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The coming-to-be of Hansen's method

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Abstract This article by Curtis Wilson is an account of the origin of Hansen's powerful systematic method for finding contributions of higher order perturbations in celestial mechanics. Hansen's method was developed in the course of improving on Laplace's treatment of the mutual perturbations of Jupiter and Saturn. This method, an entirely new way of doing celestial mechanics when it first appeared, later made possible the successful treatment of the complicated motions of our moon (see Wilson 2010). In this paper Wilson gives a brief historical introduction followed by an account of relevant technical details of the Laplacian background, an account illustrating technical details in Hansen's initial development in his Disquisitions of 1829, and a treatment illustrating details contributing to the achievement of Hansen's more refined development in his Untersuchung of 1831. These details include conditional equations Hansen provides for checking the accuracy of calculations. Wilson also includes a detailed assessment showing the extraordinary improvement in empirical accuracy of Hansen's treatment over the best earlier treatment of the Jupiter-Saturn interactions.

Communicated by: Jed Buchwald.

Curtis Wilson passed away on 24 August 2012 at the age of 91. He had completed this article some time before. Becky Wilson, Curtis' window, asked William Harper to help arrange for publication. Harper had only a printed version of the manuscript, which he digitally scanned and sent to Buchwald for the *Archive*. Buchwald then asked Springer to prepare a readable version in MSWord format, including all equations as best could be done. The result required examination by an expert, and so Buchwald asked Andreas Verdun to do so. Verdun cross-checked and fixed German citations and English translations and amended a faulty equation. Buchwald then passed Verdun's emended version to Harper, who, aided by Sree Ram Valluri, completed the necessary resetting.

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1 Introduction

"Hansen's method," an entirely new way of doing celestial mechanics when it first appeared, was the invention of Peter Andreas Hansen (1795–1874) in 1829–1830. Later, during a long, active life as both a practical and a theoretical astronomer, Hansen elaborated variants of his method; certain features, however, remained constant. He had formulated these in the course of writing a memoir entitled *Disquisitiones circa theoriam perturbationum quae motum corporum coelestium afficiunt* ("Investigations concerning the theory of the perturbations that influence the motion of the celestial bodies"). This memoir was published in two installments in the *Astronomische Nachrichten* for September and October 1829.[1]

Immediately upon completing this memoir, Hansen set about applying his new method to the mutual perturbations of Jupiter and Saturn. More than a year earlier, in July 1828, the Berlin Academy of Sciences had announced as the subject of its prize

[1] Hansen (1829). I shall hereinafter abbreviate *Astronomische Nachrichten as A.N.*

competition for 1830, "A new investigation of the mutual perturbations of Jupiter and Saturn, with special regard for the terms depending on the square and higher powers of the perturbing forces..."[2] The deadline for submission of memoirs was March 31, 1830.[3]

Hansen's memoir, *Untersuchung über die gegenseitigen Störungen des Jupiters und Saturns*,[4] reached the Berlin Academy after the deadline. Another memoir, by Philippe-Gustave, comte de Pontécoulant, had arrived earlier and was deemed worthy of the prize. Pontécoulant's memoir focused on the specific problem of calculating the second-order terms in the great inequality of Jupiter and Saturn—an issue that had been the subject of a controversy (to be described later) between Giovanni Plana and Pierre-Simon Laplace in 1825–1826. Pontécoulant thus left unaddressed the more general question set for the competition: How second- and higher-order perturbations were to be dealt with? Hansen's memoir not only addressed this question, but was judged to have resolved it in a thoroughly satisfactory and indeed admirable way. The Academy was therefore moved to award to Hansen a prize equal to that given Pontécoulant, an Akademie prize medal valued at 50 ducats.[5]

Pontécoulant was a student and close associate of Siméon-Denis Poisson, the foremost celestial mechanician in Paris after Laplace's death. Pontécoulant's memoir is essentially a clarification and correction of a Laplacian procedure on the basis of an ordering principle suggested by Poisson. Hansen's memoir, in contrast, struck out along a hitherto untrodden path.

Who was this Hansen? Born in Tonder, Denmark, near the Schleswig border, he was the son of a silver- and goldsmith. At his father's insistence, he had learned the trade of a clockmaker and practiced it into his twenty-fifth year.[6] Denied the possibility of attending university, he was an assiduous autodidact; achieving fluency in Latin and French as well as German, Danish, and Swedish, and proficiency in advanced mathematics.

Hansen's turn to astronomy came in 1820, when the family physician put young Hansen in touch with the professor of astronomy at the University of Copenhagen, Heinrich Christian Schumacher (1780–1850). Schumacher at this time was primarily engaged in the geodetic survey of Schleswig and Holstein; in the following year, he would become the survey's director, and also be named the first director of the Danish Royal Observatory, then in process of being constructed at Altona. Hansen became Schumacher's assistant, thus winning release from a life of artisan toil.

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2 The Preisfrage was stated in full as follows: "Die Akademie wünscht eine neue Untersuchung der gegenseitigen Störungen des Jupiters und Saturns zu erhalten, mit besonderer Berücksichtigung der von dem Quadrate und den höheren Potenzen der störenden Kräfte abhängigen Glieder, wodurch zugleich die Verschiedenheit der von den Herren Laplace und Plana gefundenen Werthe erklärt, und das richtige Resultat bewiesen wird." For this quotation, I am indebted to Dr. Wolfgang Knobloch of the Berlin-Brandenburgische Akademie der Wissenschaften. The reference in the final clause of the quotation to the differing values found by Laplace and Plana will be explained in Sect. 2.3.

3 *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1828* (Berlin, 1831), Historische Einleitung, pp. I–II.

4 The memoir was published at the Berlin Academy of Science's press in Berlin, 1831.

5 *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1830* (Berlin, 1832). Historische Einleitung, pp. I–II.

6 For information about Hansen's early life I am indebted to Anding (1924).

Schumacher had studied astronomy under Gauss at Göttingen, and like Gauss stressed the importance of exact measurement and numerically convenient procedures; this emphasis would be characteristic of German astronomers influenced by Gauss, such as Olbers and Bessel.[7] As Schumacher's assistant, Hansen was introduced to the practical side of astronomical and geodetic operations. He became expert in computing the orbits of comets by Olbers' method, and in 1824, he discovered a new and especially precise method of determining geographic latitude, by timing the transit of a star across the first vertical (the celestial great circle from east to west through the zenith). This discovery brought recognition within the astronomical community and led in 1825 to Hansen's being appointed director of the Seeberg Observatory near Gotha, to succeed Johann Franz Encke (1791–1865), who was leaving that post to become director of the Berlin Observatory. With Encke, who like Schumacher had been a student of Gauss, Hansen would have a close epistolary relation over the next quarter century.

During his early years of geodetic and astronomical work, Hansen does not appear to have had occasion to study the problem of planetary perturbations. The earliest indication I have found of his concerning himself with this problem is in a letter to Friedrich Wilhelm Bessel (1784–1846), dated May 31, 1828. Hansen had been brought into correspondence with Bessel in 1824 through his geodetic work; in the cited letter, he addresses Bessel as "mein hochverehrter Freund!" (As we shall see later, Hansen in his study of perturbations will employ Bessel functions and make use of mechanical quadratures as advocated by Bessel.) Amidst other topics it touches upon, Hansen's letter contains the following remarks concerning the mutual perturbations of Jupiter, Saturn, and Uranus (my translation from Hansen's original):[8]

> Much that has been discovered in recent years appears to disagree with what were previously accepted as true laws of nature, but in my opinion no judgment should be made before we have tested what pertains to the older established views. To this end I believe I shall make a contribution if I subject the theories of Jupiter, Saturn, and Uranus to a new revision, since among the older planets these present the most complicated phenomena. I have been occupied with this work since last summer, and it is now nearly completed. The method I have applied differs from Laplace's. In the motion of Uranus I have found two inequalities, of which the one has a coefficient of 35 and the other a coefficient of 7 sexagesimal

[7] See Grattan-Guinness (1990). Vol. 1, pp. 386–387 ("Interlude 641.1").

[8] In the Akademiearchiv of the Berlin-Brandenburgische Akademie der Wissenschaften are 76 letters from Hansen to Encke, and 33 from Hansen to Bessel. The German of the passage of which I give the translation is as follows (quoted with permission): Es ist in den letzten Jahren so manches entdeckt, welches gegen die bisher als wahr angenommenen Naturgesetze zu streiten scheint, allein wir dürfen kein Urtheil fällen[,] bevor wir das Alte gehörig geprüft haben. Einen Beitrag hierzu glaube ich zu liefern[,] wenn ich die Theorie des Jupiters, Saturns und Uranus einer neuen Durchsicht unterwerfe, da gerade unter den ältern Planeten diese die verwickel[t]sten Erscheinungen darbieten. Diese Arbeit hat mich schon seit vorigem Sommer beschäftigt und ist jetzt beinahe vollendet. Ich habe eine andere Methode als la Place angewandt. In der Bewegung des Uranus habe ich zwei Ungleichheiten gefunden, deren eine 35" und deren andere 7" (sex.) zum Coefficienten hat. Diese können aber die bekannte Anomalie nicht fortschaffen, indess finde ich es mehr als wahrscheinlich anzunehmen[,] dass es ausserhalb des Uranus einen oder mehrere Planeten sind, durch welche diese Anomalie hervorgebracht ist. In der Bewegung des Jupiters und des Saturns habe ich auch mehrere nicht unbedeutende Störungen gefunden[,] welche von la Place übergangen sind....

seconds. These cannot be reduced to the known anomalies, and I find it more than likely to assume that outside Uranus there are one or more planets that cause these anomalies. In the motion of Jupiter and Saturn I have also found several not inconsiderable perturbations which were passed over by Laplace....

Hansen has apparently derived the new inequalities he has found in Uranus from observations. He goes on to incorporate these and the newly found perturbations of Jupiter and Saturn in expressions for the secular changes in the eccentricities and perihelia of the three planets and to compare his results with those given by Laplace.

Without knowing the method that Hansen applies here, we can be fairly sure it differs importantly from the method of the *Disquisitiones* that he will be elaborating a year later. In the letter just cited, Hansen is obtaining perturbations of two orbital elements, the eccentricity and longitude of the perihelion, whereas in the method of the *Disquisitiones*, he will be computing the perturbations, not of *orbital elements*, but of the planetary *coordinates*. For Hansen, the announcement of the prize contest no doubt stimulated a new and penetrating consideration of the planetary problem; the central ideas of the *Disquisitiones* may not have come into view for him till *after* the announcement.

The opening section of the *Disquisitiones* outlines the inconveniences of the earlier methods of computing perturbations; evidently Hansen was acquainted with the literature of celestial mechanics. But the *Disquisitiones* does not *apply* the method it describes; it sets forth a program of action—a program the success of which, as Hansen later acknowledged, was at the time of his writing of the *Disquisitiones* still uncertain.[9] The prize contest that led to his writing of his prize paper, the *Untersuchung*, was entirely welcome as offering an opportunity to test his ideas.

After publication of the *Disquisitiones* and *Untersuchung* , wider recognition of the power and efficacy of Hansen's method was slow in coming. In 1834, in a note inserted in the *Connaissance des tems* for 1836, Pontécoulant questioned the accuracy of Hansen's calculation of the great inequality of Jupiter;[10] Hansen in his reply demonstrated that Pontécoulant's critique was badly flawed.[11] In a memoir in the *Connaissance des tems* for 1837, Pontécoulant claimed to give "une idée juste et claire" of Hansen's method;[12] Hansen found the account inadequate and misleading, in particular because it failed to mention the rigorous applicability of the method to perturbations of higher order than the first—"one of its chief advantages."[13] John Lubbock in Part IV of his *Theory of the Moon* advised Hansen to turn his efforts to evaluating the correctness of the analytic formulas of Plana's lunar theory, recently published.[14] The advice, Hansen countered, showed total incomprehension of his endeavor. Plana's analytic formulas, in Hansen's view, were worse than useless because of their slow

[9] See Hansen 1838b.

[10] de Pontécoulant (1836), pp. 21–22.

[11] *A.N.*, Vol. XI (1834), No. 262, cols. 389–396.

[12] de Pontécoulant (1837), pp. 40–62.

[13] *A.N.*, Vol. XV (1838), No. 348, cols. 201–216.

[14] Lubbock (1834), 70.

convergence; his own procedures avoided this difficulty by substituting numerical values at an early stage.[15]

Hansen's *Fundamenta nova investigationis orbitae verae quam luna perlustrat*, applying his method to the lunar theory, appeared in 1838.[16] The initial response appears to have been slight. At last, over a decade later, the work received critical and laudatory comment from two mathematicians. C.G.J. Jacobi wrote two letters to Hansen, which were published posthumously in Jacobi's *Mathematische Werke* (1851). A few of Hansen's formulations, Jacobi there showed, were improper from a formal mathematical standpoint, but he went on to praise "the easy and elegant way in which you have arrived at a unique and remarkable form of the perturbational equations."[17] In line with Jacobi's remarks, Arthur Cayley in a paper of 1857 undertook "to exhibit, in as clear a form as may be, the investigation of the remarkable equations for the motion of the moon established in Hansen's *Fundamenta nova*..."[18] In the measurement of longitudes in the varying plane of a planet's or satellite's orbit, and in the determination of the position of the lunar orbit with respect to the varying plane of the Earth's path, most earlier derivations, Cayley asserted, were very imperfect. "I except always Hansen's *Fundamenta Nova* where the points referred to are treated in a perfectly rigorous manner."[19] Cayley's reformulation of the basic propositions of Hansen's method has been followed by later expositors.[20]

In 1857, Hansen's *Tables de la lune construites d'aprés le principe newtonien de la gravitation universelle* were published by the British Government. Tested against observations, they proved the most accurate of the available tables. They were consequently adopted as the basis of the lunar ephemerides in the British *Nautical Almanac and* in the French *Connaissance des tems* beginning with the volumes for 1862; they replaced the semi-empirical tables of J.K. Burckhardt, published in 1812. The *American Ephemeris and Nautical Almanac* from its inception in 1855 had used Benjamin Peirce's lunar tables drawn from a revision of Plana's theory, but in 1883, the Americans followed the European example in adopting Hansen's tables. These tables, with minor adjustments by Simon Newcomb, would remain the basis of the national ephemerides through 1922.

Hansen's method was used in one more major application during the nineteenth century. When Simon Newcomb became head of the US Nautical Almanac Office in 1877, he had conceived the aim of reforming all the planetary tables, using a consistent system of constants. (Leverrier's tables, then in process of appearing, were deficient in this regard.) The most difficult of the problems was presented by the mutual perturbations of Jupiter and Saturn. For its resolution, Newcomb turned to George William Hill, a colleague of his at the Nautical Almanac Office, whom he regarded as "easily...the greatest master of mathematical astronomy during the last quarter of

[15] *A.N.*, Vol. XIX (1842), No.s 435–437, cols. 33–92.

[16] See Hansen (1838a).

[17] See Jacobi (1851), Vol. II, pp. 322–351, especially pp. 322 and 340.

[18] Cayley (1857), pp. 112–125; 1890, Vol. III, pp. 13–24.

[19] Cayley (1859, 1890), Vol. III, pp. 270–292.

[20] See Brown (1896) (1960), 160–193, and Brouwer and Clemence (1961) 416–464.

the nineteenth century."[21] Hill's application of Hansen's method to Jupiter and Saturn was carried out entirely by himself and occupied the years from 1882 to 1890.

The one defect in Hill's makeup, Newcomb proclaimed, was his lack of the teaching faculty.

> Had this been developed in him, I could have learned very much from him that would have been to my advantage. In saying this I have one especial point in mind. In beginning my studies in celestial mechanics, I lacked the guidance of some one conversant with the subject on its practical side. Two systems of computing planetary perturbations had been used, one by Leverrier, while the other was invented by Hansen. The former was, in principle, of great simplicity, while the latter seemed to be very complex and even clumsy. I naturally supposed that the man who computed the direction of the planet Neptune before its existence was known, must be a master of the whole subject, and followed the lines he indicated. I gradually discovered the contrary, and introduced modified methods, but did not entirely break away from the old trammels. Hill had never been bound by them, and used Hansen's method from the beginning. Had he given me a few demonstrations of its advantages, I should have been saved a great deal of time and labor.[22]

Newcomb's complaint is not the only testimony to Hill's "lack of the teaching faculty."[23] But the demand for "a few demonstrations of the advantages" of Hansen's method is not easy to fulfill. Hansen himself seems to have "lacked the teaching faculty": his weakness as an expositor has been frequently complained of.[24]

In the present essay, I seek to shed light on the original conception of Hansen's method. My central concern is with the train of Hansen's mathematical thought and the interconnection of his main ideas, rather than with the full technicality of his method. In order to provide the appropriate context, I begin with a sketch of Laplace's method for planetary perturbations as described in the *Mé canique céleste*, undertaking to show why it took the form it did, and why it had "inconveniences" (Hansen's term).

2 The Laplacian background

2.1 Laplace's method for perturbations and its application to the mutual perturbations of Jupiter and Saturn

In Book II, Chapter VI of the *Mécanique cé leste*, Laplace sets forth the method he uses for determining the planetary perturbations of the zeroth and first order with respect to the orbital eccentricities—the inequalities he determines first. Earlier, in Section 20 of Chapter III, he has derived the following standard expressions for the elliptical motion of a planet subject solely to the gravitational force of the Sun:

21 Newcomb, 1903, 218.

22 *Ibid.*

23 Schlesinger (1937), pp. 7–8.

24 See, for instance, Brouwer and Clemence, 416.

$$nt = E - e \cdot \sin E, \tag{2.1a}$$

$$\tan(v/2) = \sqrt{\frac{1+e}{1-e}} \tan(E/2), \tag{2.1b}$$

$$r = \frac{a\left(1-e^2\right)}{1+e\cos(v-\varpi)}. \tag{2.1c}$$

Here n is the rate of mean motion and t the time, so that nt counted from perihelion is the mean anomaly; E is the eccentric anomaly (putting E for Laplace's u), r the radius vector in the orbital plane, a the semi-transverse axis, ae the orbital eccentricity, ϖ the longitude of the perihelion, and v the true anomaly. Equation (2.1a) ("Kepler's equation") is transcendental, that is, not soluble in closed algebraic form. Laplace obtains for E the infinite series

$$\begin{aligned} E = nt + e \cdot \sin \cdot nt + \frac{e^2}{2!2} 2 \cdot \sin \cdot 2nt + \frac{e^3}{3!2^2} \left\{3^2 \cdot \sin \cdot 3nt - 3 \cdot \sin \cdot nt\right\} \\ + \frac{e^4}{4!2^3} \left\{4^3 \cdot \sin \cdot 4nt - 4 \cdot 2^3 \cdot \sin \cdot 2nt\right\} \\ + \frac{e^5}{5!2^4} \left\{5^4 \cdot \sin \cdot 5nt - 5.3^4 \cdot \sin \cdot 3nt + \frac{5.4}{1.2} \cdot \sin \cdot nt\right\} + \cdots \end{aligned} \tag{2.2}$$

With E calculated to the order of some chosen power of the eccentricity, v can be found from substitution of E into (2.1b), and r/a can be found from substitution of v into (2.1c). For the analytic development of his method, however, Laplace uses the following series for r/a and v:[25]

$$\begin{aligned} \frac{r}{a} = 1 + \frac{e^2}{2} - \left[e - \frac{3e^3}{8}\right] \cdot \cos \cdot (nt + \varepsilon - \varpi) - \left[\frac{e^2}{2} - \frac{e^4}{3}\right] \\ \cdot \cos \cdot 2(nt + \varepsilon - \varpi) - \&c\cdot; \end{aligned} \tag{2.3}$$

$$\begin{aligned} v = nt + \varepsilon + \left[2e - \frac{e^3}{4}\right] \cdot \sin \cdot (nt + \varepsilon - \varpi) + \left[\frac{5e^2}{4} - \frac{11e^4}{24}\right] \\ \cdot \sin \cdot 2(nt + \varepsilon - \varpi) + \&c. \end{aligned} \tag{2.4}$$

Here ε is the mean longitude at epoch, and ϖ the longitude of the perihelion, so that $nt + \varepsilon$ is the mean longitude, and $nt + \varepsilon - \varpi$ the mean anomaly. The orbital eccentricities of the planets being relatively small, Laplace assures his readers that the first few terms from these series are generally sufficient to match the precision attainable in observations. Equations (2.3) and (2.4) are presupposed in Laplace's calculation of planetary perturbations in Chapter VI.

In order to obtain the simplest expression of the planetary perturbations, Laplace employs the "perturbing function" introduced by Lagrange: a potential function whence the perturbative forces in the direction of any coordinate can be derived by differentiation with respect to that coordinate. He calls it R, and for a single perturbing planet defines it by

[25] Laplace (1966), Vol. I., p. 377.

$$R = \frac{m' \cdot (xx' + yy' + zz')}{\left(x'^2 + y'^2 + z'^2\right)^{3/2}} - \frac{m'}{\left[(x'-x)^2 + (y'-y)^2 + (z'-z)^2\right]^{1/2}}, \quad (2.5)$$

where m' is the mass of the perturbing planet (to be evaluated eventually as a fraction of the Sun's mass M taken as the unit); x, y, z are the rectangular coordinates of the perturbed planet, and (x', y', z') those of the perturbing planet.[26] The function R proves to be independent of the position of the xy plane. Shifting to polar coordinates, with r, r' as the radii vectores, and the orbital plane of m assumed to lie in the xy plane at epoch, so that z remains small and of the order of the perturbing force, Laplace expands R as an infinite series. The expansion is "very converging," because the orbits are nearly circles and the orbital planes little inclined to one another. A typical term is of the form

$$m'k \cdot \cos \cdot \left(i'n't - int + i'\varepsilon' - i\varepsilon - g_1\varpi - g_2\varpi' - g_3\theta - g_4\theta'\right),$$

where θ, θ', are the longitudes of the ascending nodes of the two orbits in the xy plane; k is a constant depending on the eccentricities, inclinations, nodes, perihelia, transverse axes, etc., and i, i', g, g_2, $g_3 g_4$ are integers. The differential equations giving the perturbed values of the radius vector r, the longitude v, and the tangent of the latitude s are then expressed in terms of derivatives of R.

In Chapter VI, Laplace limits his solution of these differential equations to the order of the first power of the perturbing forces. That is, both m and m' are initially conceived as moving in ellipses in accordance with Eqs. (2.3) and (2.4) above, and only the effects on m produced by m' are determined. Let (r), (v), and (s) represent the elliptical values of the radius vector, longitude, and tangent of the latitude of the perturbed planet. In a calculation to the first power of the perturbing forces, we obtain the perturbed coordinates of m in the form $r = (r) + \delta r$, $v = (v) + \delta v$, $s = (s) + \delta s$, where δr, δv, δs are the changes produced by perturbation due to m' (which is still conceived as moving in an unperturbed elliptical orbit).

Were the calculation to be carried to the second power of the perturbing forces, it would be necessary to determine additional effects: m, moving in accordance with its first-order perturbations, is perturbed differently than it would be if its motion were purely elliptical, and m', having been perturbed by m, perturbs m differently than it would if its motion were purely elliptical. According to Laplace, "in the theory of the planets and comets, we may neglect these squares and products [of the perturbing forces] except in a few terms of that order, which are rendered sensible by particular circumstances..."[27] The chief "particular circumstance" Laplace had in mind here, as we shall see, was a near-commensurability between the motions of the perturbed and the perturbing planets.

From the fundamental equations of motion, Laplace obtains for δr the Equation

$$0 = \frac{\mathrm{d}^2 \cdot r\delta r}{\mathrm{d}t^2} + \frac{\mu \cdot r\delta r}{r^3} + 2\int \mathbf{d}R + r \cdot \frac{\partial R}{\partial r}. \quad (2.6)$$

[26] *Ibid.*, Vol. 1, p. 504.

[27] *Ibid.*, Vol. 1, p. 509.

Here $\mu = M + m$, but ordinarily we may put $M + m \approx 1 = a^3n^2$, that is, m may be neglected in relation to M. The boldface d in dR signifies differentiation solely with respect to the coordinates of the perturbed planet. (In practice, when the coordinates have been expressed as series in sines or cosines of the mean longitude, this means differentiation with respect to t in the product nt.) From (2.6), Laplace derives for δr the expression:

$$\delta r = -a\delta u \cdot \left[1 + \frac{3e^2}{4} + 2e \cdot \cos \cdot (nt + \varepsilon - \varpi) + \frac{9e^2}{4} \cdot \cos \cdot (2nt + 2\varepsilon - 2\varpi)\right],$$

where δu is determined by the differential equation:[28]

$$\begin{aligned} 0 =& \frac{\mathrm{d}^2 \cdot \delta u}{\mathrm{d}t^2} + n^2 \delta u \\ & - \frac{1}{a^2}\left\{1 + \frac{e^2}{4} - e \cdot \cos \cdot (nt + \varepsilon - \varpi) - \frac{e^2}{4} \cdot \cos \cdot 2\,(nt + \varepsilon - \varpi)\right\} \\ & \cdot \left\{2 \int \mathbf{d}R + r \cdot \frac{\partial R}{\partial r}\right\} - \frac{2e}{a^2} \cdot \int n\mathrm{d}t \\ & \cdot \left[\sin \cdot (nt + \varepsilon - \varpi) \cdot [1 + e \cdot \cos \cdot (nt + \varepsilon - \varpi)] \cdot \left\{2 \int \mathbf{d}R + r \cdot \frac{\partial R}{\partial r}\right\}\right]. \end{aligned} \tag{X'}$$

Once δr has been determined with the aid of equation (X'), δv can be found by the equation[29]

$$\delta v = \frac{1}{\sqrt{1-e^2}} \left\{\frac{2r \cdot \mathrm{d} \cdot \delta r + \mathrm{d}r \cdot \delta r}{a^2 \cdot n\mathrm{d}t} + \frac{3a}{1+m} \cdot \iint n\mathrm{d}t \cdot \mathbf{d}R + \frac{2a}{1+m} \cdot \int n\mathrm{d}t \cdot r \cdot \frac{\mathrm{d}R}{\mathrm{d}r}\right\}. \tag{Y}$$

Finally, δs is given by the formula

$$\delta s = -a\delta u' \left\{1 + \frac{3e^2}{4} + 2e \cdot \cos \cdot (nt + \varepsilon - \varpi) + \frac{9e^2}{4} \cdot \cos \cdot (2nt + 2\varepsilon - 2\varpi)\right\},$$

where $\delta u'$ is given by the differential equation[30]

$$\begin{aligned} 0 =& \frac{\mathrm{d}^2 \cdot \delta u'}{\mathrm{d}t^2} + n^2 \delta u' \\ & - \frac{1}{a^2}\left\{1 + \frac{e^2}{4} - e \cdot \cos \cdot (nt + \varepsilon - \varpi) - \frac{e^2}{4} \cdot \cos \cdot (2nt + 2\varepsilon - 2\varpi)\right\} \cdot \frac{\partial R}{\partial z} \\ & - \frac{2e}{a^2} \cdot \int n\mathrm{d}t \cdot \left[\sin \cdot (nt + \varepsilon - \varpi) \cdot [1 + e \cdot \cos \cdot (nt + \varepsilon - \varpi)] \cdot \frac{\partial R}{\partial z}\right] \end{aligned} \tag{Z'}$$

28 *Ibid.*, Vol. 1, p. 521.

29 *Ibid.*, Vol. 1, p. 514.

30 *Ibid.*, Vol. 1, p. 521.

According to Laplace, "the system of equations (X'), (Y), (Z') will give in a very simple manner the motion of m, taking notice only of the first power of the disturbing force. The consideration of the terms of this order, is very nearly sufficient in the theory of the planets..."[31] To terms independent of the eccentricities or proportional to their first powers, Laplace undertakes to apply equations (X'), (Y), (Z') systematically, down to terms in the longitude with coefficients of 0.081 of a sexagesimal arcsecond (=0.25 centesimal second).

For terms of the second and higher orders with respect to the eccentricities and with respect to γ (the tangent of the mutual inclination of the orbits of the perturbing and perturbed planets), a similar procedure, Laplace urges, would be wasteful of time and labor. As he explains,

> The great number of inequalities depending on the squares of the eccentricities, and of the inclinations, makes it troublesome to compute all of them...[32]

The difficulty here is to select, prior to detailed computation, the terms likely to prove observationally detectable. Small divisors resulting from the integration of the equations influence the size of the coefficients of these terms in a nonlinear way.

To illustrate: Many of the terms in R leading to terms depending on the squares and products of the eccentricities and inclinations of the orbits are of the form:[33]

$$R = M \cdot \cos \cdot \left\{ i \cdot \left(n't - nt + \varepsilon' - \varepsilon\right) + 2nt + K \right\}.$$

Here M is a multinomial constant, the individual terms of which are proportional to e^2, ee', e'^2, γ^2; K is a constant depending on ε, ϖ, ϖ', and $\prod$ (the longitude of the ascending node of the orbit of m' on the orbit of m), and i takes on all integral values, positive, zero, and negative. On substituting this term in Eq. (2.6), and integrating, Laplace obtains[34]

$$\frac{r\delta r}{a^2} = \frac{\left\{ \begin{array}{l} \frac{3n^2}{2} \left\{ \begin{array}{l} (F+G) \cdot e^2 \cdot \cos \cdot \left[i \cdot \left(n't - nt + \varepsilon' - \varepsilon\right) + 2nt + 2\varepsilon - 2\varpi\right] \\ + H \cdot ee' \cdot \cos \cdot \left[i \cdot \left(n't - nt + \varepsilon' - \varepsilon\right) + 2nt + 2\varepsilon - \varpi - \varpi'\right] \end{array} \right\} \\ + \left\{ \frac{2\cdot(2-i)\cdot n}{in' + (2-i)\cdot n} \cdot aM + a^2 \frac{\partial M}{\partial a} \right\} \cdot n^2 \cos \cdot \left[i \cdot \left(n't - nt + \varepsilon' - \varepsilon\right) + 2nt + K\right] \end{array} \right\}}{\left[i \cdot n' + (3-i)\cdot n\right] \cdot \left[in' + (1-i)\cdot n\right]} \tag{2.7}$$

He then obtains for the corresponding term of δv[35]

[31] *Ibid.*

[32] *Ibid.*, Vol. III p. 17.

[33] *Ibid.*, Vol. III, p. 5.

[34] *Ibid.*, Vol. III, p. 7,

[35] *Ibid.*, Vol. III, p. 9.

$$\delta v = \frac{1}{\sqrt{1-e^2}} \times \left\{ \begin{array}{l} \frac{2d\cdot(r\delta r)^2}{a^2\mathrm{ndt}} - \frac{1}{2}\left\{ \begin{array}{l} (F+G)\cdot e^2 \cdot \sin\cdot\left[i\cdot\left(n't - nt + \varepsilon' - \varepsilon\right) + 2nt + 2\varepsilon - 2\varpi\right] \\ +H\cdot ee'\cdot \sin\cdot\left[i\cdot\left(n't - nt + \varepsilon' - \varepsilon\right) + 2nt + 2\varepsilon - \varpi - \varpi'\right] \end{array}\right\} \\ + \left\{ \begin{array}{l} \frac{(6-3i)\cdot n^2}{[in' + (2-i)\cdot n]^2}\cdot aM \\ + \frac{2na^2\frac{\partial M}{\partial a}}{in' + (2-i)\cdot n} \end{array}\right\} \cdot \sin\cdot\left[i\cdot\left(n't - nt + \varepsilon' - \varepsilon\right) + 2nt + K\right] \end{array}\right\} \tag{2.8}$$

In (2.7) and (2.8), denominators containing the integer i influence the size of the coefficients in which they occur. If the perturbed planet is Jupiter and the perturbing planet Saturn, then $n':n \approx 2:5$. If in the expression $[i \cdot n' + (3-i)\cdot n]$ which appears in the denominator of (2.7) we substitute $i = 5$, it becomes $5n' - 2n$, which is about $n/74$. In this case, therefore, $r\delta r/a^2$ becomes especially large. As a result, the first term of δv, whose value depends on $r\delta r/a^2$, will also be large. For the same planets, the term in δv with the denominator $[i \cdot n' + (2-i)\cdot n]^2$ has its largest value when $i = 3$. Thus, for any given pair of perturbed and perturbing planets, each term must be examined to see how different choices of i affect its size.

For terms proportional to powers or products of higher dimensions of the eccentricities and inclinations, the problem of identifying all terms that will be observationally significant presents similar difficulties.

It was in the mid-1780s, in a renewed attempt to resolve a major unexplained anomaly in planetary motion, that Laplace came to focus on the role of small divisors. Since Kepler's time, the mean motion of Jupiter had appeared to be slowly increasing, and that of Saturn slowly decreasing. Halley in his planetary tables, published posthumously in 1749, had postulated a secular acceleration for Jupiter and a secular deceleration for Saturn—a term in each case proportional to the time squared. Early in the 1770s, Laplace had shown that, if a term proportional to t^2 was assumed to be produced in the mean motion of a planet by gravitational interaction with a perturbing planet, its coefficient, calculated to the order of the first power of the perturbing masses and the third power of the eccentricities, would be identically zero. He concluded, unwarrantedly as later became clear, that the apparent changes in the mean motions of Jupiter and Saturn were not due to their gravitational interaction. Lagrange in a memoir of 1774—not directly responding to Laplace's memoir—pointed out that apparent changes in the mean motion, inferred from observations covering a limited period, or widely separated in time but fortuitously chosen, could *appear* to be proportional to t^2 without being strictly so; they could arise from an oscillatory inequality of long period.

This possibility, as Laplace at length discovered in 1785, proved to be the key to the anomalous apparent changes in the mean motions of Jupiter and Saturn. He announced his resolution of the problem to the Paris Academy of Sciences in November of that year. The small divisor involved was $(5n' - 2n)$.[36] A perturbational term proportional to $M \cdot \sin(5n't - 2nt + A)$, where M and A are constants, goes through its full cycle

[36] For an extended account of the history of this problem, see Wilson (1985).

of values in $360^\circ/(5n' - 2n) \approx 900$ years. For either planet, seventeenth-century and early eighteenth-century observational values could appear to fit a monotonic secular change proportional to t^2.

The coefficient M can be shown to consist of terms proportional to e^3, e^2e', ee'^2, e'^3, $e\gamma^2$, $e'\gamma^2$, where e, e' are the orbital eccentricities of the two planets, and γ is the inclination of the two orbital planes to one another. These products are small fractions, equal to about 1/8,000 or less. Because of their smallness, and the formidable labor of computing the parts of the coefficient M, a term of the form $M\sin(5n't - 2nt + A)$ had never been computed before Laplace did so in 1785. He saw that, because of the double-integration involved in solving the differential equations, the coefficient of such a term would have for divisor the small factor $(5n' - 2n)^2$; hence, M *might* prove large enough to make the term observationally detectable. Only an extended calculation could decide the question. Carrying it out, Laplace found for Jupiter $20'49''.5\sin(5n't - 2nt + A)$ and for Saturn $-48'44''\sin(5n't - 2nt + A)$, the coefficients being for the epoch 1750.[37] (The coefficients are subject to secular change, owing to secular changes in the orbital elements.) The above terms represent the largest perturbational inequalities in the solar system and are referred to as "the great inequality of Jupiter and Saturn."

Laplace incorporated these inequalities into the mean motions of the two planets. He justified this step by showing that, if elliptical values are substituted for r and v in equations (X') and (Y), and account is taken solely of terms leading to a double-integration, including those with $(5n' - 2n)^2$ as divisor, then δr and δv prove to be given by the equations:[38]

$$\begin{aligned} \delta r &= \left(\frac{\mathrm{d}r}{n\mathrm{d}t}\right)\cdot\frac{3a}{M+m}\int n\mathrm{d}t\int \mathbf{d}R, \\ \delta v &= \left(\frac{\mathrm{d}v}{n\mathrm{d}t}\right)\cdot\frac{3a}{M+m}\int n\mathrm{d}t\int \mathbf{d}R. \end{aligned} \tag{2.9}$$

Now $(\frac{dr}{ndt})$ is the elliptical value of the rate of change of the radius vector r per unit of change in the mean motion, and $(\frac{\mathrm{d}v}{n\mathrm{d}t})$ is the elliptical value of the rate of change of the true longitude v per unit change of the mean motion. In both cases, a rate is multiplied by the factor $[3a/(M + m)]\int n\mathrm{d}t\int \mathbf{d}R$, evaluated for the particular term of R yielding the denominator $(5n' - 2n)^2$. The product must give the change in r or v due to the particular term of R considered.

The incorporation of $\frac{3a}{M+m}\cdot\int n\mathrm{d}t\int \mathbf{d}R$ for $R = M\left[\left(5n' - 2n\right)t + A\right]$ into the mean motion produces increments in (r) and (v) equal to those obtained by computing δr and δv from (2.9). But the procedure has a further effect. The mean motion becomes a variable quantity, varying with only *some* of the inequalities, those of very long period. Laplace continues to calculate the other inequalities, those of shorter period, in his usual way, adding them to (r) and (v). These inequalities are, at the end of the

[37] See Laplace, *Oeuvres Completes*, XI, p. 177.

[38] Wilson (1985), pp. 258–259. The quantity to be added to the mean motion is there given as $3am'\int n\mathrm{d}t\int dR$, but the perturbing mass m' is in fact included in R.

calculation, stated in terms of the mean motion, which is no longer simply proportional to the time, but incorporates a sinusoidal variation.

The first to question this procedure on logical grounds appears to have been G.W. Hill, in the 1890s. Was Laplace's procedure for long-period inequalities consistent with his procedure for computing ordinary, short-period inequalities?

> In the *Mécanique céleste* Laplace had determined all long-period inequalities as if they were to be applied to the mean longitude, and had so directed they should, while the short-period ones were derived as if they were to be added to the true longitude. There is, therefore, a want of congruity, and even of rigor, in this way of proceeding. For Laplace has nowhere shown how these two modes of application can be employed in unison. It is plain there would be as many methods of perturbations as there were opinions as to the dividing line separating long from short-period inequalities.[39]

It was Hill's belief that Hansen, too, had been struck by the incongruity of Laplace's treating long-period inequalities in one way, and short-period inequalities in another, and had been thus led to the invention of his new method in celestial mechanics; but evidence on this point is lacking. The objection to Laplace's procedure on logical grounds was repeated by Brouwer and Clemence in their *Methods of Celestial Mechanics*.[40]

No such objection, certainly, occurred to Laplace in 1785. He was excited by the prospect of a stunning success. If he could identify and calculate all the divisors small enough to produce observationally detectable terms, he would resolve a long-standing anomaly, demonstrate at last that universal gravity was sufficient to account for the motions of the celestial bodies, and achieve the most accurate theory of Jupiter and Saturn ever constructed. Laplace, I suspect, was intensely focussed on the achievement of a *coup*.

With insight and effort, he achieved this *coup*. In his *Théorie de Jupiter et de Saturne*, published in 1788,[41] he reduced the errors in the theories of these two planets to less than 2 arcminutes. The extent of his triumph is evident from a table he gives, comparing the errors of his theory of Saturn with those of Hailey's tables, for 43 oppositions of Saturn observed from 1582 to 1786.[42] The maximum deviation of Hailey's tables from the observations is $-22'17''$, while that of Laplace's theory is $+1'52''$. If we compute the standard deviations of the errors, we find $545''.9$ for Hailey's tables and $60''.7$ for Laplace's theory, a ninefold improvement. In a comparison of his theory with a Babylonian observation of 228 B.C., Laplace found a deviation of only $-55''.5$.

In presenting a draft of his memoir to the Paris Academy on May 10, 1786, Laplace noted that the earlier oppositions with which he had compared his theory, and on the basis of which he had determined its constants, had not been corrected for the aberration of light or nutation, or for mistakes in the star positions from which the planetary

[39] Hill (1890, 1906), pp. 12–13.

[40] Brouwer and Clemence 1961, p. 416.

[41] Laplace (1788), pp. 33–160; 1835, XI, 95–207.

[42] Laplace (1835), XI, pp. 200–201.

positions were computed. The astronomer Jean-Baptiste-Joseph Delambre (1749–1822) offered to carry out the necessary corrections, and the result was Delambre's *Tables de Jupiter et de Saturne*, published in 1789, a book of 109 pages that includes an account of the procedures used; the tables alone were republished in Vol. I of the third edition of Lalande's *Astronomie* in (1792).[43] Delambre's values for the aphelia, eccentricities, and mean longitudes of the two planets in 1750, as compared with modern values for the same date, show moderate to considerable improvements over Laplace's values. In the Lalande volume, it is claimed that positions of Jupiter deduced from the tables never differ from observation by as much as 30 arcseconds.[44]

The improvement effected by Delambre from the side of the observations no doubt encouraged Laplace to push further the refinement of the theory. The comparison with observations, to be sure, could not guarantee the exactitude of any particular term. What was compared with observations was the *totality* of the theoretical terms. The presumption was that still greater closeness of fit could be achieved from either side—by selecting or obtaining observations of superior precision, and bringing them to bear more incisively on the determination of the constants that have to be established empirically, and by a more accurate deduction of the consequences of the gravitational theory.

And thus, when preparing Volume III of the *Mécanique céleste*—the volume that contains the theories of all the planets, numerically calculated—Laplace set out to revise and refine his theory of Jupiter and Saturn. Alexis Bouvard (1767–1843) assisted him in the new calculations. A number of perturbational terms hitherto omitted were now computed. These included terms proportional to products of five dimensions in the eccentricities and inclinations of the orbits[45] and a number of terms of the second order with respect to the perturbing forces.[46]

Having incorporated the "great inequalities" of Jupiter and Saturn into their respective mean motions, Laplace represents the new mean motions, thus rendered variable, by ζ, ζ'. Each of these comprises a part strictly proportional to the time and a part varying sinusoidally in accordance with the "great inequality." Thus, we may write $\zeta = n_0 t + \delta\zeta$ and $\zeta' = n_0' t + \delta\zeta'$, where n_0, n_0' are the truly mean rates of mean motion, and $\delta\zeta, \delta\zeta'$ are the sinusoidally varying parts, given by

$$\delta\zeta = \frac{3a}{M+m}\int n\mathrm{d}t \int \mathbf{d}R, \quad \delta\zeta' = \frac{3a'}{M+m'}\int n'\mathrm{d}t \int \mathbf{d}'R'. \tag{2.10}$$

Between the two quantities $\delta\zeta,\ \delta\zeta'$, Laplace obtains a quantitative relation, such that when he has computed the value of one of them, the value of the other is derivable in a single step. To obtain this relation, he starts from the equation stating the conservation of *forces vives* and neglects all terms except those which are of the order of the squares or products of the masses m and m', and have $(i'n' - in)^2$ as divisor. He thus finds that

43 See Wilson (1985), pp. 277–280.

44 Lalande (1792), Vol. I, Tables, p. 146.

45 Laplace (1966), Vol. III, pp. 289–291.

46 In particular, see *Ibid.* Vol. III, pp. 129–143.

$$m \cdot \int \mathbf{d}R + m' \cdot \int \mathbf{d}'R' = 0. \tag{2.11}$$

Here, as before, $\mathbf{d}R$ signifies the derivative of R solely with respect to the coordinates of m', and $\mathbf{d}'R'$ signifies the derivative of R' solely with respect to the coordinates of m'. Now R, and therefore $\int \mathbf{d}R$ as well, is computed only to the order of m'; similarly, R' and $\int \mathbf{d}'R'$ are computed only to the order of m. It follows, according to Laplace, that in evaluating these integrals, we may suppose the elements of the elliptical motion to be constant.[47] Hence

$$m \cdot \int\int \mathrm{d}t \cdot \mathbf{d}R = \frac{m \cdot \int\int an \cdot \mathrm{d}t{\cdot}\mathbf{d}R}{an}$$

$$\text{and } m \cdot \int\int \mathrm{d}t \cdot \mathbf{d}'R' = \frac{m' \cdot \int\int a'n' \cdot \mathrm{d}t \cdot \mathbf{d}'R'}{a'n'}$$

By multiplying the terms of (2.11) by 3, integrating them with respect to t, and making the foregoing substitutions, he can thus write

$$\frac{3m \cdot \int\int an \cdot \mathrm{d}t{\cdot}\mathbf{d}R}{an} + \frac{3m' \cdot \int\int a'n' \cdot \mathrm{d}t \cdot \mathbf{d}'R'}{a'n'} = 0. \tag{2.12}$$

Replacing the integrals in (2.12) by means of (2.10) yields

$$\delta\zeta' = -\frac{m \cdot (M+m) \cdot a'n'}{m' \cdot (M+m') \cdot an} \cdot \delta\zeta.$$

Using the relations $n^2a^3 = M + m$ and $n'^2a'^3 = M + m'$ to eliminate n and n', and then neglecting m and m' relative to M, Laplace finally obtains

$$\delta\zeta' = -\frac{m \cdot \sqrt{a}}{m' \cdot \sqrt{a'}} \cdot \delta\zeta. \tag{2.13}$$

It was this relation, Laplace tells us, that first led him to conclude that gravitational interaction was the cause of the apparent secular changes in the mean motions of Jupiter and Saturn; Halley's values for these changes were very nearly in the ratio given by (2.13).[48]

The general applicability of (2.13) in the derivation of perturbations of the second order with respect to the perturbing forces will become a question in the 1820s, as we shall see. Of the terms of the second order that Laplace computes, we focus here on those related to the great inequality; it is concerning these that questions will later be raised.

First, he computes a second-order term that the great inequality implies, which has the argument $10\zeta' - 4\zeta$, hence half the period of the great inequality. To obtain

[47] *Ibid.*, Vol. I, 647: "If we neglect the squares and products of the disturbing masses, we may, in the integration of these terms, suppose the elements of the elliptical motion to be constant..."

[48] *Ibid.*, Vol. I, p. 652.

it, he returns to the first equation of (2.10) and considers in R a term of the form $m'k \cdot \cos\cdot(5\zeta' - 2\zeta + A)$, where A is a constant, and k is a function of the elements of the elliptic motions of m and m'. (Since k is a function of the elliptical elements, it like them is subject to a secular change, which we here leave out of account.) The derivative $\mathbf{d}R$ will be given by $(\partial R/\partial\zeta)(\mathrm{d}\zeta/\mathrm{d}t)\mathrm{d}t$. Evidently, $\partial R/\partial\zeta = 2m'k \cdot \sin(5\zeta' - 2\zeta + A)$; but what is $\mathrm{d}\zeta/\mathrm{d}t$? Laplace writes $\zeta = \int n\mathrm{d}t$ and $\mathrm{d}\zeta = n\mathrm{d}t$, as if n were a variable rate, but then proceeds to treat it as a constant, which, to be sure, it approximately is. He therefore finds $\mathrm{d}R = 2m' \cdot k \cdot n\mathrm{d}t \cdot \sin\cdot(5\zeta' - 2\zeta + A)$. His task then becomes the integration of

$$\begin{aligned}\partial\zeta &= \frac{3 \cdot 2m'}{M+m} \int\int ak \cdot n^2 \mathrm{d}t^2 \cdot \sin\cdot(5\zeta' - 2\zeta + A) \\ &\approx 6m'akn^2 \int\int \mathrm{d}t^2 \cdot \sin\cdot(5\zeta' - 2\zeta + A). \qquad (2.14)\end{aligned}$$

In the second line of (2.14), I have followed Laplace in neglecting m relative to M, setting $M = 1$, and transferring the quantities that are constant in the elliptic motion from the integrand into the coefficient.

The first-order integral of (2.14) is obtained if we substitute for ζ, ζ' their average values nt and $n't$ (I here, with Laplace, take n, n' to be the constant parts of the two rates of 'mean motion'), then integrate. Thus

$$\delta\zeta = -\frac{6m'akn^2}{(5n' - 2n)^2} \cdot \sin\cdot(5n't - 2nt + A) \qquad (2.15)$$

Then by (2.13),

$$\delta\zeta' = \frac{6makn^2}{(5n' - 2n)^2} \cdot \frac{\sqrt{a}}{\sqrt{a'}} \cdot \sin\cdot(5n't - 2nt + A).$$

To proceed to the second-order value of $\delta\zeta$, we must substitute for ζ, ζ' in (2.14) not nt, $n't$, but $nt+\delta\zeta$ and $n't+\delta\zeta'$, where for $\delta\zeta$ and $\delta\zeta'$, we may use the first-order values just determined. The next steps are outlined as follows in Bowditch's commentary on the *Mécanique céleste.*[49]

For the sake of brevity, let $5n't - 2nt + A = A'$, and $\frac{6m'akn^2}{(5n'-2n)^2} = b$.

Then $\zeta = nt - b \cdot \sin\cdot A'$, and $\zeta' = n't + b \cdot \frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot \sin\cdot A'$. Therefore,

$$\begin{aligned}5\zeta' - 2\zeta + A &= 5n't - 2nt + A + \frac{2m'\sqrt{a'} + 5m\sqrt{a}}{m'\sqrt{a'}} \cdot b \cdot \sin\cdot A' \\ &= A' + \frac{2m'\sqrt{a'} + 5m\sqrt{a}}{m'\sqrt{a'}} \cdot b \cdot \sin\cdot A'.\end{aligned}$$

[49] *Ibid.*, Vol. I, p. 655, footnote 825.

Here A' is an angle that varies through the entire range from 0° to 360°; but the second term of the foregoing expression, it can be shown, is always less than 5°. Symbolizing the second term by α, we have, approximately,

$$\sin(A' + \alpha) \approx \sin \cdot A' + \alpha \cdot \cos \cdot A'.$$

Hence,

$$\begin{aligned} \sin \cdot (5\zeta' - 2\zeta + A) &= \sin \cdot A' + \left\{ \frac{2m'\sqrt{a'} + 5m\sqrt{a}}{m'\sqrt{a'}} \cdot b \cdot \sin \cdot A' \right\} \cdot \cos \cdot A' \\ &= \sin \cdot A' + \frac{2m'\sqrt{a'} + 5m\sqrt{a}}{2m'\sqrt{a'}} \cdot b \cdot \sin \cdot 2A'. \end{aligned}$$

Equation (2.14) therefore becomes

$$\delta\zeta = 6m'akn^2 \int\int dt^2 \cdot \left\{ \sin \cdot A' + \frac{2m'\sqrt{a'} + 5m\sqrt{a}}{2m'\sqrt{a'}} \cdot b \cdot \sin \cdot 2A' \right\}.$$

The integral of the first term within the brackets yields the first-order value of $\delta\zeta$, as in (2.15) above. The integral of the second term gives the second-order increment, namely

$$-\frac{9m'^2a^2n^4k^2}{2(5n' - 2n)^4} \cdot \frac{2m'\sqrt{a'} + 5m\sqrt{a}}{m'\sqrt{a'}} \cdot \sin \cdot 2(5n't - 2nt + A) \qquad (2.16)$$

The coefficient of this inequality for Jupiter in 1750, according to Laplace, was $-13''.238897$, and that for Saturn in 1750 was $30''.6889$. These terms, Laplace directs, are like the great inequality itself to be incorporated in the mean motions of their respective planets.

In describing how Laplace computes second-order contributions to the great inequality itself (that is, to the inequality with argument $5\zeta' - 2\zeta$), I shall designate by δR the second-order part of R, and by R the first-order part. Then

$$\delta R = \frac{\partial R}{\partial r} \cdot \delta r + \frac{\partial R}{\partial v} \cdot \delta v + \frac{\partial R}{\partial s} \cdot \delta s + \frac{\partial R}{\partial r'} \cdot \delta r' + \frac{\partial R}{\partial v'} \cdot \delta v' + \frac{\partial R}{\partial s'} \cdot \delta s'. \qquad (2.17)$$

Thus, each term in δR arises from a product of a partial derivative of R with respect to a given variable and a perturbation of this same variable. For instance, R contains a term representable by $M^{(0)} \cdot e'^2 \cdot \cos(3n't - nt + C)$, where $M^{(0)}$, C are constants. The partial derivative of this term with respect to r, multiplied by δr, can be represented by

$$a \cdot \frac{\partial M^{(0)}}{\partial a} \cdot e'^2 \cdot \frac{\delta r}{a} \cdot \cos \cdot (3n't - nt + C) \qquad (2.18)$$

But δr contains a term of the form $a \cdot F \cdot \cos\left(nt - 2n't + D\right)$, where F, D are constants; hence (2.18) will contain a product of the two cosines. By the trigonometric identity

$$\cos A \cdot \cos B = \frac{1}{2} \cdot \cos(A + B) + \frac{1}{2} \cdot \cos(A - B),$$

the product of the two cosines will give two terms of which one will be proportional to $\cos(5n't - 2nt + G)$, that is, a term contributing to the great inequality.

Laplace computes all the second-order terms deriving from a term in R proportional to $\cos(3n't - nt + \text{const.})$ and from a perturbation of a coordinate (δr, δv, $\delta r'$, etc.) proportional to $(\sin/\cos)[nt - 2n't + \text{const.}]$, and reciprocally, the terms deriving from a term in R proportional to $\cos(nt - 2n't + \text{const.})$ and from a perturbation of a coordinate proportional to $(\sin/\cos)[3n't - nt + \text{const.}]$.[50] No doubt he chooses factors with these particular arguments, because the first-order inequalities involving these same arguments are large for both planets. In the case of Saturn, he asserts, the sum of all these contributions constitutes "the most sensible term of the great inequality..., depending on the square of the disturbing force."[51]

Letting $T_5 = 5n't - 2nt + 5\varepsilon' - 2\varepsilon$, we can represent his net result for the longitude of Saturn in 1750 by[52]

$$\delta v' = -3''.8165 \sin \cdot T_5 + 42''.9203 \cos \cdot T_5. \tag{2.19}$$

As the corresponding term for Jupiter, he gives[53]

$$\delta v = 1''.64166 \cdot \sin \cdot T_5 - 18''.46195 \cos \cdot T_5 \tag{2.20}$$

Has Laplace considered all the second-order terms large enough to affect predictions significantly? The text of the *Mécanique céleste* provides no basis for answering this question. Laplace asserts that one term or another must be taken into account, or shows that such and such another term is negligible, without providing any discussion of still other terms neglected. In the 1830s, Pontécoulant defended Laplace's "marvelous tact in distinguishing, amidst all these inequalities, the multiplicity of which constitutes the principal difficulty of the theory of the planetary motions, those which must acquire detectable values and those which will remain forever negligible...;"[54] but at the same time, he acknowledged that Laplace gave the wrong sign to the major second-order term, and failed to compute other terms that needed to be taken into account.[55] Bowditch in his commentary could say that "several terms, omitted by the author,... are quite as important as those which he has retained."[56] As we shall see, (2.19) and (2.20)

[50] *Ibid.*, Vol. III, pp. 131–143.

[51] *Ibid.*, Vol. III, p. 143.

[52] *Ibid.*, Vol. III, p. 306.

[53] *Ibid.*, Vol. III, p. 291, footnote 2667.

[54] de Pontécoulant (1835), p. 395.

[55] *Ibid*, pp. 396, 406, 487.

[56] Laplace (1966), Vol. III, p. 130, footnote 2502.

are badly in error. Years later G.W. Hill remarked that, "In all previous investigations [prior to Hansen's] it is impossible to form a conception of the probable magnitude of the terms passed over on account of the habit of the investigator of selecting here and there a term to be computed."[57]

2.2 Bouvard's tables for Jupiter and Saturn

In 1808, Bouvard published his *Nouvelles Tables de Jupiter et de Saturne*;[58] they were based essentially on the theory of the *Mécanique Céleste*. The fate of these tables is briefly related in the first paragraph of the Introduction to a revised set of tables prepared by Bouvard and published in 1821:

> A short time after the printing of my *Tables de Jupiter et de Saturne*, M. de Laplace, on reviewing the theory of these two planets, found that the great inequality depending on the fifth powers of the eccentricities had been taken analytically with the wrong sign. This equation has a large influence on the value of the elliptic elements, and notably on the mean longitude at epoch and on the mean motion. Consequently, my *Tables* would not long continue to represent the observations accurately. In such a circumstance, there was only one course to follow, and I did not hesitate. I took up my work once again, to bring it to the level of the changes that theory had confirmed, and I have profited from the occasion by introducing other improvements that experience has indicated to me.[59]

In his first tables, Bouvard had employed the oppositions of Jupiter and Saturn deduced from the observations of La Caille, Bradley, and Maskelyne from 1747 to 1804, and those of the Observatoire de Paris from 1800 to 1804. For the second tables, he made use as well of all the quadratures furnished by the same observers, and in addition, all the oppositions and quadratures observed in Paris from 1804 to 1814. And for the first time, in the differential correction of the elliptical elements, he applied the method of least squares. On pages vii–xii of his Introduction, he supplies lists of the two sets of equations of condition, for Jupiter and for Saturn, where an equation of condition corresponds to each observation, and the error—the difference between the observation and the theoretical position—is given in each case. Of the latter Bouvard says:

> The last column presents the errors of the Tables, expressed in centesimal seconds [one centesimal second = 0.324 sexagesimal second]. For Jupiter, only one goes as high as 45″ [≈ 16 sexagesimal seconds], 14 are less than 40″, and the others, much smaller, are alternately positive or negative for several years running. In Saturn's case, the errors are smaller; three only are between 30″ and 40″; the others are very small, and generally behave like those of Jupiter.[60]

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57 Hill (1890, 1906), p. 13.

58 Paris, Bureau des Longitudes, 1808.

59 Bouvard (1821), introduction, p. i.

60 *Ibid.*, Introduction, p. vi.

Thus, the maximum errors of Bouvard's tables are about half those of Delambre's tables, a significant reduction. On the other hand, the fact that the errors go positive or negative for several years running—that there are "runs" of some length—is an indication that the errors are not purely random, but include systematic error. It follows that the theory has not been developed to the point of accounting for all the variations detected observationally. Bouvard seems unaware of, or at (east untroubled by, this consequence.

To exhibit the non-random character of the residuals, I conducted a "Runs Test" on each of the lists of residuals.[61] This test is based on the theory of Bernoulli trials and requires as data only the number of positive residuals, the number of negative residuals, and the number of runs, that is, sequences of successive residuals that are all positive or all negative. The result of the test is given as a "p value," which tells in what proportion of cases, assuming randomness, such a result would occur. For Jupiter I found:

Number of + residuals	79	
Number of − residuals	67	
Number of run	48	p value 0.00002

For Saturn:

Number of + residuals	72	
Number of − residuals	65	
Number of run	48	p value 0.00031

On the assumption of randomness, therefore, the number of runs for Jupiter should occur in only two out of 100,000 cases and those for Saturn in only 31 out of 100,000 cases.

The Bouvard tables were nevertheless used for the ephemerides of the British *Nautical Almanac* from 1834 to 1877 in the case of Jupiter, and from 1834 to 1 879 in the case of Saturn. They were then replaced by Leverrier's tables, which were in turn replaced by G.W. Hill's Hansenian tables in 1900.[62]

2.3 Plana's critique, Laplace's response; clarifications by Poisson and Pontécoulant

Giovanni Plana (1781–1864), formerly a student at the Ecole Polytechnique in Paris, from 1811 professor of astronomy at the University of Turin, and from 1813 director of the observatory in Turin, on December 9, 1825, presented to the Astronomical Society of London a *Mémoire sur différens Points relatifs à la Théorie des Perturbations des Planètes exposée dans la Mécanique céleste.*[63] Each of its five chapters critiques a

[61] I used the software supplied with Ross (2000). Ross explains and illustrates the "Runs Test" on pp. 494–499. In this matter, I gratefully acknowledge the guidance of Professor Gary Fowler of the Department of Mathematics of the United States Naval Academy, Annapolis, MD.

[62] See Seidelmann (1992), Chap. 13.

[63] Plana (1826).

different passage in Laplace's *Mécanique céleste*, entering in each case into the detail of the relevant computations. The fourth chapter treats the second-order contributions to the coefficient of the great inequality of Jupiter and Saturn.[64]

After giving Laplace's numerical results for these contributions [see Eqs. (2.19) and (2.20) above],[65] Plana remarks on the "distance" between these results and the analytic formulas of section 16 (=*Celestial Mechanics*, III, 129–147) from which they have presumably been derived: Laplace has suppressed the intermediary calculations. Also, in section 16, Laplace has omitted, without explanation, terms that, analytically speaking, are of the same order as those he includes. Finally, he has used the relation given by equation (2.13) above to derive (2.20) from (2.19) above, thus avoiding the direct calculation of the second-order term for Jupiter. But Plana claims to have found, a posteriori, that relation (2.13) does not hold for all perturbations of the second order with respect to the perturbing forces. For these several reasons, Plana undertakes a direct calculation of the second-order terms for Jupiter that he believes large enough to require being taken into account.

First, he computes the perturbation for which Laplace had obtained (2.20), but without using the relation (2.13).[66] The Laplacian result for Jupiter as perturbed by Saturn, with Saturn's mass corrected (see footnote 64), is

$$\delta\zeta = 1''.5703 \cdot \sin \cdot T_5 - 17''.6598 \cdot \cos \cdot T_5.$$

Plana's direct computation yielded

$$\delta\zeta = -1''.2462 \cdot \sin \cdot T_5 + 4''.8141 \cdot \cos \cdot T_5. \tag{2.21}$$

Here the first coefficient differs from Laplace's value by $-2''.81\,65$, and the second by $+22''.4739$: not a negligible difference!

Next, Plana considers a second-order term that Laplace had neglected.[67] Each second-order perturbation, we recall, arises from a sum of products, the factors of each product being a partial derivative of R with respect to a given variable, and a perturbation of this same variable (see Eq. 2.17 above). The cosines or sines contained in the two factors must have arguments such that their sum or difference gives $T_5 = (5n't - 2nt + 5\varepsilon' - 2\varepsilon)$, and be of the lowest order possible with respect to the eccentricities and inclinations, so that the sum of the exponents of these factors equals 3. In the second-order perturbation computed by Laplace, the two arguments were $3n't - nt +$ const., and $nt - 2n't +$ const. In the second-order perturbation now computed by Plana, the two arguments are $4n't - nt +$ const., and $n't - nt +$ const., and he obtains as the sum of all the contributions to this perturbation,

64 Chapter 4 occupies pp. 368–406 of the memoir cited in the preceding note.

65 After publication of the *Mécanique Céleste*, Laplace had authorized a reduction in the mass of Saturn from 1/3359.4 to 1/3512 of the mass of the Sun. Plana, therefore, to correct the coefficients for Jupiter (as given in equation 1.20 above), sets out to multiply them by the fraction 3359.4/3512. His result, $1''.5703 \sin .T_5 - 18''.0710 \cos .T_5$, is in error because the second coefficient should be -17.6598.

66 The computation occupies pp. 371–392 of Plana's memoir.

67 This computation occupies pp. 392–397 of Plana's memoir.

$$\delta\zeta = -0''.6738 \cdot \sin \cdot T_5 + 0''.7434 \cdot \cos \cdot T_5. \tag{2.22}$$

Other terms could be obtained, Plana points out, by using other arguments; for instance ($2n't$+const.) and ($3n't - 2nt$ + const.), or ($3n't$+const.) and ($2n't - 2nt$ + const.). "But one would have a numerical result much smaller than the preceding; thus we can dispense with carrying out this difficult calculation." Adding together his two results, he obtains:

$$\delta\zeta = -1''.9200 \cdot \sin \cdot T_5 + 5''5575 \cdot \cos \cdot T_5. \tag{2.23}$$

Such, he concludes, *is the effect of the square of the perturbing force on the great inequality of Jupiter.*[68] The first coefficient of (2.23) differs from Laplace's by $-3''.4903$, and the second by $+23''.6285$.

Finally, Plana undertakes to compute the second-order perturbation of Saturn ($\delta\zeta'$) corresponding to (2.23), without relying on (2.13) as Laplace had done in computing $\delta\zeta$ from $\delta\zeta'$.[69] Equation (2.13), Plana says, is not uniformly correct, because the statement from which it is derived, $m \int \mathrm{d}R + m' \int \mathrm{d}'R' = 0$ (2.11), is not strictly correct: the expression on the left, when only terms having $(5n' - 2n)^2$ for denominator are taken into account, is not equal to zero, but to a quantity of the third order with respect to m and m'. Plana sets about determining the detailed changes in $\int \mathrm{d}R$ that are necessary to turn it into $\int \mathrm{d}'R'$. He finds that (2.13) is applicable to the transformation of some of the constituent terms, but not to all.

The outcome of his analysis can be summarized as follows. Laplace has expanded R as a sequence of infinite series, the first such series being

$$\frac{m'}{2} \cdot \sum A^{(i)} \cdot \cos \cdot (i \cdot (n't - nt + \varepsilon' - \varepsilon)),$$

where i goes from $-\infty$ to $+\infty$, and $A^{(-i)} = A^{(i)}$. Others of the infinite series involve derivatives of $A^{(i)}$ with respect to the mean solar distances a and a'. Laplace expands R' in entirely analogous series, and—with one exception—the $A^{(i)}$ are exactly the same in the two expansions. Where the $A^{(i)}$ are the same in the two expansions, Laplace's (1–13) is applicable. The exception is $A^{(1)}$, which in the case of R, used in calculating the perturbations of m due to m', is given by

$$A^{(i)} = \frac{a}{a'^2} - \frac{1}{a'} \cdot b^{(1)}_{1/2},$$

$b^{(1)}_{1/2}$ being a constant;[70] but in the case of R' used in calculating the perturbations of m' due to m, is given by[71]

68 Plana's memoir, p. 397.

69 Plana's determination of $\delta\zeta'$ from $\delta\zeta$; occupies pp. 398–406 of his memoir.

70 See Laplace (1966), Vol. I, p. 540.

71 See *Ibid.*, Vol. I, pp. 543–544.

$$\frac{a'}{a^2} - \frac{1}{a'} \cdot b_{1/2}^{(1)}.$$

The difference between the $A^{(1)}$ used in R and that used in R' is evidently

$$\frac{a'}{a^2} - \frac{a}{a'^2}. \tag{2.24}$$

Plana's final result for the second-order perturbation of the great inequality of Saturn is[72]

$$\delta\zeta' = +25''.1036 \cdot \sin \cdot T_5 - 12''.8932 \cdot \cos \cdot T_5. \tag{2.25}$$

Laplace's result had been

$$\delta\zeta' = -3''.8165 \cdot \sin \cdot T_5 + 42''.9203 \cdot \cos \cdot T_5.$$

Plana's two coefficients thus differ from Laplace's by $+28''.92016$ and $-55''.8135$.

Laplace, now in the last year of his life (his death would come on March 5, 1827), wrote a response intended for inclusion in the *Connaissance des Tems pour l'an 1829*. In a letter to Plana dated June 15, 1826, he spoke of the volume of the *C.^e des Tems* in which the memoir was to appear as being "now in press," and said that he was having the publisher send Plana a copy of the memoir.

Plana received the copy in the early days of August 1826. The object of the memoir, he found, was to establish a modification of (2.13), namely

$$3a'n' \textstyle\int \mathbf{d}' \cdot \delta R' = -\dfrac{ma'n'}{m'an} \cdot 3an \textstyle\int \mathbf{d} \cdot \delta R + (m' - m) \cdot 3a'n' \textstyle\int \mathbf{d}' R', \tag{Z}$$

where R' is of the first order with respect to the perturbing masses, and δR and $\delta R'$ are of the second order. According to Laplace, Plana's results for the second-order perturbations of Jupiter and Saturn [given in (2.23) and (2.25) above] fail to accord with the general relation (Z), so that some error must be supposed to have entered into Plana's calculation.[73]

Plana quickly set about formulating a reply, and by early September had completed his "Note sur un Mémoire de M. de Laplace, ayant pour titer *Sur les deux grandes inégalités de Jupiter et Saturne* imprimé dans le volume de la C.^e des Tems pour l'année 1829." He sent several copies to Paris, for distribution to members of the Institut de France; later the "Note" would be published in the *Memorie* of the Accademia della Scienza of Turin, Vol. 31 (1826–1827), pp. 359–370. The gist of his reply was that Laplace's deduction of (Z) was based on a mistaken premise. Plana concluded that the relation of the two perturbations $\delta\zeta$ and $\delta\zeta'$ was "far from being able to be expressed by the very simple formula Laplace gives..." He reiterated his conviction that it was

[72] Plana's memoir, p. 405.

[73] The information in this paragraph is from Plana 1827, pp. 401–408.

best to compute each of these perturbations directly, rather than attempting to deduce one from the other. He allowed that, despite all the efforts he had made to avoid error, his results might not be completely accurate. "But it appears to me to be demonstrated that the objection raised by Laplace against my coefficients does not rest on a solid basis..."[74]

At last, apparently in late January 1827, Plana received a copy of the *Connaissance des Tems pour l'année 1829*, containing Laplace's memoir on pp. 236–244. But the text of the memoir, Plana now discovered, had been revised so as to eliminate the premise that he, Plana, had taken exception to; thus his Note, already printed for inclusion in the Turin *Memorie*, was rendered irrelevant. Laplace, following a new route, reached essentially the same result as before, but he now wrote it in its integrated form:

$$\delta\zeta' = -\frac{m\sqrt{a}}{m'\sqrt{a'}} \cdot \delta\zeta + (m' - m) \cdot \zeta'. \qquad (Z')$$

Moreover, apparently owing to Plana's critique, he now stipulated that this relation be restricted to terms not having the arguments zero or $5n't - 2nt$. The relation (Z'), like (Z), gave a result greatly differing from Plana's. "It therefore appears to me," Laplace concluded, "that the values of $[\delta\zeta]$ and $[\delta\zeta']$ determined by this learned mathematician have need of correction."[75]

Plana's response to Laplace's new derivation took the form of an "Addition," which he read at the meeting of the Accademia della Scienza on February 4, 1827, and which was published on pp. 401–408 of the same volume of the Turin *Memorie* as that containing his earlier "Note." After remarking on Laplace's introduction of the new restriction in the application of (Z')—a restriction not present in the *Mécanique Céleste*—Plana continued:

> All the efforts I have made to convince myself of the justice of the old or the new relation...employed by M. Laplace have proved fruitless. The relative size of the numerical coefficients...of the neglected terms has no role in the reasonings by which such a relation is established, and it is to be feared that these factors...introduce sensible [empirically detectable] modifications. In the midst of my doubts I prefer the direct method that is expounded in my memoir.... I have preferred to carry out calculations fearsome by their length rather than trust purely theoretical approximations that seem to me to offer only illusory advantages.[76]

For the determining of the square and higher powers of the perturbing force, Plana concluded that the general theory of perturbations was in a very imperfect state of development. He hoped that the difficulty of computing these perturbations might be mitigated by future mathematical advances, and expressed regret that, in the meantime, this difficulty too often deterred astronomers from a deeper study of the consequences hidden in the differential equations implied by universal gravitation.

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[74] *Ibid.*, pp. 367–368.

[75] Laplace (1826), p. 243.

[76] Plana (1827), p. 408.

That the foregoing remarks by Plana played a role in the Berlin Academy's selection of its topic for the contest of 1830, is evident from the Academy's statement of the prize problem, quoted earlier. As Bouvard and Poisson reported in the *Connoissance des Tems pour l'année 1832* (Paris 1829), the problem set by the Academy included "the comparison of the results of the *Mécanique céleste* [for the second-order perturbations of the great inequality of Jupiter and Saturn] with those that M. Plana has published."[77]

Plana's "Addition" presumably reached Paris only after Laplace's death. In any case, a response came, not from Laplace, but from Siméon-Denis Poisson (1781–1840). In a *Mémoire sur plusieurs points de la Mécanique céleste*, published in the *Connaissance des Terns pour l'année 1831* (Paris, 1828, pp. 23–48), Poisson proposed to clarify, "as much as I can, the difficulties and doubts that several results in the *Mécanique céleste* have presented to the estimable mathematician [Plana]."

The problem of the second-order perturbations contributing to the great inequality of Jupiter and Saturn is dealt with on pp. 40–48 of Poisson's memoir. As I explained in connection with equation (2.17) above, these terms arise from a product of two factors, one a partial derivative of R or R' with respect to a variable and the other a perturbation (indicated by the operator δ) of this same variable. Let μt be the argument of the sine or cosine contained in one of these factors, and $\mu' t$ be the argument of the sine or cosine contained in the other. If, said Poisson, we suppose that $n't$ is always preceded by a + sign in both μt and $\mu' t$, then to obtain a product with the argument $\left(5n' - 2n\right) t$, we must have

$$\mu + \mu' = 5n' - 2n. \tag{2.26a}$$

This statement in fact requires correction, as Poisson later recognized; in "Additions" to his memoir, inserted in the *Connaissance des Tems pour l'annee 1832* (Paris, 1829, pp. 94–98),[78] he pointed out that such a product could also be obtained if

$$\mu - \mu' = 5n' - 2n \tag{2.26b}$$

Poisson lists the following combinations as being the only ones in agreement with (2.26a):

$$\begin{array}{lll}
(1) & \mu = 0, & \mu' = 5n' - 2n \\
(2) & \mu = n' - n, & \mu' = 4n' - n \\
(3) & \mu = 2n' - 2n, & \mu' = 3n' \\
(4) & \mu = n', & \mu' = 4n' - 2n \\
(5) & \mu = 2n' - n, & \mu' = 3n' - n \\
(6) & \mu = 3n' - 2n, & \mu' = 2n'
\end{array}$$

[77] "Rapport fait à l'Académie sur un Mémoire de M.G. De Pontécoulant présenté le 16 février 1829, et relatif à la partie des inégalités à longues périodes, résultant de l'action de Jupiter et de la Terre [!], qui dêpend du carré de la force perturbatrice. Commissaires MM. Bouvard et Poisson (1829), *Additions*, p. 24."

[78] See *Ibid.*, p. 98.

These six pairings furnish twelve kinds of terms, since in each pairing, the arguments can be exchanged between the factors. Plana had calculated the terms corresponding to pairings (5) and (2). Poisson proposed to show that, among the terms neglected by Plana, one at least was of comparable size with the terms Plana had considered.

For this purpose, he calculated terms of $\delta\zeta$ and $\delta\zeta'$ belonging to pairing (1). The term he calculated in $\delta\zeta$ employed the argument μ in $\partial R/\partial r$ and the argument μ' in δr; the term he calculated in $\delta\zeta'$ employed the argument μ in $\partial R'/\partial r'$ and the argument μ' in $\delta r'$. (It should be understood that Poisson was by no means computing all the terms belonging to pairing (1): there are eleven more for each of $\delta\zeta$ and $\delta\zeta'$, as the reader will realize on consulting Eq. (2.17) with its six terms in the right-hand member, and remembering that μ and μ' can be exchanged between the factors.) Poisson found:

$$\begin{aligned}\delta\zeta' &= 0''.0878 \cdot \sin(5n't - 2nt + \beta),\\ \delta\zeta &= 8''.3405 \cdot \sin(5n't - 2nt + \beta');\end{aligned}$$

quantities, he urged, which cannot be neglected relative to the values of $\delta\zeta$ and $\delta\zeta'$ found by Plana. "It is therefore to be desired that M. Plana take up his calculations again, and determine the complete values of $\delta\zeta$, and $\delta\zeta'$ corresponding to the argument $5n't - 2nt$, without omitting any part..." To be included among these perturbations were the perturbations of the longitude of the epoch, which form part of the mean longitude of each planet.

(The necessity of including the perturbation of the epoch had been mentioned by Laplace in his memoir; Plana in his "Addition" had carried out the algebraic part of the calculation, promising to publish the numerical results later.)

Poisson argued at length for the correctness of Laplace's formula (Z'), but suggested that it applied only to the sum of the second-order perturbations, not to individual terms entering into that sum. Its lack of agreement with Plana's results was likely due to Plana's omission of important terms.

In the "Additions" that Poisson supplied to his memoir in the *Connaissance des Terns pour i'annee 1832*, he mentioned briefly the combinations of (2.26b), giving as their general formula

$$\mu = 5n' - 2n + i(n' - n),\ \mu' = i(n' - n),$$

where i can be any positive whole number, so that there is an infinite number of such combinations. But this infinity is not troublesome, because

> Among the inequalities that will result, the most sensible, in general, will be those that correspond to the least values of i, and that come, for example, from $6n't - 3nt$ and $n't - nt$, or from $7n't - 4nt$ and $2n't - 2nt$, etc.[79]

As we shall see, however, Poisson's "general formula" fails to include several infinities of combinations, some terms of which may need to be computed.

[79] *Ibid.*

Plana did not take up the challenge that Poisson had laid down. He was immersed in the analytic developments required for his huge three-volume work, *Théorie du mouvement de la lune* (Turin, 1832). However, the extensive calculations that Poisson had indicated as necessary were undertaken by a friend and former pupil of Poisson at the École Polytechnique, Comte Philippe-Gustave Doulcet de Pontécoulant (1795–1874).

To resolve the questions at issue, Pontécoulant remarked, the calculation of all the coefficients that could contribute to the whole must be re-done from the beginning; for there was no a priori way of determining which terms would be sizable, and which could be neglected because of their smallness. But in an initial foray, limiting himself to the terms that Laplace and Plana had considered, Pontécoulant was able to show that the second-order perturbations given by both were in error. Those in the *Mécanique Céleste* were mistaken as to sign; the original numerical computation had been done by Bouvard, who was able to review his manuscript calculations and detect the origin of the mistake.[80] In Plana's case, the detail of the computations had been published; Pontécoulant identified the places where Plana had gone astray, and Plana accepted his corrections. In a memoir read to the Académie des Sciences on February 16, 1829, Pontécoulant thus could assert that, for the particular terms considered, his own and Plana's computations, as well as those of the *Mécanique Céleste*, had been brought into agreement and that they agreed approximately as well with Laplace's relation (Z'), which was thus verified a posteriori.

These particulars were first published in summary form in the *Connaissance des Tems pour l'année 1833* (Paris, 1830), *Additions*, pp. 86–104. Earlier, in May 1829 and February 1830, Pontécoulant had submitted to the Berlin Academy the two parts of an account of what he believed to be all the sizable second-order perturbations having the argument $5n't - 2nt$, and as previously reported, for this he was awarded a prize in the Academy's 1830 contest.[81] His final account of this inequality appeared in 1835 as a 122-page memoir in Volume 6 of the collection called (for brevity) *Savans étrangers*.[82]

Here he examined in detail the cases Poisson had listed as falling under equation (2.26a) above. Poisson had indicated only one case as falling under equation (2.26b); Pontécoulant added three more, yielding four in all:

$$\begin{aligned}
&(1)\ \mu = 5n' - 2n + i(n' - n) \text{ and } \mu' = i(n' - n),\\
&(2)\ \mu = 5n' - 3n + i(n' - n) \text{ and } \mu' = i(n' - n) - n,\\
&(3)\ \mu = 5n' - 4n + i(n' - n) \text{ and } \mu' = i(n' - n) - 2n,\\
&(4)\ \mu = 5n' - 5n + i(n' - n) \text{ and } \mu' = i(n' - n) - 3n.
\end{aligned}$$

Of these types Pontécoulant calculated the terms depending on two combinations of type (1), namely $7n't - 4nt$ with $2n't - 2nt$, and $6n't - 3nt$ with $n't - nt$, and one combination of type (2), namely $7n't - 5nt$ with $2n't - 3nt$. In these three cases, he obtained the following paired results:

80 *Ibid., Additions*, p. 24.

81 *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin aus dem Jahre 1830* (Berlin, 1832), *Historische Einleitung*, pp. 1–11.

82 de Pontécoulant (1835).

$$\begin{Bmatrix} \delta\zeta' = -16''.06895 \cdot \sin\cdot T_5 + 1''.95914 \cdot \cos\cdot T_5 \\ \delta\zeta \;= +6''.62968 \cdot \sin\cdot T_5 - 0''.80830 \cdot \cos\cdot T_5 \end{Bmatrix} \tag{2.27}$$

$$\begin{Bmatrix} \delta\zeta' = +6''.04586 \cdot \sin\cdot T_5 + 2''.23454 \cdot \cos\cdot T_5 \\ \delta\zeta \;= -2''.49438 \cdot \sin\cdot T_s - 0''.92192 \cdot \cos\cdot T_5] \end{Bmatrix} \tag{2.28}$$

$$\begin{Bmatrix} \delta\zeta' = -0''.54808 \cdot \sin\cdot T_5 + 1''.29603 \cdot \cos\cdot T_5 \\ \delta\zeta \;= +0'.22613 \cdot \sin\cdot T_5 - 0''.53472 \cdot \cos\cdot T_5 \end{Bmatrix} \tag{2.29}$$

Of the pair (2.27), Pontécoulant remarked that "they have a sizable value, although no mathematician till now has undertaken to determine them, and their existence seems to have escaped even Laplace's inquiring mind."[83] In (2.27)–(2.29), Pontécoulant used relation (2.13) to deduce $\delta\zeta$ from $\delta\zeta'$ this relation being valid when $A^{(1)}$ and its derivatives are absent from the expansions of R and R'.[84] Other combinations of arguments failing under (2.26b), he opined, would be much smaller; but he allowed that some of them might be significant. Adding together all the perturbations of types (2.26a) and (2.26b) he had calculated, he obtained

$$\begin{Bmatrix} \delta\zeta \;= +3''.76027 \cdot \sin\cdot T_5 + 14''.72268 \cdot \cos\cdot T_5 \\ \delta\zeta' = +10''.72775 \cdot \sin\cdot T_5 - 32''.58055 \cdot \cos\cdot T_5 \end{Bmatrix} \tag{2.30}$$

To these perturbations must be added those of the epochs of the two planets, depending on the same argument. Pontécoulant found these to be

$$\begin{Bmatrix} \delta\varepsilon \;= +1''.310268 \cdot \sin\cdot T_s + 0''.136284 \cdot \cos\cdot T_5 \\ \delta\varepsilon' = -9''.82477 \cdot \sin\cdot T_5 - 1''.02670 \cdot \cos\cdot T_5 \end{Bmatrix} \tag{2.31}$$

Pontécoulant gives a new proof of Laplace's relation (Z'), at the same time pointing out its limitations. It applies to the results given by (2.30), provided that the particular values of $\delta\zeta$ and $\delta\zeta'$ computed for the combination of the argument $5n't - 2nt$ with the argument 0 (yielding a constant term) are first subtracted. The paired values of $\delta\zeta$ and $\delta\zeta'$ in agreement with (2.13) cancel out in (Z'), so that the only way of determining whether all the sizable perturbations in agreement with this relation have been computed is by direct calculation, term by term. Both (2.13) and (Z') are useful relations, (2.13) obviating the detailed computation of one of the terms of any pair to which it applies, and (Z') making possible a check on the calculation of all the terms to which (2.13) does not apply, with the exception of the terms deriving from the pairing of the argument $5n't - 2nt$ with the argument 0.

The entire discussion of second-order perturbations summarized in this section had for its starting point the idea that perturbations proportional to a sine or cosine with the argument $5n't - 2nt$, and having a coefficient with the denominator $(5n' - 2n)^2$, were likely to be sizable. Might not some second-order perturbations with other arguments be of comparable size? Addressing this question term by term would be an infinite

[83] *Ibid.*, p. 487.

[84] The first coefficient in the value of $\delta\zeta$, in (2.29) is given incorrectly in the *Savans étrangers*, but the correct value is given in *Connaissance des Tems pour l'année 1833, Additions*, p. 96.

and therefore incompletable task, but might there not be a doable yet systematic way of addressing it?

This issue assumed increasing importance as the nineteenth century progressed. The refinement in the theoretical derivation was driven by improvements in observational precision. These resulted from new instruments and techniques: Fraunhofer's achromatic objectives, and the large heliometer he produced for the Königsberg Observatory; more accurately machined telescope mountings; more accurately calibrated circles; Bessel's discovery of and accounting for the observer's "personal equation," and later the kymograph for registering transits; the method of least squares for summarizing observational results, and so on. Observational astronomy was approaching an accuracy of thousands of a sexagesimal arcsecond. The mathematical astronomers must achieve a like precision in their predictions. How to proceed?

3 Hansen's *Disquisitiones*

3.1 Introductory

The first two sections of Hansen's treatise review the current state, in 1829, of the theory of perturbations. The following excerpt helps us to understand the context and motive of Hansen's project:

> The illustrious La Place in his *Mécanique Céleste* gives finite expressions for the perturbations of the first order [with respect to the masses] of the true longitude, radius vector, and latitude; of these, the latter two involve the perturbing forces, while the first requires the radii vectores to be calculated.... [He] distributed [these] perturbations...in certain orders with respect to the eccentricities and inclinations, and determined those that are independent of these quantities, and those containing the first powers, accurately. The formulas for the second-order perturbations with respect to the eccentricities and inclinations presuppose the computation of the first order perturbations. He computes only a few terms of the third order, and those by peculiar methods. To obtain the perturbations of the second order with respect to the masses he makes use of the theory of the variation of arbitrary constants, but because of the huge number of terms neglected or truncated, it appears that here he succeeded less well....
> What the astronomer requires are true values of the polar coordinates of the celestial bodies. The primary business [of the celestial mechanician] is to derive the perturbations in such a way that they can be computed with the least labor. For this reason the method of variation of arbitrary constants is esteemed less than those methods by which the perturbations of the coordinates are immediately computed....[85]

In brief, Laplace's finite expressions for the coordinates of the celestial bodies supply systematic results only for perturbations of the first-order with respect to the perturbing forces. Where perturbations of higher order must be taken into account,

[85] *A.N.*, Vol. 7 (1829), Nr. 166, cols. 418–420.

Laplace invoked the method of variation of arbitrary constants; but this is more laborious and in other ways less convenient. This method was developed to its highest point by Lagrange, who gave the time rates of change of the six orbital elements of a celestial body as functions of the elements and of the derivatives of the perturbing function with respect to the elements. The perturbations of the coordinates could then be obtained from the perturbations of the orbital elements with the help of Taylor's theorem. But the coordinates being only three, the method at the outset requires the computation of double that number of quantities. And a deeper difficulty then emerges; for the perturbations of the orbital elements are larger than those of the coordinates, so that smaller quantities have to be determined from the differences of larger ones—a treacherous undertaking. "[A] perfect calculation of the perturbations of the second order and higher orders with respect to the masses requires an industry and patience not small, since each term consists of many parts, of which the larger, even when there is no suspicion of this, may often be hidden."[86]

3.2 Preliminaries

In seeking to develop a new method, Hansen starts from six Lagrangian—type formulas for the time rates of change of the orbital elements.[87] These formulas, when integrated, would give the orbital elements of the perturbed orbit in terms of the time together with certain constants that represent the orbital elements of an unperturbed ellipse. In practice, the constants would be evaluated by means of observations, and the Lagrangian formulas, when integrated, would yield the elements with a first-order correction for perturbations. Higher-order corrections could then be obtained by successive approximations. As Hansen points out, however, the Lagrangian-style formulas can be transformed so as to yield, not the orbital elements, but any functions of the orbital elements and the time, and among such functions are the coordinates of the perturbed body, for instance, the longitude and the radius vector in the orbital plane. Hansen appears to be the first to attempt to capitalize on this possibility. The desirable thing, he observes, would be formulas similar to those provided by Kepler's laws for a single planet orbiting about the Sun, where the longitude and radius vector are ultimately dependent only on the mean anomaly (see Eqs. 2.1a, 2.1b, 2.1c above).

To proceed in this direction, Hansen takes as his starting point the following stipulations.

Let x, y, z be orthogonal coordinates of the perturbed body m, having their origin in the center of gravity of the primary body, and directed arbitrarily in space. Further, let

a = the semi-major axis of the orbit of m in astronomical units;
n = the mean motion of m in fractions of a circle per year;

[86] *Ibid.*, col. 421.

[87] The equations Hansen starts from are derived from those given by Lagrange in his "Mémoire sur la théorie des variations des éléments des planètes...," read on 22 August 1808 at the Institut de France, and first published in *Mémoires de la première classe de l'Institut de France, année 1808*. See Lagrange (1873), pp. 713–768, where the formulas cited by Hansen are given on p. 760.

ae = the eccentricity of the orbit of m;
θ = the longitude of the ascending node in the xy plane;
i = the inclination of the orbit to the same plane;
ε = the mean longitude at epoch, expressed as θ plus the orbital arc from the ascending node to the mean position of m at epoch;
ϖ = the longitude of the perihelion of the orbit, expressed as θ plus the orbital arc from the ascending node to the perihelion;
r = the radius vector in the orbital plane;
v = the true orbital longitude of m (namely, θ plus the orbital arc from the ascending node to m);
m = the mass of m in parts of the mass of the primary body as the unit, and $\mu = 1+m$.

By Kepler's third law, we may put $a^3n^2 = \mu \approx 1$.

The letters a, n, e, θ, i, ε, ϖ, r, v, with prime marks affixed, designate the corresponding quantities for a second planet m'.

The distance between m and m' and the perturbing function for a single perturbed planet m perturbed by m' are given, respectively, by:

$$\Delta = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2};$$
$$\Omega = \frac{m'}{\mu}\left\{\frac{1}{\Delta} \frac{xx'+yy'+zz'}{r'^3}\right\}$$

The disturbing function Ω is a function of the true longitudes of the perturbed and the perturbing planets, their radii vectores, the mutual inclination of their orbital planes, and the position of the nodal line in which their planes intersect. As Hansen will explain later, this function is to be expanded as a series of sinusoidal terms having the mean anomalies of m and m' for arguments. The partial derivatives of Ω will also be expressible as series in terms of the mean anomalies, and hence of the time.

With these definitions, the Lagrangian-style equations with which Hansen begins may be written as follows, where "ln" signifies the logarithm (Hansen, with an eye to computation, introduces logarithms from the start):

$$\frac{\mathrm{d}\ln a}{\mathrm{d}t} = 2na\frac{\partial\Omega}{\partial\varepsilon};$$
$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -2na^2\frac{\partial\Omega}{\partial a} + na\frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{\partial\Omega}{\partial e} + na\frac{\tan(i/2)}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial i};$$
$$\frac{\mathrm{d}e}{\mathrm{d}t} = -na\frac{\sqrt{1-e^2}}{e}\frac{\partial\Omega}{\partial\varpi} - na\frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{\partial\Omega}{\partial\varepsilon};$$
$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = na\frac{\sqrt{1-e^2}}{e}\frac{\partial\Omega}{\partial e} + na\frac{\tan(i/2)}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial i}. \tag{3.1a}$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{na}{\sqrt{1=e^2}}\left\{\csc i\frac{\partial\Omega}{\partial\varepsilon} + \tan(i/2)\frac{\partial\Omega}{\partial\varpi} + \tan(i/2)\frac{\partial\Omega}{\partial\varepsilon}\right\};$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = na\frac{\csc\ i}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial i}. \tag{3.1b}$$

I have separated the six equations into two groups. The equations in (3.1a) give the time rates of change of (ln a), e, ε, ϖ, the elements most directly involved in determining the motion of m in the instantaneous orbital plane. The equations in (3.1b) give the time rates of change of θ and i, the elements most directly involved in determining the position of the instantaneous plane. The two groups are not strictly separable; for instance, both $\mathrm{d}\varepsilon/\mathrm{d}t$ and $\mathrm{d}\varpi/\mathrm{d}t$ involve the inclination i. Hansen focuses initially on the motion in the instantaneous plane, and in the first approximation ignores the terms in $\mathrm{d}\varepsilon/\mathrm{d}t$ and $\mathrm{d}\varpi/\mathrm{d}t$ that depend on i.

3.3 Longitude and radius vector in the instantaneous orbital plane

Hansen now introduces two functions λ and ρ of the elements a, e, ε, ϖ, and an initially indeterminate quantity τ. In λ, the elements a, e, ε, and ϖ are to be conjoined with τ in precisely the same way as, in the true longitude v of a constant elliptical orbit, they are conjoined with the time, t. Thus,

$$n\tau + \varepsilon - \varpi = E - e\sin E,$$
$$\tan\frac{\lambda - \varpi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \tag{3.2}$$

where E is the eccentric anomaly [compare equations (2.1a), (2.1b)]. The variable λ can be expanded in a series as a function of the quantity $n\tau + \varepsilon - \varpi$, which becomes the mean anomaly when τ is changed into t. (For the comparable expansion of v, see equation [2.4].) As for ρ, the elements a, e, ε, and ϖ are to be conjoined with τ in the same way as, in the radius vector r of the fixed elliptical orbit, they are conjoined with the time, t. Thus,

$$\rho = \frac{a\left(1 - e^2\right)}{1 + e\,\cos(\lambda - \varpi)} \tag{3.3}$$

where ρ is related to τ through λ, as determined in (3.2), and can be expanded in a series analogous to the series for r given in (2.3).

The relations just mentioned determine how λ and ρ are algebraically related to τ, but meanwhile the orbital elements can undergo change in accordance with (3.1a) and (3.1b). By the rules of partial differentiation, the variations of λ and ρ caused by the variations of the elements are

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = a\frac{\partial\lambda}{\partial a}\frac{\mathrm{d}\ln a}{\mathrm{d}t} + \frac{\partial\lambda}{\partial\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{\partial\lambda}{\partial e}\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\partial\lambda}{\partial\varpi}\frac{d\varpi}{\mathrm{d}t}; \tag{3.4}$$
$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}t} = a\frac{\partial\ln\rho}{\partial a}\frac{\mathrm{d}\ln a}{\mathrm{d}t} + \frac{\partial\ln\rho}{\partial\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{\partial\ln\rho}{\partial e}\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\partial\ln\rho}{\partial\varpi}\frac{\mathrm{d}\varpi}{\mathrm{d}t}. \tag{3.5}$$

Each term on the right-hand side of (3.4 or (3.5) is composed of two factors, the first a derivative of λ or In ρ written with curly deltas and the second a time rate of change of an orbital element as given by (3.1a). The derivatives written with curly deltas are partial derivatives: the differentiation is with respect to a designated variable in λ or ln ρ, every other variable being held constant. (Hansen sometimes uses parentheses enclosing a derivative to indicate that the differentiation is partial; but he does not always give parentheses this signification.) The derivatives $\mathrm{d}\lambda/\mathrm{d}t$ and $(\mathrm{d}\ln\rho)/\mathrm{d}t$ in (3.4) and (3.5) give the variations of λ and $(\ln \rho)$ with t only insofar as the elements a, e, ε, and ϖ vary with t. In the integration of (3.4) and (3.5) with respect to t, arbitrary constants must be added. In the integration of (3.4), Hansen adds as arbitrary constant the true longitude in a constant elliptical orbit, with elements determined observationally for the epoch. After integration, substituting the variable t for the quantity τ, we obtain an expression for λ incorporating both sorts of variation with time that λ undergoes, the variation caused by the variation of the elements, and the variation caused by the motion of m on its varying orbit.

Hansen has yet to introduce his key idea: the construal of λ as a function of a single variable ζ such that, when ζ is substituted for τ in (3.2), we obtain the true perturbed value of λ. Evidently ζ must be a function of both t and τ, and be equal to τ when $t = 0$ (that is, at the epoch); moreover, its determination must incorporate the integration of (3.4) already described. If we view λ as a function of the single variable ζ, its variation with t will be given by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial\lambda}{\partial\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t} \tag{3.6}$$

Here $\mathrm{d}\lambda/\mathrm{d}t$ is to be understood as the variation arising solely from the variation of the orbital elements, as given in (3.4); hence, the expressions for $\mathrm{d}\lambda/\mathrm{d}t$ in (3.4) and (3.6) can be equated:

$$\frac{\partial\lambda}{\partial\zeta}\frac{d\zeta}{\mathrm{d}t} = a\frac{\partial\lambda}{\partial a}\frac{\mathrm{d}\ln a}{\mathrm{d}t} + \frac{\partial\lambda}{\partial\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{\partial\lambda}{\partial e}\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\partial\lambda}{\partial\varpi}\frac{\mathrm{d}\varpi}{\mathrm{d}t} \tag{3.7}$$

To obtain an expression for $\mathrm{d}\zeta/\mathrm{d}t$, Hansen writes

$$\frac{\partial\lambda}{\partial\tau} = \frac{\partial\lambda}{\partial\zeta}\frac{\partial\zeta}{\partial\tau}, \text{ implying } \frac{\partial\lambda}{\partial\zeta} = \frac{\partial\lambda/\partial\tau}{\partial\zeta/\partial\tau}$$

Here $\partial\lambda/\partial\tau$ is a partial derivative determined by the relation between λ and τ defined by the equation (3.2); but what about $\partial\zeta/\partial\tau$? As yet, we have no formula for ζ that could be differentiated partially with respect to τ; as we shall see, Hansen will develop such a formula through successive approximations, giving $\partial\zeta/\partial\tau$ the value 1 if no perturbations have as yet occurred, and allowing it to diverge from 1 as perturbations occur. Using the foregoing relation to eliminate $\partial\lambda/\partial\zeta$, from (3.7), Hansen obtains

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \left\{\frac{\partial\zeta/\partial\tau}{\partial\lambda/\partial\tau}\right\}\left\{a\frac{\partial\lambda}{\partial a}\frac{\mathrm{d}\ln a}{\mathrm{d}t} + \frac{\partial\lambda}{\partial\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{\partial\lambda}{\partial e}\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\partial\lambda}{\partial\varpi}\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right\}. \tag{3.8}$$

Since the value of ζ, if the perturbing forces vanish, is τ, the value of the arbitrary constant to be added to (3.8) when integrated will also be τ. If after integration we change τ into t, we shall have a function, which, substituted for t in the purely elliptical expression for v, gives the perturbed value of v. This function—the integral of (3.8) with τ changed into t—will be designated by the letter z.

Turning to the integration of (3.5), Hansen proposes to calculate the arbitrary constant, to be added after integration, from the purely elliptical formulas with z in place of t, or in effect, with the true perturbed longitude. But the value of z has been determined so as to give, in the primordial ellipse—the ellipse with orbital elements determined at epoch—the true perturbed value of λ, not the true perturbed value of ($\ln \rho$). An error is thus introduced.

To correct for it, Hansen subtracts the quantity $\frac{\partial \ln \rho}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t}$ from the right-hand side of (3.5), so that the integral of that side is reduced by $\int \frac{\partial \ln \rho}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\mathrm{d}t$. Now ($\ln \rho$), Hansen points out, contains τ in two ways, implicitly through ζ, and also through its dependence (3.3) on λ and so on (3.2) (Hansen speaks of the latter dependence as explicit.) He lets $\ln(\rho)$ designate the part of ($\ln \rho$) that contains τ in the latter more explicit way, and so finds

$$\frac{\partial \ln \rho}{\partial \tau} = \frac{\partial \ln \rho}{\partial \zeta}\frac{\partial \zeta}{\partial \tau} + \frac{\partial \ln(\rho)}{\partial \tau},$$

whence

$$\frac{\partial \ln \rho}{\partial \zeta} = \left\{\frac{\partial \ln \rho}{\partial \tau} - \frac{\partial \ln(\rho)}{\partial \tau}\right\} \div \frac{\partial \zeta}{\partial \tau}$$

This expression, multiplied by $\mathrm{d}\zeta/\mathrm{d}t$, gives the quantity to be subtracted from (3.5); with a substitution into the first term from (3.8), Hansen obtains it in the form

$$\frac{\frac{\partial \ln \rho}{\partial \tau}}{\frac{\partial \lambda}{\partial \tau}}\left\{a\frac{\partial \lambda}{\partial a}\frac{\mathrm{d}\ln a}{\mathrm{d}t} + \frac{\partial \lambda}{\partial \varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{\partial \lambda}{\partial e}\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\partial \lambda}{\partial \varpi} + \frac{\mathrm{d}\varpi}{\mathrm{d}t}\right\} - \frac{\frac{\partial \ln(\rho)}{\partial \tau}}{\frac{\partial \zeta}{\partial \tau}}\frac{\mathrm{d}\zeta}{\mathrm{d}t}. \tag{3.9}$$

When (3.9) is subtracted from (3.5), the remainder gives the part of $[\mathrm{d}(\ln \rho)/\mathrm{d}t]$ that contains τ explicitly, its indirect dependence on τ through ζ having been eliminated:

$$\begin{aligned}\frac{\mathrm{d}\ln \rho}{\mathrm{d}t} = &\left\{a\frac{\partial \ln \rho}{\partial a} - a\frac{\partial \lambda}{\partial a}\frac{\frac{\partial \ln \rho}{\partial \tau}}{\frac{\partial \lambda}{\partial \tau}}\right\}\frac{\mathrm{d}\ln \rho}{\mathrm{d}t} + \left\{\frac{\partial \ln \rho}{\partial \varepsilon} - \frac{\partial \lambda}{\partial \varepsilon}\frac{\frac{\partial \ln \rho}{\partial \tau}}{\frac{\partial \lambda}{\partial \tau}}\right\}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\\ &+\left\{\frac{\partial \ln \rho}{\partial e} - \frac{\partial \lambda}{\partial e}\frac{\frac{\partial \ln \rho}{\partial \tau}}{\frac{\partial \lambda}{\partial \tau}}\right\}\frac{\mathrm{d}e}{\mathrm{d}t} + \left\{\frac{\partial \ln \rho}{\partial \varpi} - \frac{\partial \lambda}{\partial \varpi}\frac{\frac{\partial \ln \rho}{\partial \tau}}{\frac{\partial \lambda}{\partial \tau}}\right\}\\ &\times\frac{\mathrm{d}\varpi}{\mathrm{d}t} + \frac{\frac{\partial \ln(\rho)}{\partial \tau}}{\frac{\partial \zeta}{\partial \tau}}\frac{\mathrm{d}\zeta}{\mathrm{d}t}.\end{aligned} \tag{3.10}$$

In a further transformation, Hansen makes a number of substitutions in (3.8) and (3.10), using relations among the partial differentials of the longitude and radius vector in an elliptical orbit. They are derivable from the series exhibiting v and r as functions of the mean anomaly. Thus, let v be represented by

$$v = nt + \varepsilon + f(nt + \varepsilon - \varpi). \quad \text{(Compare equation (2.4).)}$$

Then, supposing this ellipse to become variable, and remembering that n and a, whether they vary or not, are always related by the equation $a^3n^2 \approx 1$, so that $\mathrm{d}n/\mathrm{d}a = -3n/2a$, we find the derivatives of v with respect to t, a, ε, and ϖ to be

$$\begin{aligned}
\frac{\partial v}{\partial t} &= n + nf'(nt+\varepsilon - \varpi), \\
\frac{\partial v}{\partial a} &= t\frac{\mathrm{d}n}{\mathrm{d}a} + t\frac{\mathrm{d}n}{\mathrm{d}a}f'(nt+\varepsilon - \varpi). \\
\frac{\partial v}{\partial \varepsilon} &= 1 + f'(nt+\varepsilon - \varpi). \\
\frac{\partial v}{\partial \varpi} &= -f'(nt+\varepsilon - \varpi).
\end{aligned}$$

By eliminating $f'(nt + \varepsilon - \varpi)$ between the first and the other three of these equations, and substituting $-(3/2)a^{-5/2}$ for $\mathrm{d}n/\mathrm{d}a$, we obtain formulas for $\partial v/\partial a$, $\partial v/\partial \varepsilon$, and $\partial v/\partial \varpi \mu$ in terms of $\partial v/\partial t$, and an entirely analogous set of formulas applies to λ and τ:

$$\begin{aligned}
&(A)\quad a\frac{\partial v}{\partial a} = -\frac{3}{2}t\frac{\partial v}{\partial t}; \quad (A')\quad a\frac{\partial \lambda}{\partial a} = -\frac{3}{2}\tau\frac{\partial \lambda}{\partial \tau}; \\
&(B)\quad \frac{\partial v}{\partial \varepsilon} = \frac{1}{n}\frac{\partial v}{\partial t}; \quad (B')\quad \frac{\partial \lambda}{\partial \varepsilon} = \frac{1}{n}\frac{\partial \lambda}{\partial \tau}; \\
&(C)\quad \frac{\partial v}{\partial \varpi} = 1 - \frac{1}{n}\frac{\partial v}{\partial t}; \quad (C')\quad \frac{\partial \lambda}{\partial \varpi} = 1 - \frac{1}{n}\frac{\partial \lambda}{\partial \tau}.
\end{aligned} \tag{3.11a}$$

In a similar way, we can represent ($\ln r$) by

$$\ln r = \ln a + \ln[1 + F(nt + \varepsilon - \varpi)]. \quad \text{[Compare equation (2.3).]}$$

Taking the partial derivatives of this expression with respect to t, a, ε, and ϖ, then eliminating $F'(nt+\varepsilon-\varpi)/[1+F(nt+\varepsilon-\varpi)]$ between the first and each of the other three of the resulting equations, we obtain expressions for $(\partial \ln r)/\partial a$, $(\partial \text{ in } r)/\partial \varepsilon$, and $(\partial \ln r)/\partial \varpi$ in terms of $(\partial \ln r)/\partial t$, and entirely analogous expressions apply to the corresponding derivatives of $\ln \rho$:

$$\begin{aligned}
&(D)\quad a\frac{\partial \ln r}{\partial a} = 1 - \frac{3}{2}t\frac{\partial \ln r}{\partial t}; \quad (D')\quad a\frac{\partial \ln \rho}{\partial a} = 1 - \frac{3}{2}\tau\frac{\partial \ln \rho}{\partial \tau}; \\
&(E)\quad \frac{\partial \ln r}{\partial \varepsilon} = \frac{1}{n}\frac{\partial \ln r}{\partial t}; \quad (E)\quad \frac{\partial \ln \rho}{\partial \varepsilon} = \frac{1}{n}\frac{\partial \ln \rho}{\partial \tau}; \\
&(F)\quad \frac{\partial \ln r}{\partial \varpi} = -\frac{1}{n}\frac{\partial \ln r}{\partial t}; \quad (F')\quad \frac{\partial \ln \rho}{\partial \varpi} = \frac{1}{n}\frac{\partial \ln \rho}{\partial \tau}.
\end{aligned} \tag{3.11b}$$

Substituting the above expressions into (3.8) and (3.10) and rearranging terms so that derivatives with respect to ζ appear only in the left-hand members, we obtain

$$\frac{\frac{\partial \zeta}{\partial t}}{\frac{\partial \zeta}{\partial \tau}} = \left\{ -\frac{3}{2}\tau \frac{\mathrm{d}\ \ln\ a}{\mathrm{d}t} + \frac{1}{n}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{\frac{\partial \lambda}{\partial e}}{\frac{\partial \lambda}{\partial \tau}}\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{1}{\frac{\partial \lambda}{\partial \tau}}\frac{\mathrm{d}\varpi}{\mathrm{d}t} - \frac{1}{n}\frac{\mathrm{d}\varpi}{\mathrm{d}t} \right\}, \tag{3.12}$$

$$\frac{\mathrm{d}\ \ln\ \rho}{\mathrm{d}t} - \frac{\frac{\partial\ \ln(\rho)}{\partial \tau}}{\frac{\partial \zeta}{\partial \tau}}\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \frac{\mathrm{d}\ \ln a}{\mathrm{d}t} + \frac{\frac{\partial\ \ln \rho}{\partial t}\frac{\partial \lambda}{\partial \tau} - \frac{\partial\ \ln \rho}{\partial \tau}\frac{\partial \lambda}{\partial e}}{\frac{\partial \lambda}{\partial \tau}}\frac{\mathrm{d}e}{\mathrm{d}t} - \frac{\frac{\partial\ \ln \rho}{\partial \tau}}{\frac{\partial \lambda}{\partial \tau}}\frac{\mathrm{d}\varpi}{\mathrm{d}t}. \tag{3.13}$$

In the left-hand member of (3.12), I have followed Hansen in changing the "d's" in $\mathrm{d}\zeta/\mathrm{d}t$ (as this derivative appears in 2.8) into curly deltas; What does the change signify? As yet we have no algebraic formula for ζ, and indeed, we shall find it to be determinable only by successive approximations. Presumably $\partial\zeta/\partial t$ means the variation of ζ with t, that is solely due to the variation of the orbital elements, the dependence of ζ on t through τ having been set aside. In (3.8), $\mathrm{d}\zeta/\mathrm{d}t$ means exactly the same thing before integration, but in the integration, with the change of τ to t, we would obtain ζ's total dependence on t, both through the variation of the elements and through τ. The use of curly deltas in (3.12) signifies that ζ prior to integration is simultaneously a function of t and τ.

Equations (3.12) and (3.13), being functions of t and τ, can be differentiated with respect to either of these variables, provided that all the quantities varying with each are simultaneously treated as variable. In differentiations with respect to t, all the orbital elements must be supposed variable. In differentiations with respect to τ, the quantities λ and ρ must be supposed variable.

Hansen proceeds to differentiate (3.12) with respect to τ, obtaining

$$\frac{\mathrm{d}\cdot\left\{\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial\tau}\right\}}{\mathrm{d}\tau} = -\frac{3}{2}\frac{\mathrm{d}\ \ln\ a}{\mathrm{d}t} + \frac{\frac{\partial^2\lambda}{\partial e\partial\tau}\frac{\partial\lambda}{\partial\tau} - \frac{\partial^2\lambda}{\partial\tau^2}\frac{\partial\lambda}{\partial e}}{\left[\frac{\partial\lambda}{\partial\tau}\right]^2}\frac{\mathrm{d}e}{\mathrm{d}t} - \frac{\frac{\partial^2\lambda}{\partial\tau^2}}{\left[\frac{\partial\lambda}{\partial\tau}\right]^2}\frac{\mathrm{d}\varpi}{\mathrm{d}t}. \tag{3.14}$$

When (3.14) is integrated with respect to τ, the arbitrary constant to be added will be a function of τ so determined that, when τ is changed into t, the whole integral vanishes. This is necessary because, before the orbital elements have undergone change, that is, before integration with respect to t, $\partial\zeta/\partial\tau = 1$, but $\partial\zeta/\partial t = 0$, so that the bracketed expression on the left of (3.14) is equal to zero. After integration with respect to t, $\partial\zeta/\partial t$ is no longer equal to zero; only then, normally, is τ changed into t.

In the further development, it is the integration of (3.14) and (3.13) that Hansen will be concerned with. We observe that ζ occurs in (3.13) only in the form of the ratio obtained by integrating (3.14) with respect to τ.

Once (3.14) is integrated with respect to τ, the way of obtaining $\partial\zeta/\partial\tau$ by successive approximations becomes discernible. In the first approximation, we can put $\partial\zeta/\partial\tau = 1$; we thus obtain an expression for $\partial\zeta/\partial t$ that can be integrated with respect to t to give an expression—the first approximation—for ζ. The latter expression can be differentiated partially with respect to τ to give an improved value of $\partial\zeta/\partial\tau$. This value can be used in the second approximation, whence improved values of $\partial\zeta/\mathrm{d}t$ and so of ζ, and $\partial\zeta/\partial\tau$ will be obtained, and so on. The successive values of $\partial\zeta/\partial\tau$ will have the form: 1 + terms multiplied by the mass of the perturbing planet. This is so because the added terms are proportional to the time rates of change of orbital elements, and these are linearly dependent on partial derivatives of Ω, and Ω is proportional to the mass of the perturbing planet.

In further preparation of (3.14) and (3.13) for integration, Hansen takes the following steps. First, in the expressions for the time rates of change of the orbital elements (3.1a), the partial derivatives of Ω with respect to a, e, ε, and ϖ are replaced by partial derivatives of Ω with respect to v and r, in accordance with the formulas

$$\begin{aligned}\frac{\partial\Omega}{\partial a} &= \frac{\partial\Omega}{\partial v}\frac{\partial v}{\partial a} + \frac{\partial\Omega}{\partial r}\frac{\partial r}{\partial a},\\ \frac{\partial\Omega}{\partial e} &= \frac{\partial\Omega}{\partial v}\frac{\partial v}{\partial e} + \frac{\partial\Omega}{\partial r}\frac{\partial r}{\partial e},\\ \frac{\partial\Omega}{\partial \varepsilon} &= \frac{\partial\Omega}{\partial v}\frac{\partial v}{\partial \varepsilon} + \frac{\partial\Omega}{\partial r}\frac{\partial r}{\partial \varepsilon},\\ \frac{\partial\Omega}{\partial \varpi} &= \frac{\partial\Omega}{\partial v}\frac{\partial v}{\partial \varpi} + \frac{\partial\Omega}{\partial r}\frac{\partial r}{\partial \varpi}.\end{aligned} \tag{3.15a}$$

Hansen's basic idea is to express Ω as a function of the longitudes, radii vectores, inclinations, and nodes; here, however, he neglects the dependence of $\partial\Omega/\partial\varepsilon$ and $\partial\Omega/\partial\varpi$ on the inclination i—a neglect he proposes to redress in the second approximation.

Next, he makes substitutions for the derivatives of v and ($\ln r$) with respect to t and e, and for the analogous derivatives of λ and ($\ln \rho$) with respect to τ and e. The equivalences employed are derivable from the elementary formulas for elliptical motion (as enumerated, for instance, in Gauss's *Theoria Motus*[88]).

$$(H)\quad \frac{\partial v}{\partial t} = n\frac{a^2}{r^2}\sqrt{1-e^2};\quad (H')\quad \frac{\partial \lambda}{\partial \tau} = n\frac{a^2}{\rho^2}\sqrt{1-e^2};$$

$$(J)\quad \frac{\partial v}{\partial e} = \left\{\frac{1}{1-e^2} + \frac{a}{r}\right\}\sin(v-\varpi);$$

$$(J')\,\frac{\partial \lambda}{\partial e} = \left\{\frac{1}{1-e} + \frac{a}{\rho}\right\}\sin(\lambda-\varpi); \tag{3.15b}$$

$$(K)\quad \frac{\partial \ln r}{\partial t} = n\frac{a}{r}\frac{e\,\sin(v-\varpi)}{\sqrt{1-e^2}};\qquad (K')\quad \frac{\partial \ln \rho}{\partial \tau} = n\frac{a}{\rho}\frac{e\,\sin(v-\varpi)}{\sqrt{1-e^2}};$$

$$(L)\quad \frac{\partial \ln r}{\partial e} = -\frac{a}{r}\cos(v-\varpi);\qquad (L')\quad \frac{\partial \ln \rho}{\partial e} = -\frac{a}{\rho}\cos(v-\varpi), \tag{3.15c}$$

All the derivatives here are partial derivatives, but note that in (L), r varies not only with e as it explicitly occurs in the formula defining r in terms of the true anomaly $v-\varpi$ (for example (2.1c)), but also through the variation of v with e, as shown in (J). An analogous statement applies to (L').

The second-order partial derivatives of λ in (3.14) are easily obtained by differentiation of (H') and (J') with respect to τ, followed by substitution of the value of $\partial\rho/\partial\tau$

[88] See Gauss (1963), p. 9.

derived from (K'). When all these substitutions are made, (3.14) and (3.13) take the forms

$$\frac{\mathrm{d}.\left[\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial\tau}\right]}{\mathrm{d}\tau} = \frac{an}{\sqrt{1-e^2}}\left\{\frac{2\rho}{r}\cos(v-\lambda) - 1 + \frac{2\rho}{a\left(1-e^2\right)}\left[\cos(v-\lambda)-1\right]\right\}\frac{\partial\Omega}{\partial v}$$
$$+\frac{2an}{\sqrt{1-e^2}}\frac{\rho}{r}\sin(v-\lambda)\,r\frac{\partial\Omega}{\partial r} \tag{3.16}$$

$$\frac{\ln(\rho)}{\mathrm{d}t} - \frac{\frac{\partial\ln(\rho)}{\partial\tau}}{\frac{\partial\zeta}{\partial\tau}}\frac{\partial\zeta}{\partial t}$$
$$= -\frac{an}{\sqrt{1-e^2}}\left\{\frac{\rho}{r}\cos(v-\lambda) - 1 + \frac{\rho}{a(1-e^2)}\left[\cos(v-\lambda)-1\right]\right\}\frac{\partial\Omega}{\partial v}$$
$$-\frac{an}{\sqrt{1-e^2}}\frac{\rho}{r}\sin(v-\lambda)\cdot r\frac{\partial\Omega}{\partial r} \tag{3.17}$$

In the *Disquisitiones*, Hansen does not undertake to integrate these equations, but indicates how it may be done. The disturbing function Ω must first be developed as a series in terms of the sines and cosines of the multiples of the mean anomalies $nt + \varepsilon - \varpi$ and $n't + \varepsilon' - \varpi'$ of the perturbed and perturbing planets. In the first approximation, a, n, e, ε, and ϖ, and the corresponding orbital elements of m' are given their unperturbed elliptical values. In the integral of (3.16) with respect to τ, $\partial\zeta/\partial\tau$ is set equal to 1, and the resulting equation is integrated with respect to t to give an expression for ζ, and therewith, numbers being substituted for the constants, a value for z. Given z and hence nz, the corresponding values of λ and ($\ln\rho$) can be calculated; the latter, we recall, is the arbitrary constant to be added in the integration of (3.17). In this integration, the term $-[\frac{\partial\ln(\rho)/\partial\tau}{\partial\zeta/\partial\tau}]\frac{\partial\zeta}{\partial t}$ can be neglected in the first approximation, since it is of the second order; in later approximations, it can be calculated from the expressions for ζ and $\ln(\rho)$ obtained in the preceding approximation.

The integration of (3.17) does not require a special calculation, because of a relation between Eqs. (3.16) and (3.17). This relation, which can be verified by a direct comparison of the two equations, is as follows:

$$\frac{\mathrm{d}\ln(\rho)}{\mathrm{d}t} - \frac{\frac{\partial\ln(\rho)}{\partial\tau}}{\frac{\partial\zeta}{\partial\tau}}\frac{\partial\zeta}{\partial t} = \frac{an}{2\sqrt{1-e^2}}\frac{\partial\Omega}{\partial v} - \frac{\mathrm{d}\cdot\left[\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial\tau}\right]}{2\mathrm{d}\tau}. \tag{3.18}$$

A first approximation to the perturbations of $\ln(\rho)$, Hansen points out, can be obtained if (3.18) is integrated with respect to t, with $\partial\zeta/\partial\tau$ set equal to 1, and the second term on the left ignored. The second term on the right of (3.18), when $\partial\zeta/\partial\tau$ is set equal to 1, becomes $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial\zeta}{\partial\tau})$; here $\partial\zeta/\partial\tau$ is to be obtained by differentiating ζ, as computed according to (3.16), and before the change of τ into t, with respect to τ; τ is then changed to t. The same procedure can be used in the second and higher approximations, but with the difference that the second term on the left of (3.18) must be calculated;

the fact that $\partial\zeta/\partial\tau$ now differs from 1 must be taken into account both there and in the second term on the right.

The preceding summary gives the main result of the *Disquisitiones* for the calculation of the longitude and radius vector in the instantaneous plane. This result, we emphasize, is a deductive consequence of the Lagrangian formulas, the only departures from deductive rigor being such as successive approximations will progressively correct for. The procedure for successive approximations is explicitly spelled out, leaving no room for intuitive guesses of the Laplacian kind.

It will be noted that Eqs. (3.16) and (3.18) can be used to check each other. This relation and another conditional equation that we shall exhibit later enable Hansen to confirm his calculation of the numerical values of the perturbational terms resulting from his theory. As Hansen will later urge, this possibility of checking his numerical calculations along an independent route is a major advantage of his method as compared with earlier methods. For this possibility, he informs us, he is indebted to the introduction of the quantity τ.

3.4 Position of the instantaneous orbital plane

The treatment of this topic in the *Disquisitiones* is flawed. Hansen discovered the error in the course of working on his *Untersuchung über die gegenseitigen Störung des Jupiters und Saturns*,[89] and there introduced an appropriate revision.[90] Here we review the valid part of the earlier treatment, which Hansen presupposes in the later work, and indicate where the original treatment went astray.

The position of the orbital plane is determined by i, the inclination of that plane to the fixed xy plane, and θ, the position of its ascending node on the xy plane. Under perturbation, these orbital elements become variable, in accordance with the equations (3.1b) for $\mathrm{d}i/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$. These time derivatives depend in turn on the partial derivatives of the disturbing function Ω with respect to ε and ϖ. In the *Disquisitiones* , Hansen seeks, in a first approximation, to express the orientation of the orbital plane independently of $\partial\Omega/\partial\varepsilon$ and $\partial\Omega/\partial\varpi$.

He begins by introducing new variables:

$$p = \sin i \cdot \sin\theta, \quad q = \sin i \cdot \cos\theta \tag{3.19}$$

(A similar change of variables had first been introduced by Lagrange in 1774;[91] but Lagrange had used $\tan i$ rather than $\sin i$ in the expressions for p and q. Laplace followed Lagrange in this respect.) For the perturbing planet m', the corresponding variables p', q' are defined analogously. From equations (3.1b) and the rules of differentiation, it follows that

[89] Hansen (1831), *Vorwort*, pp. x–xi.

[90] *Ibid.*, Art.10, pp. 37–43.

[91] *Mém. de l'Acad. des Sci.* [Paris], année 1774; *Oeuvres de Lagrange*, 6, 635–709.

$$\frac{dp}{dt} = \frac{na\cos i}{\sqrt{1-e^2}}\left\{\frac{\partial\Omega}{\partial q} - \frac{p}{1+\cos i}\left(\frac{\partial\Omega}{\partial\varpi} + \frac{\partial\Omega}{\partial\varepsilon}\right)\right\},$$
$$\frac{dq}{dt} = -\frac{na\cos i}{\sqrt{1-e^2}}\left\{\frac{\partial\Omega}{\partial p} + \frac{q}{1+\cos i}\left(\frac{\partial\Omega}{\partial\varpi} + \frac{\partial\Omega}{\partial\varepsilon}\right)\right\}, \tag{3.20}$$

The position of the xy plane at the epoch is a matter of arbitrary choice. Hansen proposes to choose it so that it coincides initially with the orbital plane of the perturbed planet. Then $i = 0$, whence $p = q = 0$, and equations (3.20) reduce to

$$\frac{dp}{dt} = \frac{an}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial q}, \quad \frac{dq}{dt} = -\frac{an}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial p} \tag{3.21}$$

Equations of this form, Hansen asserts, can always be applied: since the quantity Ω is independent of the position of the plane to which the coordinates are referred, we are free to choose this plane at any stage of our computations so as to render evanescent the variables p, q of the planet under investigation. The advantage is a reduction in computational labor.

If planet m is perturbed by planet m', the perturbations of the orbital plane of m will be most directly determined by reference to the mutual inclination I (capital I) of the orbital planes of m and m', and the ascending node Θ of the orbital plane of m' on the orbital plane of m; when by choice of the initial conditions i is made equal to zero, $i' \to 1$ and $\theta' \to \Theta$. Let $P = \sin I \sin\Theta$, $Q = \sin I\cos\Theta$.

To determine P, Q in a general way in terms of p, q and p', q', we revert to an arbitrary orientation of the xy plane. Consider the spherical triangle formed on the unit sphere by the xy plane and the orbital planes of m and m'. The side of this triangle measured in the xy plane will be $\theta' - \theta$, and the opposite angle will be I; the side measured in the orbital plane of m will be $\Theta - \theta$, and the opposite angle will be $180° - i'$; the third angle will be i. Then, by spherical trigonometry, we shall have

$$\begin{aligned}\sin I\sin(\Theta-\theta) &= \sin i'\sin(\theta'-\theta);\\ \sin I\cos(\Theta-\theta) &= \sin i'\cos i\cos(\theta'-\theta) - \cos i'\sin i.\end{aligned} \tag{3.22}$$

Substituting from the definitions of P, Q, p, q, p', q' into (3.22), we obtain the two equations

$$\begin{aligned}&Pq - Qp = p'q - q'p;\\ &Pp + Qq = (p'p + q'q)\sqrt{1-p^2-q^2} - (p^2+q^2)\sqrt{1-p'^2-q'^2},\end{aligned} \tag{3.23}$$

and from (3.23), we can deduce that

$$
\begin{aligned}
P &= p' - p - p\frac{p'p + q'q}{1 + \sqrt{1 - p^2 - q^2}} + p\frac{p'^2 + q'^2}{1 + \sqrt{1 - p'^2 - q'^2}},\\
Q &= q' - q - q\frac{p'p + q'q}{1 + \sqrt{1 - p^2 - q^2}} + q\frac{p'^2 + q'^2}{1 + \sqrt{1 - p'^2 - q'^2}}
\end{aligned}
\tag{3.24}
$$

Next, Hansen sets about finding values for $\partial\Omega/\partial p$ and $\partial\Omega/\partial q$ in (3.21) in terms of $\partial\Omega/\partial P$ and $\partial\Omega/\partial Q$, taking the orbital plane of the perturbed planet to be identical with the *xy* plane at the epoch, so that at that time $p = q = 0$. It is here in the *Disquisitiones* that his error occurs. When p and q change, so that i and θ change, the change in the longitude of the ascending node causes a change in v, the true longitude, and $\partial\Omega/\partial p$ and $\partial\Omega/\partial q$ therefore prove to be functions not only of i and θ, but of v as well. Hansen in the *Disquisitiones*, assuming that $\mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}q/\mathrm{d}t$ are independent of v, obtains

$$
\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{an}{\sqrt{1 - e^2}}\cos I\frac{\partial\Omega}{\partial Q},\\
\frac{\mathrm{d}q}{\mathrm{d}t} &= -\frac{an}{\sqrt{1 - e^2}}\cos I\frac{\partial\Omega}{\partial P}
\end{aligned}
\tag{3.25}
$$

but these results are incomplete, as we shall later see.

Since equations (3.24) express only the spherical trigonometry embodied in equations (3.22), they can be used to find the P, Q pair expressing the inclination and node relating any two intersecting planes, if the pairs p, q and p', q' for these planes, relating them to some third plane, are known. Suppose, for instance, that the p, q pair relates the orbital plane of the perturbed planet to the plane of the ecliptic, considered as fixed at a chosen epoch of time. Suppose further that the pair $[p]$, $[q]$ relates the orbital plane of the Earth (the mobile ecliptic, which moves because the Earth's motions are subject to perturbation) to the same fixed ecliptic. Then in (3.24), substituting $[p]$ for p' and $[q]$ for q', we shall obtain in the pair P, Q the relation of the orbital plane of the perturbed planet to the mobile ecliptic—the plane to which astronomers customarily refer the motions of the planets.

The preceding Sects. 3.3 and 3.4 have introduced the main formulas of Hansen's method as developed in the *Disquisitiones.* In this work, Hansen also describes how, by means of numerical integration, Ω can be expanded as a series in terms of the mean anomalies g and g', and he deduces special conditional equations that enable him to check the correctness of the massive calculations that his method entails. I shall describe these features of Hansen's method in the course of explaining how they are applied in the *Untersuchung,* to which I now turn.

4 Hansen's *Untersuchung:* first-order perturbations

The *Untersuchung* is a treatise of 318 pages, pp. 193–315 being occupied by some 47 tables, which are altogether unintelligible without the preceding mathematical derivations and explanatory digressions. Hansen's mathematical explanations can seem obscure. My account of the *Untersuchung* will seek to elucidate the logical structure of the central argument.

4.1 Vorwort

In a foreword, Hansen acknowledges that his solution of the problem posed by the Berlin Academy is not yet complete. The greater and more arduous part of the work, he states, has been accomplished: the calculation of the first-, second-, and third-order perturbations of Saturn due to Jupiter, except for a few small terms of the third order, and the calculation of the first-order perturbations of Jupiter due to Saturn. Unwilling to forego entering the contest as the deadline arrives, Hansen submits his nearly finished essay, promising to carry out the remaining calculations forthwith.

In a *Nachschrift* at the end of the *Untersuchung,* Hansen tells us that the Berlin Academy, at his request, has granted him additional time to prepare his prize essay for publication. In the interval, he has completed the calculation of the second-order perturbations of Jupiter, and with their help has carried out a check on the accuracy of the second-order perturbations of Saturn. The printed work, however, does not contain the perturbations of Jupiter beyond those of the first order. Rather, it gives tables for the perturbations of Saturn in longitude, radius vector, and latitude, including all of the third-order terms except certain ones which Hansen characterizes as "few" and "small," and which would require the prior computation of the third-order perturbations of Jupiter.

Hill later raised the question why Hansen failed to continue his work on Jupiter and Saturn to the point of preparing new tables for Jupiter and Saturn. Was Hansen, as Hill suggested, "carried away with the ambition of applying his peculiar method of treatment to the lunar theory"?[92] Hill's remark seems to imply that Hansen's decision not to develop the theory of Jupiter and Saturn to the point of providing new tables was unprofessional. Let us recall, however, that already in 1828, Hansen had arrived at the hypothesis that Uranus was perturbed by some as yet undiscovered planet(s) farther from the Sun and that a complete theory of Jupiter and Saturn would have required an account of their perturbations due to Uranus. In the letters of Hansen to Encke[93]—Encke was in charge of the *Berliner Jahrbuch*, in which the annual lunar and planetary ephemerides of the Berlin Academy were published—I have found no indication that the Berlin Academy encouraged Hansen to prepare new tables of Jupiter and Saturn. The *Untersuchung* was primarily regarded, it appears, as a demonstration of a new and controversial method, which Hansen was called on to defend. Accurate tables of Jupiter and Saturn had less urgency than an accurate lunar theory.

To return to Hansen's *Vorwort*: the chief innovation of his new method, Hansen explains, is to compute the perturbations of the *mean* longitude rather than those of the *true* longitude. The advantage is in efficiency and accuracy. Thus, to obtain a

92 Hill (1890, 1906), p. 13.

93 The Berlin-Brandenburgische Academie der Wissenschaften possesses in its archive a collection of some 76 letters from Hansen to Encke. The corresponding letters of Encke to Hansen do not appear to be extant.

precision of $0''.1$ in the first-order perturbations, Hansen has found that, by the old procedure, he must compute 49 terms in the case of Saturn, and 54 in the case of Jupiter; the corresponding numbers in the new procedure are 38 and 43. The reason is the more rapid convergence of the infinite series giving perturbations of the mean longitude, as compared with the series giving perturbations of the true longitude. The terms neglected below a given lower bound form a smaller sum; "hence the totality of the perturbations...is obtained more exactly by my method than by the old method."[94]

The only existing alternative to Hansen's new method for the systematic development of higher-order perturbations was to employ the Lagrangian formulas for the perturbations of the orbital elements. But, Hansen argues, this alternative leads to much less convergent series. When used for obtaining the perturbations of the longitude and radius vector, the Lagrangian formulas regularly require determination of higher-order terms from differences of lower-order terms, with consequent uncertainty in the results. Moreover, the labor involved is far greater. Hansen finds that for the second-order perturbations, the Lagrangian method requires the computation of 78 products, while his new method requires the computation of only 25. Finally, the Lagrangian method provides no controls for checking the correctness of the calculations; Hansen's method explicitly provides such controls.

The Berlin Academy of Sciences, in setting the prize problem, had required an explanation of the difference between Laplace's and Plana's results for the higher-order perturbations having the argument $(5g' - 2g)$. As we have seen, Poisson and Pontécoulant listed 12 different relations whence sizable terms of this kind might arise, but dismissed some of these and an infinity of other relations as leading to negligible contributions. According to Hansen, however, the number of such relations can be properly limited only by setting a lower bound to the absolute numerical magnitude of the terms considered. He has carried the calculation further than anyone earlier and shown that the sources of sizable contributions cannot be reduced to the 12 relations designated by Poisson and Pontécoulant. On the other hand, Plana and Pontécoulant have adequately explained the cause of the differences between Laplace's and Plana's results, and Hansen therefore passes this question over.

4.2 First-order perturbations of the longitude: "Bessel functions" introduced into the *Untersuchung*

Equation (3.16) of Sect. 3.3, we recall, is Hansen's basic equation for perturbations of the longitude; it is applicable to perturbations of all orders with respect to the perturbing force. Its integral yields a value of ζ; in turn, ζ, becomes z when τ, in all its occurrences in the integral, is changed into t. (As this is done, $\lambda \to v$ and $\rho \to r$.) The true perturbed value of the longitude is then obtained by substituting z in place of t in the Keplerian formula for the purely elliptical longitude.

In the first nine sections of the *Untersuchung* (pp. 3–37), Hansen develops a modified form of (3.16), applicable only to first-order perturbations. He was led to this

94 *Untersuchung*, Vorwort, p. viii.

modification by Bessel's "Untersuchung des Theils der planetarischen Storungen, welcher aus der Bewegung der Sonne entsteht," published in 1826.[95] In this memoir, Bessel employed series for derivatives of (r^2/a^2) in the expansion of the part of the disturbing function that Hansen labels Ω_2 (see Sect. 4.5); Bessel was here introducing "Bessel functions"—misleadingly so called, since Bessel was not the first to use them. The advantage for Hansen was the obtaining of more rapidly converging series. Hansen applies these series to the elaboration of the *coefficients* which, in Hansen's differential equations, multiply the partial derivatives of Ω.

To do this, he first re-expresses $\frac{d.[\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial \tau}]}{d\tau}$, hereafter designated T, as a function of $\frac{\partial\Omega}{\partial\varepsilon}, r\frac{\partial\Omega}{\partial r}$, rather than of $\frac{\partial\Omega}{\partial v}, r\frac{\partial\Omega}{\partial r}$. Thus $\partial\Omega/\partial v$ must be eliminated from (3.16), by means of the substitution

$$\frac{\partial\Omega}{\partial v} = \frac{r^2}{a^2\sqrt{1-e^2}}\frac{\partial\Omega}{\partial\varepsilon} - \frac{re\,\sin(v-\pi)}{a(1-e^2)}r\frac{\partial\Omega}{\partial r},$$

which is derivable by way of formulas given in (3.11a, b) and (3.15b, c). With simplification of the coefficient of $r(\partial\Omega/\partial r)$, the result is

$$\begin{aligned} T &= an\left\{\frac{2\rho r\cos(v-\lambda)-r^2}{a^2(1-e^2)} + 2\frac{\rho r^2[\cos(v-\lambda)-1]}{a^3(1-e^2)^2}\right\}\frac{\partial\Omega}{\partial\varepsilon} \\ &+ an\left\{2\frac{\rho r\sin(v-\lambda)}{a^2(1-e^2)^{3/2}} + 3\frac{re\sin(v-\pi)}{a(1-e^2)^{3/2}} - 4\frac{\rho e\sin(\lambda-\pi)}{a(1-e^2)^{3/2}}\right\}r\frac{\partial\Omega}{\partial r} \end{aligned} \tag{4.1a}$$

In a further transformation, Hansen introduces the mean anomaly $g = nt + \varepsilon - \pi$ into the coefficient of $\partial\Omega/\partial\varepsilon$ by differentiating this coefficient with respect to g, then integrating with respect to g. Equation (4.1a) thus becomes

$$\begin{aligned} T &= an\left\{\begin{matrix} 2\frac{\rho\cos(\lambda-\pi)}{ae} - 3\frac{r^2}{a^2(1-e^2)} - 2\frac{r^2\rho\cos(\lambda-\pi)}{a^3e(1-e^2)} \\ +2\frac{\rho\sin(\lambda-\pi)}{a(1-e^2)^{3/2}}\int\left\{2\frac{r}{a}\cos(v-\pi)+3e\right\}dg \end{matrix}\right\}\frac{\partial\Omega}{\partial\varepsilon} \\ &+ an\left\{2\frac{\rho r\sin(v-\lambda)}{a^2(1-e^2)^{3/2}} + 3\frac{re\sin(v-\pi)}{a(1-e^2)^{3/2}} - 4\frac{\rho e\sin(\lambda-\pi)}{a(1-e^2)^{3/2}}\right\}r\frac{\partial\Omega}{\partial r} \end{aligned} \tag{4.1b}$$

Here, the coefficient of $\partial\Omega/\partial\varepsilon$ has been so configured that the constant of integration of the integral there occurring is zero. This equation, says Hansen

> has the advantage over all known formulas, that it does not require the coefficients in the series expansion of the equation of center, but only the coefficients for the expansion of the square of the radius vector, and it leads therefore to much more rapidly converging series...[96]

[95] Bessel (1826). Hansen already refers to this memoir in the *Disquisitiones, A.N.*, nr. 168 (1829), col. 478.

[96] *Untersuchung*, p. 9.

Next, Hansen makes substitutions in Eq. (4.1b) in accordance with the following formulas, derivable from (3.15c), and holding only for unperturbed motion or motion subject to first-order perturbation:

$$\begin{aligned}
\frac{\partial(r^2/a^2)}{\partial g} &= 2\frac{re\sin(v-\pi)}{a\sqrt{1-e^2}},\\
\frac{\partial(r^2/a^2)}{\partial e} &= -2\frac{r}{a}\cos(v-\pi),\\
\frac{\partial(\rho^2/a^2)}{\partial \gamma} &= 2\frac{\rho e\sin(\lambda-\pi)}{a\sqrt{1-e^2}},\\
\frac{\partial(\rho^2/a^2)}{\partial e} &= -2\frac{\rho}{a}\cos(\lambda-\pi)
\end{aligned}$$

Here γ is the same function of τ that g is of t: $\gamma = n\tau + \varepsilon - \pi$. The result of the substitutions is

$$\begin{aligned}
T = {}& \frac{an}{1-e^2}\left\{\begin{array}{l} \frac{(r^2/a^2)-1+e^2}{e}\frac{\partial(\rho^2/a^2)}{\partial e} - 3\frac{r^2}{a^2} \\ -\frac{1}{e}\frac{\partial(\rho^2/a^2)}{\partial\gamma}\int\left\{\frac{\partial(r^2/a^2)}{\partial e} - 3e\right\}dg \end{array}\right\}\frac{\partial\Omega}{\partial\varepsilon}\\
&+\frac{an}{1-e^2}\left\{\begin{array}{l} \frac{3}{2}\frac{\partial(r^2/a^2)}{\partial g} - 2\frac{\partial(\rho^2/a^2)}{\partial\gamma} \\ -\frac{1}{2e}\frac{\partial(r^2/a^2)}{\partial g}\frac{\partial(\rho^2/a^2)}{\partial e} \\ +\frac{1}{2e}\frac{\partial(\rho^2/a^2)}{\partial\gamma}\frac{\partial(r^2/a^2)}{\partial e} \end{array}\right\} r\frac{\partial\Omega}{\partial r}
\end{aligned} \tag{4.1c}$$

The quotients r^2/a^2 and ρ^2/a^2 are now replaced by the series expansions

$$\begin{aligned}
\frac{r^2}{a^2} &= 1 + \sum_{k=-\infty}^{+\infty} R_k \cos kg,\\
\frac{\rho^2}{a^2} &= 1 + \sum_{k=-\infty}^{+\infty} R_\kappa \cos \kappa\gamma.
\end{aligned}$$

The coefficients R_k and R_κ are functions of the eccentricity e, given, except when the index k or κ is zero, by a formula Hansen gets from Bessel:[97]

$$R_k = -\frac{\frac{2}{k^2}\left(\frac{ke}{2}\right)^k}{k!}\left\{\begin{array}{l} 1 - \frac{1}{k+1}\left(\frac{ke}{2}\right)^2 + \frac{1}{1\cdot 2(k+1)(k+2)}\left(\frac{ke}{2}\right)^4 \\ -\frac{1}{1\times 2\times 3(k+1)(k+2)(k+3)}\left(\frac{ke}{2}\right)^6 \pm \text{etc.} \end{array}\right\},$$

and the formula for R_κ is exactly parallel, with κ replacing k. In the case where the index is zero, Bessel shows by a special process that $R_0 = 3e^2/2$. The convergence is rapid, and Hansen calculates the R_k only to $k = 8$.

97 *Untersuchung*, pp. 44–46.

When these series are introduced into (4.1c), the result is

$$T=\frac{an}{1-e^2}\sum_k\sum_\kappa\left\{\begin{array}{l}\left[\frac{R_k}{e}\frac{\partial R_\kappa}{\partial e}-\kappa\frac{R_\kappa}{e}\frac{\partial R_k}{k\partial e}\right]\cos(\kappa\gamma+kg)\\+e\frac{\partial R_\kappa}{\partial e}\cos\kappa\gamma-3R_k\cos kg-3\end{array}\right\}\frac{\partial\Omega}{\partial\varepsilon}$$
$$+\frac{an}{1-e^2}\sum_k\sum_\kappa\left\{\begin{array}{l}\left[\frac{kR_k}{2e}\frac{\partial R_\kappa}{\partial e}-\frac{\kappa R_\kappa}{2e}\frac{\partial R_k}{\partial e}\right]\sin(\kappa\gamma+kg)\\+2\kappa R_k\ \sin\ \kappa\gamma-\frac{3k}{2}R_k\sin\ kg\end{array}\right\}r\frac{\partial\Omega}{\partial r}. \tag{4.1d}$$

As before, the indicated summations extend in principle from $-\infty$ to $+\infty$. To express Eq. (4.1d) more succinctly, Hansen introduces the definitions:

$$A_{\kappa,k}=\frac{1}{1-e^2}\left\{\frac{R_k}{e}\frac{\partial R_\kappa}{\partial e}-\kappa\frac{R_\kappa}{e}\frac{\partial R_k}{k\partial e}\right\},$$
$$B_{\kappa,k}=\frac{1/2}{1-e^2}\left\{\frac{kR_k}{e}\frac{\partial R_\kappa}{\partial e}-\frac{\kappa R_\kappa}{e}\frac{\partial R_k}{\partial e}\right\}.$$

These definitions hold except where k or κ is zero, in the latter cases, the following special definitions apply:

$$A_{\kappa,0}=\frac{5}{2}\frac{e}{1-e^2}\frac{\partial R_\kappa}{\partial e},\quad A_{0,k}=0,\ A_{0,0}=-3,$$
$$B_{\kappa,0}=\frac{1}{2}\frac{\kappa}{1-e^2}R_\kappa,\quad B_{0,k}=0$$

With these stipulations, (4.1d) can be written as

$$T=an\sum_\kappa\sum_k\left[A_{\kappa,k}\cos(\kappa\gamma+kg)\right]\frac{\partial\Omega}{\partial\varepsilon}+an\sum_\kappa\sum_k\left[B_{\kappa,k}\sin(\kappa\gamma+kg)\right]r\frac{\partial\Omega}{\partial r}. \tag{4.1e}$$

The next step is to integrate (4.1e) with respect to τ. Since T is $\frac{\mathrm{d}[\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial\tau}]}{\mathrm{d}\tau}$; the integral will yield $[\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial\tau}]$; or rather, since Hansen in the first approximation sets $\partial\zeta/\partial\tau=1$, it will give $\partial\zeta/\partial t$.

Now τ is not present in Ω or its derivatives; hence, the integration will affect only the coefficients of $\partial\Omega/\partial\varepsilon$ and of $r(\partial\Omega/\partial r)$, and in these coefficients, τ appears only in $\gamma=n\tau+\varepsilon-\pi$. For terms in which neither κ nor k is zero, let us set $C_{\kappa,k}=A_{\kappa,k}/\kappa$; $D_{\kappa,k}=-B_{\kappa,k}/\kappa$; $C_{0,0}=A_{0,0}$. The integral then becomes

$$\frac{\partial\zeta}{\partial t}=\int Td\tau=a\sum_\kappa\sum_k\left[C_{\kappa,k}\sin(\kappa\gamma+kg)\right]\frac{\partial\Omega}{\partial\varepsilon}+aC_{0,0}(n\tau-nt)\frac{\partial\Omega}{\partial\varepsilon}$$
$$+a\sum_\kappa\sum_k\left[D_{\kappa,k}\cos(\kappa\gamma+kg)\right]r\frac{\partial\Omega}{\partial r}. \tag{4.2a}$$

As Hansen shows, three further stipulations are necessary in order that the integral should vanish when τ is set equal to t (we omit the proofs):

$$C_{0,k} = -\frac{5}{2}\frac{e}{1-e^2}\frac{\partial R_k}{k\partial e}, \quad D_{0,k} = -\frac{1}{2}kC_{0,k}, \quad D_{0,0} = -\frac{2-(3/4)e^2}{1-e^2}.$$

The next step is to obtain ζ, by integrating (4.2a). This requires that Ω be expanded as a series in terms of the mean anomalies of the perturbed and perturbing planets. Hansen's processes here will be described later; the result may be written

$$\Omega = \frac{m'}{\mu}\sum_{i=-\infty}^{+\infty}\sum_{i'=0}^{+\infty}\left\{(i,i';c)\cos(ig+i'g') + (i,\,i';s)\sin(ig+i'g')\right\},$$

where $(i,i';c)$ and $(i,i';s)$ represent the coefficients in the expansion. Since $g = nt+\varepsilon-\pi$, so that $\partial\Omega/\partial\varepsilon = \partial\Omega/\partial g$, the partial derivative $\partial\Omega/\partial\varepsilon$ will be given by

$$\frac{\partial\Omega}{\partial\varepsilon} = \frac{m'}{\mu}\sum_i\sum_{i'}\left\{-i\left(i,i';c\right)\sin\left(ig+i'g'\right) + i\left(i,i';s\right)\cos\left(ig+i'g'\right)\right\}$$

Hansen's formula for $r(\partial\Omega/\partial r)$ may be written

$$r\frac{\partial\Omega}{\partial r} = \frac{m'}{\mu}\sum_i\sum_{i'}\left\{\left[i,i';c\right]\cos\left(ig+i'g'\right) + \left[i,i';s\right]\sin\left(ig+i'g'\right)\right\},$$

where $[i,i';c]$ and $[i,i';s]$ are the special coefficients required in this expansion. When these expressions are introduced into (4.2a), the result may be written as

$$\begin{aligned}\frac{\partial\zeta}{\partial t} &= \frac{am'}{\mu}\sum_{i,i',\kappa,k}\left\{C_{\kappa,k}\,(i-k)\left(i-k,i';c\right)+D_{\kappa,k}\left[i-k,i';c\right]\right\}\cos\left(\kappa\gamma+ig+i'g'\right)\\ &+\frac{am'}{\mu}\sum_{i,i',\kappa,k}\left\{C_{\kappa,k}\,(i-k)\left(i-k,i';s\right)+D_{\kappa,k}\left[i-k,i';s\right]\right\}\sin\left(\kappa\gamma+ig+i'g'\right)\\ &+\frac{am'}{\mu}C_{0,0}\sum_{i,i'} i\left(i,i';s\right)(n\tau-nt)\cos\left(ig+i'g'\right)\\ &-\frac{am'}{\mu}C_{0,0}\sum_{i,i'} i\left(i,i';c\right)(n\tau-nt)\sin\left(ig+i'g'\right) \qquad (4.2b)\end{aligned}$$

In the argument of the cosine and sine in the first two terms, the integer k has been absorbed into i, so that i in the coefficient becomes $(i-k)$. The integration of (4.2b) yields

$$\begin{aligned}
n\zeta &= n\,(1-c)\,\tau \\
&\quad + \frac{am'}{\mu} \sum_{i,i',\kappa,k} \frac{n}{in+i'n'} \left\{(i-k)\left(i-k,i';c\right) C_{\kappa,k} + \left[i-k,i';c\right] D_{\kappa,k}\right\} \\
&\quad \sin\left(\kappa\gamma + ig + i'g'\right) \\
&\quad - \frac{am'}{\mu} \sum_{i,i',\kappa,k} \frac{n}{in+i'n'} \left\{(i-k)\left(i-k,i';s\right) C_{\kappa,K} + \left[i-K,i';s\right] D_{\kappa,k}\right\} \\
&\quad \cos\left(\kappa\gamma + ig + i'g'\right) \\
&\quad + \frac{am'}{\mu} C_{0,0} \sum_{i,i'} \left(\frac{n}{in+i'n'}\right)^2 i\left(i,i';c\right) \sin\left(ig+i'g'\right) \\
&\quad - \frac{am'}{\mu} C_{0,0} \sum_{i,i'} \left(\frac{n}{in+i'n'}\right)^2 i\left(i,i';s\right) \cos\left(ig+i'g'\right) \\
&\quad + \frac{am'}{\mu} C_{0,0} \sum_{i,i'} \frac{n}{in+i'n'} i\left(i,i';s\right)(n\tau - nt) \sin\left(ig+i'g'\right) \\
&\quad + \frac{am'}{\mu} C_{0,0} \sum_{i,i'} \frac{n}{in+i'n'} i\left(i,i';c\right)(n\tau - nt) \cos\left(ig+i'g'\right).
\end{aligned} \tag{4.3}$$

The last four terms on the right arise from the last two terms of (4.2b) through integration by parts. The rule for this operation is $\int P\mathrm{d}Q = PQ - \int Q\mathrm{d}p$, where P and Q are two functions. Thus, in the first of the two terms of (4.2b) containing the coefficient $C_{0,0}$, we may set $P = (n\tau - nt)$, so that $\mathrm{d}P = -n$; $\mathrm{d}Q$ is then $\cos(ig + i'g')$, so that Q will be $\frac{\sin(ig+i'g')}{in+i'n'}$ (since $g = nt + \varepsilon - \pi$,and $g' = n't + \varepsilon' - \pi'$). On calculating $PQ - \int Q\mathrm{d}p$, we obtain the third and the second of the four resulting terms of (4.3). In an entirely similar way, the fourth and first of the last four terms in (4.3) arise from the last term of (4.2b).

How to understand the first term on the right of (4.3), namely $n(1-c)\tau$? As Hansen explains, the integration leading to (4.3) ceases to be correct when $i = i' = 0$. To obtain the terms arising in that case, we must return to (4.2b), set $i = i' = 0$, and integrate; the result is

$$\begin{aligned}
n\zeta &= \frac{am'}{\mu} \sum_{\kappa,k} \left\{(-k)(-k,0;s)C_{\kappa,k} + [-k,0;s]D_{\kappa,k}\right\} nt \sin\kappa\gamma \\
&\quad + \frac{am'}{\mu} \sum_{\kappa,k} \left\{(-k)(-k,0;c)C_{\kappa,k} + [-k,0;c]D_{\kappa,k}\right\} nt\ \cos\kappa\gamma.
\end{aligned} \tag{4.4}$$

This multitude of terms must therefore be added to (4.3). Among the terms to be added are those in which, not only $i = i' = 0$, but also $\kappa = 0$:

$$\frac{am'}{\mu}\sum_k \left\{(-k)\,(-k,0;c)\,C_{0,k} + [-k,0;c]\,D_{ok}\right\} nt.$$

This term unites with the mean motion of the planet; or rather, it causes the mean motion as determined by astronomical observations to differ from the mean motion that would obtain in the absence of perturbation. Hansen gives to the coefficient of this term the symbol c:

$$c = \frac{am'}{\mu}\sum_k \left\{(-k)\,(-k,0;c)\,C_{0,k} + [-k,0;c]\,D_{o,k}\right\}. \tag{4.5}$$

Adding cnt to (4.3), he thus obtains

$$n\zeta = n\tau - c\,(n\tau - nt) + \text{ periodic perturbations,}$$

and when $\tau \to t$, it follows that

$$nz = nt + \text{ periodic perturbations,}$$

where n is to be understood as the mean motion obtained from observations.[98]

Hansen's theory for the longitude implies an infinity of perturbational terms. How does he choose for calculation the terms that will be sizable enough to matter?

He is aiming to achieve an accuracy of $0''.03$ or $0''.04$;[99] but he sets the precision in individual calculations at $0''.001$. Assume the expansion of Ω to have been carried out far enough so as to include all terms with coefficients greater than or equal to $0''.001$. For illustration, Hansen explains the steps involved in computing the coefficient of the term in nz that is multiplied by $\sin(3g + 2g')$. In (4.3), where τ has not yet been changed to t, the contributions to this coefficient come from the second line, where the part we have to be concerned about is

$$\left\{(i-k)(i-ki';c)C_{\kappa,k} + [i-k,i';c]D_{\kappa,k}\right\}\sin\left(\kappa\gamma + ig + i'g'\right).$$

Here $i' = 2$ throughout our computation, but the integer multiplying g when $\tau \to t$ and consequently $\gamma \to g$ results from the sum $\kappa + i$. Thus, all the following arguments of the sine function in (4.3) lead to the same final argument $3g + 2g'$:

$$0\gamma + 3g + 2g',\ \ -\gamma + 4g + 2g', -2\gamma + 5g + 2g', etc.$$
$$\gamma\ +2g +\ 2g', 2\gamma + g + 2g',\ \text{etc.}$$

98 The question as to how the mean motion is influenced by perturbation had been raised by Plana. See Plana 1826.

99 As he states at *Untersuchung*, p. 69.

In each of these cases, where a particular value of κ is under consideration, the various possible values of k must also be considered. Thus, the coefficient of the term multiplied by $\sin(-\gamma + 4g + 2g')$ arises from the sum

$$\begin{aligned}
&\{4\,(4, 2; c)\, C_{-1,0} + [4, 2; c] D_{-1,0}\} \\
&\quad + \{5\,(5, 2; c)\, C_{-1,-1} + [5, 2; c]\, D_{-1,-2}\} \\
&\quad + \{6\,(6, 2; c)\, C_{-1,-2} + [6, 2; c]\, D_{-1,-2}\} \\
&\quad + \text{etc.} \\
&\quad + \{3\,(3, 2; c)\, C_{-l,1} + [3, 2; c]\, D_{-1,1}\} \\
&\quad + \{2\,(2, 2; c)\, C_{-1,2}, + [2, 2; c]\, D_{-1,2}\} \\
&\quad + \text{etc.}
\end{aligned}$$

The indices κ and k are thus to be varied in the positive and in the negative directions "until we arrive at terms that can be neglected."[100]

It is evident that, in the reckoning of coefficients belonging to the argument $(ig + pg')$, the coefficients of terms in the expansion of Ω that play a role are those for which $i' = p$, and only those. The major divisions in Hansen's calculation are therefore determined by the choice of p. For a given choice of p, he obtains a sequence of arguments for the sinusoidal function that extends indefinitely in two directions, involving multiples of g decreasing toward $-\infty g$ and increasing toward $+\infty g$:

$$\begin{aligned}
&(-2g + pg'), \\
&(-g + pg'), \\
&(\quad pg') \\
&(g + pg'), \\
&(2g + pg'), \\
&\text{etc.}
\end{aligned}$$

According to Hansen, the terms corresponding to these arguments, continued forward and backward from a point where the terms have maximal values, become negligible.[101] This, he acknowledges, is an empirical finding, not a result provable a priori. He recommends determining the region of larger terms by trial and error, then continuing the calculation of terms in both directions. This way of judging the completeness of the calculation is new with Hansen.

A very considerable alleviation of the calculational labor is brought about by the following analytic result, which Hansen proves.[102] Let the main terms of (4.3) be expressed by a sum of terms of the form

[100] *Untersuchung*, p. 22.

[101] *Untersuchung* , p. 23.

[102] *Untersuchung*, pp. 25–29.

$$n\zeta = \sum_{\kappa} \left\{ \alpha_s^{(\kappa)} \sin\left(\kappa\gamma + ig + i'g'\right) + \alpha_c^{(\kappa)} \cos\left(\kappa\gamma + ig + i'g'\right) \right\}, \tag{4.6}$$

where κ takes on all integral values, positive and negative, and the coefficients labeled α are functions of κ. Then, for a positive κ, Hansen shows that

$$\begin{aligned} \alpha_s^{(\kappa+1)} &= \eta^{(\kappa)} \cdot \alpha_s^{(\kappa)} + \theta^{(\kappa)} \cdot \alpha_s^{(-\kappa)}. \\ \alpha_c^{(\kappa+1)} &= \eta^{(\kappa)} \cdot \alpha_c^{(\kappa)} + \theta^{(\kappa)} \cdot \alpha_c^{(-\kappa)}, \end{aligned} \tag{4.7a}$$

while for a negative κ,

$$\begin{aligned} \alpha_s^{(\kappa-1)} &= \eta^{(\kappa)} \cdot \alpha_s^{(\kappa)} + \theta^{(\kappa)} \cdot \alpha_s^{(-\kappa)}, \\ \alpha_c^{(\kappa-1)} &= \eta^{(\kappa)} \cdot \alpha_c^{(\kappa)} + \theta^{(\kappa)} \cdot \alpha_c^{(-\kappa)}. \end{aligned} \tag{4.7b}$$

Here

$$\eta^{(\kappa)} = \frac{1}{2}\left(\frac{R_{\kappa+1}}{R_\kappa} + \frac{\frac{\partial R_{\kappa+1}}{(\kappa+1)\partial e}}{\frac{\partial R_\kappa}{\kappa \partial e}} \right), \quad \theta^{(\kappa)} = \frac{1}{2}\left(\frac{R_{\kappa+1}}{R_\kappa} - \frac{\frac{\partial R_{\kappa+1}}{(\kappa+1)\partial e}}{\frac{\partial R_\kappa}{\kappa \partial e}} \right).$$

These coefficients can be expressed as series in terms of the eccentricity:

$$\begin{aligned} \eta^{(1)} &= \frac{e}{4} - \frac{e^3}{16} + 0e^5 + \text{etc.}, \ \theta^{(1)} = \frac{e^3}{96} - \frac{e^5}{364} + \text{etc.} \\ \eta^{(2)} &= \frac{e}{2} - \frac{e^3}{8} - \frac{e^5}{2304} + \text{etc.}, \ \theta^{(2)} = \frac{e^3}{96} - \frac{3e^5}{640} + \text{etc.} \end{aligned}$$

For $e \ll 1$, as κ increases in absolute value, the α converge sharply. (In Saturn's case, $e \approx 0.056$; in Jupiter's case, $e \approx 0.048$.)

4.3 The first-order perturbations of the radius vector

Hansen determines the first-order perturbations of the logarithm of the radius vector by means of the relation I have numbered (3.18). For first-order perturbations, (3.18) can be simplified by setting

$$\frac{\partial \zeta}{\partial \tau} = 1, \ \frac{an}{\sqrt{1-e^2}} \frac{\partial \Omega}{\partial v} = \frac{\mathrm{d}S}{\mathrm{d}t}, \quad \text{and}$$

$$\frac{\frac{\partial \ln(\rho)}{\partial \tau}}{\frac{\partial \zeta}{\partial \tau}} \frac{\partial \zeta}{\partial t} = 0.$$

The equation then takes the form

$$2\frac{\mathrm{d}\ \ln(\rho)}{\mathrm{d}t}=\frac{\mathrm{d}S}{\mathrm{d}t}-\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau\mathrm{d}t}. \tag{4.7c}$$

Integrating with respect to t yields

$$2\ln(\rho)=S-\frac{\mathrm{d}(\zeta)}{\mathrm{d}\tau}. \tag{4.8}$$

Hansen here writes (ζ) in place of ζ, to indicate that the constant of integration τ, normally added when ζ, is integrated with respect to t, is to be omitted. The only constant of integration required for (4.8) is the value of (ln ρ) computed from the purely elliptical formula, with ζ in place of τ, as explained in Sect. 3.3.

Next, Hansen differentiates (4.8) with respect to τ, so that S, which does not contain τ, disappears: $2\frac{\mathrm{d}\ln(\rho)}{\mathrm{d}\tau}=-\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}$.

Hansen's way of indicating that he has changed τ into t in a quantity is to draw a bar over the quantity, for instance $\overline{\frac{\mathrm{d}\ln(\rho)}{\mathrm{d}\tau}}=\frac{\mathrm{d}\ln(r)}{\mathrm{d}t}$. The preceding equation, with τ changed into t, is

$$\frac{\mathrm{d}\ln(r)}{\mathrm{d}t}=-\frac{1}{2}\overline{\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}},$$

$$\text{whence in(r)}=-\frac{1}{2}\int\overline{\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}}+\text{Const.} \tag{4.9}$$

What is the added constant in (4.9)? It is not the constant just mentioned as the only constant to be added to the integral (4.8). Any constant present in (4.8), we observe, would have been eliminated by the differentiation with respect to τ. The constant in (4.9), if it differs from zero, must be identical with whatever constant is present in the value of ln(r) derivable from (4.8). Now the value of ln(r) derivable from (4.8) is obtained by changing τ into t; it is

$$\ln(r)=\frac{1}{2}S-\frac{1}{2}\frac{\mathrm{d}(z)}{\mathrm{d}t}. \tag{4.10}$$

Comparing this value of ln(r) with that given by (4.9), we find

$$-\frac{1}{2}\int\overline{\frac{\mathrm{d}^2\zeta}{d\tau^2}}+\text{Const.}=\frac{1}{2}S-\frac{1}{2}\frac{\mathrm{d}(z)}{\mathrm{d}t}.$$

Neither the first term on the left nor the first term on the right can contain the constant in question; hence Const. $=-\frac{1}{2}\frac{\mathrm{d}(z)}{\mathrm{d}t}$.

But what is d(z)/dt? To understand this, we must go back to (4.2b) and integrate it so as to obtain $n(\zeta)$ rather than $n\zeta$; in other words, without adding the integrative constant which, as we see from (4.3), turns out to be $n(l-c)\tau$. Taking account of

(4.5), we find that $n(\zeta)$ = cnt + periodic terms. Consequently, the constant part of $\mathrm{d}(\zeta)/\mathrm{d}t = c$. But (constant part of $\mathrm{d}(\zeta)/\mathrm{d}t$) = (constant part of d(z)/dt), since the change of τ into t affects only the periodic terms on the right-hand side of (4.3). Thus Const. $= (1/2)\mathrm{d}(z)/\mathrm{d}t = -c/2$

Hansen introduces one further refinement. The final constant to be added to obtain the true value of ln(r) is the value of ln(ρ) from the elliptical formula, with $n\zeta$ replacing the mean motion. Into that elliptical formula there enters a value of the mean solar distance a, or, in Hansen's formulation, (ln a). The value of a to be used here is a_0, determined from the mean motion which would obtain if perturbations were absent; as we have seen, this mean motion is $n_0 = n(1-c)$. The two theoretical numbers, n_0 and a_0, are related to the observational values, n and a, by the Keplerian equation $a_0^3 n_0^2 = a^3 n^2 = a^3 n_0^2 (1-c)^2$. It follows that $a_0 = a\,(1-c)^{2\backslash 3}$. Since Hansen proposes to use (ln a) in place of (ln a_0), he must introduce the correction (ln a_0) − (ln a). But by the result just given, this is (2/3) ln$(1-c)$. If $c \ll 1$, it may be approximated as (2/3)c. Therefore, with this correction, we shall have

$$\text{Const.} = -\frac{1}{2}\frac{\mathrm{d}(z)}{\mathrm{d}t} + \frac{2}{3}c = \frac{1}{6}c.$$

From (4.9), it then follows that

$$\text{In(r)} = \frac{1}{2}\int \overline{\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}}\,\mathrm{d}t + \frac{1}{6}\text{c}. \tag{4.11}$$

To determine $n(\zeta)$ and its first and second derivatives with respect to τ, we may return to (4.3), deleting the added constant $n(1-c)\tau$, adding the terms required by (4.4), and then differentiating. We observe that, with the exception of the last two terms of (4.3), τ occurs on the right-hand sides of (4.3) and (4.4) only in $\gamma = n\tau + \varepsilon - \pi$. The result of the differentiations takes the form

$$\begin{aligned}\frac{1}{n}\overline{\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}} &= \eta_s \sin(ig + i'g') + \eta_c \cos(ig + i'g')\\ &\quad + \vartheta_s nt \sin(ig + i'g') + \vartheta_c nt \cos(ig + i'g').\end{aligned}$$

Then, integrating by parts, we find

$$\begin{aligned}\int \overline{\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}}\mathrm{d}t &= -\left\{\frac{n\eta_s}{in + i'n'} - \frac{n^2\vartheta_c}{(in + i'n')^2}\right\}\cos(ig + i'g')\\ &\quad + \left\{\frac{n\eta_c}{in + i'n'} + \frac{n^2\vartheta_s}{(in + i'n')^2}\right\}\sin(ig + i'g')\\ &\quad - \frac{n\vartheta_s}{in + i'n'}nt\cos(ig) + \frac{n\vartheta_c}{in + i'n'}nt\sin(ig).\end{aligned} \tag{4.12}$$

4.4 Two conditional equations for checking the numerical calculations of v and $\ln(r)$

The determination of $\ln(r)$ by (4.11) and (4.12) is entirely independent of $\partial\Omega/\partial v$. However, Hansen recommends checking the calculation by computing $\ln(r)$ by the entirely separate route of (4.10), which entails the somewhat laborious determination of

$$S = \int \frac{an}{\sqrt{1-e^2}} \frac{\partial \Omega}{\partial v} \mathrm{d}t \tag{4.13}$$

The agreement of the two calculations shows the correctness of both the expansion of Ω and the summations involved in the perturbation coefficients. This is one of two conditional equations on which Hansen puts great store, for they permit a rigorous check on the correctness of the numerical calculation of v and $\ln(r)$. In earlier perturbational methods, no such rigorous control was available.

The second conditional equation, like the first, depends ultimately on Hansen's introduction of the quantity τ.[103] Since the first derivatives of the coordinates with respect to the time have the same form whether they are perturbed or unperturbed, the orbital elements where they enter into this differentiation can be supposed either variable or invariable. If we differentiate the perturbed coordinate v with respect to t directly, we are manifestly supposing the orbital elements to be variable and hence to be functions of t. But if we differentiate ζ, with respect to τ, and afterward write t for τ, the coordinate v is differentiated as if the orbital elements were invariable, since τ is not present in the perturbed values of the elements. Differentiating in the first way yields

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t},$$

while differentiating in the second way yields

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial \lambda}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t},$$

where the quantity τ that is present in the derivatives on the right is to be changed into t. The two foregoing equations must agree, whence

$$\frac{\partial \lambda}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = \frac{\partial v}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

But it is evident that, after τ is changed into t,

$$\frac{\partial \lambda}{\partial \zeta} = \frac{\partial v}{\partial z}.$$

103 Hansen describes this conditional equation in his *Disquisitiones*, Section 17, cols. 446–448. It seemed best to postpone our account of it till after we had arrived at Hansen's expression for a typical term of ζ, as in (4.3).

Hence, we obtain the conditional equation

$$\frac{d\zeta}{d\tau} = \frac{dz}{dt}, \tag{4.14}$$

where in the left-hand member we are to understand that, after the differentiation, τ is changed into t.

By a similar process, Hansen obtains the conditional equation

$$\frac{\partial \ln(\rho)}{\partial \tau} = \frac{d \ln(r)}{dt}, \tag{4.15}$$

where, again, following on the indicated differentiation on the left, τ is to be changed into t.

To illustrate the use of such equations, I shall show how (4.14) is applied. A typical term in ζ, when there is only one perturbing planet, is of the form (compare equation (4.3) above)

$$\begin{aligned}
&\alpha \sin\left(ig+i'g'\right)+\beta \sin\left[\gamma+(i-1)\,g+i'g'\right]+\delta \sin\left[2\gamma+(i-2)\,g+i'g'\right]+\text{etc.}\\
&\quad+\varepsilon \sin[-\gamma+(i+1)g+i'g']+\theta \sin[-2\gamma+(i+2)\,g+i'g']+\text{etc.}\\
&\quad+\eta\,(n\tau-nt)\cos\left(ig+i'g'\right)\\
&\quad+\alpha' \cos\left(ig+i'g'\right)+\beta' \cos\left[\gamma+(i-1)\,g+i'g'\right]\\
&\quad+\delta' \cos[2\gamma+(i-2)\,g+i'g']+\text{ etc.}\\
&\quad+\varepsilon' \cos\left[-\gamma+(i+1)\,g+i'g'\right]+\theta' \cos\left[-2\gamma+(i+2)\,g+i'g'\right]+\text{ etc.}\\
&\quad+\eta'\,(n\tau-nt)\sin\left(ig+i'g'\right).
\end{aligned} \tag{4.16}$$

Now τ occurs in (4.16) only in γ, which is given by $\gamma - n\tau + \varepsilon\ - \varpi$, so that when τ is turned into t, γ becomes g, and (4.16) becomes z, namely

$$\left\{\begin{matrix}\alpha+\ \beta+\delta+\text{etc.}\\ +\varepsilon+\theta+\text{etc.}\end{matrix}\right\}\sin(ig+i'g')+\left\{\begin{matrix}\alpha'+\ \beta'+\delta'+\text{etc.}\\ +\varepsilon'+\theta'+\text{etc.}\end{matrix}\right\}\cos(ig+i'g'). \tag{4.17}$$

Carrying out the differentiations required for (4.14), replacing τ by t in $d\zeta/d\tau$, and then separately equating the coefficients of the sine terms and those of the cosine terms, we find

$$\begin{aligned}
\frac{n}{in+i'n'}\left\{\begin{matrix}\beta+2\delta+\text{etc.}+\eta\\ -\varepsilon-2\theta-\text{etc.}\end{matrix}\right\} &= \left\{\begin{matrix}\alpha+\ \beta+\delta+\text{etc.}\\ +\varepsilon+\theta+\text{etc.}\end{matrix}\right\},\\
\frac{n}{in+i'n'}\left\{\begin{matrix}\beta'+2\delta'+\text{etc.}+\eta'\\ -\varepsilon'-\theta'-\text{etc.}\end{matrix}\right\} &= \left\{\begin{matrix}\alpha'+\ \beta'+\delta'+\text{etc.}\\ +\varepsilon'+\theta'+\text{etc.}\end{matrix}\right\}
\end{aligned}$$

Thus, Hansen concludes, "a large number of conditional equations, by which this or that part of the numerical computation can be confirmed, are derivable from our method..."

4.5 The first-order perturbations of the latitude

As stated earlier, in the *Untersuchung*, Hansen has come to recognize that his account of the latitudes in the *Disquisitiones* was flawed: the position of the instantaneous orbital plane cannot be treated independently of the longitude. In his revised treatment in the *Untersuchung*, he introduces an explicit dependence of $\mathrm{d}\Omega/\mathrm{d}p$ and $\mathrm{d}\Omega/\mathrm{d}q$ on $\mathrm{d}\Omega/\mathrm{d}v$, in

$$\begin{aligned}\frac{\partial\Omega}{\partial p} &= -\cos I\frac{\partial\Omega}{\partial P} - \left[\frac{\partial\Omega}{\partial v} + Q\frac{\partial\Omega}{\partial P} - P\frac{\partial\Omega}{\partial Q}\right]\frac{Q}{1+\cos I},\\ \frac{\partial\Omega}{\partial q} &= -\cos I\frac{\partial\Omega}{\partial Q} + \left[\frac{\partial\Omega}{\partial v} + Q\frac{\partial\Omega}{\partial P} - P\frac{\partial\Omega}{\partial Q}\right]\frac{P}{1+\cos I}.\end{aligned} \tag{4.18}$$

Here, as in Sect. 3.4, $P = \sin I \sin\Theta$, $Q = \sin I\cos\Theta$, where I is the mutual inclination of the orbital planes of the perturbing and perturbed planets, and Θ the ascending node in which the orbital plane of the perturbing planet cuts the orbital plane of the perturbed planet. Meanwhile, we still have that $p = \sin i \sin\theta$, $q = \sin i\cos\theta$, where i is the inclination of the orbital plane of the perturbed planet to a fixed xy plane and θ is the ascending node in which the orbital plane of the perturbed planet cuts the xy plane. In Eq. (3.21), we recall, the fixed xy plane was identified with the orbital plane of the perturbed planet at epoch, so that, at epoch, $p = q = 0$. Substitution of (4.14) in (3.21) yields immediately

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{an}{\sqrt{1-e^2}}\left[-\cos I\frac{\partial\Omega}{\partial Q} + \frac{P}{1+\cos I}\left\{\frac{\partial\Omega}{\partial v} + Q\frac{\partial\Omega}{\partial P} - P\frac{\partial\Omega}{\partial Q}\right\}\right],\\ \frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{an}{\sqrt{1-e^2}}\left[\cos I\frac{\partial\Omega}{\partial P} + \frac{Q}{1+\cos I}\left\{\frac{\partial\Omega}{\partial v} + Q\frac{\partial\Omega}{\partial P} - P\frac{\partial\Omega}{\partial Q}\right\}\right].\end{aligned} \tag{4.19}$$

Equations (4.19) can be simplified ("without loss of rigor," Hansen assures us) by stipulating that all longitudes be measured from the node Θ. With this stipulation, we find

$$P = 0, \quad Q = \sin I, \qquad \frac{\partial\,\Omega}{\partial\,P} = \frac{\partial\,\Omega}{\partial\,\Theta}\frac{1}{\sin I}, \quad \frac{\partial\,\Omega}{\partial\,Q} = \frac{\partial\,\Omega}{\partial\,I}\frac{1}{\cos I},$$

and equations (4.19) become

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{an}{\sqrt{1-e^2}}\cos I\frac{\partial\Omega}{\partial\,Q}\\ &= -\frac{an}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial\,I};\\ \frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{an}{\sqrt{1-e^2}}\left[\frac{\partial\Omega}{\partial\,P}\cos I + \tan\left(\frac{I}{2}\right)\frac{\partial\Omega}{\partial v}\right]\\ &= \frac{an}{\sqrt{1-e^2}}\frac{1}{\sin I}\frac{\partial\Omega}{\partial\Theta} + \frac{\mathrm{d}S}{\mathrm{d}t}\tan\left(\frac{I}{2}\right).\end{aligned} \tag{4.20}$$

where $\mathrm{d}S/\mathrm{d}t$ is determined by the definition given at the start of Sect. 4.3. The required partial derivatives of Ω will be given by

$$\frac{\partial\Omega}{\partial I} = \frac{m'}{\mu}\sum_{i,i'}\left\{\frac{\mathrm{d}(i,i';c)}{\mathrm{d}I}\cos(ig+i'g') + \frac{\mathrm{d}(i,i';s)}{\mathrm{d}I}\sin(ig+i'g')\right\},$$

$$\frac{\partial\Omega}{\partial\Theta} = \frac{m'}{\mu}\sum_{i,i'}\left\{\frac{\mathrm{d}(i,i';c)}{\mathrm{d}\Theta}\cos(ig+i'g') + \frac{\mathrm{d}(i,i';s)}{\mathrm{d}\Theta}\sin(ig+i'g')\right\}.$$

Integration then yields

$$p = \frac{m'}{\mu}\frac{a}{\sqrt{1-e^2}}\sum_{i,i'}\frac{n}{in+i'n'}\left\{-\frac{\mathrm{d}(i,i';c)}{\mathrm{d}I}\sin(ig+i'g')+\frac{\mathrm{d}(i,i';s)}{\mathrm{d}I}\cos(ig+i'g')\right\},$$

$$q = \frac{m'}{\mu}\frac{a}{\sqrt{1-e^2}}\sum_{i,i'}\frac{n}{in+i'n'}\left\{\frac{\mathrm{d}(i,i';c)}{\sin I d\Theta}\sin(ig+i'g')-\frac{\mathrm{d}(i,i';s)}{\sin I d\Theta}\cos(ig+i'g')\right\}$$
$$+S\cdot\tan\left(\frac{I}{2}\right). \tag{4.21}$$

When $i = i' = 0$, the result is

$$p = -\frac{m'}{\mu}\frac{a}{\sqrt{1-e^2}}\frac{\mathrm{d}(0,0;c)}{\mathrm{d}I}nt,$$
$$q = -\frac{m'}{\mu}\frac{a}{\sqrt{1-e^2}}\frac{\mathrm{d}(0,0;c)}{\sin \mathrm{I}\,\mathrm{d}\Theta}nt + S_{0,0}\tan\left(\frac{I}{2}\right). \tag{4.21a}$$

Here $S_{o,o}$ is the term of S multiplied by nt.

4.6 The expansion of the disturbing function Ω

Hansen divides the disturbing function for m perturbed by m' into two parts:

$$\Omega_i = \frac{m'}{\mu}\frac{1}{\Delta} \tag{4.22a}$$

$$\text{where}\quad \Delta = \left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2},$$

$$\text{and}\quad \Omega_2 = -\frac{m'}{\mu}\frac{xx'+yy'+zz'}{r'^3} \tag{4.22b}$$

The corresponding formulas for m' perturbed by m are written Ω_1' and Ω_2', and are the same except that m/μ' replaces m'/μ in both parts, and r^3 replaces $(r')^3$ in Ω_2'. In the *Untersuchung*, the primed quantities normally refer to Saturn, the unprimed to Jupiter.

The numerical value of the disturbing function is independent of the coordinate system in which it is expressed. Hansen chooses a coordinate system with origin at the Sun, x-axis coinciding with the line of intersection of the two orbital planes, and xy plane bisecting the angle of inclination I of these two planes. Setting $u = v - \Theta$, $u' = v' - \Theta'$, he obtains

$$x = r\cos u, \quad x' = r'\cos u',$$
$$y = r\sin u\cos\left(\frac{I}{2}\right), \quad y' = r'\sin u'\cos\frac{I}{2},$$
$$z = r\sin u\sin\frac{I}{2}, \quad z' = -r'\sin u'\sin\frac{I}{2}.$$

It follows that

$$\Omega_1 = \frac{m'}{\mu}\left[r^2+r'^2 - 2rr'\cos^2\left(\frac{I}{2}\right)\cos(u-u') - 2rr'\sin^2\left(\frac{I}{2}\right)\cos(u+u')\right]^{-1/2},$$
$$\Omega_2 = -\frac{m'}{\mu}\frac{r}{r'^2}\left\{\cos^2\left(\frac{I}{2}\right)\cos(u-u') + \sin^2\left(\frac{I}{2}\right)\cos(u+u')\right\}. \tag{4.23}$$

These two parts of Q, and such of their derivatives as are needed, are to be expanded as series in cosines and sines of the argument $(ig + i'g')$, where g, g' are the mean anomalies of m and m' and i, i' are integers. Thus, putting y for the quantity to be expanded, we are to seek an expression of the form

$$y = \sum_{i,i'}(i, i'; c)\cos(ig + i'g') + \sum_{i,i'}(i, i'; s)\sin(ig + i'g'), \tag{4.24}$$

where the summations are to be extended to enough successive values of i and i' to give an adequate approximation to values of y. By what we now call

Fourier analysis (Hansen had no name for it), the coefficients are given by double-integrals:

$$(i, i'; c) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} y\cos(ig + i'g')dg\cdot dg',$$
$$(i, i'; s) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} y\sin(ig + i'g')dg\cdot dg'. \tag{4.25}$$

Hansen proposes to evaluate these integrals by numerical integration—"mechanical quadratures" as they were called in his day. It was his view that, whenever integrals are expressible by infinite series, but these prove complicated, the integrations by mechanical quadratures are to be preferred.[104]

In the eighteenth century, the one notable astronomical application of mechanical quadratures had been Clairaut's, in determining the perturbations influencing the date of the 1759 return of Halley's Comet.[105] The orbital eccentricity of Halley's Comet

[104] *Disquisitiones, A.N.* Nr. 168, cols. 474–474.

[105] See Wilson (1993).

is close to 1, and consequently, trigonometric series with successive terms varying as successive powers of the eccentricity converge too slowly to be useful; mechanical quadratures are indispensable. In the late 1820s, a number of astronomers undertook the numerical integrations required to predict the 1835 return of Halley's Comet.

With respect to the calculation of planetary perturbations, Euler had advocated the use of mechanical quadratures in the 1770s, in particular for the Venusian perturbations of the Earth. A comparison of the calculation by mechanical quadratures with that by the analytically derived trigonometric series, however, showed the latter to be as reliable as the former and less laborious to apply.[106] Hansen appears to have been the first since Euler to apply mechanical quadratures to the calculation of planetary perturbations.

Bessel in several memoirs had advocated mechanical quadratures for the investigation of periodic phenomena; Hansen cites two of these memoirs.[107] For Hansen, the resort to mechanical quadratures, to replace the analytic development of planetary perturbations endorsed by Lagrange and Laplace, had the special advantage of admitting of multiple cross-checks, whereby to insure the correctness of the calculation. Fifty years later, G.W. Hill would follow Hansen's lead in this matter.

To commence his calculation, Hansen adopts the orbital elements that Bouvard had assigned to Jupiter and Saturn for the end of 1799.[108] As Hansen observes, orbital elements calculated directly from observations require correction because the observations involve the perturbations. Thus, if we were starting "from scratch," it would be necessary first to compute orbital elements from observations uncorrected for perturbation, to compute perturbations using these approximate elements, and thirdly, having subtracted the perturbations out of the observations, and then to re-compute the orbital elements. This approximative process might have to be repeated several times. Hansen assumes, however, that Bouvard's orbital elements have already been corrected in this manner, and differ negligibly from those that would obtain if perturbations were absent.

With these elements, and the standard elliptic formulas as in (2.1a, b, c), (2.3), (2.4) above, Hansen can compute the solar distance of Jupiter or Saturn for a given angle of either true or mean anomaly, and thence determine the distance (Δ) between the two planets by the cosine law of trigonometry. From Δ, he can compute the corresponding numerical values of Ω_1 and Ω'_1. Likewise, after determining x, y, z and x', y', z' from the values of r, r', u, u', and I, he can compute Ω_2 and Ω'_2. In this case, however, Hansen first computes the quantity

$$\psi = -rr'\left\{\cos^2\left(\frac{I}{2}\right)\cos(u-u') + \sin^2\left(\frac{I}{2}\right)\cos(u+u')\right\}; \tag{4.26}$$

Ω_2 and Ω'_2 are then given, respectively, by $m'\psi/\mu r'^3$ and $m'\psi/\mu' r^3$.

[106] See Wilson (1980), 193–196.

[107] Namely, Bessel's (1828) and his 1826. Bessel had written earlier on this same topic; Whittaker and Robinson (1967) [a republication of the fourth edition (1944)], in their discussion of "Practical Fourier Analysis" (p. 264), specifically cite Bessel's account in the *Königsberger Beobachtungen,* I Abt. p. iii (1815).

[108] *Untersuchung*, p. 64. These elements are given in Bouvard's tables of 1821.

Hansen now divides each of the two orbits into 32 parts, with each part consisting of $360°/32 = 11°15'$ of mean anomaly, and the division beginning at perihelion in each orbit. He draws up a table giving the numerical values of $\log r$, r^2, u, $\log r'$, r'^2, u'.[109] (The quantities u, u', we recall, are $v - \Theta$, $v' - \Theta'$, where v, v' are the true longitudes, and Θ, Θ', 180° apart, are the ascending nodes of the one orbit on the other.) He is thus in a position to compute the numerical values of any periodical quantity, such as Ω_1. or ψ, corresponding to any pair of mean anomalies, g, g' among the selected dividing points. Early in his calculation, Hansen concluded that the division of Saturn's orbit could be reduced to 16 points without prejudice to the precision he was aiming at. The total number of pairs of points was thus $32 \times 16 = 512$.

For mechanical quadratures, the numerical values of the periodic quantity y must be computed for all 512 of these pairs of points. Next, the two integrations indicated in the double-integrals (4.25) must be carried out in sequence. One of the planets is assigned a fixed position, while the position of the other is allowed to vary. Consider, for instance, the case in which Saturn is assigned its perihelion position, for which $i' = 0$, while i, the index for Jupiter, goes from 0 to 31. Let the values of y for these 32 pairs of values be designated $Y_{0,0}, Y_{1,0}, Y_{2,0}, \ldots, Y_{31,0}$. Then by the precepts for mechanical quadrature,[110] the values of y, for any position of Jupiter when Saturn is at perihelion, are given by an expression of the form

$$y = \sum_{i=0} (i; c)_0 \cos ig + \sum_{j=1} (i; s)_0 \sin ig, \tag{4.27}$$

where the coefficients are given by

$$\begin{aligned}
(0; c)_0 &= \frac{1}{32} \sum_{i=0}^{31} Y_{i,0}, \\
(1; c)_0 &= \frac{1}{32} \sum_{i=0}^{31} Y_{i,0} \cos\left(\frac{1 \cdot 2i\pi}{32}\right), \ldots \\
(r; c)_0 &= \frac{1}{32} \sum_{i=0}^{31} Y_{i,0} \cos\left(\frac{r \cdot 2i\pi}{32}\right) \\
(1; s)_0 &= \frac{1}{32} \sum_{i=1}^{31} Y_{i,0} \sin\left(\frac{1 \cdot 2i\pi}{32}\right), \ldots \\
(r; s)_0 &= \frac{1}{32} \sum_{i=1}^{31} Y_{i,0} \sin\left(\frac{r \cdot 2i\pi}{32}\right).
\end{aligned} \tag{4.28}$$

109 *Untersuchung*, p. 193.

110 *A.N.*, Vol. 7 (1829), Nr. 168, sections 25–26, cols. 473–480; *Untersuchung*, pp. 49–63. A more easily accessible explanation will be found in Whittaker and Robinson (1967), p. 264ff.

Hansen computes these coefficients for integral values of $r = 1$ to $r = 16$. (A graph of y against g shows y varying smoothly between its maximum and its minimum, without apparent minor wobbles; the first terms of (4.23) are thus much the largest, and later coefficients in the series diminish rapidly to negligible size.) For $r = 2, 3$, etc., a set of algorithms, based on the periodic recurrence of values in the sequence of cosines and sines of successive multiples of the angle $2\pi/32$, reduces the labor of the computation.

The same task is to be repeated for the cases in which Saturn is held fixed in its other 15 pre-assigned positions, from $i' = 1$ to $i' = 15$. These computations yield the following 16×16 array of coefficients:

$$
\begin{array}{c}
(0; c)_0\,,\ (1; c)_0\,,\ (2; c)_0\,, \ldots \text{ to } (16; c)_0 \\
(0; c)_1\,,\ (1; c)_1\,,\ (2; c)_1\,, \ldots \text{ to } (16; c)_1 \\
(0; c)_2\,,\ (1; c)_2\,,\ (2; c)_2\,, \ldots \text{ to } (16; c)_2 \\
\ldots \text{ to} \\
(0; c)_{15}\,,\ (1; c)_{15}\,,\ (2; c)_{15}\,, \ldots \text{ to } (16; c)_{15}
\end{array}
$$

A similar array is formed by the coefficients $(i; s)_{i'}$, except that the coefficients $(16; s)_i$ are all zero.

Turning now to the second integration, Hansen forms the following coefficients, where i in each formula is a fixed integer, but there are as many formulas of the form $(i, r; c, c')$ or $(i, r; c, s')$ as values of i, namely 32:

$$
\begin{aligned}
(i, 0; c, c') &= \frac{1}{16} \sum_{i'=0}^{16} (i; c)_{i'}, \\
(i, 1; c, c') &= \frac{1}{16} \sum_{i'=0}^{16} (i; c)_{i'} \cos\left(\frac{1 \cdot 2i'\pi}{16}\right), \ldots, \\
(i, r; c, c') &= \frac{1}{16} \sum_{i'=0}^{16} (i; c)_{i'} \cos\left(\frac{r \cdot 2i'\pi}{16}\right); \\
(i, 1; c, s') &= \frac{1}{16} \sum_{i'=0}^{16} (i; c)_{i'} \sin\left(\frac{1 \cdot 2i'\pi}{16}\right), \ldots, \\
(i, r; c, s') &= \frac{1}{16} \sum_{i'=0}^{16} (i; c)_{i'} \sin\left(\frac{r \cdot 2i'\pi}{16}\right).
\end{aligned}
\tag{4.29}
$$

In an entirely similar manner are formed the coefficients $(i, i' : s, c')$ and $(i, i'; s, s')$, starting from the coefficients $(i; s)_{i'}$. The numerical integration has thus yielded the coefficients that would be expressed analytically by

$$
(i, i'; c, c') = \frac{1}{(2\pi)^2} \int\int y \cos ig \cdot \cos i'g' \cdot \mathrm{d}g \cdot \mathrm{d}g',
$$

$$
\begin{aligned}
(i,i';c,s') &= \frac{1}{(2\pi)^2}\int\int y\cos ig\cdot\sin i'g'\cdot dg\cdot dg',\\
(i,i';s,c') &= \frac{1}{(2\pi)^2}\int\int y\sin ig\cdot\cos i'g'\cdot dg\cdot dg',\\
(i,i';s,s') &= \frac{1}{(2\pi)^2}\int\int y\sin ig\cdot\sin i'g'\cdot dg\cdot dg'. \qquad (4.30)
\end{aligned}
$$

Hansen's aim, however, is to express y by a formula of the form of (4.24), with coefficients that would be expressed analytically by (4.25). But

$$
\begin{aligned}
(i,i';c) &= \frac{1}{(2\pi)^2}\int\int y\cos ig\cdot\cos i'g'\cdot dg\cdot dg' - \frac{1}{(2\pi)^2}\\
&\quad\times\int\int y\sin ig\cdot\sin i'g'\cdot dg\cdot dg',\\
(i,i';s) &= \frac{1}{(2\pi)^2}\int\int y\sin ig\cdot\cos i'g'\cdot dg\cdot dg' + \frac{1}{(2\pi)^2}\\
&\quad\times\int\int y\cos ig\cdot\sin i'g'\cdot dg\cdot dg',
\end{aligned}
$$

so that

$$
\begin{aligned}
\left(i,i';c\right) &= \left(i,i';c,c'\right) - \left(i,i';s,s'\right),\\
\left(i,i':s\right) &= (i\cdot i';s,c') + \left(i,i';c,s'\right).
\end{aligned}
$$

4.7 Bouvard's and Hansen's first-order perturbations of the longitude compared

After Hansen has described in detail his procedures for computing the first-order perturbations of Jupiter and Saturn, he presents, on pp. 73–79 of the *Untersuchung*, his calculated results for these perturbations. Are they accurate? How do they compare with Bouvard's results, published just 10 years earlier?

The following comparison is restricted to terms in the longitude. Bouvard supplies, in addition to the great inequality, 23 perturbational terms for Jupiter due to Saturn and 17 for Saturn due to Jupiter; each term is expressed as a sine term with an amplitude and a phase constant (in a few cases, the latter is equal to zero). Hansen at this point, however, presents each perturbational term as a sine–cosine pair of terms, without phase constants; of these pairs, we find 54 for Jupiter and 52 for Saturn. The net that Hansen is here casting thus has finer mesh.

To assess the accuracy of these results, a convenient and accurate standard is available in the perturbational formulas given by G.W, Hill in his *A New Theory of Jupiter and Saturn*, published 59 years after Hansen's *Untersuchung*.[111] Hill's theory was adapted to a large number of observations and became the basis of the tables of Jupiter

[111] Hill (1890, 1906).

and Saturn in the American *Nautical Almanac* from 1900 to 1959. In developing his theory, Hill follows Hansen's method. As we might have anticipated, however, in one respect, Hill's results are not fully comparable with Bouvard's and Hansen's: Bouvard and Hansen started their respective calculations with the same values (Bouvard's) for the planetary masses and for the orbital elements at the epoch of January 0, 1800, whereas Hill's calculation was based on later, improved values of these constants, with the epoch taken as January 0, 1850. As to the differences in orbital elements, no simple or convenient way exists for determining their effect on perturbational coefficients. Since the differences are small, we are probably safe in supposing the effects small.

In another respect, the difference in computational basis can be compensated for. As compared with Bouvard's values, Hill has increased Jupiter's relative mass in the ratio $1070.5/1047.9 = 1.02159$ and Saturn's relative mass in the ratio $3512/3501.6 = 1.00297$. The masses are linear factors in the perturbations that each planet causes in the other; hence, by multiplying Bouvard's and Hansen's perturbational coefficients for Saturn by 1.02159 and their perturbational coefficients for Jupiter by 1.00297, we shall obtain coefficients more nearly comparable to Hill's.

In the comparison, the perturbational terms of the three authors must be expressed in the same units, and as functions of the same variables.

Bouvard expresses perturbations in centesimal units, that is, in centesimal degrees of which there are 100 per quadrant, centesimal minutes of which there are 100 per centesimal degree, and centesimal seconds of which there are 100 per centesimal minute. Hansen and Hill use the traditional sexagesimal system, with 90 degrees per quadrant, 60 min/degree, and 60 s/min. (The French astronomers had adopted the centesimal system shortly after the founding of the Bureau des Longitudes in 1795, but officially abandoned it in the late 1820s; G.B. Airy called this the most retrograde step that had ever been taken in astronomy.) To turn centesimal degrees into sexagesimal degrees, the multiplying factor is 0.9; to turn centesimal arcseconds into sexagesimal arcseconds, the multiplying factor is 0.324.

In Bouvard's representation of perturbations, the variables are the mean longitudes of the planets, and for Jupiter and Saturn, these mean longitudes incorporate the "great inequality," and in Saturn's case a long-term perturbation due to Uranus. Thus, Bouvard gives the mean longitudes of Jupiter and Saturn as

$$\varphi = \varepsilon + nt + A, \quad \varphi' = \varepsilon' + n't + A',$$

where ε, ε' are the mean longitudes of the two planets at epoch; n, n' are their rates of mean motion, and A, A' are the long-term inequalities which Bouvard, following Laplace, includes in the mean motions. Hansen and Hill represent the perturbations in terms of the mean anomalies, which are given for each planet by the mean longitude minus the longitude of the perihelion. Hansen's mean anomalies for Jupiter and Saturn are consequently

$$g = \varepsilon + nt - \varpi, \quad g' = \varepsilon' + n't - \varpi',$$

where ε, ε', n, n' have the same meanings as for Bouvard, and ϖ, ϖ' are the longitudes of the two perihelia at epoch. It follows that

$$\varphi = g + \varpi + A, \quad \varphi' = g' + \varpi' + A'.$$

Using Bouvard's values for w, tu' and A, A' at epoch, that is at the beginning of 1800, and expressing these values in sexagesimal units, we find

$$\varphi = g + 11.^\circ 4558, \quad \varphi' = g' + 88.^\circ 3490.$$

These equations enable us to put Bouvard's perturbations in Hansen's form.

Bouvard gives no indication as to which of the terms in his tables incorporate second-order perturbations. In the *Mécanique Céleste*, Vol. 111, Laplace names, besides the great inequality, two inequalities in the theory of Jupiter and two inequalities in the theory of Saturn, as containing second-order perturbations;[112] we shall exclude those terms from our comparison.

Following Hansen, we give the coefficients in sexagesimal arcseconds to three decimal places (Hill gives four decimal places). The Bouvardian and Hansenian perturbations for Jupiter, with their errors, are as follows:

Arg.	Bouvard		Hansen		Bouvard–Hill		Hansen–Hill	
	sin	cos	sin	cos	sin	cos	sin	cos
$g' - g$	16.752	79.119	15.625	79.087	1.390	0.155	0.263	0.123
$2g' - 2g$	181.731	−84.910	178.977	−77.969	2.579	−8.092	−0.175	−1.151
$3g' - 3g$	10.382	12.676	8.891	13.715	1.629	−1.094	0.138	−0.055
$4g' - 4g$	−2.295	2.985	−2.225	2.794	−0.048	0.225	0.022	0.034
$5g' - 5g$	−1.660		−1.004		−0.666	0.130	−0.010	0.003
$6g' - 6g$	0.080	−0.401	0.091	−0.364	−0.017		−0.006	
$7g' - 7g$	0.165	−0.005	0.140	0.001	0.026	−0.010	0.001	−0.004
$2g' - g$	132.675	−3.872	123.446	3.989	9.130	−8.326	−0.099	0.465
$4g' - 2g$	−0.992	17.315	−2.260	16.446	1.328	0.879	0.060	0.010
$3g' - 2g$	−46.258	69.390	−49.869	66.287	3.862	3.430	0.251	0.327
$6g' - 4g$	0.546	1.485	0.431	1.488	0.121	0.000	0.006	0.003
$5g' - 3g$	−160.167	25.390	−154.954	12.613	−5.405	13.862	−0.192	1.085
$4g' - 3g$	14.220	5.695	13.777	6.417	0.518	−0.814	0.075	−0.092
$2g' - 3g$	11.028	−6.077	2.487	−1.428	8.543	−4.659	−0.018	−0.010
$3g' - g$	7.469	−5.879	12.427	−8.053	−5.001	2.115	−0.043	−0.059
g'	−7.673	7.959	−8.445	4.369	0.752	3.589	−0.020	−0.001
$2g'$	3.803	3.478	−1.269	1.277	5.069	2.203	−0.003	0.002
$g' - 2g$	−0.737	5.088	−0.244	1.385	−0.487	3.705	0.006	0.002
$3g' - 4g$	0.839	0.753	0.359	0.234	0.489	0.502	0.009	−0.017
$g' + g$	−0.760	0.178	−0.266		−0.495	0.253	−0.001	0.001
$6g' - 5g$	−0.878	0.158	−0.811	0.017	−0.066	0.151	−0.001	0.010
			RMS Error		4.525	4.132	0.107	0.368

[112] *Celestial Mechanics*, Vol. III, pp. 292–293, 309–310.

The corresponding numbers for Saturn are as follows:

Arg.	Bouvard		Hansen		Bouvard-Hill		Hansen–Hill	
	sin	cos	sin	cos	sin	cos	sin	cos
$g' - g$	−29.485	0.599	−6.345	−0.081	−22.969	1.520	0.171	0.840
$2g' - 2g$	−28.497	10.659	−29.493	12.817	1.024	−1.968	0.028	0.190
$3g' - 3g$	−4.237	−5.172	−3.900	−5.325	−0.385	0.171	−0.048	0.018
$4g' - 4g$	1.221	−1.587	1.275	−1.449	−0.067		−0.013	−0.016
$5g' - 5g$	0.645	0.293	0.583	0.318	0.067	−0.032	0.005	−0.007
$6g' - 6g$	−0.055		−0.070	0.241	0.015	−0.513	0.000	0.001
$7g' - 7g$	−0.120	0.004	−0.098		0.021	−0.009	0.001	0.004
$2g' - g$	−427.065	0.990	−423.899		−2.976	15.255	0.190	0.854
$4g' - 2g$	42.946	−680.062	78.400	−652.387	−37.975		−2.521	−1.152
$3g' - g$	−42.974	23.864	−28.067	20.248	−14.817	3.721	0.091	0.105
$-g$	1.411	11.399	1.057	11.696	0.395	−0.272	0.041	0.025
$4g' - 3g$	−4.631	−1.853	−4.280	−1.787	−0.369	−0.036	−0.018	0.030
$g' - 2g$	−2.549	−1.702	−0.855	−2.526	−1.704	0.845	−0.010	0.021
$5g' - 3g$	−2.948	0.508	−2.983	0.391	0.030	0.136	−0.005	0.019
$5g' - 4g$	0.223	−1.429	0.249	−1.296	−0.037	−0.136	−0.011	−0.003
				RMS Error	12.117	8.498	0.655	0.433

The root-mean-square errors, given in the final row of each table, show the greater accuracy of Hansen's perturbational coefficients. Besides the terms in these tables and the three Bouvardian terms excluded from each table, Hansen has calculated additional first-order terms, 30 in the case of Jupiter and 33 in the case of Saturn. The coefficients of these additional terms are close in value to Hill's coefficients: in the case of Jupiter, the root-mean-square average of the differences (Hansen–Hill) for the sine terms is $0''.005$ and for the cosine terms, $0''.009$. The corresponding numbers for Saturn are $0''.009$ and $0''.019$. Grouping these additional sine–cosine pairs according to the size of the largest coefficient in each, we find the following:

	$c > 1''.0$	$1''.0 > c > 0''.1$	$0''.1 > c > 0''.01$	$0''.01 > c$
Jupiter	4	13	12	1
Saturn	4	12	14	3

Hill gives, in addition to the terms given by Bouvard and Hansen, 50 sine–cosine pairs each for Jupiter and Saturn, a large majority of them with coefficients less than $0''.01$.

Laplace, in calculating the perturbations of Jupiter and Saturn, aimed to achieve a final precision of 1 centesimal arcsecond or $0''.324$ sexagesimal arcseconds.[113] By a comparison of Bouvard's perturbational terms with Hill's, I find that Bouvard has

[113] *Ibid.*, Vol. III, 275, 299.

omitted 10 terms in the theory of Jupiter and 6 terms in the theory of Saturn, with coefficients exceeding this lower bound. All these terms are included by Hansen, along with others still smaller.

The chief advantage of Hansen's method, according to Hansen, was in introducing systematic procedures for the calculation of second- and higher-order perturbations. However, his calculation of first-order perturbations was also superior to the Laplace–Bouvard calculation, here too because of its systematic character.

5 Hansen's *Untersuchung:* second-order perturbations

5.1 Second-order perturbations of the longitude

For the calculation of the second-order perturbations, Hansen returns to equation (3.16), here repeated as (5.1):

$$-\frac{\mathrm{d}.\left[\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial\tau}\right]}{\mathrm{d}\tau}=\left\{\frac{2\rho}{r}\cos(v-\lambda)-1+\frac{2\rho}{a(1-e^2)}[\cos(v-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial v}$$
$$+\frac{2\rho}{r}\sin(v-\lambda)\frac{an}{\sqrt{1-e^2}}r\frac{\partial\Omega}{\partial r}. \tag{5.1}$$

The expression on the left-hand side of this equation has previously been symbolized as T. Hansen now divides T into two parts:

$$T=T_1+T_2,$$

where T_1 represents quantities of the first order—already determined in the earlier calculation—and T_2, quantities of the second order, now to be determined.

As (5.1) shows, T, and hence T_2, is a function of λ, v, ρ, r, and the expressions involving a and $(1-e^2)$; moreover, through the partial derivatives of Q, it is also a function of v', r' and of the two variables P and Q, which determine the intersection and inclination of the orbital planes of the perturbed planet m and the perturbing planet m' (For the sake of simplicity, we consider in this account only one perturbing planet, whereas Hansen's account allows for a plurality of perturbing planets.)

The variable λ, as determined in the first-order calculation, is a function of $n\zeta$, which has undergone the increment $n(\zeta)=n\zeta-n\tau$ due to perturbation. In a parallel way, the variable v is a function of nz, which as determined in the first-order calculation has undergone the increment $n(z)=nz-nt$. Similarly, the variable v' is a function of $n'z'$, which through first-order perturbation has undergone the increment $n'(z')=n'z'-n't$, and this increment will have an effect on the second-order perturbations of m. The variables $\ln\rho$, $\ln r$, $\ln r'$, owing to perturbation, have undergone the increments $\ln(\rho)$, $\ln(r)$, and $\ln(r')$, respectively. Taking account of all the increments due to the first-order perturbations, and remembering that $\gamma=n\tau+\varepsilon-\tilde{\omega}$, so that $\mathrm{d}\cdot n\tau=\mathrm{d}\gamma$, and that $g=nt+\varepsilon-\tilde{\omega}$, so that $\mathrm{d}\cdot nt=\mathrm{d}g$, Hansen infers by Taylor's theorem that

$$
\begin{aligned}
T_2 = {} & \frac{\mathrm{d}T_1}{\mathrm{d}\gamma} n(\zeta) + \frac{\mathrm{d}T_1}{\mathrm{d}g} n(z) + \frac{\mathrm{d}T_1}{dg'} n'(z') \\
& + \frac{\mathrm{d}T_1}{\mathrm{d}\ln\rho} \ln(\rho) + \frac{\mathrm{d}T_1}{\mathrm{d}\ln r} \ln(r) + \frac{\mathrm{d}T_1}{\mathrm{d}\ln r'} \ln(r') \\
& + \frac{\mathrm{d}T_1}{\mathrm{d}P} \delta\mathrm{P} + \frac{\mathrm{d}T_1}{\mathrm{d}Q} \delta Q \\
& + \frac{\mathrm{d}T_1}{\mathrm{d.ln.}\frac{an}{\sqrt{1-e^2}}} \delta.\mathrm{ln.}\frac{an}{\sqrt{1-e^2}} + \frac{\mathrm{d}T_1}{\mathrm{d.ln.}\frac{1}{a(1-e^2)}} \delta.\mathrm{ln.}\frac{1}{a(1-e^2)}
\end{aligned}
\tag{5.2}
$$

For the numerical calculation of (5.22), a number of preparations are necessary. Hansen first calculates separately the parts of $\frac{\mathrm{d}T_1}{\mathrm{d}\ln r}$, $\frac{\mathrm{d}T_1}{\mathrm{d}\ln r'}$ that come from the differentiation of Q, calling them V and V':

$$
\begin{aligned}
V = {} & \left\{ \frac{2\rho}{r} \cos(v-\lambda) - 1 + \frac{2\rho}{a(1-e^2)} [\cos(v-\lambda) - 1] \right\} \frac{an}{\sqrt{1-e^2}} r \frac{\partial^2\Omega}{\partial v \partial r} \\
& + \frac{2\rho}{r} \sin(v-\lambda) \frac{an}{\sqrt{1-e^2}} \left\{ r^2 \frac{\partial^2\Omega}{\partial r^2} + r \frac{\partial\Omega}{\partial r} \right\}, \\
V' = {} & \left\{ \frac{2\rho}{r} \cos(v-\lambda) - 1 + \frac{2\rho}{a(1-e^2)} [\cos(v-\lambda) - 1] \right\} \frac{an}{\sqrt{1-e^2}} r' \frac{\partial^2\Omega}{\partial v \partial r'} \\
& + \frac{2\rho}{r} \sin(v-\lambda) \frac{an}{\sqrt{1-e^2}} r r' \frac{\partial^2\Omega}{\partial r \partial r'}.
\end{aligned}
\tag{5.3}
$$

He shows that the derivatives of Ω present in V' need not be computed separately once those present in V have been computed, since by Euler's theorem for homogeneous functions

$$
-\Omega = r\frac{\partial\Omega}{\partial r} + r'\frac{\partial\Omega}{\partial r'};
$$

whence

$$
\begin{aligned}
& -\frac{\partial\Omega}{\partial v} = r\frac{\partial^2\Omega}{\partial r \partial v} + r'\frac{\partial^2\Omega}{\partial r' \partial v} \\
\text{and} \quad & -r\frac{\partial\Omega}{\partial r} = \left\{ r^2 \frac{\partial\Omega}{\partial r^2} + r\frac{\partial\Omega}{\partial r} \right\} + rr'\frac{\partial^2\Omega}{\partial r \partial r'}.
\end{aligned}
$$

When the number of perturbing planets is greater than one, the calculation can again be simplified by means of Euler's theorem.[114] Introducing the definition

$$
U = 2n\frac{\rho}{(1-e^2)^{3/2}} [\cos(v-\lambda) - 1] \frac{\partial\Omega}{\partial v},
\tag{5.4}
$$

[114] *Untersuchung*, pp. 83–85.

and reminding us that by equation (4.6) above

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{an}}{\sqrt{1-e^2}} \frac{\partial \Omega}{\partial v},$$

Hansen next sets forth the following results, which are easily derived:

$$\begin{aligned} \frac{\mathrm{d}T_1}{\mathrm{d}\ln\rho} &= \rho\frac{\mathrm{d}T_1}{\mathrm{d}\rho} = T_1 + \frac{\mathrm{d}S}{\mathrm{d}t}, \\ \frac{\mathrm{d}T_1}{\mathrm{d}\ln r} &= r\frac{\mathrm{d}T_1}{\mathrm{d}r} = -T_1 + U - \frac{\mathrm{d}S}{\mathrm{d}t} + V, \\ \frac{\mathrm{d}T_1}{\mathrm{d}\ln r'} &= r'\frac{\mathrm{d}T_1}{\mathrm{d}r'} = V', \\ \frac{\mathrm{d}T_1}{\mathrm{d}.\ln\left(\frac{1}{a(1-e^2)}\right)} &= U, \qquad \frac{\mathrm{d}T_1}{\mathrm{d}.\ln\left(\frac{an}{\sqrt{1-e^2}}\right)} = T_1. \end{aligned} \tag{5.5}$$

According to Hansen, the increments $\delta.\ln.\frac{an}{\sqrt{1-e^2}}$, $\delta.\ln.\frac{1}{a(1-e^2)}$ are, respectively, equal to $-S$ and $-2S$. For proof, he expresses the time derivatives of the logarithms in terms of the time derivatives of $\ln a$ and e:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{\mathrm{an}}{\sqrt{1-\mathrm{e}^2}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{an}{\sqrt{a(1-e^2)}}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{1}{a(1-e^2)}\right) \\ &= -\frac{1}{2}\frac{\mathrm{d}\ln a}{\mathrm{d}t} + \frac{e}{1-e^2}\frac{\mathrm{d}e}{\mathrm{d}t}. \end{aligned}$$

The final expression here can then be shown, by way of our equations (3.1a), (3.11a,b), (3.15a), to be reducible to $-\frac{an}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial v} = -\frac{\mathrm{d}S}{\mathrm{d}t}$. The increments of the logarithmic expressions are given by the time integral of $(-\mathrm{d}S/\mathrm{d}t)$.

As indicated in Sect. 4.4, the partial derivatives of Ω with respect to P and Q are

$$\frac{\partial\Omega}{\partial P} = \frac{\partial\Omega}{\partial\Theta}\frac{1}{\sin I}, \qquad \frac{\partial\Omega}{\partial Q} = \frac{\partial\Omega}{\partial I}\frac{1}{\cos I};$$

which expressions can be differentiated with respect to v and r. The derivatives of T_1 with respect to P and Q are therefore

$$\begin{aligned} \frac{\mathrm{d}T_1}{\mathrm{d}P} &= \left\{\frac{2\rho}{r}\cos(v-\lambda) - 1 + \frac{2\rho}{a(1-e^2)}[\cos(v-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\frac{\partial^2\Omega}{\partial v\partial\Theta}\frac{1}{\sin I} \\ &+\frac{2\rho}{r}\sin(v-\lambda)\frac{an}{\sqrt{1-e^2}}\ r\frac{\partial^2\Omega}{\partial r\partial\Theta}\frac{1}{\sin I}, \\ \frac{\mathrm{d}T_1}{\mathrm{d}Q} &\left\{\frac{2\rho}{r}\cos(v-\lambda) - 1 + \frac{2\rho}{a(1-e^2)}[\cos(v-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\frac{\partial^2\Omega}{\partial v\partial I}\frac{1}{\cos I} \\ &+\frac{2\rho}{r}\sin(v-\lambda)\frac{an}{\sqrt{1-e^2}}\ r\frac{\partial^2\Omega}{\partial r\partial I}\frac{1}{\cos I}. \end{aligned} \tag{5.6}$$

Finally, we must identify δP and δQ in (5.2). The variables P and Q are defined in terms of the mutual inclination I of the orbital planes of the perturbing and perturbed planets, and the ascending node Θ of the orbital plane of the perturbing planet (m') on the orbital plane of the perturbed planet (m): $P = \sin \mathrm{I} \sin \Theta$, $Q = \sin \mathrm{I} \cos \Theta$. Equation (3.24) of Sect. 3.4 gives P and Q in terms of p, q and p', q' the variables that relate the orbital planes of m and m' to a fixed xy plane. The larger (first-order) terms of (3.24) are

$$P = p' - p, \quad Q = q' - q; \tag{α}$$

the smaller terms, which are of the third order, can according to Hansen be neglected, because the angle of inclination of the orbital planes of Jupiter and Saturn is so small.

As the choice of the xy plane is arbitrary, this plane can be identified with the orbital plane of m at the epoch. With this identification, p, q come to express the *change* in orientation of this plane since the epoch, because at the epoch $p = q = 0$. In order to express the latitude of the second planet m' in a similar way, Hansen would like to express by the symbols p', q' the *change* in orientation of the orbital plane of m' since the epoch. But p', q' as used in (3.24) and (α) above relate the orbital plane of m' to a fixed xy plane, just now identified with the orbital plane of m at the epoch. A change of symbols is therefore required. In a first step, Hansen replaces the p', q' of (3.24) and (α) by new symbols, x' and y'. Equations (α) become

$$P = x' - p, \quad Q = y' - q. \tag{β}$$

Next, he reintroduces the symbols p' and q', so defined that, at the epoch, $p' = q' = 0$. To relate x', y' to the new p', q', we must take account of the angle (I) between the two orbital planes at the epoch, and the orientation at the epoch of the nodal line in which they intersect; as the measure of the latter, we choose the longitude of the ascending node of the orbit of m on the orbit of m', symbolized by (Θ'). Let the variables determining these relations be (P) and (Q), defined by $(P) = \sin(I)\sin(\Theta')and(Q) = \sin(I)\cos(\Theta')$. Then the new p', q' will be given to first order, in accordance with (3.24), by

$$p' = x' - (P), \quad q' = y' - (Q). \tag{γ}$$

Using (γ) to replace x', y' in (β), we obtain

$$P = (P) + p' - p, \quad Q = (Q) + q' - q. \tag{δ}$$

Since (P) and (Q) are constants, and p', q' like p, q now measure changes since the epoch, we can express the changes in P and Q since epoch by

$$\delta P = p' - p, \quad \delta Q = q' - q. \tag{ε}$$

But Hansen would like the longitudes of both m and m' to be measured from the same starting point, and this can be accomplished by shifting the node used in x', y', and hence in p', q', 180° from the ascending node θ' of the orbit of m on the orbit of m' to the ascending node θ of the orbit of m' on the orbit of m. This shift changes p' and q' into $-p'$ and $-q'$. Hansen's final form for (ε) is thus:[115]

$$\delta P = -(p' + p), \quad \delta Q = -\left(q' + q\right). \tag{5.7}$$

With these preparations, (5.2) can be put in the form

$$\begin{aligned} T_2 = {} & \frac{\mathrm{d}T_1}{\mathrm{d}\gamma} n\,(\zeta) - \frac{1}{2}\left(T_1 + \frac{\mathrm{d}S}{\mathrm{d}t}\right)\frac{\partial(\zeta)}{\partial\tau} + \frac{\mathrm{d}T_1}{\mathrm{d}g} n(z) + \frac{\mathrm{d}T_1}{\mathrm{d}g'} n'\left(z'\right) \\ & \times\left(V - T_1 + U - \frac{\mathrm{d}S}{\mathrm{d}t}\right)\ln(r) \\ & + V'\ln\left(r'\right) + \left(\frac{1}{2}\frac{\mathrm{d}S}{\mathrm{d}t} - \frac{T_1}{2} - 2U\right)S - \frac{\mathrm{d}T_1}{\mathrm{d}P}\left(p' + p\right) - \frac{\mathrm{d}T_1}{\mathrm{d}Q}\left(q' + q\right) \end{aligned} \tag{5.8}$$

We must now see how T_2 will affect the determination of ζ. From $\frac{\mathrm{d}[\frac{\delta\zeta}{\delta t}/\frac{\delta\zeta}{\delta\tau}]}{\mathrm{d}\tau} = T_1 + T_2$ it follows that

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \frac{\delta\zeta}{\delta\tau}\int \mathrm{T}_1\mathrm{d}\tau + \frac{\delta\zeta}{\delta\tau}\int T_2\mathrm{d}\tau.$$

But $\dfrac{\delta\zeta}{\delta\tau} = 1 + \dfrac{\delta(\zeta)}{\delta\tau}.$

Hence $$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = \int T_1 d\tau + \int T_2\mathrm{d}\tau + \frac{\delta(\zeta)}{\delta\tau}\int T_1\mathrm{d}\tau + \frac{\delta(\zeta)}{\delta\tau}\int T_2\mathrm{d}\tau. \tag{5.9}$$

In (5.9), the first term on the right, $\int T_1\mathrm{d}\tau$, gives all and only the first-order perturbations in the longitude; this term can therefore be ignored in the second-order calculation. But to obtain *all* the second-order terms, a further differentiation with respect to x is necessary; the result is

$$\begin{aligned} \frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau\mathrm{d}t} = {} & T_2 + \frac{\partial^2\zeta}{\partial\tau^2}\int T_1 d\tau + \frac{\partial(\zeta)}{\partial\tau}T_1 \\ & + \frac{\partial^2\zeta}{\partial\tau^2}\int T_2 d\tau + \frac{\partial(\zeta)}{\partial\tau}T_2. \end{aligned} \tag{5.10}$$

The second and third terms on the right-hand side are second-order terms emerging from the second differentiation. (Note that, because $\mathrm{d}(\zeta)/\mathrm{d}\tau = 1 + \mathrm{d}\zeta/\mathrm{d}\tau$, the second derivatives of ζ and (ζ) with respect to x are identical.) The last two terms on the right-hand side of (5.10) are of the third order, hence to be neglected in the present calculation.

[115] I have entered into considerable detail to explain Hansen's derivation of δP and δQ; his own account is obscure.

Introducing the terms of T_2 given in (5.8) into (5.10), we obtain

$$\frac{d^2\zeta}{d\tau dt} = \frac{\partial^2\zeta}{\partial\tau^2}\int T_1 d\tau + \frac{1}{2}\frac{\partial(\zeta)}{\partial\tau}T_1 - \frac{1}{2}\frac{\partial(\zeta)}{\partial\tau}\frac{dS}{dt} + \frac{dT_1}{d\gamma}n(\zeta) + \frac{dT_1}{dg}n(z)$$
$$+\left\{V - T_1 + U - \frac{dS}{dt}\right\}\ln(r) + \left\{\frac{1}{2}\frac{dS}{dt} - \frac{1}{2}T_1 - 2U\right\}S$$
$$+\frac{dT_1}{dg'}n'\left(z'\right) + V'\ln\left(r'\right) - \frac{dT_1}{dP}\left(p'+p\right) - \frac{dT_1}{dQ}\left(q'+q\right). \qquad (5.11)$$

Every term on the right-hand side of (5.11) is a product of two factors. To prepare (5.11) for numerical calculation, Hansen reduces each factor to the form $\sum[p_s\sin(\kappa\gamma + ig + i'g') + p_c\cos(\kappa\gamma + ig + i'g')]$, where the coefficients p_s and p_c are generally constants but may involve t or t^2. The factors T_1, $n(z)$, $\ln(r)$ have already been reduced to the required form in the first-order calculation. Moreover, from the series expansion for T_1, the derivatives $dT_1/d\gamma, dT_1/dg, dT_1/dg'$ can be immediately derived. The required formulas for (ζ) and its first and second partial derivatives are obtainable from equations (4.3) and (4.4) in Sect. 4.2. The first-order perturbations of the perturbing planet m' as expressed in $n'(z')$, $\ln(r')$, p', q' are presupposed in (5.11), and so must be calculated. Starting from the definition of S in (4.8), namely $S = 2\ln(\rho) + d(\zeta)/d\tau$, Hansen develops the series expression $S = \sum[N_c\cos(ig + i'g') + N_s\sin(ig + i'g')]$, from which dS/dt can be immediately derived. For the reduction in certain further quantities—of V, V' as given in (5.3), of U as given in (5.4), and of dT_1/dP, dT_1/dQ as given in (5.6)—to the proper form, Hansen has recourse to the Besselian expansions introduced in Sect. 4.2 above. We omit the details.

Once the factors are in the required form, they must be multiplied to form the products indicated in (5.11). Each factor being in the form of a potentially infinite but apparently converging series, the product will consist of a potential infinity of product terms. Suppose a term in one factor of one of these products is $(p_c\cos a + q_s\sin a)$, and a term in the other factor is $(q_s\sin b + q_c\cos b)$, their product will be

$$\left(\frac{1}{2}p_c q_s + \frac{1}{2}p_s q_c\right)\sin(a+b) + \left(\frac{1}{2}p_c q_c + \frac{1}{2}p_s q_s\right)\cos(a+b)$$
$$+\left(\frac{1}{2}p_c q_s - \frac{1}{2}p_s q_c\right)\sin(-a+b) + \left(\frac{1}{2}p_c q_c + \frac{1}{2}p_s q_s\right)\cos(-a+b). \qquad (5.12)$$

On forming this product, Hansen adds together the logarithms of the numbers entering into the products contained in the coefficients; and in the sum, he includes the logarithm of 206,265, the number of arcseconds per radian, so that the coefficient will be given in arcseconds. The numerical result in each case shows whether the particular product leads to a significant result, larger than the number that Hansen has set as a lower bound. The first product calculated will be that of the initial and hence largest terms in the series expression of either factor; the products later calculated will be of the successively smaller terms in either factor. As Hansen remarks, "...here there is no seeking out of terms that can give a significant result, but those that cannot be neglected emerge as though of themselves."[116] It is the converging in the component series, and the continuity of Hansen's process in forming the term-by-term sub-products, that underlie his confidence in not missing any significant term.

As already mentioned, some of the coefficients p_s, p_c, q_s, q_c will contain t or t^2 as a factor; the result will be a term in the product whose coefficient is subject to secular variation. Another circumstance in which t^2 can emerge outside the arguments of sine or cosine is when the two terms multiplied to form a product both contain the same sinusoidal factor, call it cos a or sin a. The product will contain $\cos^2 a = (1 + \cos 2a)/2$ or $\sin^2 a = (1 - \cos 2a)/2$; in either case a constant, say α, appears in (5.11). The double-integration that (5.11) requires then leads to a term αt^2 in ζ. As Hansen points out, when all such terms are added up, the net result should be zero. For, a term of the form αt^2 in ζ would imply a term proportional to the time in the semi-major axis; but such a consequence is impossible. Poisson first, and later Lagrange and Laplace, proved that, to the order of the squares and products of the perturbing forces, no such term can arise. Hansen proposes that all such terms produced in the calculation be collected and added together, as a check on the accuracy of the calculation. The sum cannot be expected to be exactly zero, but it should approach zero to within the limits of accuracy set for the calculation.

In this calculation, Hansen stresses the importance of systematic procedures guided by clear schemata and multiple cross-checks on the correctness of the calculation.[117] Mitigating the labor is the fact that the second-order contributions to perturbational terms are smaller than the first-order contributions, so that fewer significant figures need be retained. That the second-order contributions should be smaller is not *a priori* necessary, but the calculator finds it so in the perturbations encountered in the solar system. Where he not to find it so, Hansen observes, the method of successive approximations would not be applicable.[118]

Equation (5.11), after the reductions described, has the general form[119]

$$\begin{aligned}\frac{d^2\zeta}{d\tau dt} &= n\alpha_s \sin(ig + i'g') + n\alpha_c \cos(ig + i'g')\\ &+n\beta_s \sin(\kappa\gamma + ig + i'g') + n\beta_c \cos(\kappa\gamma + ig + i'g') n\varepsilon_s (n\tau - nt)\\ &\sin(\kappa\gamma + ig + i'g') + n\varepsilon_c (n\tau - nt) \cos(\kappa\gamma + ig + i'g'). \qquad (5.13)\end{aligned}$$

This equation, when integrated by parts with respect to x, yields

$$\begin{aligned}\frac{d\zeta}{dt} &= -\left\{\frac{1}{\kappa}\beta_s - \frac{1}{\kappa^2}\varepsilon_c\right\} \cos(\kappa\gamma + X) + \left\{\frac{1}{\kappa}\beta_c + \frac{1}{\kappa_2}\varepsilon_s\right\} \sin(\kappa\gamma + X)\\ &+\alpha_c (n\tau - nt) \cos X + \alpha_s (n\tau - nt) \sin X\end{aligned}$$

[116] *Untersuchung*, p. 103.

[117] See especially *Untersuchung*, pp. 101–108.

[118] *Untersuchung* , p. 93.

[119] *Untersuchung*, p. 113. Equations (5.14) and (5.15) are found on p. 113, and equations (5.16) and (5.16a) on the following page.

$$-\frac{1}{\kappa}\varepsilon_s(n\tau - nt)\cos(\kappa\gamma + X) + \frac{1}{\kappa}\varepsilon_c(n\tau - nt)\sin(\kappa\gamma + X) + \text{Const.}, \tag{5.14}$$

where X stands for $ig+i'g'$. The additive constant must be such that, when τ is changed into t in the preceding terms of (5.14), the whole of (5.14) vanishes. Therefore, the constant can be obtained by changing τ into t in the preceding terms and reversing their signs. The constant having been thus determined, (5.14) takes the form

$$\begin{aligned}\frac{\mathrm{d}\zeta}{\mathrm{d}t} &= A_c \cos(\kappa\gamma + X) + A_s \sin(\kappa\gamma + X) \\ &\quad + B_c(n\tau - nt)\cos(\kappa\gamma + X) + B_s(n\tau - nt)\sin(\kappa\gamma + X) \\ &\quad + C_c nt\cos(\kappa\gamma + X) + C_s nt \sin(\kappa\gamma + X) \\ &\quad + D_c(n\tau - nt)nt\cdot\cos(\kappa\gamma + X) + D_s(n\tau - nt)nt\cdot\sin(\kappa\gamma + X).\end{aligned} \tag{5.15}$$

Integrating (5.15) by parts, Hansen obtains

$$\begin{aligned} n\zeta &= \left\{fA_c - f^2B_s + f^2C_s + 2f^3D_c\right\}\sin(\kappa\gamma + X) \\ &\quad - \left\{fA_s + f^2B_c - f^2C_c + 2f^3D_s\right\}\cos(\kappa\gamma + X) \\ &\quad + \left\{fB_c + f^2D_s\right\}(n\tau - nt)\sin(\kappa\gamma + X) - \left\{fB_s - f^2D_c\right\} \\ &\quad \times (n\tau - nt)\cos(\kappa\gamma + X) \\ &\quad + \left\{fC_c - f^2D_s\right\} nt\cdot\sin(\kappa\gamma + X) - \left\{fC_s + f^2D_c\right\} nt\cdot\cos(\kappa\gamma + X) \\ &\quad + fD_c(n\tau - nt)nt\cdot\sin(\kappa\gamma + X) - fD_s(n\tau - nt)nt\cdot\cos(\kappa\gamma + X),\end{aligned} \tag{5.16}$$

where f has been written for $n/(in + i'n')$. In the particular case where $i - i' = 0$, so that $X = 0$, (5.16) ceases to hold, and the correct integral is

$$\begin{aligned} n\zeta &= A_s nt\cdot\sin\kappa\gamma + A_c nt\cdot\cos\kappa\gamma \\ &\quad + B_s(n\tau - nt)nt\cdot\sin\kappa\gamma + B_c(n\tau - nt)nt\cdot\cos\kappa\gamma \\ &\quad + \frac{1}{2}\{C_s + B_s\}n^2t^2\sin\kappa\gamma + \frac{1}{2}\{C_c + B_c\}n^2t^2\cos\kappa\gamma.\end{aligned} \tag{5.16a}$$

Just as in the first-order integration leading from $\mathrm{d}\zeta/\mathrm{d}t$ to ζ [see Eq. (4.3] of Sect. 4.2), so here, when not only i and i' are zero, but κ is also zero, there emerge terms in (5.16a) of the form (coefficient) $\cdot nt\cdot\cos 0 = c'nt$, where c' is a constant. The effect is to cause the observational value of the mean motion to differ from the mean motion that would obtain in the absence of perturbation. To (4.3), Hansen added the integrative constant $n(1-c)\tau$, in order that n could be assigned its observational value. With the same end in view here, he treats c' as a correction to c, and in (5.16) makes the additive constant $-c'n\tau$.

5.2 Second-order perturbations of the radius vector

We have seen how, in determining the first-order perturbations of the logarithm of the radius vector (in Sect. 4.3), Hansen made use of Eq. (3.18) from Sect. 3.3 above, first deleting the second term on the left and setting $\partial\zeta/\partial\tau = 1$. For the second-order perturbations, he again turns to Eq. (3.18); but now he deletes nothing, and allows for the increment $\partial(\zeta)/\partial\tau$ in the value of $\partial\zeta/\partial\tau$. The equation can be written

$$2\frac{\partial \ln(\rho)}{\partial\tau} - 2\frac{\frac{\partial \ln(\rho)}{\partial\tau}\frac{\partial\zeta}{\partial t}}{\frac{\partial\zeta}{\partial\tau}} = \frac{\mathrm{d}S}{\mathrm{d}t} - \frac{\frac{\partial^2\zeta}{\partial\tau\partial t}}{\frac{\partial\zeta}{\partial\tau}} + \frac{\frac{\partial^2\zeta}{\partial\tau^2}\frac{\partial\zeta}{\partial t}}{\left(\frac{\partial\zeta}{\partial\tau}\right)^2}, \tag{5.17}$$

where the last two terms on the right are the result of carrying out the differentiation with respect to τ indicated in (3.18) by $\frac{\mathrm{d}.[\frac{\partial\zeta}{\partial t}/\frac{\partial\zeta}{\partial\tau}]}{\mathrm{d}\tau}$.

Multiplying through by $\partial\zeta/\partial\tau$, and rearranging terms, we can put (5.17) in the form

$$\left\{2\frac{\mathrm{d}.\ln(\rho)}{\mathrm{d}t} - \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\frac{\partial^2\zeta}{\partial\tau\partial t}}{\frac{\partial\zeta}{\partial\tau}}\right\}\frac{\partial\zeta}{\partial\tau} = \left\{2\frac{\partial.\ln(\rho)}{\partial\tau} + \frac{\frac{\partial^2\zeta}{\partial\tau^2}}{\frac{\partial\zeta}{\partial\tau}}\right\}\frac{\partial\zeta}{\partial t}.$$

This equation—since S is free of τ—can be rewritten as

$$\frac{\mathrm{d}.\left\{2\ln(\rho) - S + \ln.\frac{\partial\zeta}{\partial\tau}\right\}}{\mathrm{d}t}\frac{\partial\zeta}{\partial\tau} = \frac{\mathrm{d}.\left\{2\ln(\rho) - S + \ln.\frac{\partial\zeta}{\partial\tau}\right\}}{\mathrm{d}\tau}\frac{\partial\zeta}{\partial t}. \tag{5.18}$$

Equation (5.18) is integrable, yielding

$$2\ln(\rho) - S + \ln.\frac{\partial\zeta}{\partial\tau} = \phi(\zeta). \tag{5.19}$$

With a view to determining the nature of $\phi(\zeta)$, Hansen considers that, in the absence of perturbing forces, $S = 0$, and $\partial\zeta/\partial\tau = 1$, so that $\ln(\partial\zeta/\partial\tau) = 0$ and $\zeta = \tau$. Consequently, $\phi(\zeta) = \phi(\tau)$ and $2\ln(\rho) = \phi(\tau)$. According to Hansen,

> Here $\ln(\rho) = 0$, or rather is equal to a constant, because we have stipulated that in the calculation of $\ln(\rho)$, the value of ($\ln a$) to be applied is not the purely elliptical value, but rather the value that is given by the observationally determined value of n.[120]

Let us first observe that, besides ($\ln a$), $\ln(\rho)$ contains two other terms:

$$\ln(\rho) = \ln a + \ln\left(1 - e^2\right) - \ln(1 + e\cos[\lambda - \varpi]).$$

120 *Untersuchung*, p. 116.

Hansen passes over the second and third terms on the right without mention. The second differs little from zero; the third is confined to values between $\ln(1+e)$ and $\ln(1-e)$, never far from zero. According to the passage just quoted, the first term, ln a, differs from zero because a, derived from the observationally determined value of n, differs from a_0 the value of the semi-transverse axis that would obtain if perturbation were absent (Hansen has evidently normalized a_0 to 1.) The first term outweighs the others, and its constancy is the ground Hansen gives for asserting that, in the absence of perturbation, $\ln(\rho)$ is a constant.

From the constancy of $\ln(\rho)$, Hansen infers that $\phi(\tau)$ is a constant and concludes that $\phi(\zeta)$ must also be a constant.[121] Hence,

$$\ln(\rho) = \frac{1}{2}S - \frac{1}{2}\ln.\frac{\delta\zeta}{\delta\tau} + \text{Const.} \tag{5.20}$$

Now

$$\frac{\partial\zeta}{\partial\tau} = 1 + \frac{\partial\,(\zeta)}{\partial\tau}, \text{ and } \ln\left(1+\frac{\partial(\zeta)}{\partial\tau}\right) = \frac{\partial\zeta}{\partial\tau} - \frac{1}{2}\left(\frac{\partial\,(\zeta)}{\partial\tau}\right)^2 + \frac{1}{3}\left(\frac{\partial(\zeta)}{\partial\tau}\right)^3 \pm \text{etc.}$$

Hansen substitutes the first two terms of this series—omitting the third-order and higher-order terms—for the logarithm on the right- hand side of (5.20), and further asks us to suppose that the constant has been incorporated in S:

$$\ln(\rho) = \frac{1}{2}S - \frac{1}{2}\frac{\partial(\zeta)}{\partial\tau} + \frac{1}{4}\left(\frac{\partial(\zeta)}{\partial\tau}\right)^2. \tag{5.21}$$

Then, he eliminates S by differentiating (5.21) with respect to τ:

$$\frac{d.\ln(\rho)}{dt} = -\frac{1}{2}\frac{\partial^2(\zeta)}{\partial\tau^2} + \frac{1}{2}\frac{\partial(\zeta)}{\partial\tau}\frac{\partial^2\zeta}{\partial\tau^2}.$$

Changing τ into t in this equation yields

$$\frac{\mathrm{d}.\ln(\rho)}{\mathrm{d}t} = -\frac{1}{2}\overline{\frac{\partial^2\zeta}{\partial\tau^2}} + \frac{1}{2}\overline{\frac{\partial(\zeta)}{\partial\tau}}\,\overline{\frac{\partial^2\zeta}{\partial\tau^2}}. \tag{5.22}$$

The integral of (5.22) is

$$\ln(r) = -\frac{1}{2}\int\overline{\frac{\partial^2\zeta}{\partial\tau^2}}\mathrm{d}t + \frac{1}{2}\int\overline{\frac{\partial(\zeta)}{\partial\tau}}\,\overline{\frac{\partial^2\zeta}{\partial\tau^2}}\mathrm{d}t + \text{const.} \tag{5.23}$$

To determine the constant of integration in (5.23), Hansen proceeds as in the first-order calculation (see Sect. 4.3). The value of $\ln(r)$ obtainable from (5.21) by changing τ into t is

121 Hansen apparently considers that ϕ's constancy derives from its form alone.

$$\ln(r) = \frac{1}{2}S - \frac{1}{2}\frac{\mathrm{d}(z)}{\mathrm{d}t} + \frac{1}{4}\left(\frac{\mathrm{d}(z)}{\mathrm{d}t}\right)^2. \tag{5.24}$$

The constant incorporated in S, we recall, is (ln a), which cannot be the constant of integration in (5.23). Therefore, comparing (5.23) with (5.24), Hansen concludes that the constant in (5.23) is

$$\text{Const.} = -\frac{1}{2}\frac{\mathrm{d}(z)}{\mathrm{d}t} + \frac{1}{4}\left(\frac{\mathrm{d}(z)}{\mathrm{d}t}\right)^2.$$

To determine d(z)/dt, we return to the equation for $n\zeta$. The perturbations have been shown to produce an addition to the mean motion equal to $n(c + c')t$. Therefore, the constant added when $n\zeta$ is obtained by integration is $n(1 - c - c')\tau$; this choice being such that, when $\tau \to t$, nz is given by nt + periodic perturbational terms. To find $n(\zeta)$, we must delete the added constant from $n\zeta$. Then, the part of $n(\zeta)$ and hence of $n(z)$ that is directly proportional to t will be $n(c + c')t$, and hence

$$\frac{\mathrm{d}(z)}{\mathrm{d}t} = c + c'; \quad \left(\frac{\mathrm{d}(z)}{\mathrm{d}t}\right)^2 = c^2 + k,$$

where k is the constant emerging from the periodic perturbational terms when d(z)/dt is squared. We therefore have

$$\text{Const.} = -\frac{1}{2}c - \frac{1}{2}c' + \frac{1}{4}c^2 + \frac{1}{4}k.$$

As in Sect. 4.3, Hansen adjusts the constant to take account of the difference between (ln a_0) and (ln a), namely to quantities of the second order,

$$\ln a_0 - \ln a = -\frac{2}{3}\ln\left(1 - c - c'\right) \cong \frac{2}{3}c + \frac{2}{3}c' + \frac{1}{3}c^2.$$

Adding these terms to the above expression for the constant, we shall have

$$\text{Const.} = \frac{1}{6}c + \frac{1}{6}c' + \frac{7}{12}c^2 + \frac{1}{4}k.$$

But the first term in the constant was already produced in the first-order calculation of ln(r); Hansen therefore deletes it in writing the second-order contribution to ln(r):

$$\ln(r) = -\frac{1}{2}\int \overline{\frac{\partial^2\zeta}{\partial\tau^2}}\mathrm{d}t + \frac{1}{2}\int \overline{\frac{\partial(\zeta)}{\partial\tau}}\,\overline{\frac{\partial^2\zeta}{\partial\tau^2}}\mathrm{d}t + \frac{1}{6}c' + \frac{7}{12}c^2 + \frac{1}{4}k. \tag{5.25}$$

In calculating ln(r), then, we are directed to return to Eqs. (5.16) and (5.16a) for $n\zeta$, and to differentiate ζ, twice with respect to τ as required for the right-hand side of

(5.22). Substituting the result into (5.22) and changing τ where it occurs into t, we shall have an expression of the form

$$\frac{\mathrm{d}\ln(r)}{\mathrm{d}t} = nA_s \sin X + nA_c \cos X + nB_s nt \sin X + nB_c nt \cos X$$
$$+nC_s n^2t^2 \sin X + nC_c n^2t^2 \cos X. \tag{5.26}$$

The integral with respect to t will then be

$$\begin{aligned}\ln(r) = &-\left\{fA_s - f^2B_c - 2f^3C_s\right\}\cos X + \left\{fA_c + f^2B_s - 2f^3C_c\right\}\sin X\\ &-\left\{fB_s - 2f^2C_c\right\}nt\cos X + \left\{fB_c + 2f^2C_s\right\}nt\sin X\\ &-fC_sn^2t^2\cos X + fC_cn^2t^2\sin X\\ &+\frac{1}{6}c' + \frac{7}{12}c^2 + \frac{1}{4}k.\end{aligned} \tag{5.27}$$

In (5.26) and (5.27) as earlier, X stands for $ig + i'g'$, and in (5.27) f stands for $n/(in + i'n')$. Equation (5.27) is no longer valid when $g = g' = 0$; the terms resulting from this condition are of the form $A_c nt + (1/2)B_c n^2t^2$.

5.3 Second-order perturbations of the latitude

Differential equations embodying the first-order perturbations of the latitude were given in equations (4.19) for $\mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}q/\mathrm{d}t$ (Sect. 4.5). By taking Θ as the origin of longitudes, equations (4.19) were simplified to yield equations (4.20), and the latter equations were integrated to yield equations (4.21) and (4.21a), giving p and q varying in time as the first-order perturbations require. In all these formulas, Hansen took account only of perturbations due to a single perturbing planet. In deriving the second-order perturbations (*Untersuchung*, pp. 123–131), however, he includes in his formulas terms representing the perturbations due to additional perturbing planets. These complications will here be omitted: we shall confine our endeavor to sketching the main steps of Hansen's procedure for the second-order perturbations produced by a single perturbing planet.

Hansen begins, as in deriving the second-order perturbations of ζ, by applying Taylor's theorem. Let $\mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}q/\mathrm{d}t$ remain the symbols for the time rates of change of p and q due to first-order perturbations, as given in (4.15), and let us introduce special symbols (Hansen supplies none) for the time rates of change of p and q due to second-order perturbations, namely $(\frac{\mathrm{d}p}{\mathrm{d}t})_2$, $(\frac{\mathrm{d}q}{\mathrm{d}t})_2$. To obtain formulas for the latter, we must take account of increments to $\mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}q/\mathrm{d}t$ due to the first-order increments

$$\delta.\ln.\frac{an}{\sqrt{1-e^2}} = -S,\quad n(z),\quad n'\left(z'\right),\quad \ln\left(r'\right),\quad \delta P \text{ and } \delta\mathrm{Q}\,.$$

The expressions for $\mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}q/\mathrm{d}t$ in (4.15) contain $\frac{an}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial v} = \frac{\mathrm{d}s}{\mathrm{d}t}$; and since Hansen has already computed the value of S as affected by first-order perturbations (as a way of checking his calculation of $\ln(r)$: see Eq. (4.13)), and $\mathrm{d}S/\mathrm{d}t$ is immediately obtainable from S by differentiation, it is convenient to separate the parts of equations (4.15) that do not contain $\mathrm{d}S/\mathrm{d}t$ from those that do. Hansen introduces special symbols for the parts *not* containing $\mathrm{d}S/\mathrm{d}t$; we shall use $\mathrm{d}\hat{p}/\mathrm{d}t$ and $\mathrm{d}\hat{q}/\mathrm{d}t$ for this purpose:

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}t} - \frac{P}{1+\cos \mathrm{I}}\frac{\mathrm{d}S}{\mathrm{d}t}, \quad \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t} - \frac{Q}{1+\cos I}\frac{\mathrm{d}S}{\mathrm{d}t}.$$

Finally, he designates the second-order perturbations in S by S_2. He then obtains the following formulas for the second-order perturbations:

$$\begin{aligned}
\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)_2 &= -\frac{\mathrm{d}\hat{p}}{\mathrm{d}t}S + \frac{\partial \mathrm{d}\hat{p}}{\partial g \mathrm{d}t}n(z) + \frac{\partial \mathrm{d}\hat{p}}{\partial g' \mathrm{d}t}n'\left(z'\right) + r\frac{\partial \mathrm{d}\hat{p}}{\partial r \mathrm{dt}}\ln(r) + r'\frac{\partial d\hat{p}}{\partial r' \mathrm{d}t}\ln\left(r'\right) \\
&\quad - \left[\frac{\partial \mathrm{d}\hat{p}}{\partial P \mathrm{d}t} + \frac{\mathrm{d}S}{\mathrm{d}t}\frac{\mathrm{d}.\frac{P}{1+\cos I}}{\mathrm{d}P}\right]\left(p+p'\right) \\
&\quad - \left[\frac{\partial \mathrm{d}\hat{p}}{\partial Q \mathrm{d}t} + \frac{\mathrm{d}S}{\mathrm{d}t}\frac{\mathrm{d}.\frac{P}{1+\cos I}}{\mathrm{d}Q}\right]\left(q+q'\right) \\
&\quad + \frac{(P)}{1+\cos(I)}\frac{\mathrm{d}S_2}{\mathrm{d}t}; \\
\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)_2 &= -\frac{\mathrm{d}\hat{q}}{\mathrm{d}t}S + \frac{\partial \mathrm{d}\hat{q}}{\partial g \mathrm{d}t}n(z) + \frac{\partial \mathrm{d}\hat{q}}{\partial g' \mathrm{d}t}n'\left(z'\right) + r\frac{\partial \mathrm{d}\hat{q}}{\partial r \mathrm{dt}}\ln(r) + r'\frac{\partial \mathrm{d}\hat{q}}{\partial r' \mathrm{d}t}\ln\left(r'\right) \\
&\quad - \left[\frac{\partial \mathrm{d}\hat{q}}{\partial P \mathrm{d}t} + \frac{\mathrm{d}S}{\mathrm{d}t}\frac{\mathrm{d}.\frac{Q}{1+\cos I}}{\mathrm{d}P}\right]\left(p+p'\right) \\
&\quad - \left[\frac{\partial \mathrm{d}\hat{q}}{\partial Q \mathrm{d}t} + \frac{\mathrm{d}S}{\mathrm{d}t}\frac{\mathrm{d}.\frac{Q}{1+\cos I}}{\mathrm{d}Q}\right]\left(q+q'\right) \\
&\quad + \frac{(Q)}{1+\cos(I)}\frac{\mathrm{d}S_2}{\mathrm{d}t};
\end{aligned} \tag{5.28}$$

Just as in (5.2), so in (5.28), the terms on the right-hand side can be characterized as consisting of two factors, the first being a derivative of some part of $\mathrm{d}p/\mathrm{d}t$ or $\mathrm{d}q/dt$ with respect to a variable and the second being an increment in the said variable. The first term on the right-hand side, in each case, may appear to be an exception but is not. To take the case of $-\frac{\mathrm{d}\hat{p}}{\mathrm{d}t}S$, it is the result of the following calculation:

$$\frac{\partial\left(\frac{\mathrm{d}\hat{p}}{\mathrm{d}t}\right)}{\partial\left(\ln.\frac{an}{\sqrt{1-e^2}}\right)}\delta.\ln.\frac{an}{\sqrt{1-e^2}} = \frac{an}{\sqrt{1-e^2}}\frac{\partial\left(\frac{\mathrm{d}\hat{p}}{\mathrm{d}t}\right)}{\partial\left(\frac{an}{\sqrt{1-e^2}}\right)}(-S) = -\frac{\mathrm{d}\hat{p}}{\mathrm{d}t}S.$$

For the line of argument proving that $\delta.\ln.\frac{an}{\sqrt{1-e^2}} = -S$ see Sect. 5.1.

As in the case of equations (4.19), the equations (5.28) are simplified somewhat by stipulating that all longitudes be measured from $\Theta = 0$. By a lengthy sequence of reductions, Hansen puts the right-hand members of equations (5.28) in the form

$$nH_c \cos X + nH_s \sin X + nH_c'nt \cos X + nH_s'nt \sin X,$$

where, as earlier, $X = ig + i'g'$. The integral, except when $X = 0$, is of the form

$$\left\{fH_c + f^2H_s'\right\} \sin X - \left\{fH_s - f^2H_c'\right\} \cos X$$
$$+fH_s'nt \sin X - fH_s'nt \cos X,$$

where, as before, $f = n/(in + i'n')$. When $X = 0$, the integral is of the form

$$H_C nt + \frac{1}{2}H_c'n^2t^2.$$

5.4 Hansen's perturbations in the longitude of Saturn compared to Hill's

Hansen in the *Untersuchung* lists his results for second-order perturbations only in the case of Saturn (*Untersuchung*, pp. 166–170). The means we have used previously for assessing the accuracy of Hansen's results—namely comparison with the corresponding results provided by Hill—are not available here, for the terms Hill gives as "second-order perturbations of Saturn" are not solely of the second order. As Hill explains:[122]

> Being now in possession of the several factors of the terms of δT and $\delta T'$, we could proceed immediately to the calculation of the terms, strictly of the second order, which arise from these quantities. But the more important parts of these functions are the terms coming from the secular variations of the elements.
> This prominence is kept up in the terms of the third, and apparently of all higher orders. And it is, perhaps, the most surprising instance in the planetary theories of a lack of convergence that the secular variations of the eccentricities and places of the perihelia of Jupiter and Saturn are augmented about a fourth part by the terms of the second order with respect to disturbing forces. Since the mass of Jupiter is less than 1/1000 of that of the Sun it would naturally be supposed that the ratio of the second to the first-order terms would be somewhere in the neighborhood of this fraction. It is, however, 250 times larger.
> By far the larger portions of these second-order terms arise from the terms of δT and $\delta T'$, having severally the arguments γ and γ'. By computing these portions at the outset, and annexing them to the first-order terms corresponding to the same arguments before proceeding to the general calculation of δT and $\delta T'$, we shall include in the determination of the second-order terms the more notable

[122] *A New Theory of Jupiter and Saturn* as cited in note 38, p. 249.

> portion of the third-order terms. In like manner, on arriving at the general computation of the latter, we shall first compute the terms having the arguments γ and γ', and annexing them to the second-order terms, shall then be able to include the more remarkable portion of the fourth-order terms in that of the third....

In sum, Hill's processes after calculation of the first-order perturbations cease to be strictly comparable with Hansen's.

To give an idea of the accuracy Hansen achieves, I shall compare his final results for the non-secular perturbations in the longitude of Saturn (that is, the terms not factored by $n't$ or its powers) and compare them with the corresponding terms in Hill's theory. Hansen's values are given in the *Untersuchung* on pp. 189–190; Hill's values are given in *A New Theory of Jupiter and Saturn* on pp. 560–563.

To make Hansen's coefficients comparable with Hill's, I multiply, as earlier, the first-order coefficients by 1070.5/1047.9 = 1.021 59, the factor by which Hill's value for the mass of Jupiter is greater than Hansen's. The second-order terms fall into two groups, according as they are factored by the square of the mass of Jupiter or by the product of the mass of Jupiter by the mass of Saturn. Hansen, recognizing that the values of the masses will need to be corrected, gives the terms of the two groups in two separate lists. The coefficients of the first group must be multiplied by $(1.02159)^2 = 1.043646$, and those of the second group by $1.02159 \times 1.00297 = 1.024624$, where 1.00297 is the factor by which Hill's value for the mass of Saturn is greater than Hansen's. The third-order perturbations are of three kinds, according as they are factored by m^3 (the cube of Jupiter's mass), m^2m', or mm'^2; Hansen has calculated only perturbations of the first two varieties, and unfortunately does not list them separately. Terms in the two groups need to be increased by the factors 1.06618 and 1.04675, respectively; I have used instead the average of these two factors. Only six of Hansen's terms in the following table include contributions from third-order perturbations, namely those with the arguments $4g'-2g$, $5g'-2g$, $6g'-2g$, $8g'-4g$, $9g'-4g$, and $10g'-4g$. The perturbations are given in the form $k_0 \sin(\chi + K_0)$, where χ is the argument $ig' - ig$.

χ	Hansen		Hill		Hansen–Hill
	k_0	K_0	k_0	K_0	Δk_0
$-2g'-g$	0″.191	169°.75	0″.195	165°.85	−0″.004
$-g'-g$	0″.390	142°.19	0″.362	141°.80	0″.028
$0g'-g$	12″.050	86°.55	12″.089	86°.76	−0″.039
$g'-g$	7″.214	190°.65	7″.196	189°.58	0″.018
$2g'-g$	420″.579	181°.19	421″.948	181°.43	−1″.369
$3g'-g$	33″.308	122°.14	33″.511	121°.23	−0″.203
$4g'-g$	0″.108	87°.88	0″.101	90°.31	0″.007
$-g'-2g$	0″.043	237°.43	0″.076	244°.37	-0″.033
$0g'-2g$	0″.167	103°.87	0″.164	144°.20	0″.003
$g'-2g$	2″.737	250°.49	2″.764	250°.00	−0″.027

χ	Hansen		Hill		Hansen–Hill
	k_0	K_0	k_0	K_0	Δk_0
$2g' - 2g$	32″.016	156°.46	32″.025	156°.97	−0″.009
$3g' - 2g$	25″.958	135°.26	26″.138	135°.55	−0″.180
$4g' - 2g$	684″.664	277°.18	683″.664	277°.40	1″.000
$5g' - 2g$	2916″.406	246°.89	2907″.855	247°.11	8″.551
$6g' - 2g$	1″.401	321°.55	1″.719	255°.29	−0″.318
$g' - 3g$	0″.098	259°.38	0″.139	269°.50	−0″.041
$2g' - 3g$	0″.207	134°.02	0″.190	142°.90	0″.017
$3g' - 3g$	6″.514	233°.84	6″.513	234°.38	0″.001
$4g' - 3g$	4″.617	202°.70	4″.600	203°.26	0″.017
$5g' - 3g$	3″.329	173°.83	3″.250	174°.62	0″.079
$6g' - 3g$	3″.366	156°.63	3″.339	157°.35	0″.027
$7g' - 3g$	5″.560	24°.85	6″.247	31°.40	−0″.687
$8g' - 3g$	0″.477	12°.23	0″.654	18°.17	−0″.177
$3g' - 4g$	0″.148	200°.51	0″.122	205°.35	0″.026
$4g' - 4g$	1″.930	311°.35	1″.910	312°.14	0″.020
$5g' - 4g$	1″.284	280°.68	1″.290	281°.84	−0″.006
$6g' - 4g$	0″.685	250°.11	0″.692	249°.55	−0″.007
$7g' - 4g$	0″.317	36°.54	0″.375	41°.85	−0″.058
$8g' - 4g$	1″.345	13°.18	1″.486	14°.60	−0″.141
$9g' - 4g$	8″.101	163°.33	8″.824	163°.71	−0″.723
$10g' - 4g$	23″.277	132°.79	26″.795	133°.62	−3″.518
$4g' - 5g$	0″.077	272°.97	0″.069	280°.92	0″.008
$5g' - 5g$	0″.664	28°.61	0″.661	29°.70	0″.003
$6g' - 5g$	0″.488	359°.30	0″.479	0°.12	0″.009
$7g' - 5g$	0″.223	328°.95	0″.219	332°.20	0″.004
$8g' - 5g$	0″.035	296°.57	0″.120	121°.55	−0″.085
$5g' - 6g$	0″.042	350°.31	0″.038	356°.25	0′.004
$6g' - 6g$	0″.251	106°.20	0″.251	106°.73	0″.000
$7g' - 6g$	0″.200	77°.01	0″.200	78°.48	0″.000
$8g' - 6g$	0″.098	45°.00	0″.092	50°.92	0″.006
$9g' - 6g$	0″.040	16°.14	0″.047	199°.67	−0″.007
$6g' - 7g$	0″.021	67°.17	0″.021	72°.63	0″.000
$7g' - 7g$	0″.098	180°.58	0″.099	183°.25	−0″.001
$8g' - 7g$	0″.088	154°.31	0″.086	156°.38	0″.002
$9g' - 7g$	0″.044	125°.84	0″.045	130°.15	−0″.001
$8g' - 8g$	0″.041	255°.96	0″.041	260°.32	0″.000
$9g' - 8g$	0″.045	231°.34	0″.040	233°.87	0″.005

Among the 47 terms here compared, the differences between Hansen's and Hill's coefficients exceed or equal 1 arcsecond in four cases:

Argument	Δk_0
$2g' - g$	$1''.369$
$4g' - 2g$	$1''.000$
$5g' - 2g$	$8''.551$
$10g' - 4g$	$-3''.518$

The coefficients of the first three of these terms are very large; the coefficient of the fourth is also sizable. Hansen's failure to carry out a complete calculation of the third-order perturbations may play a role in these discrepancies—the sources of which, however, will here have to remain unidentified.

Of the remaining 43 terms, two have coefficients with errors between $1''.0$ and $0''.5$, and four have coefficients with errors between $0''.5$ and $0''.1$. For the 35 terms with errors less than $0''.1$, the root-mean-square error is $0''.023$.

Hansen's and Hill's phase constants (K_0) are in roughly satisfactory agreement except in two cases, the terms with arguments $6g' - 2g$ and $9g' - 6g$. For the remaining 45 terms, the root-mean-square difference between Hansen's and Hill's phase constants is $4°.37$.

Hill's perturbations for the longitude include 53 terms omitted by Hansen. The coefficients of the five largest of these, with their arguments, are

Argument	Coefficient
$-g' - 2g$	$0''.076$
$9g' - 5g$	$0''.145$
$10g' - 5g$	$0''.129$
$11g' - 5g$	$0''.211$
$12g' - 5g$	$0''.241$

The last four of these are in a sequence such as Hansen regularly employed in order to catch all terms with coefficients above $0''.03$; his mistaken value for the term with argument $8g' - 5g$ may, by its smallness, have led him to suppose that the succeeding terms in the sequence would be negligible. As for the remaining 48 terms omitted by Hansen, the root-mean-square average of their coefficients is $0''.017$, well below Hansen's stipulated numerical lower bound.

The foregoing comparison suggests that Hansen's theory, despite a few errors that it contained, would have proved superior to Bouvard's tables as a basis for ephemerides.

6 Conclusion: what Hansen's method achieves

On May 8, 1830, a few days before at last sending on the text of the *Untersuchung* to the Berlin Akademie's prize committee, Hansen wrote to his friend Bessel, seeking to provide a clear if preliminary notion of what "Hansen's Method" achieves,[123]. This letter was a response to a letter from Bessel dated 22 January, in which Bessel admitted to not having grasped the force of Hansen's introduction of the time τ in the new signification that Hansen assigned to it.[124] The following two paragraphs give what we may take to be the key points of Hansen's response:

> Remarkable in my method are two things which belong exclusively to me, namely the choice of two of the coordinates, and the introduction of the undetermined quantity τ. Laplace in his theory of the Moon, following the pattern of the earlier astronomers, used the time and 1/r as coordinates, both expressed in terms of the true longitude; by no means did he use the two coordinates that I have used, the mean longitude and the logarithm of the radius vector. And by no means is the quantity τ [the same as] the time, for in the integration with respect to time, τ is treated as a constant; it becomes the time only when the undetermined quantities $n\zeta$, and $\ln(\rho)$ go over into those coordinates [that is, into the mean longitude and the logarithm of the radius vector just referred to].
> The first thing I brought about was to obtain rigorous differential equations (of the first order with respect to time) for the perturbations of the coordinates, and indeed such equations as, without transformation, can be developed and integrated; this had not been done previously.... Since I have rigorous differential equations, I can calculate the perturbations of any order with respect to the masses by one and the same rule. In this way I have brought unity into the method of calculation—a quality in scientific work on which you [Bessel] justly place great store....[125]

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[123] The eleven-page letter is found in Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften. Quotations from the original in footnotes 124–127 are given with express permission of the Berlin-Brandenburgische Akademie, pp. 35r–40r.

[124] Hansen reports this in his letter of May 8, 1830. Bessel's letters to Hansen appear not to be extant, so that we must rely on Hansen's letters for information about their content. Hansen's letter of January 12, 1830 (misdated 1813!) shows the two friends in a *contretemps*, stemming from misapprehensions on both sides; we need not enter into the particulars. Bessel's letter of 22 January was evidently conciliatory, but Hansen did not reply till 8 May; presumably he was preoccupied during the interval with completing the *Untersuchung*.

[125] The German of the quoted passage, appearing on pp. 36v–37r of the original letter, is as follows:

"Es sind vorzüglich in meiner Methode zwei Dinge[,] welche mir eigenthümlich sind, nämlich die Wahl zweier der Coordinaten und die Einführung der unbestimmten Grösse τ. Zwei Coordinaten sind von den bisher gebrauchten wesentlich verschieden, denn damit[,] dass Laplace nach dem Vorbild der älteren Astronomen in seiner Theorie des Mondes die Zeit und $1/r$ durch die wahre Länge ausdrückt, hat er keinesweges, wie ich es gethan habe, die mittlere Länge und den Logarithmus des Radius Vectors für zwei der Coordinaten angenommen. Die Grösse τ ist keinesweges die Zeit, denn sie muss bei den Integrationen in Beziehung auf die Zeit constant gesetzt werden, diese Grösse geht alsdann erst in die Zeit über, wenn die unbestimmten Grössen $n\zeta$ und $l(\bar{n})$ in jene Coordinaten übergehen sollen."

"Das erste was[,] ich nun bewirkt habe ist[,] dass ich strenge differentiale Gleichungen (von der ersten Ordnung in Beziehung auf die Zeit) für die Störungen der Coordinaten bekommen habe, und zwar solche Gleichungen[,] die sich ohne Umwandelung sogleich entwickeln und integriren lassen; dies ist früher nicht geleistet...."

"Da ich strenge Differentialgleichungen habe[,] so kann ich die Berechnung der Störungen[,] die in Beziehung auf die Massen von einerlei Ordnung sind, nach einer und derselben Regel berechnen, und habe also dadurch in die Berechnungsmethode die Einheit gebracht, auf welche Sie bei wissenschaftlichen Arbeiten, mit vollen Recht[,] einen grossen Werthe legen...."

The reader may here object: did not Hansen, at the start of the *Untersuchung*, reformulate his differential equation for $d\zeta/dt$ in a way applicable only in the first-order approximation—his aim being to take advantage of the superior convergence of the series for R_k? It is indeed so.

In the reformulation, Hansen reintroduces $\partial\Omega/\partial\varepsilon$, but the orbital element ε is soon replaced by the mean anomaly g which is a linear function of ε. Basically, it is one and the same differential equation for $d\zeta/dt$ that is being solved, and it is the same differential equation that is then solved at each of the successive approximations that Hansen carries through. The second- and higher-order approximations are obtained by means of Taylor's theorem, this route having proved the most efficient. In principle, however, all the approximations could have been obtained using the same procedure.

Let us compare the map of the Hansenian solution with that of the Laplacian solution. Laplace's equations (X'), (Y), and (Z') (Sect. 2.1) specifically presuppose the limitation to perturbations of the first order with respect to the perturbing forces, since they incorporate truncated series representing the unperturbed Keplerian motion of the perturbed planet (elliptical and in accordance with Kepler's area rule). If the approximating series are eliminated, the equations cease to be; the equations cannot be reconfigured so as to be made applicable to higher-order approximations. Thus, equations (X'), (Y), and (Z') are in their *essence* approximative, and *essentially* limited to perturbations of the first order. Hansen's equations, by contrast, are rigorously true of the variables they contain, and these variables retain applicability to successive orders of approximation. To be sure, their application can be pursued *only* by approximations of successive orders with respect to the perturbing forces. Laplace, by contrast, when he turns to perturbations of higher order than the first, uses special computational procedures that are incapable of being made systematic.

Prior to Hansen's development of his new method, the only systematic method available for higher-order perturbations was that provided by the Lagrangian equations for the time rates of change of the orbital elements, combined with successive applications of Taylor's theorem. (For the Lagrangian equations, see Eqs. (3.1a) and (3.1b) of Sect. 3.2.) But, as Hansen observed at the start of the *Disquisitiones*, the application of these equations entails serious drawbacks. The orbital elements being six in number and the coordinates three in number, the perturbations of six quantities have to be computed in order to obtain the perturbations of the three coordinates. Moreover, the perturbations of the orbital elements are found to be considerably larger than the perturbations of the coordinates they entail, so that the latter have to be computed from differences of much larger quantities—an unreliable procedure.

Hansen's motive for seeking a direct way of computing the perturbations of the coordinates is thus understandable: he was attempting to replace roundabout and unreliable computational procedures by direct and reliable ones. Starting from the Lagrangian formulas, he carried out a transformation that gave the perturbed mean anomaly and perturbed logarithm of the radius vector in terms of partial derivatives of the perturbing function with respect to the coordinates, thus eliminating reference to the orbital elements a, e, ε, and ϖ as variables. The only old-style orbital elements remaining as variables were those determining the position of the instantaneous orbital plane (Hansen's p and q).

Of Hansen's initial choice of the mean anomaly as a coordinate, he offers little in justification, but that little suggests that he was seeking to return, insofar as possible, to the simple and direct manner of formulating the planetary problem available to pre-Newtonian astronomers: given the time, to compute the planet's position. His choice proved fortuitous: when he had obtained formulas for $d\zeta/dt$ and $[d\ln(\rho)]/dt$ (namely, our equations (3.16) and (3.17)), there emerged a simple relation between the two equations, namely equation (3.18). Since the perturbations in the radial and trans-radial directions must be intimately related, their relation would be of interest; but the admirable simplicity of (3.18) was a discovery worth celebrating. It meant that once the perturbations of the mean anomaly ($n\zeta$) were computed to any order, the corresponding perturbations of $[\ln(\rho)]$ could be written out forthwith, without a separate calculation. For this shortcut, Hansen tells Bessei, he is indebted partly to the quantity τ and partly to the choice of the coordinates. (The choice of $[\ln(\rho)]$ in place of ρ, we note, is labor-saving only because Hansen is carrying out the numerical computations logarithmically.)

In the remainder of Hansen's account, written for Bessei, the stress is upon (1) the efficiencies that the method achieves and (2) the controls on the numerical calculation that it provides.

One of the gains in efficiency came, as just mentioned, from the relation (3.18), whereby the perturbations of the radius vector or its logarithm could be determined directly from those of the mean longitude.

Another came from the decision, departing from earlier astronomical practice, to compute the perturbations of the mean longitude rather than those of the true longitude:

> The series by which I compute the perturbation coefficients converge more rapidly than those known previously; this circumstance, which lends greater certainty to the outcome and permits a saving of time in the calculation, I owe to the choice of coordinates. I find a posteriori that, calculating these series to the precision of a fixed numerical bound, the number of terms giving the perturbations of the mean longitude is smaller than the number of terms giving the perturbations of the true longitude....[126]

Still another increment in efficiency came from eliminating the computation of the perturbations of the orbital elements e and ϖ, the eccentricity and the longitude of the perihelion:

> If in accordance with the theory of the variation of arbitrary constants one undertakes to compute the perturbations of the second order, there appear in e and ϖ perturbations of the order of $1/e$ and $1/e^2$, which are very large if e is small. These perturbations disappear if one goes over from the perturbations of the

[126] The German of the quoted passage is from NL Bessel, Nr. 246, p. 37r:
"Die Reihen[,] durch deren Numeration ich die Störungscoefficienten finden muss, convergiren weit mehr als die früher bekannten; diesen Umstand, der sowohl grössere Sicherheit ins Resultat bringt, als auchpagination Zeitersparnis bei der Berechnung der Störungen veranlasst, habe ich der Wahl der Coordinaten zu verdanken. Auch hat sich a posteriori ergeben[,] dass bis zu einer festen numerischen Grenze herab, die Zahl der Störungscoefficienten der mittleren Länge kleiner ist als die der wahren Länge."

> elements to the perturbations of the coordinates; their existence is thus a circumstance arising from the unsuitability of the mode of calculation. In my method this species of perturbation simply does not appear.[127]

Yet a further gain in efficiency, Hansen tells Bessel, is the one exhibited in equations (3.7a,b) at the end of Sect. 4.2, whereby coefficients of the sine and cosine terms in $n\zeta$ for any value of κ could be obtained from the coefficients for which $\kappa = -1, 0, +1$. All these increments in efficiency, Hansen points out, stem ultimately from his choice of the mean anomaly as a fundamental variable and from his introduction of the quantity τ.

Turning finally to the conditional equations, he establishes for the numerical calculation (see Sect. 4.4), Hansen tells Bessel that they enable him to guard against the least mistake.

> I hold this circumstance, that I can subject the entire calculation to a secure control, for an essential advantage of my method. Without the introduction of the quantity τ neither of the two chief controls would be available.[128]

From the foregoing summary, it will be evident that Hansen in devising his method was guided by a unifying theoretical idea, the aim of reducing the perturbational problem posed by the Newtonian law of gravitation to an older paradigm: given the value of a variable functioning as the time or mean longitude, to find the planet's position. The computational efficiencies and controls that accompany the method are spin-offs from the successful achievement of Hansen's unifying theoretical aim.

In the computational process that Hansen envisages, the final stage is to consist of substituting a temporal variable z into purely Keplerian formulas, in which the orbital elements have the values determined observationally for the epoch. In Hansen's analysis, as real-time elapses from the epoch, three processes are occurring: the planet's instantaneous orbital plane slowly changes in position; the planet's orbit in the instantaneous orbital plane undergoes change because the orbital elements a, ε, e, ϖ vary, and the planet moves along its instantaneous orbit. In a step that is not free of risk, Hansen initially sets the first of these processes aside, proposing to deal with it later, and focuses on the second and third.

To obtain a mathematical description of the motion of the planet along its orbit while the orbit itself is varying in time, Hansen must take account of two things at once, and he therefore introduces a variable λ which varies in two ways. On the one hand, λ depends on the quantity τ in exactly the same way that the longitude v would depend on the time t in strictly Keplerian motion. On the other hand, λ is a function of the orbital elements

[127] The German of the quoted passage is from NL Bessel, Nr. 246, p. 37v: "Wenn man nach der Theorie der Veränderung der willkührlichen Constanten die Störungen der zweiten Ordnung rechnen will, so erscheinen in e and $\tilde{\omega}$ Störungen[,] welche von der Ordnung $1/e$ und $1/e^2$ sind, und also sehr gross werden[,] wenn e klein ist. Diese Störungen heben sich auf[,] wenn man von den Störungen der Elemente zu den Störungen der Coordinaten übergeht; ihr Daseÿn ist daher ein Umstand[,] der grosse Unbequemlichkeit bei der Berechnung verursacht; bei meiner Methode erscheint diese Gattung von Störungen gar nicht."

[128] NL Bessel, Nr. 246, p. 37v: "Ich halte auch diesen Umstand, dass ich die ganze Rechnung einer sicheren Controlle unterwerfen kann[,] für einen wesentlichen Vorzug meiner Methode; ohne die Einführung der Grösse τ würden beide genannten Hauptcontrollen nicht statt finden."

$a, \varepsilon, e, \varpi$ which figure in the equations of Keplerian motion, but which change in accordance with the Lagrangian equations (3.1a), which give the time rates of change of these orbital elements under perturbation. To subsume these two variations under a single variation, Hansen makes λ a function of a new variable ζ, which is to play the role of an enhanced or perturbable mean longitude. Then ζ in turn must be a function of both t and τ. The variation of ζ, with t takes account of the variation of the orbital elements in accordance with the Lagrangian equations (3.1a). Once this latter variation has been accounted for (by integration), τ can be changed into t.

Simultaneously, Hansen stipulates that ζ is to become z, so that z is the perturbed mean longitude, substitutable into the Keplerian equations using the values of the orbital elements determined observationally for the epoch. Once the perturbed longitude is known, equation (3.18) can supply the perturbed radius vector in the instantaneous plane without further calculation.

Hansen's postponement of the problem of the position of the instantaneous plane, as suggested earlier, was potentially risky; and indeed, Hansen initially fell into error by assuming the motion of the instantaneous plane to be strictly separable from the motion in longitude (see Sect. 3.4). He thus arrived at the erroneous equation (3.25). In the *Untersuchung*, he stepped back from this mistake (see Sect. 4.5) by making $\partial\Omega/\partial p$ and $\partial\Omega/\partial q$ dependent on $\partial\Omega/\partial v$. He did not, however, supply a geometrical derivation or justification for the resulting equation (namely (4.18)). That would first be done by Cayley.[129] It remains true that Hansen, in marked contrast to Laplace, addressed the problem of perturbation in latitude in a way capable of being made systematic for successive orders of perturbation.

In sum, Hansen's method emerged out of a sustained endeavor to introduce logical rigor and unity into the practical approximative business of celestial mechanics. To bring this aspect of Hansen's undertaking into focus has been the purpose of the present study.

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