trudi felt to be on the right path, and he wanted to further pursue his studies.

In 1853, at the Academy of Naples he presented his most important paper *Su una rappresentazione geometrica immediata dell'equazione fondamentale nella teorica delle equazioni ellittiche* [On an immediate geometrical representation of the fundamental equation in the theory of elliptic functions] (Trudi 1853), printed only in 1856.[49] In a footnote on page 65, added when the paper went to press, Trudi wrote that he had just been informed by Joseph Sylvester that, in 1853, Arthur Cayley had published some notes on the same subject.[50] Trudi specified that he had not yet had the opportunity to read them, because the *Philosophical Magazine*—the journal in which Cayley published his notes—was not among those available in the Royal Library of Naples.

4.2 The important paper of 1853

In (1853) Trudi solved, through geometrical constructions, the question of addition and multiplication of elliptic integrals of the form

45 Probably Trudi knew of the formulae given by Fuss only through the paper of Jacobi.

46 As recorded in the *Rendiconto delle adunanze e dei lavori della reale Accademia delle Scienze di Napoli*, 3 (1844) pp. 196–197, Jacobi and Steiner attended at the meeting of this Academy for the 23 and 30 of April 1844. At the first meeting, Trudi read a memoir on a problem of elimination among algebraic equations in several variables by means of differentiation and integration.

47 Here Trudi confused the year in which Jacobi visited Naples: he wrote "1845" instead of "1844."

48 The proceedings of the Congress were published in 1846, we refer to as (Congresso Scienziati 1845).

49 The first volume of the *Memorie della Reale Accademia delle Scienze di Napoli* includes all papers presented to the Academy during the years 1852–1854.

50 Sylvester visited Naples in February of 1856 (see Hunger 2006b, p. 110).

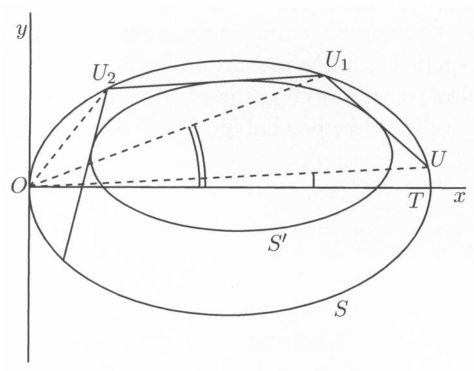

Fig. 17 Description of how Trudi proceeded in order to prove his Theorem T1

$$\int \frac{\mathrm{d}u}{\sqrt{\psi(u)}},$$

where $\psi(u)$ is a polynomial of degree four. Then, he applied the derived theorems and formulae to find the necessary and sufficient conditions under which two conics admit an inter-scribed n-gon.

First of all, Trudi proved the following (pp. 66–67).

Theorem T1 *Let S, S' be two conics and UU_1 any chord of S tangent to S'. Let x be a diameter of S and O one its intersection points with S, and define $u := \tan \widehat{UOx}$, $u_1 := \tan \widehat{U_1Ox}$. Then, u and u_1 satisfy an equation of the form*

$$Au^2u_1^2 + 2B(u+u_1)uu_1 + C(u+u_1)^2 + 2D(u+u_1) + 2Euu_1 + F = 0$$

where A, B, C, D, E, F are constants depending on the minors of order 2 of the matrices of the two conics.[51]

Trudi chose as x-axis the line x and as y-axis the tangent to S at O (Fig. 17), so that he represented the two conics S, S', respectively, by

$$y^2 = 2rx + mx^2,$$
$$ay^2 + 2bxy + cx^2 + 2dy + 2ex + f = 0.$$

Under this assumption, if $U = (x, y)$ and $U_1 = (x_1, y_1)$, is $u = x/y$ and $u_1 = x_1/y_1$, and, since both U, U_1 are on S, it follows

$$\begin{aligned} x &= \tfrac{2r}{u^2-m}, \quad x_1 = \tfrac{2r}{u_1^2-m} \\ y &= \tfrac{2ru}{u^2-m}, \quad y_1 = \tfrac{2ru_1}{u_1^2-m} \end{aligned} \tag{4.3}$$

[51] Trudi's statement says "costanti dipendenti dai determinanti delle due coniche" [constants depending on the determinants of the two conics].

He chose X, Y for the coordinates on the line UU_1 and wrote its equation in the form

$$(Y-y)(x-x_1)=(X-x)(y-y_1),$$

or, what is the same,

$$(u-u_1)Y=(uu_1+m)X+2r.$$

Then, a line $Y=pX+q$ will touch S' if (and only if)

$$(d^2-af)p^2+2(de-bf)p+(e^2+cf)+2(ae-bd)pq+(b^2-ac)q^2+2(be-cd)q=0$$

holds true. Since

$$p=\frac{uu_1+m}{u+u_1},\quad q=\frac{2r}{u+u_1},$$

by substituting these expressions into the above relation, and setting

$$\begin{array}{ll} A=d^2-af, & D=m((dc-bf)+2r(bc-cd), \\ B=de-bf, & E=m(d^2-af)+2r(ae-bd), \\ C=e^2-cf, & F=m^2(d^2-af)+4mr(ae-bd)+4r^2(b^2-ac), \end{array} \tag{4.4}$$

he obtained the result.

Trudi also proved the converse of the above theorem (pp. 70–71), precisely:

Theorem T2 *Suppose that between two variables u and u_1 there exists a relation of the form*

$$Au^2u_1^2+2B(u+u_1)uu_1+C(u+u_1)^2+2D(u+u_1)+2Euu_1+F=0, \tag{4.5}$$

and that a conic S of equation $y^2=2rx+mx^2$ is given. Then, a unique conic S' can be found such that, for any chord UU_1 of the first conic that touch the second, the trigonometric tangents $\tan\widehat{UOx}$ *and* $\tan\widehat{U_1Ox}$ *always satisfy the relation* (4.5).

In order to prove the theorem, Trudi proceeded as follows. He observed that, if such a conic S' with equation $ay^2+2bxy+cx^2+2dy+2ex+f=0$ exists, then, setting $u:=\tan\widehat{UOx}$ and $u_1:=\tan\widehat{U_1Ox}$ as above, from theorem T1 it follows that a relation similar to the (4.5), and whose coefficients are expressed by (4.3), holds true. Therefore, by comparing the coefficients of this relation with the given one, he expressed b, c, d, e, f in terms of A, B, C, D, E, F and a, so getting for S' the equation

$$\begin{aligned}&(AF-E)^2y^2+2[(BF-DE)+m(AD-BE)]xy\\&+\Big[(CF-D^2)+2m(BD-CE)+m^2(AC-B^2)\Big]x^2\\&+4r(AD-BE)y+4r\Big[(BD-CE)+m(AC-B^2)\Big]x\\&+4r^2(AC-B^2)=0.\end{aligned} \tag{4.6}$$

He also remarked that S' is the envelope of all chords of S whose elements u, u_1 satisfy the relation of theorem T1.

Then, Trudi proved the following (pp. 72–73):

Theorem T3 *The differentials* $\mathrm{d}u$ *and* $\mathrm{d}u_1$ *satisfy a relation of the form*

$$\frac{\mathrm{d}u}{\sqrt{\alpha u^4+\beta u^3+\gamma u^2+\delta u+\varepsilon}}=\pm\frac{\mathrm{d}u_1}{\sqrt{\alpha u_1^4+\beta u_1^3+\gamma u_1^2+\delta u_1+\varepsilon}},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ *are constants depending solely on the coefficients in the equations of* S *and* S'.

To prove this, he differentiated relation (4.5) with respect to u and u_1, and squared the two members so obtained, and setting $C+E=G$ he obtained the equation

$$\frac{\mathrm{d}u^2}{\psi(u)}=\frac{\mathrm{d}u_1^2}{\psi(u_1)},$$

where

$$\psi(u)=[(Au_1^2+2Bu_1+c)u+Bu_1^2+Gu+D]^2,$$
$$\psi(u)=[(Au^2+2Bu+c)u_1+Bu^2+Gu+D]^2.$$

Taking the square roots, he wrote the two differential equations

$$\frac{\mathrm{d}u}{\sqrt{\psi(u)}}=\pm\frac{\mathrm{d}u_1}{\sqrt{\psi(u_1)}}.$$

From here, with some computation, and setting

$$\begin{cases}\alpha=B^2-AC\\ \beta=2(BE-AD)\\ \gamma=E^2-AF+2(CE-BD)\\ \delta=2(DE-BF)\\ \varepsilon=D^2-CF\end{cases}\tag{4.7}$$

he finally derived the theorem.

Trudi knew [see his note at p. 74, where he referred to Euler (1794)] that the differential equation

$$\frac{\mathrm{d}x}{\sqrt{\alpha x^4+\beta x^3+\gamma x^2+\delta x+\varepsilon}}=\frac{\mathrm{d}y}{\sqrt{\alpha y^4+\beta y^3+\gamma y^2+\delta y+\varepsilon}}\tag{4.8}$$

always admits a complete integral of the form

$$Ax^2y^2+2B(x+y)xy+C(x+y)^2+2D(x+y)+2Exy+F=0\tag{4.9}$$

where

$$\begin{cases} A = 4\alpha(K+\gamma) - \beta^2 \\ B = 2\alpha\delta + K\beta \\ C = 4\alpha\varepsilon - K^2 \\ D = 2\beta\varepsilon + \delta K \\ E = \beta\delta + 2(K+\gamma)K \\ F = 4\varepsilon(K+\gamma) - \delta^2 \end{cases}$$

and K is an arbitrary constant. This means, Trudi remarked, that in theorem T2 instead of the relation (4.4) one can consider the differential equation (4.8), and so the equation (4.5) of S' can be put in the form

$$(K+\gamma)y^2+(m\beta+\delta)xy+(m^2\alpha-mK+\varepsilon)x^2+2r\beta y+2r(2m\alpha-K)x+4r\alpha^2=0.$$

Next, from his theorem 4 (p. 77), Trudi showed that the conic S, and all the conics described by the previous equations, belong to the same pencil (theorem 5, p. 78).[52]

He also brought to light a remarkable property of the relation (4.5), precisely he proved (see theorem 6, pp. 79–81):

Theorem T4 *By eliminating the variables $u_1, u_2, \ldots, u_{n-1}$ in the following system of equations in $n+1$ variables*

$$\begin{cases} Au^2u_1^2 + 2B(u+u_1)uu_1 + C(u+u_1)^2 \\ \quad +2D(u+u_1) + 2Euu_1 + F = 0, \\ Au_1^2u_2^2 + 2B(u_1+u_2)u_1u_2 + C(u_1+u_2)^2 \\ \quad +2D(u_1+u_2) + 2Eu_1u_2 + F = 0, \\ \vdots \\ Au_{n-1}^2u_n^2 + 2B(u_{n-1}+u_n)u_{n-1}u_2 + C(u_{n-1}+u_n)^2 \\ \quad +2D(u_{n-1}+u_n) + 2Eu_{n-1}u_n + F = 0 \end{cases} \tag{4.10}$$

the resulting equation in u, u_n is still of the form (4.5).

He used Euler's differential equation to perform the elimination. By applying to each equation the procedure used in the proof of theorem T3, he obtained

$$\frac{\mathrm{d}u^2}{\psi(u)} = \frac{\mathrm{d}u_1^2}{\psi(u_1)} = \frac{\mathrm{d}u_2^2}{\psi(u_2)} = \cdots = \frac{\mathrm{d}u_n^2}{\psi(u_n)}$$

where

$$\psi(u_i) = \alpha u_i^4 + \beta u_i^3 + \gamma u_i^2 + \delta u_i + \varepsilon$$

with $\alpha, \beta, \gamma, \delta, \varepsilon$ given by (4.7). Hence, one has

$$\frac{\mathrm{d}u^2}{\psi(u)} = \frac{\mathrm{d}u_n^2}{\psi(u_n)}.$$

[52] Trudi wrote "hanno le stesse secanti comuni, reali o ideali (secondo la denominazione dell'illustre Poncelet" [they have the same real or ideal chords (according the definition of the illustrious Poncelet).

Then, by integrating, he obtained an equation in u and u_n (necessarily) of the (required) form

$$A'u^2u_n^2 + 2B'(u+u_n)uu_n + C'(u+u_n)^2 + 2D'(u+u_n) + 2E'uu_n + F' = 0, \tag{4.11}$$

with

$$\begin{cases} A' = 4\alpha(K+\gamma) - \beta^2 \\ B' = 2\alpha\delta + K\beta \\ C' = 4\alpha\varepsilon - K^2 \\ D' = 2\beta\varepsilon + \delta K \\ E' = \beta\delta + 2(K+\gamma)K \\ F' = 4\varepsilon(K+\gamma) - \delta^2, \end{cases} \tag{4.12}$$

and where K, no longer an arbitrary constant, depends on the number of equations and their coefficients. Trudi remarked that the value of K can be found by equating the two values of u_n obtained, from one side, by setting $u = 0$ (or $u = \infty$) in the equations (4.10), and from the other, by setting $u = 0$ (or $u = \infty$) in the (4.11).

The coefficients of (4.11), Trudi also noticed, are in general different from the coefficients of the equations (4.10), but he proved (pp. 81–82) that:

($\star$) *if* $K = -2(CE - BD)$, *then the coefficients* A', B', C', D', E', F' *are, respectively, equal (up to a constant) to the coefficients* A, B, C, D, E, F.

In the last part of his memoir, Trudi gave geometrical and analytical applications of his results; among the first he proved Poncelet's theorem and for the second, he only considered the addition and multiplication of elliptic functions.

He started by solving the following problem (p. 83): *Let S and S' be two non-singular conics, and inscribed in S any polygon whose sides, but one, are tangent to S', find the curve enveloped by the free side.*

He considered a polygon $U, U_1, U_2, \ldots, U_n$ inscribed in a conic S and whose sides are all tangent to another conic S', except U_nU. He observed that there is no loss of generality in supposing the two conics have equations as in theorem T4. Then, if $u, u_1, u_2, \ldots, u_n$, defined as above, correspond to the vertices of the polygon, the conditions of tangency of the first n sides give n equations like (4.10), whose coefficients are expressed by means of (4.3). So, by eliminating the intermediate variables $u_1, u_2, \ldots, u_{n-1}$, he got, via theorem 4, that the equation satisfied by the parameters of the extremities of the free side of the polygon will be of the form

$$A'u^2u_n^2 + 2B'(u+u_n)uu_n + C'(u+u_n)^2 + 2D'(u+u_n) + 2E'uu_n + F' = 0.$$

Hence, he applied the corollary to theorem T2, in order to get the conic envelop of U_nU, and found

$$(K+\gamma)y^2 + (m\beta+\delta)xy + (m^2\alpha+\varepsilon-mK)x^2 + 2r\beta y + 2r(2m\alpha-K)x + 4r^2\alpha = 0,$$

where K is determined as prescribed in theorem T4.

Then, the previous results, and in particular the formulae (4.7) and (4.4), together with some computation, allowed Trudi to write the equation of the enveloped conic in terms of the coefficients of S and S', which resulted in

$$(a+\mu)y^2 + 2bxy + (c - m\mu)x^2 + 2dy + 2(e - r\mu)x + f = 0 \qquad (4.13)$$

where

$$\mu = \frac{K + 2(CE - BD)}{4r^2\Delta},$$

and $\Delta = -\det(S')$.

Trudi remarked that from equation (4.13) it follows that the enveloped conic of the free side meets the conics S and S' exactly in their common points (p. 85), that is, he added, "le due coniche date e la conica inviluppo hanno le stesse secanti comuni (reali, o ideali)"[the two given conics and the conic envelop have the same common chords (real, or ideal)].

He had proved:

The envelop of the free side of any polygon inscribed in a conic S, whose side are all but one tangent to another conic S', is a conic I, belonging to the pencil determined by S and S', whose equation is given by (4.13).

Moreover, from the same equation, it follows that $U, U_1, U_2, \ldots, U_n, U$ is circumscribed about S', if $\mu = 0$, i.e., if $K = -2(CE - BD)$, and, since he showed that the reciprocal also holds true, he had proved that $K = -2(CE - BD)$ is a necessary and sufficient condition for the existence of a polygon of n sides which is inscribed in S and circumscribed about S'.

This, Trudi stressed (p. 86), gives a proof of the closure theorem of Poncelet,[53] and provides the condition on the coefficients of S and S' that allows the existence of a polygon of $n+1$ sides inter-scribed to the two conics.

When the condition above is satisfied, by virtue of the remark ($\star$), (4.10) becomes

$$\begin{cases} Cu_1^2 + 2Du_1 + F = 0 \\ Au_1^2u_2^2 + 2B(u_1+u_2)u_1u_2 + C(u_1+u_2)^2 \\ \quad + 2D(u_1+u_2) + 2Eu_1u_2 + F = 0, \\ Au_2^2u_3^2 + 2B(u_2+u_3)u_1u_2 + C(u_2+u_3)^2 \\ \quad + 2D(u_2+u_3) + 2Eu_2u_3 + F = 0, \\ \vdots \\ Cu_n^2 + 2Du_n + F = 0 \end{cases} \qquad (4.14)$$

so the condition on the two conics S and S' that allowing the existence of a n-gon inter-scribed to them, is the equation obtained by eliminating the n variables from the above system of $n+1$ equations.

[53] In fact, the condition he found does not depend on any of the vertices of the inscribed polygon.

Summing up, Trudi had succeeded in completing Jacobi's program.

At this point, in order to illustrate his method, Trudi considered in detail the cases $n = 3, 4$ and 5, for which he obtained explicit formulae.

In case $n = 3$, the system above reduces to

$$\begin{cases} Cu_1^2 + 2Du_1 + F = 0, \\ Au_1^2u_2^2 + 2B(u_1 + u_2)u_1u_2 + C(u_1 + u_2)^2 \\ \quad + 2D(u_1 + u_2) + 2Eu_1u_2 + F = 0, \\ Cu_n^2 + 2Du_n + F = 0. \end{cases}$$

From the first and the third equation, it follows that $u_1u_2 = F/C$ and $u_1 + u_2 = -2D/C$, so substituting these values in the second one gets

$$AF + C^2 + 2EC - 4BD = 0 \tag{4.15}$$

which, by the (4.3), gives (4.1): a necessary and sufficient condition for the existence of an inter-scribed triangle expressed in terms of the 2×2 minors of the matrices of S and S'.

When the two conics are circles, then $B = D = 0$, and (4.15) reduces to $AF + C^2 + 2EC = 0$, that is readily seen to be equivalent to condition (1.1).

For the sake of space, we omit here the other two cases. Suffice to say that Trudi's formulae for circles are equivalent to those found by Fuss.

Ten years later, shortly after Poncelet had published the first volume of *Applications d'analyse et de géométrie*, Trudi returned to the subject with two memoirs that we will comment in section seven.

5 Cayley's explicit conditions

Arthur Cayley[54] had become interested in elliptic functions early in the 1840s. He approached that theory through Jacobi's *Fundamenta Nova Theoriae Functionum Ellipticarum* (1829), that he had mastered immediately after his degree (Cayley 1895, pp. xi–xii).[55]

The celebrated *Mémoire sur une propriété générale d'une classe très étendue de fonctions transcendantes* authored by Abel was published posthumously in 1841. Here, he proved the so-called addition theorem for Abelian integrals that provided the basis for the development of the modern algebraic geometry. This memoir attracted Cayley's attention quite early in his scientific career.

54 For a comprehensive biography of Arthur Cayley, we refer to Crilly (2006).

55 The prominent position occupied in the *Fundamenta* by the theory of transformation naturally attracted his interest. As early as 1844, he wrote short notes on the subject. Cayley always maintained Jacobi's point of view, and he particularly appreciated the algebraic approach. When he published the treatise on elliptic functions (Cayley 1876), where the theory of transformation is discussed at considerable length, it was already an old-fashioned work (Crilly 2006, p. 337).

An *Abelian integral* is an integral of type

$$\int R(x, y)\mathrm{d}x,$$

where $R(x, y)$ is a rational function of x, y, being $y(x)$ the algebraic function defined by $\chi(x, y) = 0$ with $\chi(x, y)$ an irreducible polynomial.

Abel's addition theorem affirms that, if $\theta(x, y; t) = 0$ is a family of plane curves, depending rationally upon the parameter t, and $(x_i(t), y_i(t))$, $i = 1, \ldots, N$, are the intersection points of the curves of the family with the curve $\chi(x, y) = 0$, then

$$\sum_1^N \int_{x_0}^{x_i(t)} R(x, y(x))\mathrm{d}x = V(t) + \log W(t),$$

where $V(t)$ and $W(t)$ are rational functions of the parameter t. In particular, if the integral is of the *first kind*, i.e., if its value remains finite when integration is carried out along any path from the initial point x_0 to the final point x_i, the above sum reduces to a constant [see for instance (Markushevich 1992; Del Centina 2003; Bottazzini and Gray 2013), and for a very careful analysis (Kleiman 2004)]. Since any elliptic integral is an Abelian integral, this theorem clearly generalizes Euler's addition theorem, which, as well known, holds for elliptic integrals

$$\int \frac{\mathrm{d}x}{\sqrt{X}},$$

where X is a polynomial of degree 3 and 4 without multiple roots, i.e., the integral is an elliptic integral of the first kind (Euler 1768).

The geometrical significance of the theorem of Abel extends to that of Euler: for instance, if X has degree 3, then one has

$$\int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{X}} + \int_{x_0}^{x_2} \frac{\mathrm{d}x}{\sqrt{X}} + \int_{x_0}^{x_3} \frac{\mathrm{d}x}{\sqrt{X}} = \text{cost.},$$

for any triple $(x_i, y_i(x_i))$, $i = 1, 2, 3$, of collinear points on the cubic curve $y^2 = X$.

Less general version of Abel's addition theorem had already appeared in Abel (1828, 1829), where he considered the particular case of hyperelliptic functions ($\chi(x, y) = y^2 - X$, with X a polynomial of degree ≥ 5).

Some years later Jacobi explained the theorem for hyperelliptic functions in his paper (Jacobi 1832) and reformulated it as follows (p. 396): let X be a polynomial in x of degree $2m$ or $2m - 1$, and set

$$\Pi(x) := \int_0^x \frac{A + A_1 x + \cdots + A_k x^k}{\sqrt{X}}\mathrm{d}x,$$

then, given m values $x_0, x_1, \ldots, x_{m-1}$ of the variable, $m - 1$ quantities $a_0, a_1, \ldots,$ a_{m-2} can be algebraically determined from these (they are roots of an algebraic equation of degree $m-1$, whose coefficients are rationally expressed in terms of $x_0, x_1, \ldots, x_{m-1}$ and $\sqrt{X(x_0)}, \ldots, \sqrt{X(x_{m-1})}$), such that

$$\Pi(x_0) + \Pi(x_1) + \cdots + \Pi(x_{m-1}) = \Pi(a_0) + \Pi(a_1) + \cdots + \Pi(a_{m-2}).$$

This geometrical feature of the addition theorem represented a powerful inspiration for Cayley in developing the program that Jacobi had drawn in 1828, i.e., to find the conditions on two conics U and V for the existence of an inter-scribed n-gon. In a series of notes, published in 1853 and in 1854, by using Abel's theorem and the development in power series of $\sqrt{\det(\xi U + V)}$ as main tools, he proved the closure theorem of Poncelet and found the required conditions on U and V. At the end of the first note (Cayley 1853a), referring to Jacobi (1828), he wrote:

> The preceding investigations were, it is hardly necessary to remark, suggested by a well-known memoir of the late illustrious Jacobi,[56] and contain, I think, the extension which he remarks it would be interesting to make of the principles in such memoir to a system of two conics.

Cayley proved PCT for triangles in the first two notes. In the third one, he extended the results to polygons of $n > 3$ sides, and in the fourth he gave explicit conditions when $n = 4, 5$.

A few year later he returned to the subject with two papers concerning the porism of an in-and-circumscribed triangle (Cayley 1857, 1858). The aim of the first was to extend what we called "main lemma," from the case of three circles, to the case of a conic and two curves of higher degree, while the aim of the second was to give new proofs of Poncelet's propositions 1–3 (here stated in subsection 2.4). We will comment these papers, which are of pure geometrical character, at the end of this section.

In (1861), Cayley republished the results that he had already achieved on polygons inter-scribed to two conics in the years 1853–1854, in a more organized and complete form. This paper (Cayley 1861) became the standard reference for Cayley's discoveries in this field.

Four years later, in a short note, he considered the problem of a triangle inscribed in and circumscribed about a (real) quartic curve (Cayley 1865).

At the meeting of the Royal Society in Liverpool in 1870, Cayley reconsidered the problem of the in-and-circumscribed polygons. It was one of his favorite problems, and he picked it up at the same point he had left it years before. He published two papers on this topic (Cayley 1871a, b). In the first, he introduced the (2, 2)-correspondences in the study of that problem; in the second, he raised, and solved, the question of the number of triangles which can be inscribed in and circumscribed about curves of degree higher than two.

We will discuss the two last-mentioned papers in section eight.

[56] Jacobi died in 1851.

5.1 Cayley's first notes (1853–1854)

In the first note, Cayley considered two non-singular conics U and V, in the projective complex plane, meeting in four distinct points. It was known that, in this case, U and V admit a unique common *self-polar triangle*, i.e., a triangle ABC such that each vertex is the pole of the opposite side of the triangle, with respect to both U and V.[57] Then, by choosing projective coordinates x, y, z so that $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$, Cayley was able to represent U and V, respectively, by the equations:[58]

$$x^2 + y^2 + z^2 = 0, \quad ax^2 + by^2 + cz^2 = 0.$$

Let $\mathcal{F}$ to denote the pencil of conics $mU + V = 0$. He observed that for any tangent t_k to the conic

$$U_k := kU + V = (a+k)^2x^2 + (b+k)^2y^2 + (c+k)^2z^2 = 0$$

there is another conic from $\mathcal{F}$ which is tangent to t_k, say $pU + V = 0$ (Fig. 18a). Then, p can be taken as parameter for the tangent t_k as well as for its point of contact T_k with U_k. A general point T_k on U_k has coordinates

$$\left(\frac{\sqrt{b-c}\sqrt{a+p}}{\sqrt{a+k}}, \frac{\sqrt{c-a}\sqrt{b+p}}{\sqrt{b+k}}, \frac{\sqrt{a-b}\sqrt{c+p}}{\sqrt{c+k}}\right)$$

and t_k is represented by the equation

$$x\sqrt{b-c}\sqrt{a+p}\sqrt{a+k}+y\sqrt{c-a}\sqrt{b+p}\sqrt{b+k}+z\sqrt{a-b}\sqrt{c+p}\sqrt{c+k}=0.$$

If t_k meets U in the points P and P' corresponding to the parameters θ, ∞ and θ', ∞, respectively, one has

$$P = \left(\sqrt{b-c}\sqrt{a+\theta}, \sqrt{c-a}\sqrt{b+\theta}, \sqrt{a-b}\sqrt{c+\theta}\right).$$

By substituting these values in the equation of t_k, Cayley got the equation

$$\begin{cases} (b-c)\sqrt{a+k}\sqrt{a+p}\sqrt{a+\theta} + (c-a)\sqrt{b+k}\sqrt{b+p}\sqrt{b+\theta} + \\ (a-b)\sqrt{c+k}\sqrt{c+p}\sqrt{c+\theta} = 0, \end{cases} \tag{5.1}$$

connecting p and θ. He rationalized this equation by putting

$$\sqrt{(a+k)(a+p)(a+\theta)} = \lambda + \mu a,$$
$$\sqrt{(b+k)(b+p)(b+\theta)} = \lambda + \mu b,$$
$$\sqrt{(c+k)(c+p)(c+\theta)} = \lambda + \mu c,$$

[57] This result appears for instance in Poncelet (1822, p. 193).

[58] We can only say that this was Cayley's reasoning behind the choice of these equations; in fact, he did not at all justify it. See also next subsection, where we will comment on (Cayley 1861).

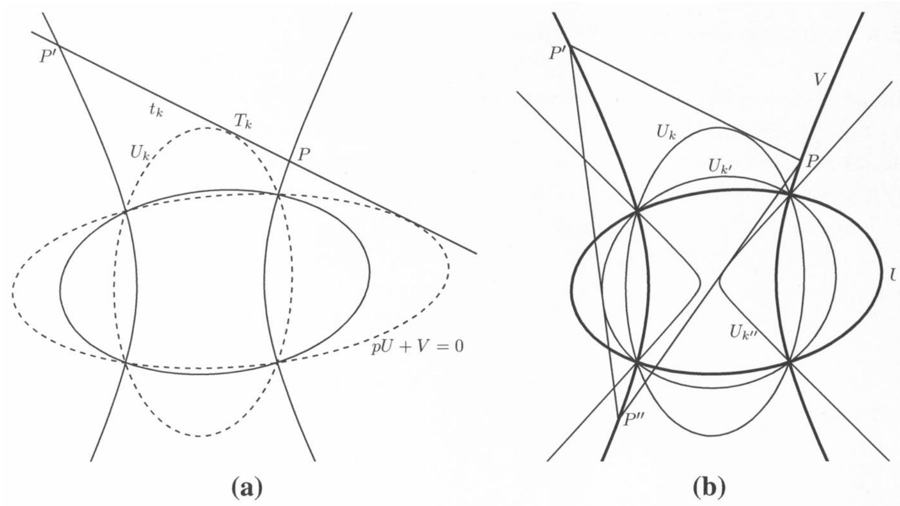

Fig. 18 Cayley's procedure for his proof of PCT in case $n = 3$. **b** if a triangle $PP'P''$ exists, which is inscribed in V and whose sides are respectively tangent to the conics U_k, $U_{k'}$ and $U_{k''}$ in the pencil, then $\Pi(k) + \Pi(k') + \Pi(k'') = 0$

values which, evidently, satisfy the equation in question. Squaring and eliminating λ and μ, he obtained

$$\begin{cases} [bc + ca + ab - (p\theta + kp + k\theta)]^2 + \\ -4(a + b + c + k + p + \theta)(abc + kp\theta) = 0 \end{cases} \tag{5.2}$$

which is the rational form of (5.1).

Cayley made the important observation that, due to the symmetry of (5.1) the same equation would have been obtained by eliminating L, M from the equations

$$\sqrt{(\zeta + a)(\zeta + b)(\zeta + c)} = L + M\zeta,$$

for $\zeta = k, p, \theta$. Then, invoking Abel's theorem,[59] that if

$$\Pi(x) := \int_\infty^x \frac{\mathrm{d}x}{\sqrt{(x + a)(x + b)(x + c)}},$$

then the algebraic relation (5.1) is equivalent to

$$\Pi(\theta) = \Pi(p) - \Pi(k).$$

There is of course a similar equation for θ' with $\Pi(k)$ taken with opposite sign:

$$\Pi(\theta') = \Pi(p) + \Pi(k).$$

59 He also noticed that the result might be verified by means of Euler's addition theorem for elliptic integrals.

The elimination of $\Pi(p)$ between the two equations gives

$$\Pi(\theta') - \Pi(\theta) = 2\Pi(k).$$

This means, remarked Cayley, that if the points P, P' on V are such that their parameters θ, θ' satisfy this equation, then the line PP' will always be tangent to the conic U_k.

If a triangle $PP'P''$ exists, which is inscribed in V and whose sides PP', $P'P''$ and $P''P$ are, respectively, tangent to the conics U_k, $U_{k'}$ and $U_{k''}$ in the pencil $\mathcal{F}$ (see Fig. 18b), one must have:

$$\begin{aligned}\Pi(\theta') - \Pi(\theta) &= 2\Pi(k),\\ \Pi(\theta'') - \Pi(\theta') &= 2\Pi(k'),\\ \Pi(\theta) - \Pi(\theta'') &= 2\Pi(k''),\end{aligned}$$

hence, by adding, one gets

$$\Pi(k) + \Pi(k') + \Pi(k'') = 0. \tag{5.3}$$

Cayley observed that when (5.3) holds true, there are infinitely many triangles inscribed in U, the sides of which touch the three conics. So one has

Proposition C *Equation* (5.3) *is a necessary and sufficient condition, on the parameters k, k', k'' of three conics $U_k, U_{k'}, U_{k''}$ in $\mathcal{F}$, for the existence of a triangle, and therefore of infinitely many, which is inscribed in V and whose sides are, respectively, tangent to $U_k, U_{k'}, U_{k''}$.*

Cayley, without explanation, also added that the same holds for a polygon of any number of sides.

In the subsequent paper (Cayley 1853b), he went further, giving an algebraic interpretation of (5.3).[60]

He let $\Box\xi$ denote the determinant $(\xi+a)(\xi+b)(\xi+c)$ of the conic $\xi U + V = 0$, and noticed that according to Abel's theorem k, k', k'' are the abscissae of the intersection points of the curve $y^2 = \Box x$ with a line $y + \beta_0 x + \beta_1 = 0$. Substituting $\sqrt{\Box x}$ in the last equation he obtained

$$\sqrt{\Box x} + \beta_0 x + \beta_1 = 0.$$

From here, it is clear that there exists a triangle inscribed U whose sides are, respectively, tangent to $U_k, U_{k'}, U_{k''}$ if and only if

$$\begin{vmatrix} 1 & k & \sqrt{\Box k} \\ 1 & k' & \sqrt{\Box k'} \\ 1 & k'' & \sqrt{\Box k''} \end{vmatrix} = 0. \tag{5.4}$$

60 When Cayley returned to London from his summer's travels in the Wales, he wrote to W.R. Hamilton thanking him for sending the compendious *Lecture on Quaternions* and enthusiastically describing his work (Cayley 1853b), that he had submitted to the *Philosophical Transactions* in July (Crilly 2006, p. 184–185).

In order to get an explicit condition on the coefficients of the two conics, Cayley considered the development in power series of the square root of the determinant $\square\xi$: $\sqrt{\square\xi} = A + B\xi + C\xi^2 + D\xi^3 + E\xi^4 + \cdots$. He substituted the corresponding expressions for $\sqrt{\square k}$, $\sqrt{\square k'}$, $\sqrt{\square k''}$, and wrote equation (5.4) in the form

$$\begin{vmatrix} 1 & k & k^2 \\ 1 & k' & k'^2 \\ 1 & k'' & k''^2 \end{vmatrix} (C + \text{terms multiplied by } k, k', k'').$$

Then, if k, k', k'' are all different, the above equation is equivalent to $(C + \text{terms multiplied by } k, k', k'') = 0$. This implies, when $k = k' = k'' = 0$, i.e., $U_k = U_{k'} = U_{k''} = U$, that

$$C = 0,$$

is the necessary, and sufficient, condition for the existence of a triangle inscribed in V and circumscribed about U.

At this point of the paper (p. 101), Cayley claimed that the same reasoning applies to polygons of any number of sides, and he stated the erroneous (see below) theorem: $(*)$ *the vanishing of the coefficient of* ξ^{n-1}, *in the development of* $\sqrt{\square\xi}$, *is the condition for the existence of a n-gons inscribed in V and circumscribed about U.*

Cayley applied his result in order to write explicit conditions for $n = 3$ and $n = 4$, and he found, respectively[61]:

$$a^2 + b^2 + c^2 - 2bc - 2ca - 2ab = 0,$$
$$(b + c - a)(c + a - b)(a + b - c) = 0.$$

He also claimed that similar relations hold for the pentagon, the hexagon, etc. Finally, he considered the case of two circles,[62] that he wrote

$$x^2 + y^2 - R^2 = 0, \quad (x - a)^2 + y^2 - r^2 = 0,$$

where a denotes the distance between their centers. He reformulated the above theorem in this case and verified that for a triangle his condition leads to equation (1.1).

Cayley corrected theorem $(*)$ in the two pages note (1853c). He began by writing:

> The two theorems in my "Note on the Porism of the in-and-circumscribed Polygon" are erroneous,[63] the mistake arising from my having inadvertently assumed a wrong formula for the addition of elliptic integrals. The first of two theorems (which, in fact, includes the other as particular case) should be as follows.

[61] We observe that Cayley in finding these conditions, inverted the role of U and V, and consequently their equations, see his footnote at p. 101.

[62] The second inside the first.

[63] Cayley was referring to theorem $(*)$ and to the equivalent form of it in case of circles.

Theorem C *The condition that there may be an infinity of n-gons, which are inscribed in the conic $U = 0$ and circumscribed about the conic $V = 0$, depends upon the development of*

$$\sqrt{\Box\xi} = A + B\xi + C\xi^2 + D\xi^3 + E\xi^4 + F\xi^5 + G\xi^6 + H\xi^7 + \cdots,$$

precisely for $n = 3, 5, 7, \ldots$, respectively, the conditions are:

$$C = 0, \quad \begin{vmatrix} C & D \\ D & E \end{vmatrix} = 0, \quad \begin{vmatrix} C & D & E \\ D & E & F \\ E & F & G \end{vmatrix} = 0, \ldots$$

and for $n = 4, 6, 8, \ldots$, respectively, the conditions are:

$$D = 0, \quad \begin{vmatrix} D & E \\ E & F \end{vmatrix} = 0, \quad \begin{vmatrix} D & E & F \\ E & F & G \\ F & G & H \end{vmatrix} = 0, \ldots$$

After having stated the corrected version, Cayley remarked that the two examples ($n = 3, 4$) he had given in the previous paper were correct, being, respectively, equal to $C = 0$ and $D = 0$.

In the second page, he worked out the case of two concentric circles ($a = 0$), along the line of his new theorem. In particular, putting for brevity $\alpha = R^2/r^2$, he verified that the condition $CE - D^2 = 0$ for the pentagon is equivalent to $\alpha^2 - 12\alpha + 16 = 0$, i.e., the well known $r/R = \cos\frac{\pi}{5} = (\sqrt{5} + 1)/4$.

In the note (Cayley 1854), he explicitly computed the conditional equations when the conics are circles up to $n = 8$ and compared them with those of Fuss and Steiner for $n = 4, 5$. He also considered the case of two concentric circles, and setting $M = R^2/r^2$ he established the following conditional equations: $M - 2 = 0$,[64] $M - 4 = 0$, $M^2 - 12M + 16 = 0$, respectively, for $n = 3, 4, 5$. Cayley also remarked that the geometrical properties of the polygons inter-scribed to two conics having a double contact are obtained from the case of concentric circles.[65]

The papers on the in-and-circumscribed triangle (1857–1858)

In the paper (Cayley 1857), he considered a triangle abc inscribed in a conic $\mathcal{S}$, whose sides ac and bc are tangent to fixed curves $\mathcal{A}$, $\mathcal{B}$, and sought to find the curve $\mathcal{C}$ that is enveloped by the free side ab, when c moves on $\mathcal{S}$. He recognized that ab, and its contact point γ with $\mathcal{C}$, could be constructed in the same way that Poncelet had done in his proof of the "main lemma" (see Fig. 19, and also Fig. 13), but he preferred to follows another route "which it may be modified so as to be applicable to curves $\mathcal{S}$ of any order," as he wrote at p. 344.

He first computed that the class of $\mathcal{C}$, i.e., the degree of the dual $\mathcal{C}^*$, is $2mn$ where m and n are, respectively, the class of $\mathcal{A}$ and $\mathcal{B}$. Then, via an ingenious geometrical construction, he computed the number of bitangents to $\mathcal{C}$, i.e., the number of ordinary

64 In a misprint, see p. 343, $M - 2$ is written $M + 2 = 0$.

65 Since Poncelet it was known that two concentric circles are the projective image of two bitangent conics.

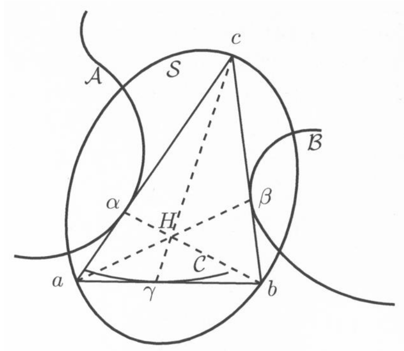

Fig. 19 In his paper (1857) Cayley considered a more general situation: a triangle inscribed in a conic S having two sides tangent to other two fixed curves A and B, which are not necessarily conics. Cayley showed that, when the triangle abc moves remaining inscribed in S and whose sides ac and bc remain tangents to curves A and B, respectively, the third side envelops a third curve C

double points of $\mathcal{C}^*$, which turned ut to be $mn(2mn - m - n + 1)$. He also proved that, in general, $\mathcal{C}$ does not have stationary tangents, i.e., that $\mathcal{C}^*$ does not have cusps. Then, through the Plücker formula,[66] he found that $\mathcal{C}$ has degree

$$2mn(2mn - 1) - 2mn(2mn - m - n + 1) = 2mn(m + n - 1).$$

Since Cayley had already examined, through a number of lemmas, the cases in which the curves and $\mathcal{A}$, $\mathcal{B}$ $\mathcal{S}$ are in particular positions, at p. 352 he stated the following:

Theorem C1 *If a triangle abc is inscribed in a conic $\mathcal{S}$, and the sides ac and bc are tangent to fixed curves $\mathcal{A}$, $\mathcal{B}$ of class m and n, respectively, the side ab will envelope a curve $\mathcal{C}$ of the class $2mn$, with in general $mn(2mn - m - n)$ double tangents, but not stationary tangents (i.e., not tangent at inflexion points), and therefore of the order $2mn(m + n - 1)$. If the curve $\mathcal{A}$ touch the conic $\mathcal{S}$, each point of contact will give rise to n double tangents of the curve $\mathcal{S}$, and so if the curve $\mathcal{B}$ touch the conic $\mathcal{S}$, each point of contact will give rise to m double tangents of the curve $\mathcal{C}$. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ intersect on the conic $\mathcal{S}$, each such intersection will give rise to a double tangent of the curve $\mathcal{C}$. The curve $\mathcal{C}$ in general touches the conic $\mathcal{S}$ in the points in which it is intersected by any common tangent of the curves $\mathcal{A}$ and $\mathcal{B}$; but if the points of contact be harmonically situated with respect to the conic $\mathcal{S}$, then $\mathcal{C}$ does not pass through the points of intersection, but the tangents to $\mathcal{S}$ at the points of intersection are stationary tangents of $\mathcal{C}$. There is of course in the above-mentioned special cases a corresponding reduction in the order of $\mathcal{C}$.*

Cayley applied the theorem above to the particular case in which the curves $\mathcal{A}$ and $\mathcal{B}$ are conics. In this case, the envelope $\mathcal{C}$ is of class 8 and, in general, of degree 24. Moreover, he considered two special cases of great interest (p. 353–354): first, $\mathcal{A}$ and $\mathcal{B}$ both have a double contact with the conic $\mathcal{S}$; second, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ all pass through the same four points.

66 For a curve C of degree d and class d', with δ nodes and κ cusps, one has $d' = d(d - 1) - 2\delta - 3\kappa$.

In the first case, Cayley showed that the curve $\mathfrak{C}$ has degree 8 and splits into four conics, each having a double contact with the conic $\mathcal{S}$. Attending only to one of these four conics, he obtained what he called "porism (homographic) of the in-and-circumscribed triangle":

If a triangle abc is inscribed in a conic, and two of the sides touch conics having double contact with the circumscribed conic to abc, then will the third side touch a conic having double contact with the circumscribed conic.

We observe that this is an extension of proposition 1 of subsection 2.4 above (Poncelet 1822, Art. 433), in which two of the sides of the triangle pass through fixed points, and the remaining side envelops a conic having a double contact with the circumscribed conic.

In the second case, he showed that the curve $\mathcal{C}$ has order 4 and splits into two conics, each passing through the point of intersection of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$. Attending only to one of these two conics, Cayley obtained what he called "porism (allographic) of the in-and-circumscribed triangle":

If a triangle abc is inscribed in a conic, and two of the sides touch conics meeting the circumscribed conic to abc in the same four points, the remaining side will touch a conic meeting the circumscribed conic in the four points.

In case of circles, the last claim is the "main lemma" of subsection 2.4.[67]

The following year Cayley published another paper on the subject, here he wrote (Cayley 1858, p. 31):

> In my former paper "On the Porism of the In-and-Circumscribed Triangle" [(Cayley 1857)], the two porisms (the homographic and the allographic) were established a priori, i.e., by means of an investigation of the order of the curve enveloped by the third side of a triangle. I propose in the present paper to give the a posteriori demonstration of these two porisms; first according to Poncelet, and then in a form not involving (as do his demonstration) the principle of projections.[68] My objection to the employment of the principle may be stated as follows: viz. that in a systematic development of the subject, the theorems relating to a particular case and which are by the principle in question extended to the general case, are not in anywise more simple or easier to demonstrate than are the theorems for the general case; consequently, that the circuity of the method can and ought to be avoided.

Likely these words were not appreciated by Poncelet.

Cayley gave two proofs of both the porisms, one according to Poncelet and one independent from the principle of continuity, that for sake of space we will not comment.

The memoir of 1861

Cayley published a complete proof of Theorem C years later in Cayley (1861). We present this proof here below, developing some details.

[67] In section eight of Bos et al. (1987), where the infinitesimal argument that Poncelet used in the proof of the main lemma is developed according to the modern theory of deformations, the authors showed that Poncelet's argument applies not only to conics but to algebraic curves in general.

[68] That is the principle of continuity.

He considered the conics $U = ax^2 + by^2 + cz^2 = 0$, $V = x^2 + y^2 + z^2 = 0$, the pencil $U + \xi V = 0$, and for $n = 3$ he proceeded as in the first notes (proposition C above). We remark that, at p. 229, he felt the need of justify the choice of these equations for U and V, by saying:

> The foregoing demonstration relates to the particular forms $U = ax^2+by^2+cz^2$, $V = x^2 + y^2 + z^2$; but observing that the function $\sqrt{(\xi + a)(\xi + b)(\xi + c)}$, which enters under the integral sign in the transcendental function $\Pi\xi$ is the square root of the discriminant of $U + \xi V$, the theory of covariants shows at once that the conclusions apply to any form whatever of U, V.[69]

For an n-gon which is inscribed in V, and whose sides touch in the order the conics $U_{k_1}, \ldots, U_{k_n}$, Cayley directly wrote the condition

$$\Pi(k_1) + \Pi(k_2) + \cdots + \Pi(k_n) = 0. \tag{5.5}$$

"By Abel's theorem," Cayley noticed (p. 230), "this transcendental equation is equivalent to an algebraical one." In fact, the $k_1, k_2, \ldots, k_n$ are the abscissae of the intersection points of the curve $y^2 = \Box x$ with some algebraic curve $\theta(x, y) = 0$. Then, extracting Cayley's paper, "if $\varphi(x)$ and $\chi(x)$ are polynomial in x with arbitrary coefficients, and if

$$\varphi(x)^2 + \chi(x)^2\Box x = A(x - k_1)(x - k_2)\cdots(x - k_n),$$

which implies that for n even the degrees of $\varphi(x)$ and $\varphi(x)$ are, respectively, $n/2$ and $(n - 4)/2$, and for n odd are, respectively, $(n - 1)/2$ and $(n - 3)/2$, the algebraical equation is that obtaining b the elimination of the arbitrary coefficients from the system of equations

$$\begin{aligned} \varphi(k_1) + \chi(k_1)\Box k_1 &= 0 \\ \varphi(k_2) + \chi(k_2)\Box k_2 &= 0 \\ &\vdots \\ \varphi(k_n) + \chi(k_n)\Box k_n &= 0 \end{aligned}$$

or, what is the same, for $n = 2p - 1$ it is

$$\{1, \theta, \ldots, \theta^{p-1}, \sqrt{\Box\theta}, \ldots, \theta^{p-2}\sqrt{\Box\theta}\} = 0,$$

and for $n = 2p$ it is

$$\{1, \theta, \ldots, \theta^{p}, \sqrt{\Box\theta}, \ldots, \theta^{p-2}\sqrt{\Box\theta}\} = 0,$$

where the expressions in { } denote, respectively, the determinants formed by substituting for θ the values $k_1, k_2, \ldots, k_n$, respectively. Thus, for $n = 3$ the equation is

[69] It seems to us that this amount to say: given two (non-singular) conics in general position, by a suitable projective transformation of the plane, their equations always can be put that form. A remark that he missed to do in his early notes.

$$\begin{vmatrix} 1 & k_1 & \sqrt{\Box k_1} \\ 1 & k_2 & \sqrt{\Box k_2} \\ 1 & k_3 & \sqrt{\Box k_3} \end{vmatrix} = 0$$

and for $n = 4$ it is

$$\begin{vmatrix} 1 & k_1 & k_1^2 & \sqrt{\Box k_1} \\ 1 & k_2 & k_2^2 & \sqrt{\Box k_2} \\ 1 & k_3 & k_3^2 & \sqrt{\Box k_3} \\ 1 & k_4 & k_4^2 & \sqrt{\Box k_4} \end{vmatrix} = 0$$

and so on.

Suppose

$$\sqrt{\Box \xi} = A + B\xi + C\xi^2 + D\xi^3 + E\xi^4 + \cdots,$$

then substituting the corresponding expressions for $\sqrt{\Box k_1}, \sqrt{\Box k_2}$, etc., the determinant will divide by $\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}$, and it may be seen without difficulty that the resulting equation, on putting therein $k_1 = k_2 = \cdots = k_n = 0$, will, according as $n = 3, 4, 5, 6$ etc., be

$$C = 0, \quad D = 0, \quad \begin{vmatrix} C & D \\ D & E \end{vmatrix} = 0, \quad \begin{vmatrix} D & E \\ E & F \end{vmatrix} = 0, \quad \begin{vmatrix} C & D & E \\ D & E & F \\ E & F & G \end{vmatrix} = 0, \quad \text{etc.},$$

which is the theorem above referred to."

Let us remark that, there exists a n-gon inscribed in V and circumscribed about U, if, and only, it is possible to find coefficients of the polynomial $\varphi(x)$ and $\chi(x)$ so that $\varphi(x) + \chi(x)\Box x$ has 0 as a root of multiplicity n.

In the remaining of the paper, Cayley applied his result to get the condition in polynomial form up to $n = 9$.

It is convenient here to change Cayley's notation. If $\sqrt{\Box x} = A + Bx + C_2 + C_3x^2 + \cdots$, then, for $n = 2m$ the condition above is equivalent to

$$\begin{vmatrix} C_3 & C_4 & \cdots & C_{m+1} \\ C_4 & C_5 & \cdots & C_{m+2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{m+1} & C_{m+2} & \cdots & C_{2m-1} \end{vmatrix} = 0, \tag{5.6}$$

for $n = 2m$, and for $n = 2m + 1$ to the condition

$$\begin{vmatrix} C_2 & C_3 & \cdots & C_{m+1} \\ C_3 & C_4 & \cdots & C_{m+2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{m+1} & C_{m+2} & \cdots & C_{2m} \end{vmatrix} = 0. \tag{5.7}$$

It is worth calling attention to the fact that Cayley did not try to explain the geometrical meaning of the above equations, nowadays called *Cayley's conditions*. This was done more than 100 years later by Griffiths and Harris.

In the following of the paper, Cayley applied his result to find explicit formulae for the existence of an in-and-circumscribed n-gon to U and V up to $n = 9$. Moreover, he considered the particular case of two circles, determining the required conditional equations that he compared with those found by J. Mention the year before (see section seven).

In his paper of (1861), Cayley did not quote Trudi. Could it be that Sylvester had not informed him of Trudi's results?

6 An algebraic approach through invariants

If two conics U and V are such that a n-gon inter-scribed to them there exists, it is obvious that a certain relation must hold among the invariants (and covariants) of the two conics. This remark induced George Salmon to produce an "elementary" proof of Cayley's result. What he had in mind was a proof built on the basics of the theory of invariants of a pair of conics, such as he had developed in his treatise (Salmon 1855), that avoids the use of the elliptic functions.

An *invariant* of an algebraic form $f(\mathbf{x})$, in two or more variables, is a polynomial $I(\mathbf{a})$ of the coefficients of $f(\mathbf{x})$, that, under a linear transformation of the variables of determinant Δ, remains unaltered up to a power of Δ, i.e., $I(\mathbf{a}') = \Delta^k I(\mathbf{a})$. If $k = 0$, the invariant is said *absolute*. A *covariant* of $f(\mathbf{x})$, is a polynomial $I(\mathbf{a}, \mathbf{x})$ of the coefficients and the variables of $f(\mathbf{x})$, which, under a linear transformation as above is such that $I(\mathbf{a}', \mathbf{x}') = \Delta^k I(\mathbf{a}, \mathbf{x})$. The theory of invariants and covariants of algebraic forms, which began to be developed in the early 1840s, with the pioneering work of George Boole, Cayley, James J. Sylvester, and Salmon, who formulated the basic concepts and developed the key techniques. Salmon also codified the theory in high-level textbooks. For the early history of the theory of invariants, we refer to Crilly (1986) and Hunger (1989, 2006a, b).

Salmon realized his program with the paper (Salmon 1857), that was published, divided into three distinct parts, in a single issue of the *Philosophical Magazine*. Here below we illustrate the content of this paper, extracting directly from it in the hope of keeping its original flavor, but also inserting some detail from Salmon's treatise on conic sections (Salmon 1855).

Let $\lambda U + V = 0$ be the general conic of the pencil generated by the pair of conics U, V. Its determinant, as a polynomial in λ, can be written

$$\det(\lambda U + V) = \Delta\lambda^3 + \Theta\lambda^2 + \Theta'\lambda + \Delta',$$

where Δ and Δ' are, respectively, the determinant of U and V, Θ and Θ' are, respectively, $\mathrm{tr}(U \cdot \mathrm{adj}(V))$ and $\mathrm{tr}(V \cdot \mathrm{adj}(U))$. We explicitly remark that Δ, Θ, Θ', Δ are of degree 3 in the coefficients of U and V. For a general theorem of the theory of invariants, all the projective invariants of the pair of conics U, V, are rational functions of Δ, Θ, Θ' and Δ'.[70]

In the first part of the paper (pp. 190–191), Salmon posed the following problem (a simplified version of Poncelet's main lemma): *find the envelop of the third side of the triangle inscribed in the conic U, and two of whose sides touch the conic V.*

To this end he argued as follows. The condition that $\lambda U + V = 0$ represents a pair of line is expressed by the condition $\Delta\lambda^3 + \Theta\lambda^2 + \Theta\lambda + \Delta' = 0$. Salmon wrote:

> Since the value of λ plainly cannot depend on the particular axes to which the equations are referred, it follows that no matter how the equations are transformed, the ratios of the coefficients of the powers of λ in the equation just written remain unaltered. Let now the sides of the triangle in any position be x, y, z, then the equations of the conics admit of being transformed into:
>
> $$U = 2xy + 2yz + 2xz = 0,$$
> $$V = l^2x^2 + m^2y^2 + n^2z^2 - 2lmxy - 2lnxz - 2mnyz - 2Axy = 0;$$
>
> and it is plain that the equation
>
> $$AU + V = 0$$
>
> represents a conic that the third side z touches.
>
> But in this case we find, if $p : l + m + n$ and $r := lmn$,
>
> $$\Delta = 2, \quad \Theta = -p^2 - 2A, \quad \Theta' = 2p(2r + An), \quad \Delta' = -(2r + Ar)^2,$$
>
> whence
>
> $$4\Theta\Delta' - \Theta'^2 = 8A(2r + An)^2,$$
>
> and the equation $AU + V = 0$ can be written
>
> $$(4\Theta\Delta' - \Theta'^2)U - 4\Delta\Delta' V = 0.$$
>
> The coefficients in this equation being invariants, it follows that the conic which we have proved is touched by the third side is a *fixed* conic. which is an equation depending on the coefficients of the two given conics.

Let us remark that, if the condition $4\Theta\Delta' - \Theta'^2 = 0$ is satisfied, is clear that the envelope of the third side of the triangle coincides with the conic V.

Next Salmon asked the question: *find the locus of the third vertex of a triangle circumscribed about V when the other two vertices move along U.*

He continued by observing:

[70] For a modern approach to the theory of invariants of a pair of conics (see Sommerville 1933; Todd 1947 or Dolgachev 2012).

In this case, the equations of the conics U and V can be transformed into

$$U = 2xy + 2yz + 2xz + Az^2 = 0,$$
$$V = l^2x^2 + m^2y^2 + n^2z^2 - 2lmxy - 2lnxz - 2mnyz = 0,$$

and we have

$$\Delta = 2 - A, \quad \Theta = -p^2 + 2lmA, \quad \Theta' = 4pr, \quad \Delta' = -4r^2.$$

Again, let $F = 0$ be the equation of the covariant conic which passes through the points of contact of the common tangents to U and V (see my "Conics," pp. 268, 288),[71] the coefficient of z^2 in its equation will be $-4rn(1 - A)$.

It is useful here to recall some facts pertaining to the theory of reciprocal (dual) conics. Let S and S' be two general conics of equation

$$S = ax^2 + a'y^2 + a''z^2 + 2bzy + 2b'yx + 2b''xz = 0,$$
$$S' = Ax^2 + A'y^2 + A''z^2 + 2Bzy + 2B'yx + 2B''xz = 0,$$

then, the reciprocal conic of the conic $S + \lambda S'$ has equation

$$\Sigma + \lambda\Phi + \lambda^2\Sigma' = 0, \tag{6.1}$$

where Σ and Σ' are, respectively, the reciprocal conics of S and S', and Φ is the polynomial

$$(a'A'' + a''A' - 2bB)x^2 + (a''A + aA'' - 2b'B')y^2 + (aA' + a'A - 2b''B'')z^2$$
$$+2(b'B'' + b''B' - aB - bA)yz + 2(b''B + bB'' - a'B' - b'A')xz$$
$$+2(bB' + b'B - a''B'' - b''A'')xy.$$

Since the conics of the original system pass through four fixed points, the conics of the reciprocal system always touch four fixed lines.

The form of equation (6.1) shows that the reciprocal always touches $4\Sigma\Sigma' - \Phi^2 = 0$. This means that this last equation is the equation of the four common tangents to Σ and Σ' and to the other conics of the reciprocal system. The form of $4\Sigma\Sigma' = \Phi^2$ also shows that Σ is touched by those four lines and that Φ passes through the points of contact. Since the same holds for Σ', it follows that the eight point of contact of the four common tangents to the two conics Σ and Σ', all lie on the conic $\Phi = 0$.

The reciprocal of the system $\Sigma + \lambda\Sigma' = 0$ is the system

$$\Delta S + \lambda F + \lambda^2\Delta' S' = 0,$$

where F is what Φ becomes when the coefficients of Φ are written in terms of the elements of the polynomials Σ and Σ', $\mathfrak{a}, \mathfrak{a}', \ldots, \mathfrak{b}''$, and $\mathfrak{A}, \mathfrak{A}', \ldots, \mathfrak{B}''$:

[71] Salmon was referring to Salmon (1855).

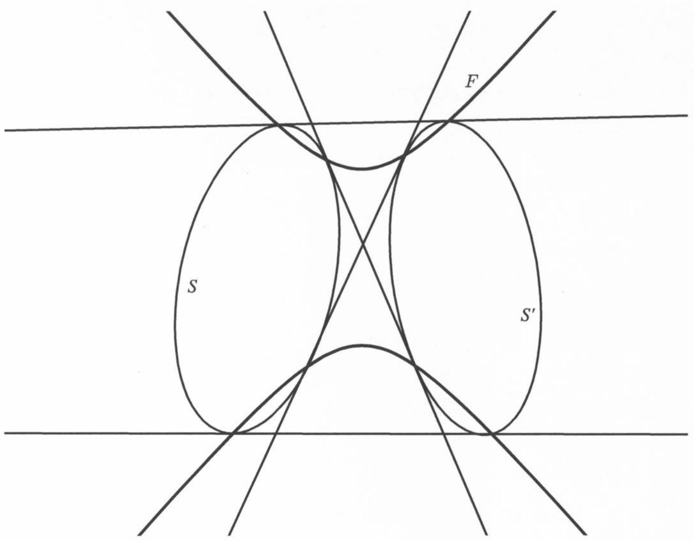

Fig. 20 The conic of equation $F = 0$ passing through the eight points of contact of the four common tangents to two conics S and S'

$$\begin{aligned}&(\mathfrak{a}'\mathfrak{A}'' + \mathfrak{a}''\mathfrak{A}' - 2\mathfrak{b}\mathfrak{B})x^2 + (\mathfrak{a}''\mathfrak{A} + \mathfrak{a}\mathfrak{A}'' - 2\mathfrak{b}'\mathfrak{B}')y^2 + (\mathfrak{a}\mathfrak{A}' + \mathfrak{a}'\mathfrak{A} - 2\mathfrak{b}''\mathfrak{B}'')z^2\\ &+2(\mathfrak{b}'\mathfrak{B}'' + \mathfrak{b}''\mathfrak{B}' - \mathfrak{a}\mathfrak{B} - \mathfrak{b}\mathfrak{A})yz + 2(\mathfrak{b}''\mathfrak{B} + \mathfrak{b}\mathfrak{B}'' - \mathfrak{a}'\mathfrak{B}' - \mathfrak{b}'\mathfrak{A}')xz\\ &+2(\mathfrak{b}\mathfrak{B}' + \mathfrak{b}'\mathfrak{B} - \mathfrak{a}''\mathfrak{B}'' - \mathfrak{b}''\mathfrak{A}'')xy.\end{aligned}$$

From the above, it follows that the conic $F = 0$ passes through the eight points of contact of the four common tangents to S and S' (see Fig. 20).

Moreover, it is easy to check that $\mathfrak{a}\mathfrak{A}' + \mathfrak{a}'\mathfrak{A} - 2\mathfrak{b}''\mathfrak{B}''$ is equal to $-4rn(1 - A)$ when $S = U$ and $S' = V$.

At this point, Salmon wrote:

it can be seen that the coefficient of z^2 vanishes identically in the equation

$$16\Delta'^2\Delta U - 4(4\Theta\Delta' - \Theta'^2)\Delta' F + (4\Theta\Delta' - \Theta'^2)^2 V = 0,$$

which is therefore the equation of the locus required. Its form shows that this conic is tangent to the four common tangents to U and V.

If $4\Theta\Delta' - \Theta'^2 = 0$, the envelope reduces to V, and the locus to U, in conformity to Mr. Cayley's theorem. It does not seem impracticable to obtain the equation of the locus by the same method when the two sides touch *different* conics.

In the second part of his paper (pp. 267–269), Salmon considered the problem of finding the locus of the fourth vertex of a quadrilateral, whose other three vertices move on V, and whose sides touch U. He put for brevity

$$\alpha := 4\Delta\Delta', \quad \beta := \Theta^2 - 4\Delta\Theta', \quad \gamma := 2\Delta\alpha + \Theta\beta,$$

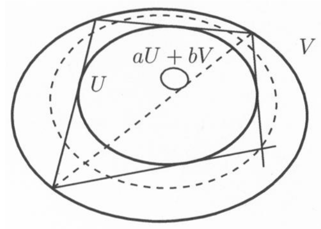

Fig. 21 Salmon's question for $n = 4$

and reduced the problem to finding the locus of the third vertex of a triangle two of whose vertices move on V, two of whose sides touch U, and the third touches $\alpha U + \beta V$. Proceeding as in the first part of his paper, he obtained for the required locus the following equation

$$\alpha^2\gamma^2 U + \alpha\gamma\beta^2 F + \Delta\Delta'\beta^4 V = 0,$$

which clearly reduces to V if $\gamma = 0$.

Finally, in the third part (pp. 337–338), Salmon considered the question: *find the locus of the free vertex of a polygon, whose sides all touch U, and whose vertices all but one move on V.*[72]

On p. 337, he wrote:

> This [the question above] is immediately reduced to the last question, since the line joining the two vertices of the polygon adjacent to that whose locus is sought, touches a conic whose equation is of the form $aU + bV = 0$ [see Fig. 21]. The locus will therefore always be of the form
>
> $$\Delta\Delta'\lambda^2 V + \lambda\mu F + \mu^2 U = 0,$$

So Salmon proceeded by induction, similarly to Poncelet in his proof of the general theorem.

The procedure allowed him to find a recursive procedure for determining the condition that should make it possible to describe a polygon inscribed in V and circumscribed about U. Precisely: if λ', μ' are the values for a polygon of $n - 1$ sides, and λ'', μ'' those for a polygon of n sides, then the values for a polygon of $n + 1$ sides are

$$\lambda''' = \mu'\mu''^2, \quad \mu''' = \Delta'\lambda'\lambda''(\alpha\mu'' - \Delta'\beta\lambda'').$$

Since for a triangle one has $\lambda' = \alpha$, $\mu' = \Delta'\beta$, for quadrilateral one has $\lambda'' = \beta^2$, $\mu'' = \alpha\gamma$, the required conditions are:

[72] Let us recall that Landen had determined this locus when U and V are real circles.

$$\begin{array}{lll} \text{triangle,} & \beta = 0, & \\ \text{quadrilateral,} & \gamma = 0, & \\ \text{pentagon,} & \delta = 0, & \delta := \alpha^2\gamma - \Delta'\beta^3, \\ \text{hexagon,} & \epsilon = 0, & \epsilon := \delta - \Delta'\gamma^2, \\ \text{heptagon,} & \phi = 0, & \phi := \alpha^2\gamma\epsilon - \delta^2, \\ \text{octagon,} & \psi = 0, & \psi := \delta\phi - \Delta'\beta^3\epsilon^2, \\ & \vdots & \end{array}$$

Salmon concluded by stating: "I suppose these values will be found to coincide with those found by a different way by Mr. Cayley in a former Number of this Journal, but I have not verified this."

Two years after Salmon's paper was printed, Francesco Brioschi proved that the formulae of Cayley and Salmon were equivalent, by showing that both descend from a common principle (Brioschi 1857).

To show this, he supposed the conic U is circumscribed to the triangle *abc* whose sides are given by $x = 0$, $y = 0$, $z = 0$. This allowed him to put the equation of U in the form

$$U = \alpha yz + \beta zx + \gamma xy = 0.$$

Then, he considered another conic

$$V = l^2x^2 + m^2y^2 + n^2z^2 - ayz - bzx - cxy = 0,$$

and observed that the lines $x = 0$, $y = 0$, $z = 0$ will be, respectively, tangents to the conics $k_1U - V = 0$, $k_2U - V = 0$ and $k_3U - V = 0$ if and only if

$$a = 2mn - \alpha k_1, \quad b = 2ln - \beta k_2, \quad c = 2lm - \gamma k_3.$$

Brioschi denoted

$$\Delta^2(k) = a_0k^3 + a_1k^2 + a_2k + a_3$$

the discriminant of the "function" $kU - V$. Setting

$$p = l\alpha + m\beta + n\gamma, \quad q = 4r - l\alpha k_1 - m\beta k_2 - n\gamma k_2, \quad r = lmn,$$

he got

$$a_0 = \alpha\beta\gamma, \quad a_1 = p^2 - \alpha\beta\gamma(k_1 + k_2 + k_3),$$
$$a_2 = 2pq + \alpha\beta\gamma(k_1k_2 + k_3k_1 + k_2k_3), \quad a_3 = q^2 - \alpha\beta\gamma k_1k_2k_3.$$

Hence, if the equation

$$k^3 + Ak^2 + Bk + C = 0$$

has solutions k_1, k_2, k_3, it will be

$$a_1 - a_0A = p^2, \quad a_2 - a_0B = 2pq, \quad a_3 - a_0C = q^2. \tag{6.2}$$

Multiplying these equations, respectively, for k_1^2, k_1, 1, adding up, and taking into account the previous equation, he got

$$a_0k_1^3 + a_1x_1^2 + a_2x_1 + a_3 = (pk_1 + q)^2.$$

Proceeding similarly for k_2 and k_3, he finally found that

$$a_0k^3 + a_1k^2 + a_2k + a_3 - (pk + q)^2 = a_0(k - k_1)(k - k_2)(k - k_3), \tag{6.3}$$

and therefore, if $\psi(x) := \int dk/\Delta(k)$, by Abel's theorem, as in Cayley (1853a), he got that k_1, k_2, k_3 must satisfy the transcendental equation

$$\epsilon_1\psi(k_1) + \epsilon_2\psi(k_2) + \epsilon_3\psi(k_3) = C,$$

which is equivalent to the following (irrational) algebraic equation

$$\begin{vmatrix} 1 & k_1 & \Delta(k_1) \\ 1 & k_2 & \Delta(k_2) \\ 1 & k_3 & \Delta(k_3) \end{vmatrix} = 0. \tag{6.4}$$

Brioschi noticed that (6.2) also leads to

$$4(a_1 - a_0A)(a_3 - a_0C) - (a_2 - a_0B)^2 = 0. \tag{6.5}$$

Now, relation (6.4) constitutes the result of Cayley ($n = 3$), while relation (6.5) constitutes the result of Salmon ($n = 3$). From (6.3), it follows that

$$p = \frac{\Delta(k_1) - \Delta(k_2)}{k_1 - k_2}, \quad q = \frac{k_1\Delta(k_2) - k_2\Delta(k_1)}{k_1 - k_2},$$

then, setting $\Delta(k) = A_0 + A_1k + A_2k^2 + \cdots$ the development in power series of the discriminant, he obtained

$$q = A_0 - k_1k_2P,$$

where

$$P = A_2 + A_3(k_1 + k_2) + A_4(k_1^2 + k_1k_2 + k_2^2) + \cdots$$

From the relations above, it follows that $k_3 = a_0^{-1}(k_1k_2P^2 - 2A_0P)$, which, if $k_1 = k_2 = 0$, i.e., the first two sides of the triangle are tangent to V, gives

$$k_3 = -\frac{2A_0A_2}{a_0} = \frac{a_2^2 - 4a_1a_3}{4a_0a_3},$$

This means, remarked Brioschi, that the triangle *abc* is circumscribed to V if and only if

$$a_2^2 - 4a_1a_3 = 0,$$

conditions equivalent, respectively, to (Cayley's) $C = 0$ and to (Salmon's) $4\Theta\Delta' - \Theta'^2 = 0$.

By considering diagonals and intermediate triangles, as Poncelet and Jacobi had done in the case of a pencil of circles (recall Fig. 11b), Brioschi showed that the same holds true first for quadrilateral, and then for a polygon of any number of sides.

Almost 100 years after the publication of Salmon's paper, John A. Todd revisited the same subject in Todd (1948). We will return on this argument in section thirteen.

7 Other contributions from 1850 to 1875

In the third quarter of the nineteenth century, many papers related to Poncelet's theorem and its generalizations appeared. In this section, we present and discuss only those which, in our opinion, are the most interesting both *per se* and from a historical point of view. We have divided them into two major branches along which the theory developed: (1) find new proofs of PCT and simplify those already known; (2) extend the theorems in higher dimension and prove other "closure theorems."

In this section, we also aim to introduce the reader to the topics discussed in the subsequent sections.[73]

7.1 New proofs of PCT

In 1849, a new paper by Jacobi's pupil Richelot was printed. In his (1849) he obtained, in the case of two nested circles, an algebraic condition for the existence of an in-and-circumscribed p-gon, p being a prime. He also gave a method for solving the problem in case of a polygon of n sides knowing the solution for polygons of $n-1$ and $n+1$ sides.

In 1860, J. Mention published *Essai sur le problème de Fuss* (Mention 1860).[74] He called "problem of Fuss," the problem of determining the relation between the data R, r, δ, n for the existence of a n-gon inter-scribed to two circles, respectively, of radii R, r, being δ the distance between their centers. To solve the problem, Mention argued, very ingeniously, as follows.

Let AB be a chord of the circle C of radius R and center O, which is tangent to the circle c of center I and radius r. Denote by $\widehat{A}$, and $\widehat{B}$, the angle between the chord and

[73] For a more extended review of the literature of this period, we refer to Loria (1889a, b, 1896), but alert the reader that many references therein are incorrectly dated or have page numbering wrong, or even present a misleading indiction of the journal that should contain the quoted paper.

[74] The memoir was read at the Academy of Saint Petersburg the 13th of Mai 1859. Very little is known about J. Mention (1821–?), probably a Russian mathematician. In the years 1845–1865, he published several short notes in *Nouvelles Annales de Mathematique*.

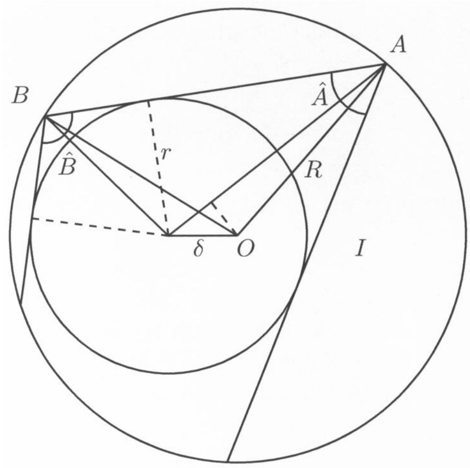

Fig. 22 Illustration of how Mention proceeded in order to determine the relation among R, r, δ, n for the existence of a n-gon inter-scribed to two circles of radii R and r, being δ the distance between their centers

the other tangent to c drawn, respectively, from A, and B (Fig. 22). We have that

$$OA = \frac{r}{\sin\frac{\widehat{A}}{2}}, \quad OB = \frac{r}{\sin\frac{\widehat{B}}{2}}.$$

Then, considering the triangles IAO, IBO and by using some trigonometry one has

$$\delta^2 = R^2 + \frac{r^2}{\sin^2\frac{\widehat{A}}{2}} - \frac{2Rr}{\sin\frac{\widehat{A}}{2}} \cos\left(\widehat{BAO} - \frac{\widehat{A}}{2}\right),$$
$$\delta^2 = R^2 + \frac{r^2}{\sin^2\frac{\widehat{B}}{2}} - \frac{2Rr}{\sin\frac{\widehat{B}}{2}} \cos\left(\widehat{ABO} - \frac{\widehat{B}}{2}\right).$$

By subtracting the two equations above, and using some trigonometry, Mention got the following equation

$$\frac{r}{2R}\left(\cot\frac{\widehat{A}}{2} + \cot\frac{\widehat{B}}{2}\right) = \cos\widehat{BAO}.$$

Now, by adding the same two equations above, and taking into account the previous one, he got

$$\frac{R^2 + r^2 - \delta^2}{2Rr} - \frac{r}{2R}\cot\frac{\widehat{A}}{2}\cot\frac{\widehat{B}}{2} = \sin\widehat{BAO}.$$

By squaring and adding, he finally obtained

$$4R^2r^2 - \frac{(R^2+r^2-\delta^2)^2}{r^4} = \cot^2\frac{\widehat{A}}{2} + \cot^2\frac{\widehat{B}}{2} + \cot^2\frac{\widehat{A}}{2}\cot^2\frac{\widehat{B}}{2} + \\ -2\cot\frac{\widehat{A}}{2}\cot\frac{\widehat{B}}{2}\left(\frac{R^2-\delta^2}{r^2}\right).$$

Setting

$$\nu = \frac{2R^2r^2 + 2R^2\delta^2 + 2r^2\delta^2 - R^4 - r^4 - \delta^4}{r^4}, \quad i = \frac{R^2-\delta^2}{r^2},$$

and $x_1 = \cot\widehat{A}/2$, $x_2 = \cot\widehat{B}/2$, Mention wrote the equation above in the form

$$\nu = x_1^2x_2^2 + x_1^2 + x_2 - 2ix_1x_2.$$

If a polygon of n sides is inscribed in C and circumscribed about c, denoting $x_1, x_2, x_3, \dots, x_n$ the cotangents of the half of the angles at the respective vertices, one has the following system:

$$\begin{cases} \nu = x_1^2x_2^2 + x_1^2 + x_2^2 - 2ix_1x_2 \\ \nu = x_2^2x_3^2 + x_2^2 + x_3^2 - 2ix_2x_3 \\ \vdots \\ \nu = x_{n-1}^2x_n^2 + x_{n-1}^2 + x_n^2 - 2ix_{n-1}x_n \\ \nu = x_n^2x_1^2 + x_n^2 + x_1^2 - 2ix_nx_1. \end{cases} \tag{7.1}$$

These equations, Mention affirmed, will be compatible only if a certain relation among ν, i and n holds true, and, vice versa, if such a relation is satisfied it will be possible to give to any initial angle an arbitrary value. Therefore, if a polygon of n-sides can be inscribed in C and circumscribed about c, then there are infinitely many such polygons.

For a triangle, one has

$$\nu = x_1^2x_2^2 + x_1^2 + x_2^2 - 2ix_1x_2$$
$$\nu = x_2^2x_3^2 + x_2^2 + x_3^2 - 2ix_2x_3$$
$$\nu = x_3^2x_1^2 + x_3^2 + x_1^2 - 2ix_3x_1,$$

from the first and the last equations one gets

$$x_2 + x_3 = \frac{2ix_1}{1+x_1^2}, \quad x_2x_3 = \frac{x_1^2-\nu}{1+x_1^2}.$$

By substituting these values into the second, after some computation, one is lead to

$$\nu + 2i + 1 = 0,$$

which is readily seen to be equivalent to $\delta^2 = R^2 \pm 2Rr$, which includes (1.1).

Proceeding similarly, Mention found the conditional equation that allows the existence of an inter-scribed n-gon for n up to 11, but for higher value of n, the difficulties in the elimination become insurmountable, and he was able to give only some recursive formula.

Let us remark that the system (7.1) can be deduced from the more general (4.10) found by Trudi.

In 1862, Poncelet published the first volume of *Applications d'analyse et de géométrie*. In the *Note historique*, inserted at the end of the book, he described the development of the theory until then.[75] Recalling (Jacobi 1828), he expressed his disagreement with the term "géométrie élémentaire" that Jacobi used in the title in connection with PCT. Then, Poncelet quoted the results of Fuss, Steiner, Richelot, Cayley, Brioschi and Mention. This latter, according to him, had the great merit of having tackled the question directly and geometrically, and, although he had only solved the problem for $n \leq 11$, he had highlighted the scale of relation between the polygons of $n, n-1$ and $n-2$ sides (see Poncelet 1862, p. 483). He continued by saying:

> C'est d'ailleurs une question de savoir si le problème, si mal résolut par Fuss en 1792, l'a été mieux depuis par d'autre, notamment en Angleterre par M. Cayley, qui, ignorant sans doute mes publications de 1817 et 1822 citées plus haut, a attribué gratuitement à cet ancien et estimable géomètre, sous le nome de *porisme*, le théorème de la p. 364 sur les cercles. Parmi les nombreux Mémoire de M, Cayley, écrits dans une langue mathématique pour moi doublement étrangère, j'entrevois bien, en effet, de belles méthodes algébriques pour passer d'un terme à un autre de la série des polygones, mais non pour franchir, sans calculs intermédiaires, l'intervalle qui sépare entre eux deux termes de rang quelconque. Ainsi, par exemple, dans son dernier Mémoire résumé, de mars 1861, il n'arrive à la formule de l'ennéagone, obtenue par M. Mention et relative au cas simple de deux cercles, qu'après avoir laborieusement calculé toutes celles qui appartiennent aux polygones d'ordre inférieur [This is also a question of whether the problem, so badly solved by Fuss in 1792, has been better solved by others later, as in England by M. Cayley, who, undoubtedly ignorant of my publications of 1817 and 1822 quoted above, assigned gratuitously to this estimated geometer, under the name of *porism*, the theorem at p. 364 on the circles. Among the memoirs of M. Cayley, which are written in a mathematical language doubly extraneous to me, I see, indeed, beautiful algebraic methods for passing from one term to another of the series of polygons, but which do not cross, without intermediate calculations, the interval between two of any rank. So, for example, in his latest and concluding memoir of March 1861, he arrives at the formula for the enneagon, obtained by M. Mention and relative to the simple case of two circles, only after having painstakingly calculated all those belonging to polygons of a lesser number of sides].

Cayley replied to Poncelet's remarks with a letter in which, after having denied wish to attribute the theorem in question to Fuss, he dealt with the criticism levelled

[75] This note makes for very interesting reading in many ways.

at his paper of 1861. He briefly reviewed his method, displayed the formulae for n up to 8 and stressed that the condition was actually and explicitly found for a polygon of any number of sides, underlying, "*sans passer par celles qui appartiennent aux polygones d'ordre inférieur*" [without passing through those belonging to polygons of a less number of sides]. Cayley's letter to Poncelet ended with this meaningful statement:

> Comme j'attache, je l'avoue, un peu d'importance à cette solution (laquelle selon l'explication que je viens de donner ne parait pas mériter la critique que vous en faites) je serais bien aise si vous voulez bien communiquer cette lettre à l'Académie [As I attach, I confess, some importance to this solution (which, according to the explanation I have just given, does not seem to deserve your criticism) I will be glad if you could communicate this letter to the Academy].

The letter was quickly published in the *Comptes rendus* (Cayley 1862).

In his historical note, Poncelet failed to quote Trudi, whose memoir of 1853 he had probably not read. This lack of recognition somewhat annoyed Trudi, who regarded his results as being more general and deeper than those of Mention.

The following year Trudi published the long memoir (Trudi 1863a), on which, returning to the question related to the existence of inter-scribed polygons to two conics, he claimed priority for the complete analytical proof of Poncelet's closure theorem. In the *Notizie storiche*, that he inserted at the beginning of his work, Trudi suggested that he had not been mentioned because the title of his memoir of 1853, *Rappresentazione geometrica immediata dell'equazione fondamentale nella teorica delle funzioni ellittiche* [Immediate geometrical representation of the fundamental equation in the theory of elliptic functions], had not alluded at all to Poncelet's theorem and related questions. To endorse his priority, he also mentioned the first studies of 1841, the memoir of 1843 he had read at the *Congresso degli Scienziati* held in Naples in 1845, and the encouragements he had received from Jacobi that led him to write the memoir of 1853.[76] Then, he added:

> Risulta da questi fatti che noi possiamo pretendere alla piccola gloria di aver dato i primi una dimostrazione analitica compiuta e diretta dei teoremi di Poncelet, e di aver dato anche i primi un metodo per la ricerca della relazione, affiché un poligono di qualsivoglia numero di lati possa esser iscritto e circoscritto a due coniche [From these facts, it appears that we can expect little glory for being the first to give a direct and complete analytical proof of the theorems of Poncelet, and for being also the first to give a method to obtain the relation under which a polygon of any number of sides can be inscribed in and circumscribed about to two conics].

The same year, Trudi published another paper on the same subject, that he entitled, more explicitly, *Su' teoremi di Poncelet relativi a' poligoni iscritti e circoscritti alle*

[76] In passing we note that Trudi's memoir of 1853 was cited by Angelo Genocchi in his paper on a construction of the theorem of Abel, in relation to the addition of elliptic functions, published in the first volume of the just founded *Annali di Matematica Pura e Applicata* (Genocchi 1858, p. 36).

coniche [On Poncelet's theorems related to inscribed and circumscribed polygons to conics] (Trudi 1863b).

These two memoirs do not add much to the previous one of 1853, so for sake of space we avoid comments. Suffice to say that Trudi (1853) and Trudi (1863a) were mentioned by Loria (1889b, 1896), while Dingeldey only quoted the second (Dingeldey 1903, p. 47). These memoirs and (Trudi 1863b) were cited in Gerbaldi (1919, p. 97), where Trudi's method was shortly presented (see our section eleven). Since then it seems that Trudi's work on the theorem of Poncelet has been forgotten until very recently (see Dragović 2011, p. 105).

Poncelet's book quoted above contained, as an appendix, a memoir by Théodore Moutard,[77] titled *Recherhes analytiques sur les polygones simultanément inscrits et circonscrits a deaux coniques* (Moutard 1862). By means of algebraic methods, Moutard wrote an equation of the curve enveloped by the last side of a n-gon inscribed in a conic $\mathfrak{A}_0$, and whose first $n-1$ sides are tangent to another conic $\mathfrak{A}$, when its vertices move along $\mathfrak{A}_0$. By means of elegant geometrical considerations, he found a simple recursive law for the formation of the conditional equations relative to the cases $3, 4, \ldots, n$. The study of this law comes down to the study of certain functional equations, whose solution leads directly to the transcendental functions Θ, H of Jacobi, of which the elliptic functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ are simple rational expressions. Moutard observed how many of the properties of these functions were related to the theorem of Poncelet.

Jakob Rosanes and Moritz Pasch, with their joint work (Rosanes and Pash 1865), also completed Jacobi's project. Generalizing the method used by Jacobi, they were able to write the relation that the coefficients of two conics, A and B, must satisfy for the existence of a polygon of n sides inter-scribed to them. In the introduction to their paper, after having recalled Euler, Fuss, Steiner, Jacobi, they quoted (Cayley 1853a, b), (Moutard 1862), briefly summarized the results therein, and added:

> Die gegenwärtige Abhandlung, deren Verfasser von den letztgenannten beiden Arbeiten bis vor kurzer Zeit keine Kenntniss hatten, scheint von diesen sowohl in Bezug auf den eingeschlagenen Weg, als die Form der Resultate, welche grosse Ächnlichkeit mit den von Jacobi gefundenen Formeln aufweist, sosehr verschieden, dass die Veröffentlichung derselben wohl gerechtfertigt erscheinen dürfe [the present memoir, whose authors until recently had no knowledge of the last works mentioned, which seem to have great resemblance, both in terms of path as in the shape of the results, with the formulae found by Jacobi, are indeed rather different, so that the publication of it probably could be justified.]

By performing a projective change of coordinates, Rosanes and Pasch put the equations of A and B in the simple form

$$x^2 + y^2 + z^2 = 0, \quad \alpha x^2 + \beta y^2 + \gamma z^2 = 0,$$

[77] Théodore Florentin Moutard (1827–1901), engineer. His mathematical work was primarily in the theory of algebraic surfaces, differential geometry and differential equations. He taught mechanics at the *École des mines*.

and considered separately four cases, according to the behavior of the intersections and the common tangents of the two conics are real or imaginary, and for each of these cases they obtained the relative condition allowing the existence of an inter-scribed polygon. Finally, they computed these relations in terms of the coefficients of the cubic polynomial

$$\delta_0\lambda^3 + \delta_1\lambda^2 + \delta_2\lambda + \delta_3 = \det(B - \lambda A)$$

for $n = 3, 4$, getting, respectively

$$\delta_0^2 = 4\delta_1\delta_2, \quad \delta_2^3 + 8\delta_0\delta_3^2 = 4\delta_1\delta_2\delta_3.$$

As we know, these relations were already obtained by Salmon years before.

Four years later, in their paper (Rosanes and Pash 1869), Rosanes and Pasch recognized that the above question (and in fact an entire class of geometrical problems) could be put in the following form.

Let there be given a symmetric doubly quadratic equation

$$f(t_0, t_1) = at_0^2t_1^2 + 2bt_0t_1(t_0^2 + t_1^2) + c(t_0 + t_1)^2 + 2dt_0t_1 + 2e(t_0 + t_1) + f.$$

For a fixed value of t_1, there are two values of the first variable satisfying the equation, say t_0 and another value t_2. Fixed t_2, there is another value other than t_1, say t_3, satisfying the same equation. Proceeding in this way one gets a sequence $t_0, t_1, t_2, \ldots, t_n$, such that $f(t_0, t_1) = f(t_1, t_2) = \cdots = f(t_{n-1}, t_n) = 0$. The question is: for a given $n > 2$ is $t_0 = t_n$ and at the same time $t_{n+1} = t_1$, and in general $t_{n+h} = t_h$?

At this point (p. 169), the two authors affirmed that t_0 and t_n satisfy an equation of the same form as above, that is, by eliminating the intermediate variables $t_1, \ldots, t_{n-1}$ one obtains

$$a_{n-1}t_0^2t_n^2 + 2b_{n-1}t_0t_n(t_0^2 + t_n^2) + c_{n-1}(t_0 + t_n)^2 + 2d_{n-1}t_0t_n + 2e(t_0 + t_n) + f_{n-1}.$$

Let us remark that this does not appear completely justified, and in fact, Trudi felt the need to prove it by means Euler's differential equation.

Rosanes and Pasch asked for the condition under which $t_0 = t_n$ for a certain $n > 2$. By means of an elaborate algebraic computation, they proved that a necessary and sufficient condition is given by the vanishing of the function q_n, defined recursively as follows:

$$q_0 = 0, q_1 = q_2 = 1, q_3 = \sigma$$
$$q_{n-2}q_{n+2} + q_{n-1}q_{n+1} = \sigma q_n^2,$$
$$q_{n-2}q_{n+1}^2 + q_{n+2}q_{n-1}^2 = q_n(\lambda q_n^2 + 2\delta q_{n-1}q_{n+1}),$$

if n is even, or

$$q_{n-2}q_{n+2} + \lambda q_{n-1}q_{n+1} = \sigma q_n^2,$$
$$q_{n-2}q_{n+1}^2 + q_{n+2}q_{n-1}^2 = q_n(q_n^2 + 2\delta q_{n-1}q_{n+1}),$$

if n is odd.

Clearly, q_n is an entire function of the coefficients of $f(t_0, t_1)$, of degree $n^2/4 - 1$ or $(n^2 - 1)/4$ according if n is even or odd.

Here (p. 173) Rosanes and Pasch, without further explanation, claimed that: if $P_0, P_1, \ldots, P_n$ is a polygonal line inscribed in A and circumscribed about B, the corresponding parameters $t_0, t_1, \ldots, t_n$ of these points satisfy, in pairs (t_0, t_1), (t_1, t_2), etc., to a symmetric biquadratic equation of the previous type, whose coefficients depend on those of A and B. Hence, the condition for the closure of the polygonal line, i.e, for the existence of a n-gon inter-scribed to A and B, is given by $q_n = 0$. Then, expressing λ, σ and δ in terms of the coefficients of the discriminant $\det(B - \alpha A) = \delta_0\alpha^3 + \delta_1\alpha^2 + \delta_2\alpha + \delta_3$ [as in Rosanes and Pash (1865), section 9], they wrote down, respectively, for a triangle a quadrangle and a pentagon, the following conditions:

$$q_3 = \delta_2^2 - 4\delta_1\delta_3 = 0, \quad q_4 = 2[8\delta_0\delta_3^2 + (\delta_2^2 - 4\delta_2\delta_3)] = 0,$$
$$q_5 = q_3^2 - 16\delta_0\delta_3^2 q_4 = 0,$$

which are readily seen to be equivalent to that given by Salmon.

Hence, on the base of induction, Rosanes and Pasch claimed: q_n *is an entire function of the coefficients of the two conics, which is of degree*

$$3\left(\frac{n^2}{4} - 1\right) \quad or \quad 3\frac{n^2 - 1}{4}, \tag{7.2}$$

according if n is even or odd.

The question of determining the degree of the invariant whose vanishing guaranties the existence of an inter-scribed n-gon to the two given conics, was to be studied in depth by Gerbaldi 50 years later.

In a footnote, Rosanes and Pasch gave notice of dissertation (Simon 1867) by Max Simon.[78] In his thesis, Simon presented a new proof of PCT by means of the emerging theory of the Weierstrass $\wp$-function, instead of the classical elliptic functions of Jacobi, and expressed the conditional equation in terms of the invariant of the pencil of conics. He also noticed the relation between PCT and biquadratic binary equations (Simon 1867, pp. 8–12). It is worth to say that in 1864–1865, while still student in Berlin, Simon participated in a seminar dealing with these topics. An enlarged version of the thesis was published years later (Simon 1876). This approach to PCT was later codified by Halphen, in the second volume of his treatise on elliptic functions (Halphen 1888). We will return on this in section ten.

7.2 New closure theorems

Several closure theorems were proposed by Steiner after 1832 [see the appendix of Steiner (1832)]. In one of these, he considered two (real) circles C_1 and C_2, the second lying inside the interior of the first, and a sequence of circles $c_1, c_2, \ldots, c_n$

[78] After graduating, Max Simon (1844–1918) moved in Strasburg where he taught from 1871 until 1912. His research dealt mainly with the history of mathematics.

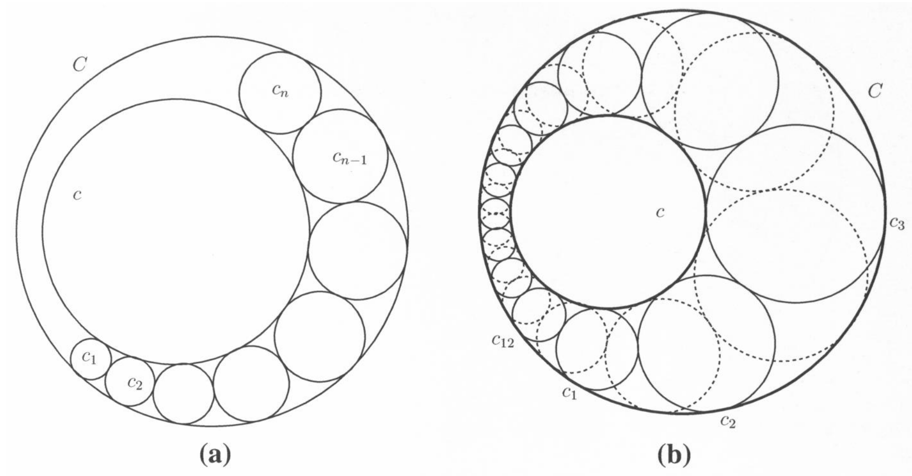

Fig. 23 Steiner's closure theorem for circles (1832)

such that each of them is tangent to both C_1 and C_2, and c_i is tangent to c_{i-1} for every $i = 2, \ldots, n$ (see Fig. 23a).

Steiner claimed that either the chain never closes whatever n is, i.e., c_n is never tangent to c_1 for any n, or the chain closes, i.e., c_n is tangent to c_1 for some n. In this case, the same happens for any similar chain of n circles whatever is the first circle c_1 one considers (see Fig. 23b).

Denoting R_1, R_2 the radii of the two given circles, A the distance between their centers, and m the number of times the chain wraps around C_2, Steiner gave the following conditional equation, *Bedingungsgleichung* (Steiner 1832, pp. 318–320), allowing the existence of the relative closed chain:

$$(R_1 - R_2)^2 - 4R_1R_2 \tan^2\left(\frac{m}{n}\pi\right) = A^2.$$

To prove the claim when C_1 and C_2 are concentric poses no difficulty. Since in this case the figure is completely symmetric, it is enough to apply some elementary geometries (Fig. 24a), and it follows that all circles in the chain have the same diameter. It is also clear that does not matter from which position one starts: if one chain closes, then all chains close.

A suitable circular inversion allows us to pass from the case of concentric circles to the general case proposed by Steiner. In fact, such a transformation map circles into circles, lines into circles, and preserve tangency and angles (Fig. 24b).

Steiner proposed a new closure problem in Steiner (1846). Let E be a non-singular plane cubic and P, Q be two fixed point of it. Chosen a point $A_1 \neq P, Q$ on E, the line PA_1 meets E in a third point A_2. The line QA_2 meets E, other then in A_2 and Q, in another point A_3. Similarly, the line PA_3 meets E, other than in P and A_3, in another point A_4. Continuing on this way one gets a transversal $A_1A_2A_3A_4 \ldots A_{2n}A_{2n+1}$ inscribed in E. Then, he stated: *there are two possibilities, either the polygonal line never closes, or it closes, i.e.,* $A_{2n+1} = A_1$, *forming a polygon of 2n sides inscribed in the cubic; in this case, the same holds true whatever is the initial point* A_1.

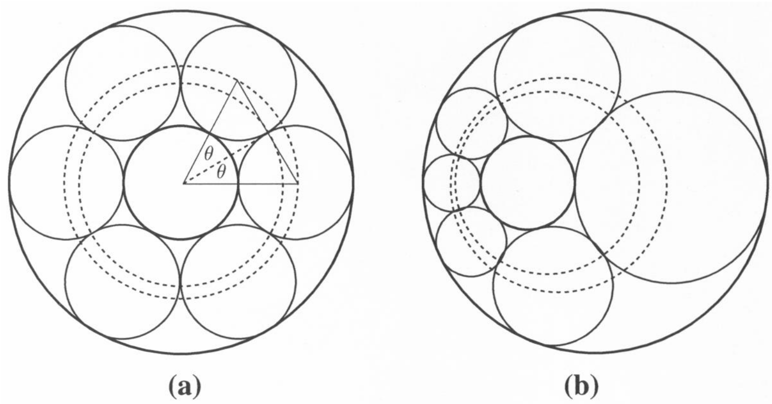

Fig. 24 **a** The case of concentric circles is easy to solve. **b** By circular inversion one pass to the general case

The theorem was proved by Clebsch as follows (see Clebsch, 1864, p. 106). If the polygonal line closes, argued Clebsch, then, according to Abel's addition theorem, one has the following equivalences (in the group law on the cubic E):

$$\begin{cases} A_1 + A_2 + P + u_0 \equiv 0 \\ A_2 + A_3 + Q + u_0 \equiv 0 \\ \vdots \\ A_{2n-1} + A_{2n} + P + u_0 \equiv 0 \\ A_{2n} + A_1 + Q + u_0 \equiv 0 \end{cases}$$

where u_0 is a constant. Adding the first, third, fifth, ...and penultimate equation, and subtracting from this sum the sum of the second, fourth, sixth,...and last equation, one gets

$$n(P - Q) \equiv 0,$$

which expresses the condition on the points P, Q which allows the existence of an inscribed $2n$-gon.

Since this condition does not depend on the choice of the initial point A_1, it follows that if P, Q satisfy it, then the transversal closes in $2n$ steps whatever is the initial point. Let us observe that if $n = 3$, then the above condition means that P, Q are both flexes of E.

A second proof of Steiner's theorem was given by the Czech Eduard Weyr in his paper (Weyr 1870). In the same paper, Weyr also proved a similar theorem for a plane curve C of degree four with two ordinary double points P, Q (see Fig. 25). Precisely: given $A_1 \in C \setminus \{P, Q\}$ the line PA_1 meets C in a point B_1, the line QB_1 meets C in a point A_2, and keep going on this way one has a sequence of points $A_1, A_2, \dots A_{2n+1}$. Then, Weyr proved that, if $A_{2n+1} = A_1$ for some n, then the sequence always closes, after $2n$ steps, for any choice of A_1 on C. This problem, as we will see later on, was revisited in Griffiths (1976, pp. 346–347).

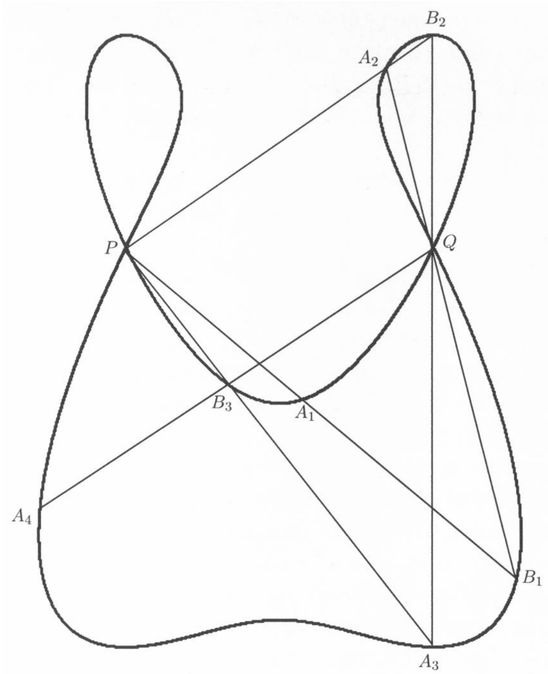

Fig. 25 Closure theorem that Steiner proposed in (1846) for a plane quartic curve with two nodes

The first attempt to extend Poncelet's theorem from conics to quadrics was made by Cayley. In his paper (Cayley 1853d), he tried to extend the reasoning of Cayley (1853a) to the case of two quadrics $Q)$ $x^2 + y^2 + z^2 + w^2 = 0$ and $Q')$ $ax^2 + by^2 + bz^2 + dw^2 = 0$, by taking chords of Q touching Q' in order to construct an inter-scribed polygonal line to the two quadrics. He was lead to consider hyperelliptic integrals of type

$$\prod x = \int \frac{\mathrm{d}x}{\sqrt{(x+a)(x+b)(x+c)(x+d)(x-k)(x-k')}},$$

where k, k' are two values of the parameter λ in $\lambda Q + Q' = 0$. After having developed some transcendental equations, Cayley soon realized that the theory of Poncelet polygons for conics could not be extended to quadrics in the simple way that one might be led to suppose.

Michel Chasles investigated the properties of n-gons which are inscribed in a given (real) ellipse (Chasles 1865). He found that among all such n-gons, there are infinitely many having maximum perimeter, and these all have their sides tangent to a second ellipse confocal to the first. Similarly, he proved that among all the n-gons circumscribed about a given ellipse, there are infinitely many having minimum perimeter, and all these are inscribed in a second ellipse confocal to the first. These results were generalized to (real) ellipsoids by Darboux.

In the short note of 1870, Darboux stated three theorems regarding polygons inscribed in an ellipsoid and circumscribed about another, which extend to ellipsoids the results of Chasles. A polygonal line $P_1, P_2, \dots, P_{n+1}$ is said *inscribed in an ellipsoid* A, if all its vertices are on A; is said *circumscribed about an ellipsoid* B, if all its sides are tangent to B. If $P_1 = P_{n+1}$, the polygonal line closes in a polygon of n-sides, which may be inscribed in A or/and circumscribed about B. The first and the third are:

There are infinitely many polygons of n sides inscribed in an ellipsoid A having maximum perimeter, and all them are circumscribed to two ellipsoids B, B_1 confocal to A;

There are infinitely many polygons of n sides circumscribed about an ellipsoid B having minimum perimeter, and all them are inscribed in an ellipsoid A and (simultaneously) circumscribed to another ellipsoids B_1 *confocal to B and A.*

The second, and more important, theorem, can be stated as follows:

Let B, B_1 *and* $A_1, A_2, \dots, A_n$ *be confocal ellipsoids. In general, there are no n-gons having their vertices on* $A_1, A_2, \dots, A_n$, *whose sides are tangent to B and* B_1. *If one such polygon exists, then infinitely many others exist enjoining the same property.*

Darboux remarked that to prove the theorems was necessary the use of hyper-elliptic functions with four periods. The proofs were actually given by means of Abel's addition theorem for hyperelliptic integrals in the second volume of his treatise *Leçons sur la théorie générale des surfaces* (Darboux 1889, pp. 303–307).[79]

More than 100 years later, these theorems, especially the second, became of great interest for physicists [see for instance (Dragović 2011) and the references therein]. In his note, perhaps foreshadowing the future, Darboux had written:"Un rayon lumineux qui se réfléchirait à l'intérieur de l'ellipsoide décrira ces polygones s'il est d'abord dirigé suivant le premier côté" [a ray of light that is reflected within an ellipsoid, will describe one of these polygons if is first directed along the first side].

In the years following Darboux continued to work on Poncelet's theorem, producing the very interesting results that we will present in section nine.

A new type of Poncelet theorem for quadrics was proved by Weyr. He considered the smooth curve E intersection of two quadric Q and Q' of rank ≥ 3 in $\mathbb{P}^3$. He fixed a ruling S on Q and a ruling Σ on Q'. If A_1 is any point of E, the line in S from A_1 intersects E in another point A_2, and the line in Σ from A_2 intersects E in another point A_3. By proceeding in this manner, one gets a skew polygonal line connecting the points $A_1, A_2, \dots A_{2n+1}$ on E. Weyr stated that if for some n the polygonal line closes, i.e., $A_{2n+1} = A_1$, then also the polygonal line constructed starting from any other point of E closes after $2n$ steps (Weyr 1870, p. 28). He obtained the result as a corollary of the analogous theorem for plane quartics with two nodes that we have recalled above.

This last theorem can be seen as the historical origin of the Poncelet theorem in space of Griffiths and Harris that will be discussed in section fourteen of this paper.

[79] At p. 307, Darboux quoted O. Staude, who in 1883 had proved the theorems by the use of the theta function with four periods (Staude 1883).

8 (2, 2)-Correspondences and closure problems

From what we have seen above it is clear that Trudi and Cayley must be credited for having pursued Jacobi's plan ahead of others, and for investigating the difficult determination of the relation between the coefficients of two conics when they admit an in-and-circumscribed n-gon. If Cayley had the great merit of having given this relation in explicit form, Trudi had that, as we will be clear shortly, of having brought to light the role that biquadratic binary equations play in the study of the Poncelet polygons and related questions.

The theory of algebraic correspondences, that arose in the 1850s in the school of Chasles, developed gradually in the second half of the nineteenth century in Chasles (1864), De Morgan (1865), Cayley (1866), Cremona (1867), Zeuthen (1871), Brill (1873) and others.[80] Here, we shortly recall some basic facts of this theory that will be useful later.

Let $f(x, y)$ be a polynomial of degree m in x and of degree n in y. The equation

$$f(x, y) = 0,$$

determines an (m, n)-*correspondence* between the variables x and y, in the sense that, to any value of x correspond n values $y_1, \ldots, y_n$ of y, while to any values of y correspond m values $x_1, \ldots, x_m$ of x. We may think of x, y as parameters fixing two points P, Q, respectively, on a line l and on a line l', or, more generally, on two *unicursal* (i.e., rational) curves C and C'.[81]

A *coincidence* of a point P with one of its correspondent points $Q_1, \ldots, Q_n$, occurs when one of the y_i is equal to the x from which it arises. Therefore, such coincidences are given by the equation

$$f(x, x) = 0,$$

which is of degree $m + n$. The *principle of correspondence* (Chasles 1864) affirms that there are, in general, $m + n$ coincidences.[82]

A *branch point* is a point such that two (or more) of its n corresponding points coincide. If we write $f(x, y) = X_0 y^n + X_1 y^{n-1} + \cdots + X_n$, where the X_i have degree m in x, this equation in y has two coincident roots if $\frac{\partial f}{\partial y} = n X_0 y^{n-1} + \cdots + X_{n-1} = 0$. Therefore, in general a (m, n)-correspondence has $2m(n-1)$ branch points.

A correspondence is said *symmetric*, if for any pair of corresponding points P, Q also Q, P is a pair of corresponding points. This means that the two polynomials $f(x, y)$ and $f(y, x)$ are identical.

[80] For a historical study of this concept (see Segre 1892; Coolidge 1940).

[81] The term *unicursal* was coined by Cayley, who also derived the fundamental properties of these curves (see Cayley 1866).

[82] This principle is also referred as *Chasles' principle of correspondence*, or even as *the Chasles–Cayley–Brill principle of correspondence*.

Symmetric (2, 2)-correspondences are associated with biquadratic (sometime also called doubly quadratic) equations of the following type

$$f(x, y) = ax^2y^2 + bxy(x + y) + c(x^2 + y^2) + dxy + e(x + y) + f = 0.$$

From above it follows that, a general symmetric (2, 2)-correspondence has four coincidence (or fixed points), corresponding to the roots of

$$ax^4 + 2bx^3 + (2c + d)x^2 + 2ex + f = 0, \tag{8.1}$$

and four branch points corresponding to the roots of

$$D(x) = Q(x)^2 - 4P(x)R(x) = 0,$$

where $D(x)$ is the discriminant of the polynomial $f(x, y) = P(x)y^2 + Q(x)y + R(x)$ (as polynomial in y).

8.1 Cayley's papers of 1871

As we have seen, the relation between Poncelet's closure theorem and symmetric (2, 2)-correspondences emerged in part through the works of Trudi (1853, 1863a, b), and of Rosanes and Pash (1869).

Cayley at the beginning of his paper (1871a) wrote:

> The porism of the in-and-circumscribed polygon has its foundation in the theory of the symmetrical (2, 2) correspondence of points on a conic; viz. a (2, 2) correspondence is such that to any given position of either point there correspond two positions of the other point; in a symmetrical (2, 2) correspondence either point indifferently may be considered as a first point and the other of them will be the second point of the correspondence. Or, what is the same thing, if x, y are the parameters which serve to determine the two points, then x, y are connected by an equation of the form[83]
>
> $$(*) \quad (x, 1)^2(y, 1)^2 = 0,$$
>
> which is symmetrical with respect to the parameters x, y.

It seems it is here that, for the first time, symmetrical (2, 2)-correspondences are explicitly associated with the construction of Poncelet's polygons.

Before continuing with the exposition of Cayley's paper, we explain what he meant by the above.

Let C and D be two non-singular conics in a plane π. One may suppose C rationally parameterized by a parameter s, so that to any point on C corresponds a value of s and vice versa. Let P be a point on C corresponding to the value x of the parameter. In

[83] This is the symbolic form that Cayley used to write a doubly quadratic equation.

the construction of Poncelet, to the point P correspond two points Q and Q' on C, of parameter value, respectively, y and y', such that the lines PQ and PQ' are tangent to D. This construction gives a (2, 2)-correspondence on C, which is clearly symmetric, since any one of Q and Q' can be chosen as first correspondent of P. Hence, the parameter values of corresponding points are connected by an equation of the form

$$ax^2y^2 + bxy(x+y) + c(x^2+y^2) + \mathrm{d}xy + e(x+y) + f = 0.$$

Cayley showed that any symmetric (2, 2)-correspondence on a conic is defined in this way. One may suppose C to be given, in parametric equations, by $(1, s, s^2)$, so if X, Y, Z are the coordinates in the plane, the line l_{xy} joining two points $(1, x, x^2)$ and $(1, y, y^2)$ on C is expressed by the equation

$$\begin{vmatrix} X & Y & Z \\ 1 & x & x^2 \\ 1 & y & y^2 \end{vmatrix} = 0,$$

that is $xyX - (x+y)Y + Z = 0$. In the dual plane π^*, this line is represented by the point $(xy, -(x+y), 1)$. Let D be any non-singular conic, represented by its tangential equation (i.e., by the dual conic) of equation

$$A\alpha^2 + B\beta^2 + C\gamma^2 + D\beta\gamma + E\alpha\gamma + F\alpha\beta = 0.$$

The line l_{xy} will be tangent to D if and only if

$$Ax^2y^2 + B(x+y)^2 + C - D(x+y) + Exy - Fxy(x+y) = 0,$$

and this represents a general symmetric (2, 2)-correspondence.

Cayley showed that, whether symmetric or not, a (2, 2)-correspondence always leads to a differential equation of the form

$$\frac{\mathrm{d}x}{\sqrt{X}} \pm \frac{\mathrm{d}y}{\sqrt{Y}} = 0,$$

where X, Y are polynomials of degree 4, respectively, in x and y, having the same coefficients if the correspondence is symmetric. In this case, he also noticed that (p. 85):

> if A and B are corresponding points, the corresponding points of B are A and a new point C; those of C are B and a new point$D$, and so on; so that the points form a series $A, B, C, D, E, F, \ldots$; and the porismatic property is that, if for a given position of A this series closes at a certain term, for instance, if $F = A$; then it will always thus close, whatever be the position of A.

This follows at once, observed Cayley, from the consideration of the differential equation $\frac{\mathrm{d}x}{\sqrt{X}} = \frac{\mathrm{d}y}{\sqrt{Y}}$ and its complete integral of the form (*).[84] In fact, since this differential equation is integrable in the form

[84] It is useful to recall (4.8) and (4.9).

$$\Pi(y) - \Pi(x) = \Pi(k),$$

by forming the equations for the corresponding points B, C; C, D;... and, assuming that the series closes after n steps, one has

$$\Pi(z) - \Pi(y) = \Pi(k),$$
$$\vdots$$
$$\Pi(x) - \Pi(u) = \Pi(k),$$

where $\Pi(x)$ in the last equation must differ by a period Ω of the integral from that in the first. Hence, by adding, Cayley got $\Omega = n\Pi(k)$, or

$$\Pi(k) = \frac{1}{n}\Omega,$$

which gives the condition on the coefficients of the equation $(*)$ for the series to close after n steps. This condition is independent of x, i.e., from the position of A.

We have to say that, although Cayley had showed how (2, 2)-correspondences come into play, in Cayley (1871a) he did not use the principle of correspondence in order to prove PCT. Nevertheless, this principle was used in another paper, which appeared in the same year 1871, to solve the following problem: compute the number of the in-and-circumscribed triangles to given curves. In that paper, he wrote (Cayley 1871b, p. 369):

> The problem of the in-and-circumscribed triangle is a particular case of that of the in-and-circumscribed polygon: the last-mentioned problem may be thus stated—to find a polygon such that the angles [vertices] are situate in and the sides touch a given curve or curves. And we may in the first instance inquire as to the number of such polygons. In the case where the curves containing the angle [vertices] and touched by the sides, respectively, are all of them distinct curves, the number of polygons is obtained very easily and has a simple expression ...But when several of the curves become one and the same curve, and in particular when the angles [vertices] are all of them situate in and the sides all touch one and the same curve, it is a much more difficult problem to find the number of polygons.

Cayley considered a triangle of vertices a, c, e, respectively, on the three curves a, c, e, whose sides B, D, F are tangent, respectively, to the three curves B, D, F (see Fig. 26a). He computed, for 52 possible cases of coincidence among the six curves involved, the number of in-and-circumscribed triangles to them (the results were listed in a table, eight pages long, inserted in the paper), by using the theory of correspondences.[85] He applied the principle of correspondence as follows (art. 1): consider the unclosed trilateral $aBcDeFg$ (see Fig. 26b), where the points a and g

[85] We have not verified these results except for conics, and only in the following cases: $a = c = e$ and $B = D$; $a = c = e$; no conditions. In these cases, the number of triangles showed in the table seems to be correct. It would be of interest to reinterpret Cayley's results in light of the modern algebraic geometry.

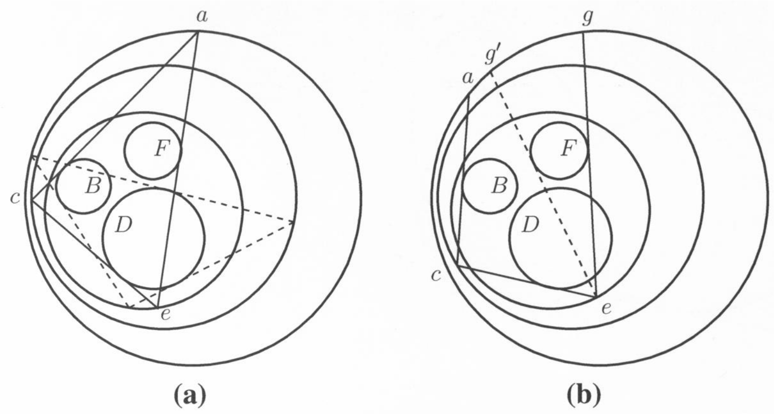

Fig. 26 In his paper (1871b) Cayley used the principle of correspondence in order to prove Poncelet's general theorem. The figures illustrate his procedure in order to get the proof

are on the same curve a. Starting from an arbitrary point a on the curve a, let ac be any one of the tangents to the curve B, touching this curve, say at the point B, and intersecting the curve c in the point c. The same for the tangents ce and eg. Suppose that for a position of the point a there correspond χ positions g, g', etc.. Similarly, suppose that starting from a position of g there correspond χ' positions of a. Then, the points a, g, over the curve a, are in a (χ, χ')-correspondence. When one of the points g coincides with one of the points a, the point $a = g$ is a coincidence point of the correspondence, and the trilateral in question becomes an in-and-circumscribed triangle. Thus, the numbers of triangles are equal to that of coincidence points. By the general theory of correspondences this numbers is, in several of the cases but not in all, equal to $\chi + \chi'$. Cayley was able to express χ and χ' in terms of the order, and of the class of the curves a, c, e and B, D, F, that he denoted, respectively, a, c, e, b, d, f, and A, C, E, B, D, F. Then, via the principle of correspondence and an accurate analysis of the possible situations of tangency (flexes, double tangents, etc.), he computed the required number.

8.2 Hurwitz's general view

Adolf Hurwitz, referring to the closure theorems, began his note (Hurwitz 1879) with the following words:

> Es giebt in der Geometrie eine grosse Anzahl von Sätzen, die aussagen, dass ein gewisses Ereiguiss unendlich oft Statt hat, sobald es nur ein Mal oder endlich oft eintritt [There is a large number of theorems in Geometry, affirming that if a certain event happens once then it happens an infinite number of times].

Hurwitz recognized that all these theorems (as those recalled in the previous section) have the common feature of being linked to (2, 2)-correspondences, and saw in the principle of correspondence the explanation for the existence of infinitely many solutions of the problem, when at least one solution exists.

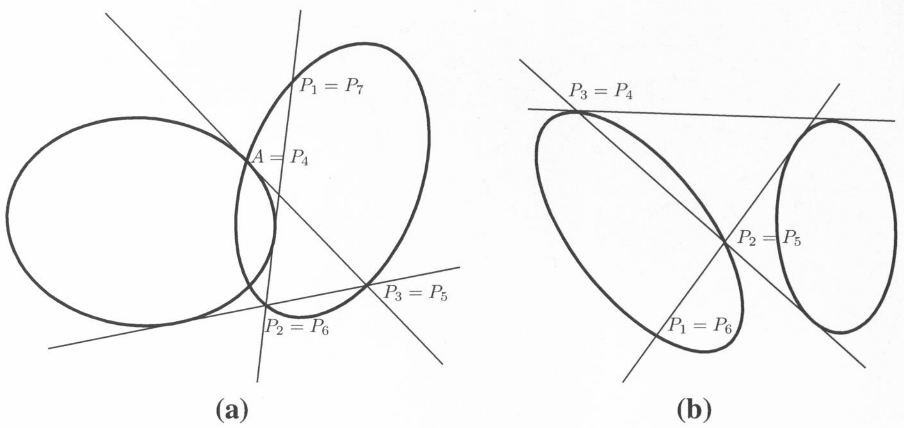

Fig. 27 Hurwitz's second example in (1879): **a** the case $n = 6$, **b** the case $n = 5$

To illustrate his thinking, Hurwitz examined various examples, and the first he considered it was that of Steiner's chain of circles (Fig. 23a, b).

Let $c_1, c_2, \ldots, c_{n+1}$ be such a chain and denote by $t_1, t_2, \ldots, t_{n+1}$ the respective points of tangency with the circle C. The law that associates t_1 to t_{n+1} is a $(2, 2)$-correspondence on C, which in general has 4 coincidences. If the chain closes, remarked Hurwitz, i.e., for some n is $t_{n+1} = t_1$, each point t_i is a coincidence point. Hence, if $n > 2$, the correspondence has more than 4 coincidence points, and then, equation (8.1) is an identity, i.e., every point on C is a coincidence point. This means that any chain of tangent circles closes in the same way, whatever is the starting point t_1 on C.

The subsequent example was Poncelet's closure theorem. As it is known, the construction of an inter-scribed transversal $P_1, P_2, \ldots, P_{n+1}$ to two conics K_1 and K_2 leads to a symmetric $(2, 2)$-correspondence on K_1. Hurwitz observed that, for any even number $n = 2m$ and any point $A \in K_1 \cap K_2$, one can construct a polygon of n sides, which is inscribed in K_1 and circumscribed about K_2. He proceeded as follows (Fig. 27a, illustrates the case $n = 6$). Let $P_{m+1} = A$, the tangent from P_{m+1} to K_2 meets K_1 in P_{m+1} and another point P_m, then the tangent to K_2 from this last point meets K_1 in P_m and another point P_{m-1}, and continuing in this way one gets a point P_1 on K_1. It is clear that taking P_1 as starting point of Poncelet's construction, one gets a closed polygon of n sides inter-scribed to K_1 and K_2. Similarly, for any odd number $n = 2m + 1$ and for any of the four contact points on K_1 of the four common tangents to the two conics, one can construct a polygon of n sides inter-scribed to them (Fig. 27b illustrates the procedure for $n = 5$). Hence, for any n there are always 4 coincidences of the correspondence. If there exists a *proper* polygon of n sides which is inter-scribed to K_1 and K_2, then, since each vertex is a coincidence point of the correspondence (which must be counted twice), there are in total $4 + 2n$ coincidences, and then, as above, it follows that every point of K_1 is a coincidence point. This brief but conclusive reasoning gave Poncelet's problem its true setting.

Hurwitz also examined the problem of *Steiner polygons* (Fig. 28a, b).

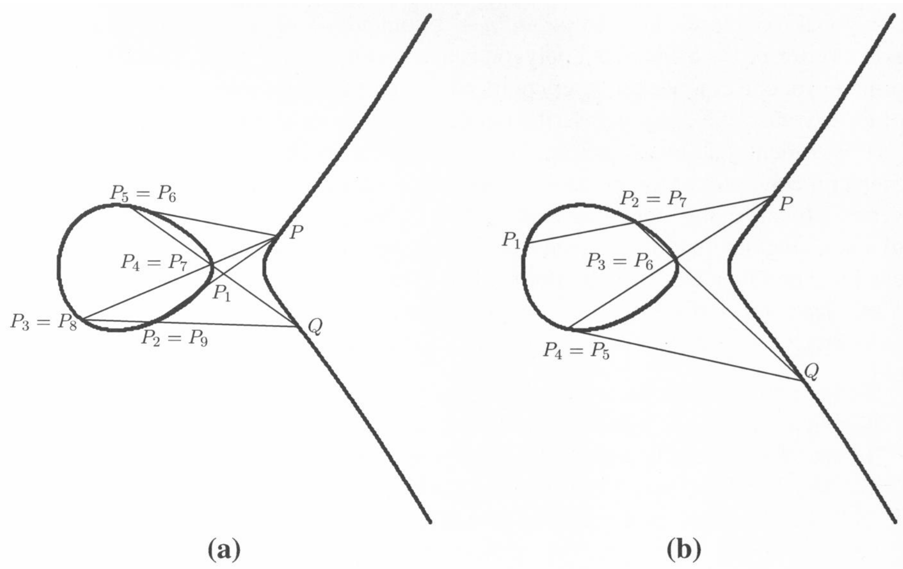

Fig. 28 Hurwitz's third example in (1879): **a** the case $n = 4$, **b** the case $n = 3$

Let E be a non-singular plane cubic, and P, Q be two points on it. Starting from any point P_1 on E, one can draw a polygonal line of vertices $P_1, P_2, P_3, \dots, P_{2n+1}$, which is inscribed in E and whose sides alternatively pass through P and Q. The map $PP_1 \mapsto PP_{2n+1}$ defines a $(2, 2)$-correspondence among the lines of the pencil through P. Hurwitz noticed that if n is even, and the line PP_{n+1} is tangent to E at P_{n+1}, then P_1 and P_{2n+1} coincide, i.e., one has a coincidence point (Fig. 27a illustrates the case $n = 4$)). The same happens if n is odd and the line QP_{n+1} is tangent to E at P_{n+1} (Fig. 27b illustrates the case $n = 3$). Then, Hurwitz remarked, whatever the parity of n may be, the correspondence always has 4 coincidences, being 4 the tangents than can be drawn from each of the points P, Q. If another (proper) polygonal line closes forming an inscribed $2n$-gon to E, then every edge of it is a coincidence point, and therefore, there are $4 + 2n > 4$ coincidence points. This means that every point of E is vertex of a similar inscribed $2n$-gon.

Finally, Hurwitz considered the problem proposed in Weyr (1870, p. 28). Let C_4 be a quartic of first species, i.e., the smooth intersection of two quadrics surfaces Q_1, Q_2 in $\mathbb{P}^3$. Let s and σ be two secants of C_4. The plane through s containing a point 1 of C_4 (not on s) intersects C_4 in another point 2 (we are using Hurwitz notation); the plane through σ containing 2 intersects C_4 in another point 3; the plane through s containing 3 intersects C_4 in another point 4, and so on. In this way, it is determined a polygonal line $1, 2, 3, 4, \dots, 2n + 1$ of $2n$-sides which is inscribed in C_4 and whose sides alternatively meet s and σ (the "odd numbered" meet s, and the "even numbered" meet σ). By associating with the plane s, 1 the plane s, $2n + 1$, one gets a symmetric $(2, 2)$-correspondence among the planes of the pencil through s. This correspondence always have 4 coincidences. They come, similarly to the case of the Steiner polygons, from the contact points of the 4 planes through s, or σ, that are tangent to C_4. If the polygonal line closes, i.e., the point $2n + 1$ coincides with 1, the point 1, and then, every vertex of the constructed polygon is a new coincidence point. Therefore, by the principle of correspondence, every point of C_4 enjoys this property, and so every point of C_4 is vertex of a $2n$-gon ,inscribed in C_4, whose sides alternatively meet s and σ.

It is evident that, by projecting C_4 from one of its points into a plane, one has the Steiner theorem for a smooth cubic. Moreover, C_4 projects doubly on a conic from the vertex S of one of the four cones in the pencil $Q_1 + \lambda Q_2 = 0$, and since the two rulings of a non-singular quadric are projected from S onto the tangents to another conic, it is clear that any Poncelet $2n$-gon inter-scribed to these two conics is the projection from S of a $2n$-gon inscribed in C_4 of the type above.

Hurwitz concluded by saying (Hurwitz 1879, p. 15):

> Scliesslich sei noch darauf hingrwiesen, dass unser Kriterium immer nur das Resultat ergiebt, dass gewisse Aufgaben unendlich Lösungen haben, wenn sie Eiene oder eine endliche Anzahl von Lösungen haben besitzen; nicht aber auch die Möglich-keit, dass dieser Umstand wirklich eintreiten kann, was in vielen Fällen nicht selbstverständlich ist [Finally, we call attention on the fact that our criterion always only gives the certitude that if a problem admits a solution, or a finite number of solutions, then the problem admits infinitely many solutions, but it does not give the possibility to verify that at least one solution actually exists; this in many cases is not obvious at all].

If Cayley was the first to explicitly recognize the link between Poncelet's polygons and symmetric (2, 2)-correspondences, Hurwitz was the first to complete the algebraic explanation of the "porismatic character" of certain questions. Loria in his remarkable work (Loria 1896), reporting on Hurwitz's paper, at the end of the paragraph concerning the closure theorems, wrote:

> Non sappiamo se più ammirare la vastità di vedute o la perfezione della forma, e colla quale poniamo termine a questa digressione, alla quale invano cercheremo chiusa più degna [we do not know whether to wonder more at the breadth of views or at the perfection of the form, and so with this we bring to an end this digression, for which we should seek in vain a close more worthy.]

8.3 Geometric interpretation of (2, 2)-correspondences

We end this section by introducing a geometrical interpretation of the (2, 2)-correspondences, suggested by the examples above, and that will be useful to have at hand in the sequel.

Any non-singular quadric is projectively equivalent to the *Segre quadric* $\mathcal{S}$, embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ via the *Segre map*: $(1, u) \times (1, v) \mapsto (1, u, v, uv)$. If (z_0, z_1, z_2, z_3) are homogeneous coordinates in $\mathbb{P}^3$, then $\mathcal{S}$ is defined by

$$z_0 z_3 - z_1 z_2 = 0.$$

A (2, 2)-correspondence

$$au^2v^2 + bu^2v + b'uv^2 + cuv + du^2 + d'v^2 + eu + e'v + f = 0,$$

is obtained intersecting $\mathcal{S}$ with the quadric $\mathcal{Q}$ defined by the equation

$$az_3^2 + bz_3z_1 + b'z_3z_2 + cz_3z_0 + dz_1^2 + d'z_2^2 + ez_1z_0 + e'z_2z_0 + fz_0^2 = 0.$$

Conversely, the intersection of any quadric $\mathcal{Q}$ with the Segre quadric defines a (2, 2)-correspondence.

Any pair of corresponding points (P, Q) under a (2, 2)-correspondence is then associated with the intersection point of the x-line corresponding to P and the y-line corresponding to Q. So, to any (2, 2)-correspondence is associated a curve $E = \mathcal{S} \cap \mathcal{Q}$ of bi-degree (2, 2) and vice versa. Generally, the curve E is non-singular, hence has genus 1, i.e., is an elliptic curve. In fact, the projection of $E \subset \mathcal{S} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, on the first factor is a 2 to 1 morphism, ramified at the four points of the intersection of E with the plane $z_1 - z_2 = 0$ (corresponding to the condition $u = v$), and by the Riemann–Hurwitz formula it follows that E has genus 1.[86]

We remark that the branch points of the (2, 2)-correspondence are associated with the lines of the two rulings of $\mathcal{S}$ which are tangent to E.

9 The theorems of Darboux

Gaston Darboux likely started to work on Poncelet's theorems and related questions in 1868. At the end of that year, he presented a memoir to the *Académie des Sciences* on an important class of curves (and surfaces) of degree four. These curves were those resulting from the intersection of a sphere with a quadric, which he proposed to call *cycliques* (Darboux 1869, p. 1311). He had extended to these curves many of the more important properties of the circle, and, as a consequence, he had found a new proof of the Poncelet closure theorem:

> On obtient, comme conséquence de ces propriétés, une demonstration, nouvelle et indépendente de la théorie de fonctions elliptiques, du théorème del Poncelet sur les polygones inscrits et circonscrits. On démontre de même un théorème qui est un peu plus général que le théorème de Poncelet [We obtain, as a consequence of these properties, a new proof, independent of the theory of elliptic functions, of the theorem of Poncelet on inscribed and circumscribed polygons. We also prove a theorem which is somewhat more general than the theorem of Poncelet].

For some reason,[87] the printing of this memoir was delayed, and Darboux decided to publish it elsewhere. The first part appeared in 1870, in the Memoirs of the Academy of Bordeaux (Darboux 1870b). In 1872, Darboux published the paper *Sur un nouveau*

[86] See for instance (Griffiths and Harris 1978b). This formula was stated by Riemann and proved by Hurwitz in (1891). For the present case: if $f : X \to Y$ is a map of degree two from the curve X, of genus $\widetilde{g}$, onto the curve Y, of genus g, then $2\widetilde{g} - 2 = 2(2g - 2) + N$, where N is the number of branch points, i.e., of those points $p \in Y$ such that $f^{-1}(p)$ contains only one point of X. Here, since $g = 0$ and $N = 4$, one has $\widetilde{g} = 1$.

[87] See the foreword of Darboux (1873a).

système de coordonnées et sur les polygones circonscrits aux conique. Here, he wrote (Darboux 1872, p. 100)[88]:

> Dans un mémoire présenté en 1868 à l'Académie des sciences, j'ai été conduit à une démonstration indirecte des théorèmes de Poncelet... Cette démonstration m'avait paru mériter d'être développée parce qu'elle donnait, sans l'emploi des coordonnées elliptiques, et au moyen d'une transformation analytique des plus simples, la proposition fondamentale de Poncelet... Depuis en examinant la méthode employée, j'ai reconnu qu'elle était susceptible d'extension, et que, par sa nature même, elle conduisait à des théorèmes ayant la plus grande analogie avec ceux de Poncelet, et qu'on peut considérer comme des généralisations des propositions de l'illustre géomètre. Si, après tant de belles démonstrations, soit analytiques, soit géométriques de ces propositions, je me permets d'en proposer une nouvelle, c'est que cell-ci me parait réellement se distinguer par quelques principes qui n'ont pas encore été employés dans l'étude de cette question [In a memoir presented in 1868 to the Academy of Sciences, I have been lead to an indirect proof of the Poncelet theorems ...This proof seemed to me worthy of being developed, because it gives, without the use of elliptic functions, and by means of very simple analytical transformations, Poncelet's fundamental proposition of ...After having examined the method used, I have recognized that it was capable of extension, and that, by its own nature, it led to theorems having a great analogy with those of Poncelet and that can be considered as generalizations of the propositions of the illustrious geometer. If, after the many beautiful proofs, both analytic and geometric, of these propositions I allow myself to present here a new one, it is because it actually seems to me to be distinguished for some principles that have not yet being used in the study of that question].

The second part of the memoir that he had presented to the Paris Academy of Sciences followed in 1873, published in the same journal where the first part had appeared. The same year, a new redaction of the whole memoir was printed, in the form of a book, by Gauthier–Villars (Darboux 1873a). Here, in the *Notes et Additions*, several in-depth studies were included. Note II, titled *Sur une démonstration analytique des théorèmes de Poncelet, et sur un nouveaux système de coordonnées dans le plane*, which we refer to as (Darboux 1873b), contained in its first five sections the paper *Sur un nouveau système de coordonnées etc.*, published the year before.

The new proofs of Poncelet's theorems were based on the properties of certain curves, today known as "Poncelet curves"or "Poncelet–Darboux curves": curves of degree n passing through the intersection points of $n+1$ tangents to a given conic.[89] The key point in Darboux's approach was the introduction of a new system of coordinates, by which every point of the plane is seen as the point of intersection of two tangents to a fixed conic. This new system of coordinates, today called "Darboux coordinates" [see

88 Darboux was referring to the proof in art. 38 of his memoir (see Darboux 1873a, p. 99).

89 These curves were called "Poncelet curves" in Böhmer (1985), but recently the name of "Poncelet–Darboux curves" seems to be preferred, see for instance (Dragović 2011). We will adopt the second name, in honor of Darboux who introduced them.

for instance (Dragović 2011)], was suggested to Darboux by the Chasles representation of a quadric as a double plane (see Darboux 1917, pp. 236–237).[90]

9.1 Darboux coordinates and Poncelet–Darboux curves

All tangents to a non-singular conic (K) can be obtained by varying the parameter m in the equation

$$\alpha m^2 + \beta m + \gamma = 0, \tag{9.1}$$

where α, β, γ are linear functions of the coordinates in the plane.[91] Then, since from any point on (K) only one tangent can be drawn to it, the conic (K) can be represented by the equation

$$\beta^2 - 4\alpha\gamma = 0. \tag{9.2}$$

If a tangent to (K) passes through a point $(\alpha', \beta', \gamma')$, then m must satisfy the equation $\alpha' m^2 + \beta' m + \gamma' = 0$. So, putting ρ, ρ' its roots, one has

$$\alpha' = \frac{\beta'}{\rho + \rho'} = \frac{\gamma'}{\rho\rho'}. \tag{9.3}$$

Vice versa if ρ, ρ' are given, the equations (9.3) determine the point $(\alpha', \beta', \gamma')$.

Darboux considered (ρ, ρ') as new coordinates in the plane (Darboux 1873, p. 184). These are called *Darboux coordinates*. Clearly, with respect to them, the conic (K) has equation

$$(\rho - \rho')^2 = 0. \tag{9.4}$$

Moreover, any conic having two double contacts with (K), i.e., given by $K - L^2 = 0$, where L is a linear form, can be written

$$d(\rho - \rho') = a\rho\rho' + b(\rho + \rho') + c,$$

where a, b, c, d are constants and that is of the form

$$A\rho\rho' + B\rho + C\rho' + D = 0.$$

An algebraic equation, of degree m in ρ and of degree m' in ρ',

$$f(\rho, \rho') = 0$$

defines an algebraic curve in the plane. It is easy to see that this curve is of degree m or $m + m'$, according as f is symmetric or not with respect to ρ and ρ'. Moreover, a count

[90] The projection of a non-singular quadric Q on a plane π from a point $A \notin Q$, not passing through A, gives a birational map of degree two, from Q onto π, which is branched along the conic $(K) := Q \cap \pi$. Under this map, all the lines of the two rulings of Q are mapped into the tangents to the conic (K).

[91] The tangent to the conic $\Gamma : f(x, y, z) = 0$ at $P \in \Gamma$ is given by $(\partial_x f)_P x + (\partial_y f)_P y + (\partial_z f)_P z = 0$. If the conic Γ is given by $y^2 - xz$, and P is $(m^2, m, 1)$, the tangent at P has equation $m^2 x + my + z = 0$. Since (K) can be mapped onto Γ by a suitable projective transformation of the plane, it is clear that by applying the inverse map the claim follows.

of constants shows that any curve of degree m is represented by an equation of this type, which is symmetric with respect to ρ and ρ', and depends on $(m+1)(m+2)/2$ arbitrary constants (p. 185–186).

Here, Darboux wrote:"Nous pouvons, à l'aide de ces seules remarques, démontrer plusieurs théorèmes généraux sur les polygones inscrits et circonscrits" [By means of only these remarks, we can prove several general theorems on inscribed and circumscribed polygons].

We will focus only on some of these theorems, the first of which is the following:

Theorem D1 *If a curve of degree n passes through the n^2 points of intersection of two systems of n tangents to the conic (K), then it contains infinitely many sets of n^2 points, each of them constituting the intersection locus of two systems of n tangents to (K).*

Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be two system of n lines in the plane. The intersection points $A_i \cap B_j$, $i, j = 1, \ldots, n$, constitute a set of n^2 points. Any curve of degree n passing through these n^2 points is given by

$$A_1 A_2 \cdots A_n - k B_1 B_2 \cdots B_n = 0. \tag{9.5}$$

If all the lines are tangent to the conic (K), and then

$$\begin{cases} A_i = \alpha a_i^2 + \beta a_i + \gamma = \alpha(a_i - \rho)(a_i - \rho') \\ \\ B_i = \alpha b_i^2 + \beta b_i + \gamma = \alpha(b_i - \rho)(b_i - \rho'), \end{cases} \tag{9.6}$$

for $I = 1, \ldots, n$, by setting

$$\begin{cases} \varphi(\rho) = (\rho - a_1)(\rho - a_2) \cdots (\rho - a_n) \\ \\ \psi(\rho) = \sqrt{k}(\rho - b_1)(\rho - b_2) \cdots (\rho - b_n), \end{cases} \tag{9.7}$$

then the equation of the curve becomes

$$\varphi(\rho)\varphi(\rho') = \psi(\rho)\psi(\rho'),$$

or

$$\frac{\varphi(\rho)}{\psi(\rho)} = \frac{\psi(\rho')}{\varphi(\rho')}. \tag{9.8}$$

Darboux observed (as he had already done in section n. 28 of his book), that this equation can be written

$$\frac{\Phi(\rho)}{\Psi(\rho)} = \frac{\Psi(\rho')}{\Phi(\rho')}, \tag{9.9}$$

where

$$\Phi(u) = m\varphi(u) + n\psi(u), \quad \Psi(u) = n\varphi(u) + m\psi(u),$$

which is of the same form as (9.8), but the roots of the polynomials are different.

Since equation (9.9) contains a new parameter that can assume any arbitrary value, Darboux declared the theorem proved (p. 187).

As example he observed that, if a conic contains the four vertices of a quadrangle circumscribed to another conic, then it contains the vertices of infinitely many other quadrangles circumscribed to the same conic.

The second theorem is the following

Theorem D2 *If a curve of degree n contains the vertices of a* $(n+1)$*-gon, whose sides are tangent to a given conic* (K)*, then it contains the vertices of infinitely many* $(n+1)$*-gons whose sides are tangent to* (K)*.*

Let $A_0, A_1, A_2, \ldots, A_n$ be $n+1$ tangents to (K). Darboux observed that every curve C of degree n, which passes through the $n(n+1)/2$ points of intersection of the tangents, is represented by an equation of the form

$$\frac{a_0}{A_0}+\frac{a_1}{A_1}+\cdots+\frac{a_n}{A_n}=0, \tag{9.10}$$

where $a_0, a_1, \ldots, a_n$ denote arbitrary constants.[92] Each line A_i, being tangent to (K), has equation

$$A_i: \quad \alpha(b_i-\rho)(b_i-\rho')=0,$$

and then (9.10) can be written

$$\sum \frac{a_i}{(b_i-\rho)(b_i-\rho')}=0, \tag{9.11}$$

or, multiplying by $\rho-\rho'$,

$$\sum \frac{a_i}{(b_i-\rho)}=\sum \frac{a_i}{(b_i-\rho')}, \tag{9.12}$$

and this equation is readily seen to be of the form

$$\frac{f(\rho)}{\varphi(\rho)}=\frac{f(\rho')}{\varphi(\rho')}. \tag{9.13}$$

Vice versa, all equations of this type can be reduced to the form (9.12), which represents a curve of degree n, containing all vertices of the polygon circumscribed to (K), and whose sides are defined by $\varphi(\rho)=0$.

Darboux concluded that, since equation (9.13) can be put in the form

$$\frac{f(\rho)}{\varphi(\rho)+kf(\rho)}=\frac{f(\rho')}{\varphi(\rho')+kf(\rho')}, \tag{9.14}$$

where k is an arbitrary constant, the theorem is proved.

92 This can be easily proved by induction on n (see Darboux 1873a, pp. 191–192).

The curves that satisfy the conditions of theorem D2 are called *Poncelet–Darboux curves* of degree n related to the conic (K).

Darboux deduced the PCT as a corollary of theorem D2, by arguing as follows.

Suppose that a conic (C) contains the $n+1$ vertices of a polygon of $n+1$ sides $A_0, A_1, \ldots, A_n$, which is circumscribed to the conic (K). One may fix the coefficients $a_0, a_1, \ldots, a_n$ in the equation (9.10) so that the curve $\mathcal{C}$ intersects (C) in other n points, besides the $n+1$ vertices of the inscribed polygon. Then, by Bezout's theorem, the curve $\mathcal{C}$ having at least $2n+1$ points in common with C decomposes in the conic (C) and another curve (C') of degree $n-2$. Therefore, from theorem D2, the curve $\mathcal{C} = (C) \cup (C')$ contains the vertices of ∞^1 polygons circumscribed to (K). It follows that the conic (C) is circumscribed to every such polygons, and the Poncelet closure theorem is proved.

Actually Darboux went further, and he showed that the curve (C') completely decomposes into conics, or in conics and a line, according to the parity of n. The conic (C) has an equation of the form

$$A(\rho^2 + \rho'^2) + B\rho\rho' + C\rho\rho'(\rho + \rho') + D(\rho + \rho') + E\rho^2\rho'^2 + F = 0, \qquad (9.15)$$

a tangent $\rho = a$ intersects (C) in two points, determined by the values ρ_1, ρ_2 of ρ. These two values are the coordinates of another vertex of the polygon, and since these two must satisfy an equation of the form above, the new vertex will move along a conic. Continuing in this way, it became clear that (C') decomposes in k conics, or in $k-1$ conics and a line, according as $n = 2(k+1)$ or $n = 2k+1$.

Up to this point, we have discussed the first five sections of Darboux (1873b), i.e Darboux (1872). In the remaining sections of his paper, Darboux proved others interesting results, such as those on hyperelliptic integrals or quartic curves with two nodes. He also gave a new proof of Chasles's theorem on confocal conics and showed a connection between Euler's differential equation and equations of type (9.15).

In his paper (Darboux 1880), he exhibited the connection between the existence of a Poncelet n-gons, and the rational transformations of elliptic integrals.

In 1917, Darboux published his last treatise *Principes de géométrie analytique*.[93] In the third part of the book, titled *Les théorèmes de Poncelet*, he gathered the many results on Poncelet's theorems and related questions that he had obtained over the years. Probably, apart from some improvements or new proofs that he could have obtained later, Darboux achieved many of these results before 1880. So we think that it is not too long a chronological leap to present them here, before commenting on the contributions to the theory that were made in the twentieth century.

9.2 Biquadratic equations and a new proof of PGT

Darboux divided part III into three chapters and devoted the first section of the first chapter to the definition of his new system of coordinates. In doing so, he introduced a few changes with respect to Darboux (1872) that it will be convenient to state here.

[93] In it Darboux presented, in coordinated form, the lectures he had delivered at intervals since 1872, either at the *Sorbonne* and at the *École Normale*.

He considered the conic (K) given by

$$y^2 - xz = 0,$$

so that any tangent to (K) is represented by the equation

$$m^2x - 2my + z = 0,$$

where m is a parameter. Then, if (x, y, z) are the homogeneous coordinates of a point P determined by the roots ρ, ρ_1 of the equation above, it follows that

$$2y = x(\rho + \rho_1), \quad z = x\rho\rho_1,$$

and the equation of (K) becomes $(\rho - \rho')^2 = 0$. Since the above formulae are symmetric with respect to ρ and ρ_1, a curve $F(x, y, z) = 0$ of degree m is represented in the new system of coordinates by the equation

$$F\left(1, \frac{\rho + \rho_1}{2}, \rho_1\right) = 0,$$

which is symmetric and of degree m in ρ, ρ_1. Vice versa, any such equation in ρ, ρ_1 represents a curve of degree m. For instance

$$A\rho\rho_1 + B(\rho + \rho_1) + C = 0$$

represents the line

$$Az + 2By + Cx = 0,$$

and

$$A\rho^2\rho_1^2 + B\rho\rho_1(\rho + \rho_1)) + C(\rho + \rho_1)^2 + 2D\rho\rho_1 + E(\rho + \rho_1) + F = 0$$

represents the conic

$$Az^2 + 2Byz + 4Cy^2 + 2Dzx + 2Exy + Fx^2 = 0.$$

In the subsequent sections of the first chapter, Darboux gave theorems D1, D2 and other theorems already published in Darboux (1873a), also giving them alternative proofs.

In the second chapter, he proved the following:

Theorem D3 *Let (C) and (K) be two conics. Suppose that from a point M on the conic (C) are drawn the two tangents to the conic (K) and that from the two new points M_1, M_{-1} of intersection of these tangents with the first conic, new tangents to (K) are drawn, and so on. It results from this construction a polygonal line inter-scribed to the two conics $\ldots M_{-h} \ldots M_{-1}MM_1 \ldots M_h \ldots$, which can be prolonged in two directions, such that the parameters ρ_i of M_i and ρ_{i+k} of M_{i+k}, $(i > 0)$, satisfy a symmetric biquadratic equation $f_{k-1}(\rho_i, \rho_{i+k}) = 0$.*

To prove this theorem, he proceeded by induction on k (Darboux 1917, n. 160). He observed that any pair of consecutive parameters ρ_i, ρ_{i+1} satisfies an equation of the form:

$$\begin{cases} f(\rho_i, \rho_{i+1}) = A\rho_i{}^2\rho_{i+1}^2 + B\rho_i\rho_{i+1}()\rho_i + \rho_{i+1}) + C(\rho_i{}^2 + \rho_{i+1}^2) \\ +D\rho i^2\rho_{i+1}^2 + E(\rho_i{}^2 + \rho_{i+1}^2) + F = 0. \end{cases} \tag{9.16}$$

By eliminating ρ_i from $f(\rho_i, \rho_{i+1}) = 0$ and $f(\rho_i, \rho_{i-1}) = 0$, he obtained an equation of bidegree 4 containing the factor $(\rho_{i-1} - \rho_{i+1})^2$. Dividing the resultant by this factor, he obtained a biquadratic equation $f_1(\rho_{i-1}, \rho_{i+1}) = 0$, symmetric with respect the two variables. Since by replacing i by $i+1$, it follows that ρ, ρ_{i+2} satisfy the equation $f_1(\rho, \rho_{i+2}) = 0$, he had proved the theorem for $k = 2$. Then, he supposed that for a certain value of k the following holds

$$f(\rho_i, \rho_{i+1}) = 0, f_1(\rho, \rho_{i+2}) = 0, \ldots, f_{k-1}(\rho_i, \rho_{i+k}) = 0.$$

The resultant of the elimination of ρ_{i+k} between

$$f_{k-1}(\rho_i, \rho_{i+k}) = 0, \quad f(\rho_{i+k}, \rho_{i+k+1}) = 0, \tag{9.17}$$

is an equation $\Phi(\rho_i, \rho_{i+k+1}) = 0$ of degree 4 with respect to both the variables. Since the second of the equations (9.17) is verified when ρ_{i+k+1} is replaced by ρ_{i+k-1}, Darboux affirmed that the same holds true for the previous equation. From this, he deduced that the first member must contain a factor $f_{k-2}(\rho_i, \rho_{i+k-1})$, which equated to zero gives the relation between ρ_1 and ρ_{i+k-1}. Hence, $\Phi = f_{k-2}(\rho_i, \rho_{i+k+1})f_k(\rho_i, \rho_{i+k+1})$, where $f_k(\rho_i, \rho_{i+k+1})$ is a biquadratic polynomial with respect to the two variables ρ_i, ρ_{i+k+1}. Then, the required relation is $f_k(\rho_i, \rho_{i+k+1}) = 0$. Since f_k is of the same form of $f_{k-1}, f_{k-2}, \ldots$ the second step of the induction holds true, and the theorem is completely proved.

Darboux observed that together with $f_{k-1}(\rho_i, \rho_{i+k}) = 0$, also holds

$$f_{k-1}(\rho_{-i}, \rho_{-i-k}) = 0,$$

and, since i is any integer, by changing i with $-i-k$ it follows that $f_{k-1}(\rho_{i+k}, \rho_i) = 0$, i.e., the primitive relation is symmetric.

From this, he deduced the following (n. 161):

Corollary *All equations $f_{k-1}(\rho_i, \rho_{i+k}) = 0$ have equal roots for the same values of ρ_i.*

Then, he gave a geometrical interpretation of the above results. Since the equations $f_{k-1}(\rho_i, \rho_{i+k}) = 0$ are symmetric with respect to the variables, the point of coordinates ρ_i, ρ_{i+k} will describe a conic (C_{k-1}).

On the other hand, the points of this conic for which the two values of ρ_{i+k} coincide are those where it is touched by one of the common tangents with (K). Since these points always correspond to equal values of ρ_i, he could claim (Fig. 29a):

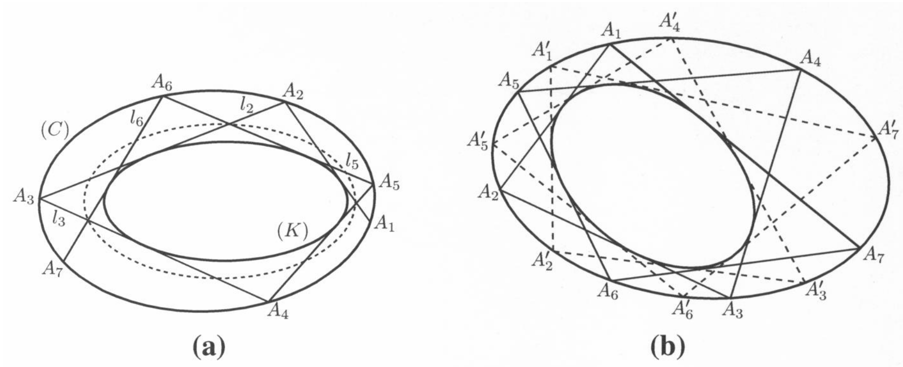

Fig. 29 **a** Theorem D4. **b** Darboux applied Theorem D4 in order to prove the Poncelet closure theorem for $n \geq 5$. The figure illustrates the case $n = 7$

Theorem D4 *If a polygonal line moves while remaining inscribed in (C) and circumscribed about (K), the intersection point of any two sides whose indexes differ by k, for instance the ith and $(i + k)$th, always describes a conic inscribed into the quadrilateral of their common tangents.*

In light of this theorem, Darboux gave the following new proof of PCT.

Since he had already proved the theorem for $n = 3, 4$ (n. 86, 150), he supposed $n \geq 5$. He let $A_1 A_2 \dots A_n$ be an inter-scribed polygon to (C) and (K), and let $A'_1 A'_2 \dots A'_{n+1}$ be a transversal constructed as above, starting from any point A'_1 on (C) (see Fig. 29b). In view of the above theorem, the intersection point of the two sides $A'_1 A'_2$, $A'_n A'_{n+1}$ describes a conic (C') which must pass through all the vertices of the inter-scribed polygon, and, there being at least five of, these the conic (C') must coincide with (C).

Darboux stressed that this proof, although less simple than the one that could be achieved directly from the previous theorem, had the advantage of highlighting the following corollary (Fig. 30):

Corollary *If a polygon moves remaining inter-scribed to the conics (C) and (K), the conic (C_{k-1}), described by the intersection point of two sides indexed by i and $i + k$, is inscribed in the quadrilateral of the common tangents to (C) and (K). Moreover, the diagonals of the polygon envelop conics belonging to the pencil defined by (C) and (K).*

Darboux devoted chapter three to discuss the general Poncelet theorem. He presented essentially two proofs of it, the first based on the theory of conic envelopes (n. 167–169), the second on the properties of Euler's differential equation (n. 170–171).

Here, for brevity, we will comment only the second, as it is of greater interest from a historical point of view.

He considered two conic (f) and (φ), represented by their tangential equations:

$$f = a_0 u^2 + a_2 v^2 + a_4 w^2 + 2a_3 vw + 2a_2 uw + 2a_1 uv = 0, \qquad (9.18)$$

$$\varphi = v^2 - 4uw = 0. \qquad (9.19)$$

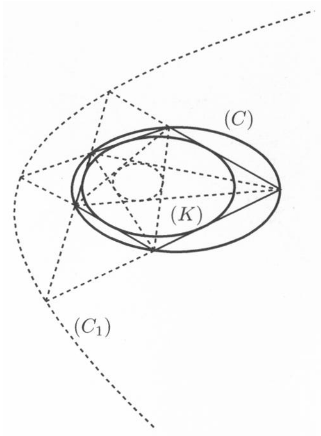

Fig. 30 An illustration of the corollary to Theorem D4 for $n = 5$

Then, the equation of the tangential pencil $f + m\varphi = 0$ is

$$F = -\begin{vmatrix} 0 & x & y & z \\ x & a_0 & a_1 & a_2 - 2m \\ y & a_1 & a_2 + m & a_3 \\ z & a_2 - 2m & a_3 & a_1 \end{vmatrix} = 0. \tag{9.20}$$

He put this equation in the form

$$F = H + Km + Lm^2 = 0, \tag{9.21}$$

where H, K, L are polynomials of degree 2 in the variables x, y, z. By adopting the coordinates ρ, ρ_1, he expressed H, K, L in terms of ρ, ρ_1 and observed that equation (9.21) becomes of degree 2 with respect to both the new variables. Putting $f(\rho) = a_0\rho^4 + 4a_1\rho^3 + 6a_2\rho^2 + 4a_3\rho + a_4$, the discriminant of (9.21) is of the form

$$K^2 - 4HL = f(\rho)f(\rho_1). \tag{9.22}$$

Considering (9.20) as an equation in ρ_1, that is $F = P\rho_1^2 + Q\rho_1 + R = 0$, he computed its discriminant, getting

$$Q^2 - 4PR = f(\rho)[4m^2 - im - j], \tag{9.23}$$

where $i = a_o a_1 - 4a_1a_3 + 3a_2^2$ and $j = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_4a_1^2 - a_2^2$.
For $F = P_1\rho^2 + Q_1\rho + R_1 = 0$, it follows

$$Q_1^2 - 4P_1R_1 = f(\rho_1)[4m^2 - im - j]. \tag{9.24}$$

After setting $\Delta(m) := 4m^2 - im - j$, he differentiated equation (9.21) with respect to m, ρ, ρ_1, obtaining

$$(2Lm + K)dm + (2P\rho_1 + Q)d\rho_1 + (2P_1\rho + Q_1)d\rho = 0. \tag{9.25}$$

Since

$$\begin{cases} 2Lm + K = \pm\sqrt{f(\rho)f(\rho_1)} \\ 2P\rho_1 + Q = \pm\sqrt{f(\rho)\Delta(m)} \\ 2P_1\rho + Q_1 = \pm\sqrt{f(\rho_1)\Delta(m)} \end{cases} \tag{9.26}$$

equation (9.25) gives

$$\frac{dm}{\sqrt{\Delta(m)}} = \pm\frac{d\rho}{\sqrt{f(\rho)}} \pm \frac{d\rho_1}{\sqrt{f(\rho_1)}}, \tag{9.27}$$

which reduces to

$$\frac{d\rho}{\sqrt{f(\rho)}} \pm \frac{d\rho_1}{\sqrt{f(\rho_1)}} = 0, \tag{9.28}$$

if one moves on one of the two conics of the pencil which pass through the point (ρ, ρ_1).[94] Darboux showed that the converse also holds true; hence, he had proved the following theorem:

Theorem D5 *If ρ and ρ_1 vary so that they satisfy one or the other (according the sign) of the above differential equations, the point (ρ, ρ_1) describes one of the conics of the pencil, and vice versa.*

This established, Darboux proceed to give a new proof of the PGT.

His reasoning was as follows. Let $A_1A_2 \ldots A_n$ be a n-gon circumscribed about the base conic of the tangential pencil, and suppose that it moves so that all its vertices but one describe other conics of the pencil. He put $\rho, \rho_1, \ldots, \rho_n$ be the parameters of the different sides, and supposed, without loss of generality, that the $n-1$ vertices $(\rho_1, \rho_2), (\rho_2, \rho_3), \ldots, (\rho_{n-1}, \rho_n)$ describe conics of the pencil. Hence, by the theorem above, the following relations must hold:

$$\frac{d\rho_1}{\sqrt{f(\rho_1)}} = \pm\frac{d\rho_2}{\sqrt{f(\rho_2)}}, \quad \ldots \quad , \frac{d\rho_{n-1}}{\sqrt{f(\rho_{n-1})}} = \pm\frac{d\rho_n}{\sqrt{f(\rho_n)}}.$$

Then, by eliminating the intermediate variables $\rho_2, \ldots, \rho_{n-1}$, he obtained

$$\frac{d\rho_1}{\sqrt{f(\rho_1)}} = \pm\frac{d\rho_n}{\sqrt{f(\rho_n)}}$$

which shows that the free vertex (ρ_1, ρ_n) of the polygon also moves along a conic of the pencil. Thus, he had proved (see Fig. 31):

Theorem D6 *If a polygon moves while remaining circumscribed about a conic (K) in such a way that all its vertices except one, describe the conics $(K_1), \ldots, (K_{n-1})$*

94 We recall that the pencil considered is a *tangential pencil.*

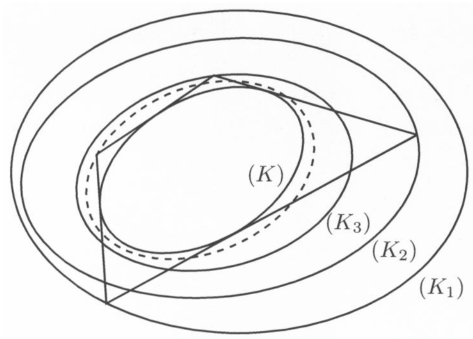

Fig. 31 An illustration of Theorem D6 for $n = 6$

all inscribed in a quadrangle circumscribed about (K), *i.e., belonging to the same tangential pencil as* (K), *then also the last vertex of the polygon describes a conic belonging to this pencil.*

Finally, Darboux observed that by a transformation by reciprocal polars (i.e., by duality), this amounts to Poncelet's general theorem.

We stress the similarity with the proof given by Trudi. Darboux never quoted Trudi, whose papers he had probably not read.

10 Poncelet polygons in Halphen's treatise

Georges Henri Halphen became interested in Poncelet polygons in the late 1870s (Halphen 1878, 1879a, b). In Halphen (1878), as an application of the results developed therein, he computed the number of conics, from a given system whose first characteristic is μ,[95] containing the vertices of a triangle, or of a quadrangle, which in turn is circumscribed about a fixed conic from the same system. Halphen used Salmon's conditions $\beta = 0$, $\gamma = 0$ (see section 6 above),[96] to show that in the first case $\alpha = 2$, $\beta = 0$, and in the second $\alpha = 3$, $\beta = 0$. So he found, respectively, 2μ and 3μ, and then for a pencil these numbers are 2 and 3.

One year later, in the short note (Halphen 1879a), he wrote:

> On sait,... pour que deux coniques A, B soient ainsi, la première inscrite, la seconde circonscrite à un polygone de m côtés, il faut et il suffit que leurs éléments satisfassent à une seule relations. Cette relation a été explicitement formée par divers géomètres pour les nombres m les plus simples, sans qu'on ait jusqu'à

[95] A *system of conics* S is given by an equation of second degree $\sum a_{ij}(\lambda)x_i x_j = 0$, whose coefficients depend on a parameter. Chasles defined *first characteristic* of the system the number μ of conics in S which pass through a point, and *second characteristic* of the system the number ν of conics in S which are tangent to a line. Perfecting Chasles' theory (see Halphen 1878, pp. 27–31), Halphen proved that the number of conics in a system of characteristics (μ, ν) which satisfy a projective condition, is, under certain hypothesis, $\alpha\mu + \beta\nu$, where α and β are positive integers depending on the condition. In particular, if the system is a pencil $\mu = 1$ and $\nu = 2$.

[96] Halphen quoted Salmon, *Higher Algebra*, in the French translation by Bazin, p. 203.

> présent découvert quelle en est la loi. Cette loi est certaiment fot compliquée et, comme on le sait d'après Jacobi, n'est autre que la loi des polyno[m]es naissant de la multiplication des fonctions elliptiques...Si l'on suppose donnée la conique B et que l'on astreigne la conique A à faire partie d'un système S, il y a parmi les coniques de ce système plusieurs solutions A. On demande le nombre. [It is known,...for two conics A and B such that the first is inscribed in, and the second circumscribed about, a polygon of m sides, it is necessary and sufficient that their coefficients satisfy only one condition. This relation has been explicitly found by several geometers for the more simple [the first] numbers m, without having yet discovered what the [general] law is. This law is certainly very complicated, and after Jacobi as is well known, it is nothing but the law of polynomials arising from the multiplication of the elliptic functions...If one supposes that the conic B be given and the conic A is forced to belong to a system S, then among the conics of this system there are several solutions A. One asks for the number.]

His claim"...without having yet discovered what the law is" sounds rather strange. Since Halphen was well aware of Cayley's result, it could be that he was only referring to the conditional equations expressed in terms of invariants, as in Salmon's *Conics*. Anyway, Halphen was here looking for the number of conics A, in a same system as a given conic B, such that there is n-gon which is inscribed in B and circumscribed about A.

According to modern literature (see Barth and Michel 1993), the conics A will be said *n-inscribed* in the conic B, and, reversing the situation, the conic B is said *n-circumscribed* about the conic A. So Halphen wanted to find the number of conics in a pencil which are n-inscribed in a given one from the same pencil. The question he was considering was an essentially new, and difficult, problem in the landscape of Poncelet's polygons.

After having recalled the result for triangles and quadrilaterals as above, Halphen continued by saying:

> Des considérations tirées de la théorie des caractéristiques conduisent aisément à conclure que, pour le cas général, le nombre cherché est toujours de la forme $M\mu$, M étant un nombre qui ne dépend que de m. Mais la détermination de ce nombre M n'est pas sans difficulté. Il ma fallu de faire une étude assez approfondie de la rélation générale, dont la loi n'est pas explicitement connue, pour lever cette difficulté. J'y suis parvenu, et je peux actuellement donner le théorème suivant [Considerations deduced from the theory of characteristics easily lead to the conclusion that, for the general case, the required number is always of the form $M\mu$, M being a number depending only on m.[97] But the determination of this number M is not at all without difficulty. It took me an in-depth study of the general relation, whose law is not yet known explicitly, to overcome this difficulty. I succeeded and now I can formulate the following theorem]

[97] Here M has the meaning of the previous α. In particular, Halphen asserted that, in these cases, one has always $\beta = 0$.

then Halphen stated that the number M is given by the following formula

$$M = \frac{1}{4}m^2\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{r^2}\right)\cdots, \tag{10.1}$$

where $p, q, r, \ldots$ are the primes in the prime factorization of $n = p^\alpha q^\beta r^\gamma \cdots$.

He neither proved nor explained this formula, which fits well in the cases $m = 3, 4$ seen above. However, since $4M$ equals the number of the *primitive m-th part of the periods* of an elliptic function (with additive group of periods Λ), i.e., of those w such that $mw \equiv 0 (\text{mod } \Lambda)$ but $kw \not\equiv 0 (\text{mod } \Lambda)$ for any k which divides m, precisely:

$$\mathcal{T}(m) = (p^2-1)p^{2(\alpha-1)}(q^2-1)q^{2(\beta-1)}(r^2-1)r^{2(\gamma-1)}\cdots, \tag{10.2}$$

we may think that Halphen used the multiplication of the argument to write down (10.2) and (10.1). In fact, this was so, as we will see in a while.

The first of the three volumes constituting Halphen's *Traité des fonctions elliptiques et de leurs applications*was printed in 1886. With this fundamental work, Halphen presented the theory of elliptic functions, in terms of the new functions introduced by Weierstrass, mainly the $\wp(u)$ and the $\sigma(u)$, that he found more suitable, especially in dealing with applications, than Jacobi's $\mathrm{sn}(u)$ and $\mathrm{cn}(u)$. The second volume, devoted to the applications to mechanics, physics, geodesy, geometry and integral calculus, followed in Halphen (1888). The third was printed posthumously in 1891. This last volume, instead of the theories of the modular equation and of the complex multiplication together with a historical survey, as Halphen had projected, contained only some unpublished manuscripts on the division of periods, an article already published on the complex multiplication, and some fragments (Halphen 1891).

In the treatise, Poncelet polygons make their first entry at the end of chapter I of the first volume, where, in discussing the geometrical interpretation of the addition formulae for the Jacobian elliptic functions, Halphen forwarded again Jacobi's proofs of the theorems of Poncelet for circles.

Halphen devoted the whole chapter X of the second volume to the same subject. In fact, he entitled it "Les polygones de Poncelet." Here, he presented the numerous results on this topic that he had probably been collecting since 1878. Among other things, he proved Poncelet's theorems by means of symmetric (2, 2)-correspondences, determined the closure conditions and re-obtained Cayley's formulae. Then, he introduced the "elliptic representation" of points of the plane to study the problem of finding the number of conics from a pencil that are n-circumscribed about a given one belonging to the same pencil: the question he had considered 10 years before.

Halphen returned on Poncelet's theorems in chapter XIV, where he gave new proofs based on the development in continued fractions of $\sqrt{X}$, with X is a polynomial in one variable of degree 3 or 4.

Gino Loria, in his historical account, did not comment on the results on Poncelet polygons contained in Halphen's treatise, published the year before, partly because, as he wrote, "facendo parte di un'opera voluminosa, non si può fotografare in poche frasi con sufficiente chiarezza" [being part of a voluminous work, it cannot be photographed in a few sentences with sufficient clarity] (Loria 1889a, p. 20).

As we have said above, the third volume of Halphen's treatise contained some unpublished fragments. Among these, we can find the proof of formula (10.2) that we summarize here below (Halphen 1891, pp. 194–201).

He denoted by w_n a (nonzero) nth part of a period of the Weierstrass's function $\wp$, with n a prime number. In this case, modulo periods, all w_n, are defined by the formula

$$w_n = \frac{2p\omega + 2p'\omega'}{n} = (p, p'),$$

where ω and ω' are half-periods, and $0 \le p, p' \le n-1$ are integers such that $p^2 + p'^2 \neq 0$. Their number is $n^2 - 1$, and they form a group, partitioned in $n+1$ cyclic group of $n-1$ elements. Halphen observed that, when n is any positive integer, the formula above gives a nth part of period if and only if p, p' and n are relatively prime (i.e., $(p, p', n) = 1$). Moreover, a cyclic group if formed by multiplying an element w_n by the integers less than n and relatively prime to n.

Then, Halphen supposed $n = a^\alpha$, with a a prime number. In this case, w_n is a n-part of a period if and only if one—at least—of p, p' is prime with a. So the number of w_n is

$$n^2 - 1 - \left[\left(\frac{n}{a}\right)^2 - 1\right] = n^2\left(1 - \frac{1}{a^2}\right).$$

Now, mw_n is a nth part of a period if m is not divisible by a. Then, taking m in the sequence $1, 2, \ldots, n-1$, the numbers $a, 2a, 3a, \ldots, (a^{\alpha-1} - 1)a$ must be excluded, and so only $a^\alpha - 1 - (a^{\alpha-1} - 1)$ numbers m remain. He set

$$\varphi(n) = a^\alpha - a^{\alpha-1} = n\left(1 - \frac{1}{a}\right).$$

Reasoning as above, he concluded that the number of groups is

$$T(n) = n\left(1 + \frac{1}{a}\right).$$

Finally, Halphen supposed that

$$n = a^\alpha b^\beta c^\gamma \cdots,$$

and he put

$$\frac{p}{n} = \frac{p_1}{a^h} + \frac{p_2}{b^k} + \frac{p_3}{c^l} + \cdots; \quad \frac{p'}{n} = \frac{p'_1}{a^{h'}} + \frac{p'_2}{b^{k'}} + \frac{p'_3}{c^{l'}} + \cdots$$

In this way, observed Halphen, the w_n are given as sum of elements of the form

$$\frac{p_1\omega}{a^h} + \frac{p'_1\omega'}{a^{h'}}; \quad \frac{p_2\omega}{b^k} + \frac{p'_2\omega'}{b^{k'}}; \ldots$$

Then, w_n will be a n-part of a period if and only if one of the exponents h, h' is equal to α, one of k, k is equal to β, and so on. Hence, w_n is the sum of elements $w_{n_1}, w_{n_2}, \ldots$, with $n_1 = a^\alpha$, $n_2 = b^\beta$, $n_3 = c^\gamma$, etc. Moreover, there will also be elements w_n belonging to different groups, because the condition $pr' - rp' \equiv 0 (\mathrm{mod}\, n)$ decomposes in $p_1 r_1' - r_1 p_1' \equiv 0 (\mathrm{mod}\, n_1)$, $p_2 r_2' - r_2 p_2' \equiv 0 (\mathrm{mod}\, n_2)$, one of which, at least, does not hold by the hypothesis. Therefore, it follows that the number of groups is

$$T(n) = T(n_1)T(n_2)\cdots = n\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\cdots,$$

the number of elements in each group is

$$\varphi(n) = \varphi(n_1)\varphi(n_2)\cdots = n\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\cdots,$$

and then the number of w_n is

$$\varphi(n)T(n) = n^2\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{b^2}\right)\cdots,$$

where $\varphi(n)$, he concluded, "est bien connue en Arithmétique, comme dénom-brant les nombres premiers à n et inférieurs à n" [is well known in Arithmetic, as the number counting those numbers which are less than n and prime to n]: the Euler totient function.

10.1 Doubly quadratic equations and closure conditions

Before entering these questions, we briefly recall the content of chapter IX, titled "Equation d'Euler." Here, Halphen introduced doubly quadratic equations (i.e., (2, 2)-correspondences), that he represented in the form

$$F = \sum (m, n) x^m y^n = 0,$$

where $m, n \in \{0, 1, 2\}$ and (m, n) denote the coefficient of the monomial $x^m y^n$. He also wrote

$$F = Ay^2 + 2By + C = A'x^2 + 2B'x + C',$$

with A, B, C and A', B', C' polynomials of degree 2, respectively, in x and y, and put $X = B^2 - AC$, $Y = B'^2 - A'C'$. The equation $F = 0$ implies the differential equation

$$\frac{\mathrm{d}x}{\sqrt{X}} \pm \frac{\mathrm{d}y}{\sqrt{Y}} = 0.$$

If F is symmetric, i.e., $(0, 1) = (1, 0)$, $(1, 2) = (2, 1)$ and $(0, 2) = (2, 0)$, the two polynomials X,Y, of degree four, are the same except for the variable, i.e., x^k and y^k

have the same coefficient for any $k = 4, \ldots, 0$. In this case, the above differential equation is Euler's differential equation (Euler 1768, 1794), previously considered by Trudi.

To any polynomial X of degree 4, there is an associated elliptic function $f(u)$ having (only) two simple poles (i.e., of degree two), such that putting $x = f(u)$ one has

$$\frac{\mathrm{d}x}{\sqrt{X}} = \mathrm{d}u.$$

The same holds for $y = f(u_1)$, so that the above differential equation becomes $du \pm du_1 = 0$ and gives $u = \pm u_1 + c$ where c is a constant. Hence, observed Halphen, every symmetric doubly quadratic equation expresses the relation between $f(u)$ and $f(u + U)$, where f is an elliptic function of degree two of the variable u and U is a constant.

In the preamble of chapter X, he wrote:

> Dès le début du Tome I, on a vu (p. 13) l'addition des arguments répésentée par une construction géométrique au moyen de deux cercles. A chaque point de l'un des cercles, on fait correspondre un argument elliptique: la corde qui joit deux points, dont la différence des arguments est constante, enveloppe le seconde cercle.
>
> Cette construction de l'addition peut être modifiée de façon que, au lieu de deux cercles, on ait à considérer deux coniques quelconques. On n'en surait douter, d'après les enseignements de la Géométrie projective. Mai il convient de présenter directement cette construction sous sa forme générale. C'est à quoi se prête merveileusement la considération des équations doublement quadratiques, object principal du Chapitre précédent [At the beginning of volume I, we have seen (p. 13) the addition of the arguments represented geometrically by means of two circles. To each point of one of these circles, there corresponds an elliptic argument: the chord joining two points, whose arguments differ by a constant, envelops the second circle. This construction of the addition can be modified in such a way that, instead of the two circles, one has two conics whatever. This is indubitable by the principles of the projective geometry. But it is convenient to present this construction in a more general form. For doing this, the doubly quadratic equations, the main object of the previous chapter, fit wonderfully].

Let $F(x, y) = 0$ be a symmetric doubly quadratic equation. Fixing any value for x, the equation gives two values for y, say x_1 one of these. For $x = x_1$, the equation gives other two values, one of which is x and the other x_2. To the latter correspond x_1 and a new one x_3, and so on. Denoting x_{-1} the second correspondent of x, x_{-2} the second correspondent of x_{-1}, keep doing this way it is established a sequence

$$\ldots, x_{-2}, x_{-1}, x, x_1, x_2, x_3 \ldots$$

Halphen fixed a non-singular conic C and denoted x a parameter that rationally determines the points on C.[98] He observed that, if in the above sequence two con-

[98] Halphen referred to C as "unicursal curve," see also back in section five.

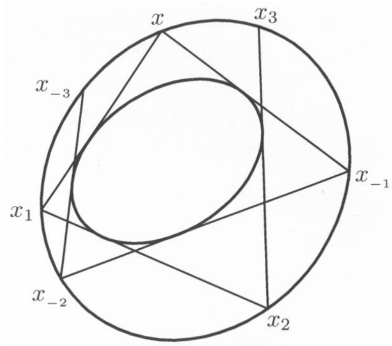

Fig. 32 Every symmetric doubly-quadratic equation $F(x, y) = 0$, once fixed a value for x, gives a sequence $\ldots x_{-2}, x_{-1}, x, x_1, x_2, \ldots$ which determines an inter-scribed polygon to two conics. The figure illustrates this situation

secutive values represent the end points of a chord of C, then the envelope of these chords is another conic D: in fact, for any point of C there are exactly two tangents to the enveloped curve, which is necessarily of the second class, i.e., a conic. Hence, Halphen claimed: *every symmetric doubly quadratic equation translates the relation among the ending points of a variable chord inscribed in a conic C and enveloping another conic D.*

In particular, any sequence as above determined a polygonal line, represented $\ldots, x_{-2}, x_{-1}, x, x_1, x_2, x_3 \ldots$, which is inscribed in the conic C and, at the same, time circumscribed about D (see Fig. 32).

There are four particular values $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ of the parameter x, to each of them corresponds a double root of y; say $\beta_0, \beta_1, \beta_2, \beta_3$ these double roots. The tangents to D from the point (of parameter) α_0 coincide, so $\alpha_0 \in D$. Hence, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are the parameters on C of the four points $C \cap D$, and $\beta_0, \beta_1, \beta_2, \beta_3$ are the parameters of the same points on D. Then, he proved (Halphen 1888, pp. 340, 374) that symmetric doubly quadratic equations are characterized by the two invariants:

$$\alpha = \frac{(\alpha_0 - \alpha_1)(\alpha_3 - \alpha_2)}{(\alpha_0 - \alpha_2)(\alpha_3 - \alpha_1)}, \quad \gamma = \frac{(\beta_0 - \beta_1)(\beta_3 - \beta_2)}{(\beta_0 - \beta_2)(\beta_3 - \beta_1)},$$

corresponding to the cross-ratios of the four points $C \cap D$, taken in the same order, on C and on D.

Among the polygonal lines $\ldots, x_{-2}, x_{-1}, x, x_1, x_2, x_3 \ldots$, there are those obtained starting from a point (of parameter) α belonging to $C \cap D$, or from a point of C where a common tangent to C and D touch C. These polygonal lines, Halphen observed, can be prolonged only in one direction. If $\alpha_1, \alpha_2, \ldots$ are the other vertices of the transversal, then, by taking x as initial point of the polygonal line, the points α_n and $\alpha_{n-1}, \alpha_{n-2}$, etc. are such that the polygonal line folds up on itself, and

$$x_n = \alpha, \quad x_{n+1} = x_{n-1}, \quad x_{n+2} = x_{n-2}, \quad \ldots \quad x_{2n} = \alpha_n.$$

In general, this does not occur drawing the polygonal line in the other direction. The same holds when α' is a point of contact of a common tangent to the two conics. To

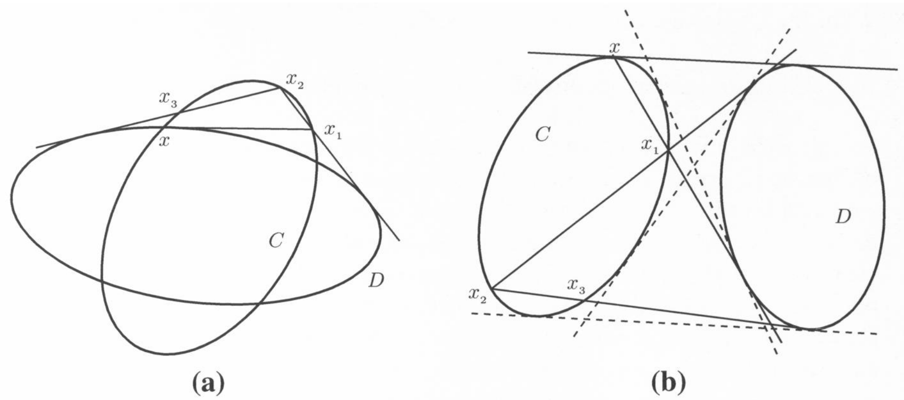

Fig. 33 Folded polygons inter-scribed to the conics C and D: **a** of the first kind, **b** of the second kind

these, Halphen gave the names of *folded polygonal lines*, respectively, of *first* and *second kind* (see Fig. 33a, b).

The closure condition for a polygonal line, in order to give a polygon of m sides, is $x = x_m$.

In chapter IX, Halphen had shown that the two roots x, x_1 of $F = 0$, corresponding to a certain value of y, can be considered, respectively, equal to $\wp(u)$ and $\wp(u+U)$. In this way, the vertices $\ldots, x_{-2}, x_{-1}, x, x_1, x_2, x_3 \ldots$ of a polygonal line have parameters given by the values of $\wp$ for $\ldots, u-2U, u-U, u, u+U, u+2U, u+3U, \ldots$. Then, according to what he had already proved, the parameters y corresponding to $x = \wp(u)$ are $\wp(u \pm U/2)$. With this "elliptic representation of the polygonal line," the closure condition is translated into the condition that mU must be a period.

By means of the theory of the function $\wp(u)$ developed in the first volume of his treatise, Halphen was able to express the condition above through the invariants α and γ. Precisely, he found that the condition for mU to be a period can be expressed by the vanishing of a polynomial involving the invariants

$$x = -\frac{[\alpha^2 - 2\gamma(2\gamma^2 - 3\gamma + 2)\alpha + \gamma^4]^2}{2^8\alpha^2(\alpha-1)^2\gamma^4(\gamma-1)^4},$$
$$y = -\frac{(\gamma^2-\alpha)(\gamma^2-2\gamma+\alpha)(\gamma^2-2\alpha\gamma+\alpha)}{2^3\alpha(\alpha-1)\gamma^2(\gamma-1)^2},$$

(to be not confused with the variables x, y of the doubly quadratic equation $F = 0$). Then, he wrote these equations explicitly for $m = 3, \ldots, 11$ (p. 377):

$$\begin{aligned}
m &= 3\ldots && x = 0,\\
m &= 4\ldots && y = 0,\\
m &= 5\ldots && y - x = 0,\\
m &= 6\ldots && y - x - y^2 = 0,\\
m &= 7\ldots && (y-x)x - y^3 = 0,\\
m &= 8\ldots && (y-x)(2x-y) - xy^2 = 0,\\
m &= 9\ldots && y^3(y-x-y^2) - (y-x)^3) = 0,\\
m &= 10\ldots && y^2(xy-x^2-y^3) - x(y-x-y^2)^2 = 0,\\
m &= 11\ldots && (xy-x^2-y^3)(y-x)^3 - xy(y-x-y^2)^2 = 0.
\end{aligned}$$

10.2 On the Cayley conditions

At page 387 of the second volume of his treatise, Halphen commented:

> Le calcul de la condition pour l'existence des polygones à m côtés se fait, au moyen des invariants x, y, comme il a été indiqué précédemment. C'est ce qu'on a de plus simple sur ce suject. On ne surait némmoins omettre un autre moyen de faire le calcul, infiniment moins commode, mais extêmement élégant. Il à été trouvé par M. Cayley [The computation of the condition for the existence of polygons of m sides is carried out by means of the invariants x, y, as previously shown. This is the simplest one we have on the subject. But we cannot omit another method of doing this computation, which is infinitely less handy to deal with, but very much more elegant. It was found by Mr. Cayley].

Halphen, recognizing the elegance of Cayley's method, also expressed the conviction that it was more difficult to handle than that he had just expounded. The reason for his opinion is not clear, but probably it was because Cayley called upon the theory of Abelian integrals, while he preferred to treat the question algebraically.

The condition for which mU is a period, observed Halphen, is equivalent to the existence of a polynomial function $M(\wp(u)) + N(\wp(u))\wp'(u)$ having a zero of order m for $u = U$. He had already noticed (p. 344) that the discriminant $F(s)$ of the pencil $sX + Y$ has roots proportional to $\wp(U + e_1)$, $\wp(U + e_2)$, $\wp(U + e_3)$, where e_1, e_2, e_3 are such that $\wp'(u)^2 = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3)$, and this allowed him to put the above polynomial function in the form $M(s) + N(s)\sqrt{F(s)}$. Since the value $u = U$ corresponds to $s = 0$, the existence of the root U of order m requires that the development in power series of $M(s) + N(s)\sqrt{F(s)}$ around $s = 0$ begins with a term of degree m. Let

$$\sqrt{F(s)} = p_0 + p_1 s + p_2 s^2 + p_3^3 + \cdots, \tag{10.3}$$

he observed that, since s has degree 2, and $M(s) + N(s)\sqrt{F(s)}$ must be of degree m, it was convenient to consider two cases according as m is even or odd.

For $m = 2n$, one has $M = a_0 + a_1 s + \cdots + a_n s^n$ and $N = b_0 + b_1 s + \cdots + b_{n-2}s^{n-2}$; then, by equating to zero the terms of degree $0, 1, \ldots, n$ in (10.3) equations are obtained containing the coefficients of M, while for the following terms only the coefficients of N occur:

$$\begin{aligned} &b_{n-2}p_3 + b_{n-2}p_4 + \cdots + b_0 p_{n+1} = 0, \\ &b_{n-2}p_4 + b_{n-3}p_5 + \cdots + b_0 p_{n+2} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &b_{n-2}p_{n+1} + b_{n-3}p_{n+2} + \cdots + b_0 p_{2n-1} = 0. \end{aligned}$$

For $m = 2n + 1$, M is the same but N has one more term $b_{n-1}s^{n-1}$, and then the equations above become:

$$\begin{aligned}
&b_{n-1}p_2 + b_{n-2}p_3 + \cdots + b_0 p_{n+1} = 0,\\
&b_{n-1}p_3 + b_{n-2}p_4 + \cdots + b_0 p_{n+2} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&b_{n-1}p_{n+1} + b_{n-2}p_{n+2} + \cdots + b_0 p_{2n} = 0.
\end{aligned}$$

It follows that the condition for the existence of a Poncelet's polygon of $2n$ sides is

$$\begin{vmatrix} p_3 & p_4 & \cdots & p_{n+1} \\ p_4 & p_5 & \cdots & p_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \end{vmatrix} = 0,$$

while for a polygon of $2n+1$ sides is

$$\begin{vmatrix} p_2 & p_3 & \cdots & p_{n+1} \\ p_3 & p_4 & \cdots & p_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n+1} & p_{n+2} & \cdots & p_{2n} \end{vmatrix} = 0.$$

Hence, he observed that for a triangle the condition is just $p_2 = 0$, for the quadrangle is $p_3 = 0$, and then, referring to the invariant x and y above, he added: "on vérifiera aisément la concordance de ces conditions avec celles qui on été trouvées précédemment" [one can easily check that these conditions are the same than those previously determined].

Halphen also affirmed that, at that point, the link with the continued fractions was evident, but that he reserved this study for a subsequent chapter.

10.3 The "elliptic representation of the plane" and the curve Π_m

From the elliptic representation of the polygonal line inter-scribed to two conics C and D, Halphen knew that the closure condition was equivalent to "mU is a period," and here he wanted to go more in depth into the question.

In the first volume of his treatise (Halphen 1886, pp. 96–103), Halphen defined the polynomial functions of $\wp(u)$ and $\wp'(u)$:

$$\psi_n(u) = \wp'(u)\left[-\tfrac{1}{2}n(\wp(u))^{\frac{n^2-4}{2}} + \cdots\right], \quad \text{for } n \text{ even},$$

$$\psi_n(u) = \left[n(\wp(u))^{\frac{n^2-1}{2}} + \cdots\right], \quad \text{for } n \text{ odd}.$$

These functions vanish exactly at the nth parts of the periods ($n \geq 2$). Then, he defined the irrational functions

$$\gamma_n(u) = \psi_n(u)\psi_2(u)^{-\frac{n^2-1}{3}},$$

$\gamma_1 = \gamma_2 = 1$, and showed that they satisfy the recursive formula

$$\gamma_{m+n}\gamma_{m-n} = \gamma_{m+1}\gamma_{m-1}\gamma_n^2 - \gamma_{n+1}\gamma_{n-1}\gamma_m^2,$$

and consequently that

$$\begin{aligned}\gamma_{2n+1} &= \gamma_{n+2}\gamma_n^3 - \gamma_{n-1}\gamma_{n+1}^3,\\ \gamma_{2n} &= \gamma_n(\gamma_{n+2}\gamma_{n-1}^2 - \gamma_{n-2}\gamma_{n+1}^2).\end{aligned}$$

Putting

$$x(u) = \gamma_3^3(u), \quad y(u) = \gamma_4(u),$$

it follows that

$$\begin{aligned}\gamma_5 &= y - x,\\ \gamma_6 &= y - x - y^2,\\ \gamma_7 &= (y-x)x - y^3,\\ \gamma_8 &= (y[(y-x)(2x-y) - xy^2],\\ \gamma_9 &= x^{\frac{1}{3}}[y^3(y-x-y^2) - (y-x)^3)],\\ &\vdots\end{aligned}$$

In chapter X of the second volume of his treatise (pages 392–404), by means of the duplication formula, Halphen expressed the functions $x(2u)$ and $y(2u)$ as *combinants* of the pencil generated by two conics f and ψ, i.e., as rational functions over $\mathbb{P}^2$ invariant under the action of the symmetric group Σ_4.[99] In particular, he proved that the expression of $\gamma_n(2u)$ does not depend on the conics f and ψ but only on the four points $f \cap \psi$.

At p. 404, Halphen wrote:

> Ici se present tout naturellement la considération du lieu géométrique défini par l'équation $\psi_n(2u) = 0$. Nous devons donner de ce lieu géométrique une définition indépendante des fonctions elliptique et mettre en lunière un fait bien remarquable: ce lieu se décompose en plusieurs lignes distinctes…[It is natural here to consider the geometrical locus defined by the equation $\psi_n(2u) = 0$. We must give to this geometrical locus a definition independent of the elliptic functions and highlight a very remarkable fact: This locus decomposes in several distinct components…]

To this end, Halphen proceeded as follows.

He considered $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, which are four points in the plane in general position, i.e., no three of them are on a line, as base points of a pencil of conics, and he associated with them the half-period ω_i, $i = 0, 1, 2, 3$, with $\omega_0 = 0$. For any given point z in the plane, he considered in the pencil with base points $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ the conic f passing through z. The cross-ratio α of the four points $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ on f (which

[99] A *combinant* of the pair f, ψ, according to Sylvester (1853), is a covariant of the forms that, besides having the ordinary character of invariance when linear substitutions are applied to the variables, possesses the same character of invariance when linear substitutions are applied to their linear combinations.

coincides with the cross-ratio of the four lines $z\alpha_0$, $z\alpha_1$, $z\alpha_2$ and $z\alpha_3$), wrote Halphen, defines the *absolute invariant* of the elliptic functions. Clearly, he was referring to the invariant $J = \frac{g_2^3}{g_2^3 - 27g_3^2}$, which can be expressed as rational function of anyone of the six cross-ratios of the four points ∞, e_1, e_2, e_3, where e_1, e_2, e_3 are the roots of the equation $\wp'^2 = 4\wp^3 - g_2\wp - g_3$. In this way, observed Halphen, the function $\wp(u)$ is a parameter for the points on f, which assumes the values ∞, e_1, e_2, e_3 for the values $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ of the argument. Let γ be the cross-ratio of the four lines through α_0 containing, in this order, $z, \alpha_1, \alpha_2, \alpha_3$. From the assumption above, it follows that

$$\gamma = \frac{\wp(u) - e_1}{\wp(u) - e_2} \frac{e_3 - e_2}{e_3 - e_1},$$

and at the same time

$$\alpha = \frac{e_3 - e_2}{e_3 - e_1}.$$

At this point, Halphen affirmed (p. 405):

> Voilà donc les fonctions elliptiques et l'argument définis, pour chaque point du plan, par les rapports anharmoniques de deux faisceaux de droits. Il s'agit maintenant de reconnaître la propriété géometrique des points qui répondent à des parties aliquotes de périodes [here then are the elliptic functions and the argument defined, for every point of the plane, by means of the cross-ratios of two pencils of lines. It is now time to recognize the geometrical property of those points which correspond to the nth parts of the periods].

Then, he considered the line $\alpha_0 z$ through α_0 and z, and the conic ψ of the pencil which is tangent (necessarily at α_0) to $\alpha_0 z$. For the pair f, ψ, he found that the argument u of the point z coincides with U. So he claimed:

> Voici donc la propriété géometrique de tout point z dont l'argument u est une m-ième partie de période: *si, parz, on mène une conique f du faisceau et que, tangentiellement à $\alpha_0 z$, on prenne une autre conique ψ du faicseau, il existe des polygones de m côtés, inscrits dans f et circonscrits à ψ*. Pour chaque entier m, il y a un lieu du point z; c'est ce lieu qui doit être examiné. [Here is the geometrical property of each point z whose argument u is the mth part of a period: *if through z is drawn a conic f of the pencil and, tangentially to $\alpha_0 z$, another conic of pencil is taken, then there exists a polygon of m sides inscribed in F and circumscribed about ψ*. For each integer m, there is a locus of the point z; it is this locus that has to be examined.]

It will be useful, for the future, to denote this locus Π_m and let Π'_m be the sub-locus of primitive mth parts of periods (as defined at the beginning of this section). The intersections of Π_m with the tangent to ψ at α_0 define the conics in the pencil which admit an inscribed polygon of m sides which, at the same time, is circumscribed about ψ. So the degree of Π_m (Π'_m) is the number of those conics in the pencil which are m-circumscribed (properly m-circumscribed) about the conic ψ.

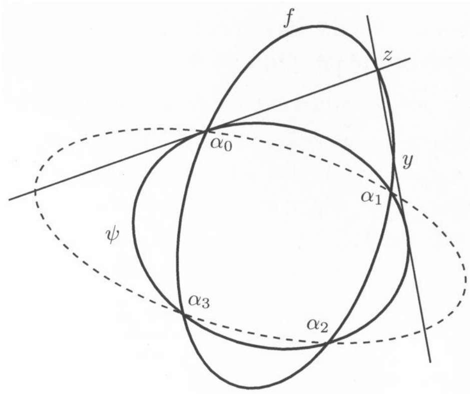

Fig. 34 Halphen's construction for doubling the argument of z on f: the corresponding point on f is the point y which is the second intersection of the second tangent to ψ from z

Halphen first considered the multiplication of the argument by 2. He knew that, f and ψ being as above, the point of the plane having the same modulus and argument twice the argument of z, is precisely the second intersection with f of the second tangent to ψ from z (Fig. 34).

Aiming to express this construction in formulae, he let

$$\alpha_0 = (1, 1, 1), \quad \alpha_1 = (-1, 1, 1), \quad \alpha_2 = (1, -1, 1), \quad \alpha_3 = (1, 1, -1).$$

Then, denoting by (x_1, x_2, x_3) the variable coordinates in the plane and putting $z = (z_1, z_2, z_3)$, he was led to the following equations for the conics f_z and ψ_z (p. 406):

$$f_z = (z_2^2 - z_3^2)x_1^2 + (z_3^2 - z_1^2)x_2^2 + (z_1^2 - z_2^2)x_3^2 = 0,$$
$$\psi_z = (z_2 - z_3)x_1^2 + (z_3 - z_1)x_2^2 + (z_1 - z_2)x_3^2 = 0.$$

He also found that the point of contact x of the second tangent from z to the conic ϕ_z has coordinates

$$x_1 = z_2z_3 - z_1z_2 - z_1z_3,$$
$$x_2 = z_3z_1 - z_2z_3 - z_2z_1,$$
$$x_3 = z_1z_2 - z_3z_1 - z_3z_2,$$

and y has coordinates

$$y_1 = z_2^2z_3^2 - z_1^2z_2^2 - z_1^2z_3^2,$$
$$y_2 = z_3^2z_1^2 - z_2^2z_3^2 - z_1^2z_2^2,$$
$$y_3 = z_1^2z_2^2 - z_3^2z_1^2 - z_2^2z_3^2.$$

From these formulae, it follows

$$\frac{y_1z_2 - z_1y_2}{(z_1 - z_2)x_3} = \frac{y_2z_3 - z_2y_3}{(z_2 - z_3)x_1} = \frac{y_3z_1 - z_3y_1}{(z_3 - z_1)x_2} = z_1z_2 + z_2z_3 + z_3z_1.$$

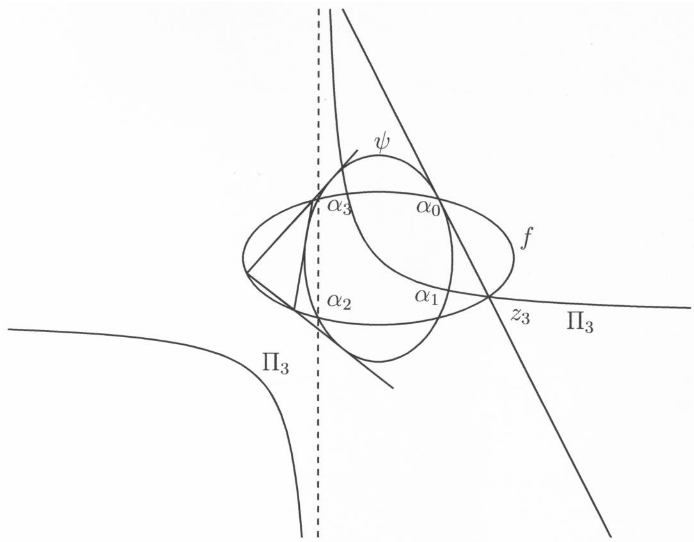

Fig. 35 An illustration of the locus Π_3 of points z whose argument is a third period: a conic

He observed that the points z such that $y = z$ are those points whose argument is a third of a period, so for the locus Π_3 he found the equation

$$z_1z_2 + z_2z_3 + z_3z_1 = 0,$$

i.e., a conic (see Fig. 35).

Similarly, Halphen showed that the condition

$$z_1z_2z_3 = 0$$

characterizes the points of the plane whose argument is a quarter period; hence, Π_4 decomposes into the three coordinate lines (see Fig. 36).

For Π_6 he argued as follows. Let $(2n\omega + 2n'\omega')/6$ the argument of z, then the integers n and n' cannot be both even; otherwise, it would be a point Π_3. Consider the point whose argument is $n\omega + n'\omega'$ is one of the $\alpha_i \neq \alpha_0$, then the difference between the arguments of α_i and z is $(2n\omega + 2n'\omega')/3$. Then, if one considers the conic ψ tangent to the line $\alpha_i z$, the corresponding argument U is a third of a period. Hence, the point z has, with respect to α_i, the same property that each point in Π_3 has with respect to α_0. By changing the sign of the coordinates, the points $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are permuted among them, and this change the conic $A_0 : z_1z_2 + z_2z_3 + z_3z_1 = 0$ into the conics

$$z_2z_3 - z_1z_2 - z_1z_3 = 0, \quad z_3z_1 - z_2z_3 - z_2z_1 = 0, \quad z_1z_2 - z_3z_1 - z_3z_2 = 0.$$

Denoting these conics, respectively, A_1, A_2, A_3, it follows that Π'_6 is the locus $A_1 \cup A_2 \cup A_3$, so it has degree 6.

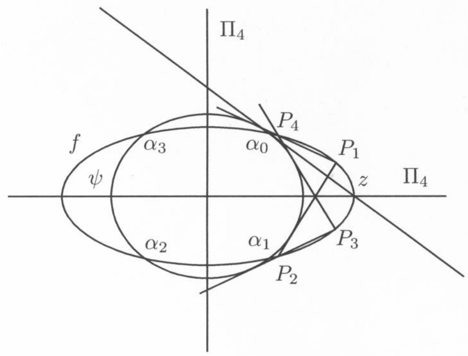

Fig. 36 The locus Π_4 of points z whose argument is a quarter period: the locus is decomposed into the three coordinate lines

If in the A_1, A_2, A_3 the z's are replaced by the y's as above, one obtains the equations of the three curves constituting the locus Π'_{12}, each of them of degree 8, so Π'_{12} has degree 24. Continuing in this way, one see that $\deg\Pi'_{24} = 96$, etc.

At this point (p. 409) Halphen recalled that the roots of the function $\psi_m(2u)$ are all the nth parts of the periods, being n any divisor of $2m$.

Then, he claimed: if m is odd, the locus Π_m contains Π_n and Π_{2n}, in total four curves, for each divisor n of m; if $m = 2^a m'$, where m' is odd, Π'_m decomposes in $3a + (3a+4)\sum n'$ curves, where by $\sum n'$ he denoted the number of the divisors $n' \neq 1$ of m', in particular if $m = 2a$ the number of distinct curves is only $3a$.

We may interpret Halphen's reasoning as follows. If m is not a prime, then Π_m decomposes in Π'_m and a number of curve Π'_n, not necessarily irreducible, one for each $n|m, n \geq 3$. Since Π_3 and Π_4 have, respectively, degree 2 and 3, the degrees of Π_m and Π'_m can be recursively computed for many values of m. We have already seen that Π'_6 decomposes into 3 conics, so is degree is 6, and $\Pi_6 = \Pi'_6 \cup \Pi_3$, so that it has degree 8. One can easily see that Π'_8 has 3 components of degree 4, and $\Pi_8 = \Pi'_8 \cup \Pi_4$, so Π_8 has in total 4 components, and total degree 15. Similarly, Π'_{12} has 3 components of degree 8, and $\Pi_{12} = \Pi'_{12} \cup \Pi'_6 \cup \Pi'_4 \cup \Pi'_3$, so it has in total 10 components and degree 35. This process can be continued.

In Gruson (1992, p. 193), it is suggested that Halphen here proved the formula

$$\deg\Pi'_n = \frac{1}{4}n^2 \prod_{p|n,\, p \text{ prime}} \left(1 - p^{-2}\right),$$

but in the second volume of Halphen's treatise there is no trace of it. Nevertheless, taking (10.2), the above reasoning leads to that formula. Moreover, one has $\deg\Pi_n = (n^2-1)/4$, if n is odd, and $\deg\Pi_n = n^2/4 - 1$, if n is even.

The construction above has clear formulation in Barth and Michel (1993), where the two authors, unaware of Halphen's result, proved the above formula in a modern algebraic–geometric setting. We will return to this in the penultimate section of our paper.

10.4 Continued fractions and Poncelet polygons

At p. 388 of the second volume of his treatise, Halphen noticed that Cayley's method for determining the closure condition "se rattache, de la manière la plus directe, à la théorie des fractions continues, ainsi qu'on le verra dans un Chapitre ultérieur" [this method, as it will see in a subsequent chapter, is connected, in the more direct way, to the theory of continued fractions].

He dealt with this question in chapter XIV, where he pursued the study, initiated by Abel and Jacobi, of the development in continued fractions of $\sqrt{X(x)}$, being $X(x)$ a polynomial in the variable x.

Abel, in his celebrated memoir of 1826, proved that the integral

$$\int \frac{\rho(x)\mathrm{d}x}{\sqrt{X(x)}},$$

where $\rho(x)$ is a polynomial, can be expressed by means of rational functions and logarithms of algebraic functions if and only if $\sqrt{X(x)}$ admits a periodic continued fraction development.

In his note (Jacobi 1831), Jacobi studied an algorithm for developing $\sqrt{X(x)}$ in continued fractions. The difficulties that he encountered in the computation forced him to abandon the algebraic route, and to consider the use of elliptic functions in order to express the partial quotients of the continued fraction. He wrote down interesting formulae in the case X has at most degree four.[100]

Halphen considered the continued fractions development of the more general element

$$\frac{\sqrt{X}+\sqrt{Y}}{x+y},$$

where $Y := X(y)(\deg X = 3, 4)$. His point of departure was the function

$$V_m := C_m \frac{[\sigma(a-u)]^{2m-1}\sigma(u+2ma+mv-w)}{[\sigma(u)\sigma(u+v)]^m},$$

where σ is the Weierstrass σ-function, and m an integer. V_m is a doubly periodic function with respect to any of the four arguments a, u, v, w. Through the development of V_m/V_{m-1} in continued fractions, Halphen established some general properties for the development of $\frac{\sqrt{X}+\sqrt{Y}}{x+y}$, e.g., symmetry and periodicity. Then, he concentrated on the development of $\sqrt{X(x)}$ and deduced recursive formulae, simpler than those found by Jacobi, for the computation of the partial terms of the continued fraction.

Afterward, on page 600, Halphen returned shortly on Poncelet's polygons. He recognized that if the polynomial $X(x)$ is of degree 3, two relations that he had found studying the development of $\sqrt{X(x)}$ were the same that "d'après M. Cayley" he had

[100] Jacobi published these formulae without proof. They were proved by Borchardt (1854), who also extended them to the case of the continued fraction development of $\sqrt{X(x)}$, with $\deg X > 4$.

already discussed in chapter X at page 389. These two relations were those expressing the conditions for the existence of a Poncelet polygon of $2n$, or $2n + 1$, sides.

With this observation, Halphen brought to light a connection between Poncelet's closure theorem and the development in continued factions of $\sqrt{X(x)}$. Thirty years later this connection was investigated in depth by Gerbaldi, who dealt with the question from an algebraic point of view.

This seems a good place to temporarily stop our story about Poncelet's porism, because with Gerbaldi and his studies we enter the twentieth century.

References

Abel, N.H. 1828. Remarques sur quelques propriétés générales d'une certaine sorte de fonctions transcendantes. *Journal für die reine und Angewandte Mathematik* 3: 313–323; also in *Oeuvres Complètes*, vol. I, pp. 444–456, Grøndhal and Søn, Christiania 1881.

Abel, N.H. 1829. Précis d'une thérie des fonctions elliptiques. *Journal für die Reine und Angewandte Mathematik* 4: 236–277, 309–348; also in *Oeuvres Complètes*, vol. I, pp. 518–617, Grøndhal and Søn, Christiania 1881.

Amodeo, F. 1924. *Vita matematica napoletana*. Napoli: Tip. Ac. Pontiniana.

Barth, W., and J. Michel. 1993. Modular curves and Poncelet polygons. *Mathematische Annalen* 295: 25–49.

Belhoste, B. 1998. De l'École polytechnique à Saratoff, les premiers travaux géométriques de Poncelet. *Bullitin de la Sabix* 19: 9–29.

Birkhoff, G. 1927. *Dynamical systems*. Providence, R.I.: American Mathematical Society.

Böhmer, Trautmann. 1985. Special instanton bundles and Pocelet curves. *Singularities, representations of algebras, and vector bundles*, Springer LNM 1273, 325–336. Berlin: Springer.

Borchardt, C.W. 1854. Applications des transcendantes abéliennes à la théorie des fractions continues. *Journal für die Reine und Angewandte Mathematik* 48: 69–104.

Bos, H.J.M., C. Kers, F. Oort, and D.W. Raven. 1987. Poncelet's closure theorem. *Expositiones Mathematicae* 5: 289–364.

Bottazzini, U., and J. Gray. 2013. *Hidden harmony—geometric fantasies, the rise of complex function theory*. New York: Springer.

Brianchon, C.J. 1810. Solution de quelques problémes de géométrie. *Journal de l'Ecole Polytechnique* 4: 1–15.

Brill, A. 1873. Über Entsprechen von Puntksystemen aud Einer Curven. *Mathematische Annalen* 6: 33–65.

Brioschi, F. 1857. Sui poligoni inscritti alle coniche. *Annali di Scienze Matematiche e Fisiche* 8: 119–124.

Cantor, M. 1901. *Vorlesungen über Geschichte der Mathematik*, vol. 3, 2nd ed., 552–553. Leipzig: Treubner.

Cauchy, A.-L. 1820. Rapport à l'académie royale des sciences. *Annales de Mathématiques Pure et Appliquèe* 11(1820–11): 69–83.

Cayley, A. 1853a. Note on the Geometrical Representation of the Integral $\int \frac{dx}{\sqrt{(x+a)(x+b)(x+c)}}$. *Philosophical Magazine* S.4(5): 281–284; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 2, 1889, Cambridge Un. Press, pp. 53–56.

Cayley, A. 1853b. Note on the Porism of the in-and-circumscribed Polygon. *Philosophical Magazine* S.4(6): 99–103; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 2, 1889, pp. 87–90.

Cayley, A. 1853c. Correction of two Theorems relating to the Porism of the in-and-circumscribed Polygon. *Philosophical Magazine* S.4(6): 376–377; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 2, 1889, Cambridge Un. Press, pp. 91–92.

Cayley, A. 1853d. On the Geometrical representation of an Abelian Integral. *Philosophical Magazine* S.4(6): 412–418; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 2, 1889, Cambridge University Press, pp. 138–144.

Cayley, A. 1854. Developments on the Porism of the in-and-circumscribed Polygon. *Philosophical Magazine* S.4(7): 339–345; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 2, 1889, Cambridge University Press, pp. 138–144.

Cayley, A. 1857. On the porism of the in-an-circumscribed triangle. *Quarterly Mathematical Journal* 1: 344–354; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 3, Cambridge University Press, 1890, pp. 67–75.

Cayley, A. 1858. On the posteriori demonstration of the porism of the in-an-circumscribed triangle. *Quarterly Mathematical Journal* 1: 31–38; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 3, Cambridge University Press, 1890, pp. 80–85.
Cayley, A. 1861. On the Porism of the in-and-circumscribed Polygon. *Philosophical Transactions of the Royal Society of London.* 151: 225–239; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 4, 1891, Cambridge University Press, pp. 292–308.
Cayley, A. 1862. Sur le problème du polygone inscrit et circonscrit. Lettre à M. Poncelet. *Comptes rendus Ac. de Sc.* 700–701; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 5, 1892, Cambridge University Press, pp. 21–22.
Cayley, A. 1865. On a triangle in-and-circumscribed to a quartic curve. *Philosophical Magazine* S.4(30): 340–342; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 5, 1892, Cambridge University Press, pp. 489–492.
Cayley, A. 1866. Note sur la correspondance de deux points sur une courbe. *Comptes Rendus Ac. Sc.* LXII: 586–590; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 6, 1893, Cambridge University Press, pp. 542–545.
Cayley, A. 1871a. On the porism of the in-and-circumscribed polygon, and the (2, 2)-correspondence of points on a conic. *The Quarterly Journal of Mathematics* 11: 83–91; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 8, 1895, Cambridge University Press, pp. 14–21.
Cayley, A. 1871b. On the problem of the in-and-circumscribed triangle. *Philosophical Transactions of the Royal Society of London* 161: 369–412; also in *The Collected Mathematical Papers of Arthur Cayley*, vol. 8, 1895, Cambridge University Press, pp. 212–257.
Cayley, A. 1876. *An elementary treatise on elliptic functions.* Cambridge: Deighton, Bell.
Cayley, A. 1895. *The collected mathematical papers of Arthur Cayley*, vol. 8. Cambridge: Cambridge University Press.
Chapple, W. 1746. An essay on the properties of triangles inscribed in, and circumscribed about two given circles. *Gentlemen's Magazine, Miscellanea Curiosa Mathematica* 4: 117–124.
Chasles, M. 1837. *Aperçu historique sur l'origine et le developpement des méthodes en géométrie.* Bruxelles: Hayez
Chasles, M. 1864. Considerations sur la méthode générale exposée dans la séance du 15 frévier. *Comptes rendus Ac. Sc.* 58: 1167–1175.
Chasles, M. 1865. *Traité sur les sections coniques.* Paris: Gauthier-Villars.
Clebsch, 1864. Ueber einen Satz von Steiner und einige Punkte der Theorie den Curven der dritten Ordnung. *Journal für die reine und angewandte Mathematik* 63: 94–121.
Coolidge, J.L. 1940. *A history of geometrical methods.* Oxford: Clarendon Press.
Cremona, L. 1867. Un teorema attorno alle forme quadratiche non omogenee fra due variabili. *Rendiconti del R. Istituto Lombardo, Serie I* IV: 199-201.
Crilly, T. 1986. The rise of Caley's invariant theory. *Historia Mathematica* 13: 241–254.
Crilly, T. 2006. *Arthur Cayley, mathematician laureate of the victorian age.* Baltimore: The Johns Hopkins University Press.
Darboux, G. 1869. Mémoire sur une classe de courbes et de surfaces. *Comptes Rendus de l'Académie des Sciences* 68: 1311–1313.
Darboux, G. 1870a. Sur les polygones inscrits et circonscrits à l'ellipsoïde. *Bulletin de la Société Philomatique de Paris* 6e serie, 7: 92–94.
Darboux, G. 1870b. Sur une classe remarkable de courbes et de surfaces algébriques et sur la théorie des imaginaires. *Mémoires de la Société des Sciences de Bordeaux* 8: 291–350, 9 (1873), pp. 1–280
Darboux, G. 1872. Sur un nouveau système de coordonnées et sur les polygones circonscrits aux coniques. *Bulletin de la Société Philomatique de Paris* 6e serie, 9: 100–107, 128–142.
Darboux, G. 1873a. *Sur une classe remarkable de courbes et desurfaces algébriques et sur la théorie des imaginaires*, Paris, Gauthier–Villars; 2nd ed. Paris, Hermann 1886.
Darboux, G. 1873b. Sur une démonstration analytique des théorèmes de Poncelet, et sur un nouveaux système de coordonnées dans le plane. In *Sur une classe remarkable de courbes et de surfaces algébriques.* Gauthier-Villars, Paris, 183–207.
Darboux, G. 1880. Sur les polygones inscrits à une conique et circonscrits à une autre conique. *Comptes Rendus de l'Académie des Sciences* 90: 85–87.
Darboux, G. 1889. *Leçons sur la théorie générale des surfaces*, vol. 2, Paris, Gauthier-Villars. 2nd ed. Paris, Gauthier-Villars 1914.
Darboux, G. 1917. *Principes de géométrie analytique.* Paris: Gauthier-Villars.

Del Centina, A. 2003. La memoria parigina di Abel e la sua importanza per la geometria. *Lettera Matematica Pristem* 47: 45–55.

De Morgan. 1865. On the correspondence of two points on a curve. *Proceedings of the London Mathematical Society* s.1(1): 77–84.

Dingeldey, F. 1903. Kekelschnitte und Kegelschnittsysteme. In *Enzyklopedie der Mathematischen Wissenschaften*, Band III, 2. Teil, 1 Hälfte, Leipzig 1903–1915, pp. 1–160.

Didion, I. 1870. Notice sur la vie et les ouvrages du général J.-V. Poncelet. *Mémoire de l'Académie de Metz* 50: 101–159.

Dolgachev, I. 2012. *Classical algebraic geometry*. Cambridge: Cambridge University Press.

Dragović, V., and M. Radnović. 2011. *Poncelet Porisms and beyond*. Basel: Birkäuser (Springer-Science).

Dragović, V., and M. Radnović. 2014. Bicentennial of the Great Poncelet's theorem, current advances. *Bulletin of the American Mathematical Society* 51: 373–445.

Euler, L. 1765. Solutio facilis problematum quorundam geometricorum difficillimorum. *Novi Comm. Ac. Sci. Imp. Petropolitanae* XI: 103–123; also in *Opera Omnia*, Leipzig, 1911. ser. I 26: 139–157.

Euler, L. 1768. Integratio equationis $\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{dy}{\sqrt{A+By+Cy^2+Dy^3+Ey^4}}$. *Novi Comm. Ac. Sci. Imp. Petropolitanae* XII: 3–16; also in *Opera Omnia*, Leipzig, 1912. ser. I 20: 302–317.

Euler, L. 1794. *Institutiones calculi integralis*, vol. 4, Petropoli, Im. Ac. Imperialis Scientiarum.

Flatto, L. 2009. *Poncelet's theorem*. Providence, RI: American Mathematical Society.

Flauti ,V. 1840–1844, *Produzioni relative al programma di tre questioni geometriche proposto da un nostro professore*, Parte I: *Programma e disputazione su di esso*, Parte II: *Memorie premiate in risposta al programma*, Napoli: Stamperia Op. Prof. Flauti.

Fuss, N. 1797. De quadrilateris quibus circulum tam inscribere quam circumscribere licet. *Nova acta Ac. Sci. Imp. Petropolitanae* 10: 103–125.

Fuss, N. 1802. De polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis. *Nova acta Ac. Sci. Imp. Petropolitanae* 13: 166–189.

Gaultier, L. 1813. Sur le moyens généraux de construire graphiquement une cercle détérminé par trois conditions, et.... *Journal de l'Ecole Polytechnique* 16e cahier: 124–214.

Genocchi, A. 1858. Sopra una costruzione del teorema di Abel. *Annali di Matematica Pura ed Applicata* 1(1): 33–40.

Gerbaldi, F. 1919. Le frazioni continue di Halphen in relazione alle corrispondenze (2,2) involutorie e coi poligoni di Poncelet. *Rendiconti del Circolo Matematico di Palermo* 43: 78–104.

Gergonne. 1810. Questions proposées. Théorème de géométrie. *Annales de Mathématiques pures et appliquées* 1(1810–1811): 62–64.

Gray, J. 2007. *Worlds out of nothing*. New-York: Springer.

Griffiths, P.A. 1976. Variations on a theorem of Abel. *Inventiones Mathematicae* 35: 321–390.

Griffiths, P.A., and J. Harris. 1977. A Poncelet theorem in space. *Commentarii Mathematici Helvetici* 52: 145–160.

Griffiths, P.A., and J. Harris. 1978a. On Cayley explicit solution to Poncelet's porism. *Enseignement Mathématique* 24: 31–40.

Griffiths, P.A., and J. Harris. 1978b. *Principles of algebraic geometry*. New York: Wiley.

Gruson, L. 1992. Un aperçu des travaux mathématiques de G.-H. Halphen. In *Complex Projective Geometry*, ed. D. Ellingsrud, C. Peskine, G. Sacchiero, S.A. Stromme, Springer Lecture Notes 179, pp. 189–198, Springer, Berlin.

Halphen, G.-H. 1878. Sur les caractéristiques des systèmes de coniques et de surfaces de second ordre. *Journal de l'Ecole Polytechnique* 28: 27–89.

Halphen, G.-H. 1879a. Sur une application de la théorie des caractéristiques pour les coniques, à une question relative aux polygones de Poncelet. *Bulletin de la Société Philomatique de Paris* 7e série, 3: 17–19.

Halphen, G.-H. 1879b. Sur certaines propriétés métriques relatives aux polygones de Poncelet. *Journal de Mathématiques Pures et Appliquées* 3me série, 5: 285–292.

Halphen, G.-H. 1886. *Traité des fonctions elliptiques, premième partie*. Paris: Gauthier-Villars.

Halphen, G.-H. 1888. *Traité des fonctions elliptiques, deuxième partie*. Paris: Gauthier-Villars.

Halphen, G.-H. 1891. *Traité des fonctions elliptiques, troisème partie*. Paris: Gauthier-Villars.

Hunger, Parshall K. 1989. Toward a history of nineteenth-century invariant theory. In *The history of modern mathematics in America*, vol. 1, ed. D.E. Rowe, and J. McCleary, 157–206. Boston: Academic Press.

Hunger, Parshall K. 2006a. The British development of the theory of invariants. *Journal of the British Society for the History of Mathematics* 21: 186–199.

Hunger, Parshal K. 2006b. *James Joseph Sylvester; Jewish mathematician in a Victorian World*. Baltimore: J. Hopkins University Press.

Hurwitz, A. 1879. Über unendlich-vieldeutige geometrische Aufgaben insbesondere über die Schiessungsprobleme. *Mathematische Annalen* 15: 8–15.

Hurwitz, A. 1891. Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten. *Mathematische Annalen* 39: 1–61.

Hurwitz, A. 1893. Über algebraische Gebilde mit eindeutigen Transformationen in sich. *Mathematische Annalen* 41: 391–430 (323–335).

Jacobi, C.G.J. 1828. Ueber die Anvendung der elliptischen Transcendenten auf ein bekanntes Problem der Elementargeometrie. *Journal für die reihe und Angewandte Mathematik* 3: 376–389 (partly translated in French in *J. de Mathématiques Pure et Appl.* 10 (1845) pp. 435–444); also in *Mathematische Werke*, vol. 1, (1881), Berlin, Von G. Reimer, pp. 277–293.

Jacobi, C.G.J. 1831. Note sur une nouvelle application de l'analyse des fonctions elliptiques à l'algèbre. *Journal für die reihe und Angewandte Mathematik* 7: 41–42; also in *Gesammelte Werke*, vol. 1, (1881), Berlin, Von G. Reimer, pp. 329–331.

Jacobi, C.G.J. 1832. Considerationes generales de transcendentibus abelianis. *Journal für die reihe und Angewandte Mathematik* 9: 394–403; also in *Gesammelte Werke*, vol. 2, (1882), Berlin, Von G. Reimer, pp. 5–16.

Jacobi, C.G.J. 1846. Extrait d'une lettre addressée à M. Hermite. *Journal für die reihe und Angewandte Mathematik* 32: 176–181; also in *Gesammelte Werke*, vol. 2, (1882), Berlin, Von G. Reimer, pp. 115–120.

Kleiman, S.L. 2004. What is Abel's theorem anyway? In *The legacy of Niels Henrik Abel*, ed. O.A. Laudal and R. Piene, 395–440. New York: Springer.

Kline, M. 1972. *Mathematical thought from ancient to modern times*. Oxford: Oxford University Press.

Kötter, E. 1901. Die Entwicklung der synthetischen Geometrie; von Monge bis auf Staudt (1847). *Jahresbericht der Deutschen mathematikervereiningung* 5(1901): 1–485.

Landen, J. 1755. *Mathematical lucubrations*. London: Nourse.

Lebesgue, H. 1921. Exposé géometrique d'un mémoire de Cayley sur les polygones de Poncelet. *Annales de la Faculté des Sciences de Toulouse* 3e série, 13: 61–91.

Legendre, A.M. 1825. *Traité des fonctions elliptiques et des intégrales eulé-riennes*, vol. 1. Paris: Huzard et Courcier.

Legendre, A.M. 1828. *Traité des fonctions elliptiques et des intégrales eulé-riennes*, vol. 3. Paris: Huzard et Courcier.

Lhuilier, S.A.J. 1810. Théorèmes sur les triangles, relatifs à la page 64 de ce Annales. *Annales de Mathématiques pures et appliquées* 1(1810–1811): 149–159.

Loria, G. 1889a. *I poligoni di Poncelet: discorso pronunziato nell'Università di Genova*. Torino: Paravia.

Loria, G. 1889b. Rassegna di alcuni scritti sui poligoni di Poncelet. *Bibliotheca Mathematica*, Serie 3(3): 67–74.

Loria, G. 1896. *Il passato e il presente delle principali teorie geometriche*. Torino: Clausen.

Mackay, J.S. 1887. Historical note on a geometrical theorem and its development. *Proceedings of the Edinburgh Mathematical Society* 5: 62–78.

Markushevich, A.I. 1992. *Introduction to the classical theory of Abelian functions*. Translations of Mathematical Monographs, vol. 96. Providence: AMS.

Mention, J. 1860. Essai sur le problème de Fuss. *Bull Ac. Imp. des Sc. St Péterrsbourg* T.1: 15–29.

Moutard, T.F. 1862. Recherhes analytiques sur les polygones simultanément inscrits et circonscrits a deaux coniques, in *Poncelet, Applications d'analyse et de géométrie*, vol. 1. Paris: Mallet-Bachelier.

Poncelet, J.-V. 1822. *Traité sur les propriétés projectives des figures*, Paris: Bachelier; 2nd edition 1865–66. Paris: Gauthier-Villars.

Poncelet, J.-V. 1862. *Applications d'analyse et de géométrie*, vol. 1. Paris: Mallet-Bachelier.

Richelot, F.J. 1830. Anwendung der elliptischen Transcedenten auf die sphäri-sche Polygone, welche zugleich einem kleinen Kreise det Kigel eingeschrieben und einem anderen umgeschrieben sind. *Journal für die reihe und Angewandte Mathematik* 5: 250–267; also translated in French in *J. de Mathématiques Pure et Appl.* 11, (1846) pp. 25–33.

Richelot, F.J. 1849. Ueber die Anwendung einiger Formeln aus der Theorie der elliptischen Functionen auf ein bekanntes Problem der Geometrie. *Journal für die reihe und Angewandte Mathematik* 38: 353–372.

Rosanes, J., and M. Pash. 1865. Ueber des einen Kegelschnitte umbeschriebene und einem andern eibeschriebene Polygon. *Journal für die reihe und Angewandte Mathematik* 64: 126–166.

Rosanes, J., and M. Pash. 1869. Ueber eine algebraischen Aufgabe, welche siner Gattung geometrischer Probleme zu Grande lieegt. *Journal für die reihe und Angewandte Mathematik* 70: 169–174.

Salmon, G. 1855. *A treatise on conic sections*, 3rd ed. Dublin: Hodges and Smith.

Salmon, G. 1857. On the problem of the In-and-circumscribed Triangle. *Philosophical Magazine* 13: 190–191, 267–269, 337–338.

Segre, C. 1892. Intorno alla storia del principio di corrispondenza e dei sistemi di curve. *Bibliotheca Mathematica* 6: 33–47.

Simon, M. 1867. *De relationibus inter constantes duarum linearum secundi ordinis, ut sit polygonum alteri inscriptum, circumscriptum alteri*. Inauguraldissertation, Berlin, Driesner.

Simon, M. 1876. Ganzzhalige Multiplication der elliptischen Functionen in Verbindung mit dem Schlessungsproblem. *Journal für die reihe und Angewandte Mathematik* 81: 301–323.

Sinai, Ya G. 1976. *Introduction to ergodic theory*. Princeton: Princeton University Press.

Sommerville, D. 1933. *Analytical conics*. London: G. Bell and Sons.

Staude, O. 1883. Geometrischen Deutung der Addition Theoreme der hyperelliptischen integrale und Functionen I. Ordung im System der confocalen Flächen 2. Grades. *Mathematische Annalen* XXII: 1–145.

Steiner, J. 1826. Einige geometrische Betrachtungen. *Journal für die reihe und Angewandte Mathematik* 1: 161–184.

Steiner, J. 1827. Aufgaben und Lehsätze (n. 57). *Journal für die reihe und Angewandte Mathematik* 2: 289.

Steiner, J. 1832. *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*. Berlin: G. Fincke.

Steiner, J. 1846. Säe über curven zwiter und dritter Ordnung. *Journal für die reihe und Angewandte Mathematik* 32: 300–304.

Sylvester, J.J. 1853. On the calculus of forms, otherwise the theory of invariants, sect. VII, "On combinants". *Dublin Mathematical Journal* VIII: 256–269; also in *The Collected Mathematical papers of James Joseph Sylvester*, vol. 1, pp. 411–422, Cambridge Univ. Press 1904.

The mathematical Questions. 1817. *The Mathematical Questions Proposed in the Ladies' Diary and Their Original Answers, together with Some New Solutions 1706–1816*, vol. I. London: Leybourn.

Todd, J.A. 1947. *Projective and analytical geometry*. New-York: Pitnam Pub.

Todd, J.A. 1948. Poncelet Poristic Polygons. *The Mathematical Gazette* 32: 274–280.

Trudi, N. 1841. *Sui poligoni iscritti e circoscritti alle curve coniche con date condizioni*, Napoli: Stamperia per le opere del prof. Flauti.

Trudi, N. 1843. Delle relazioni tra i determinati di due sezioni coniche l'una iscritta, l'altra circoscritta ad un poligono irregolare. *Rendiconti del Reale Accademia delle Scienze di Napoli* n.8: 89–93.

Trudi, N. 1853. Rappresentazione geometrica immediata dell'equazione fondamentale della teorica delle funzioni ellittiche con diverse applicazioni. *Memorie della Reale Accademia delle Scienze di Napoli* 1(1856): 63–100.

Trudi, N. 1863a. Studii intorno ad una singolare eliminazione, con applicazione alla ricerca della condizione tra gli elementi di due coniche, l'una inscritta, l'altra circoscritta ad un poligono; ed ai corrispondenti teoremi del Poncelet. *Atti della Reale Accademia delle Scienze di Torino* 1: 1–53.

Trudi, N. 1863b. Su' teoremi di Poncelet relativi a' poligoni iscritti e circoscritti alle coniche. *Giornale di Matematiche* 1: 81–90, 125–126.

Weil, A. 1980. History of Mathematics: Why and How. In *Proceedings of the International Congress of Mathematicians, Helsinki 1978*, ed. O. Lehto. Helsinki: Academia Scientiarum Fennica, vol. I, pp. 226–236.

Weyr, E. 1870. Ueber einige Sätze von Steiner und ihren Zusammenhang mit der zwei und zwiigliedrigen Verwandtschaft der Grunfgebilde ersten Grades. *Journal für die reihe und Angewandte Mathematik* 71: 18–28.

White, H.S. 1916. Poncelet Polygons. *Science* 43: 149–158.

Zeuthen, H.G. 1871. Nouvelle démonstration des théorèmes sur les séries de points correspondants sur deux courbes. *Mathematische Annalen* 3: 150–156.