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From Problems to Structures: the Cousin Problems and the Emergence of the Sheaf Concept

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Abstract Historical work on the emergence of sheaf theory has mainly concentrated on the topological origins of sheaf *cohomology* in the period from 1945 to 1950 and on subsequent developments. However, a shift of emphasis both in time-scale and disciplinary context can help gain new insight into the emergence of the sheaf *concept*. This paper concentrates on Henri Cartan's work in the theory of analytic functions of several complex variables and the strikingly different roles it played at two stages of the emergence of sheaf theory: the definition of a new structure and formulation of a new research programme in 1940–1944; the unexpected integration into sheaf cohomology in 1951–1952. In order to bring this two-stage structural transition into perspective, we will concentrate more specifically on a family of problems, the so-called Cousin problems, from Poincaré (1883) to Cartan. This medium-term narrative provides insight into two more general issues in the history of contemporary mathematics. First, we will focus on the use of problems in theory-making. Second, the history of the design of structures in geometrically flavoured contexts—such as for the sheaf and fibre-bundle structures—which will help provide a more comprehensive view of the *structuralist moment*, a moment whose algebraic component has so far been the main focus for historical work.

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1 Introduction

Sheaf theory has already been investigated from a historical viewpoint, by Gray (1979), Fasanelli (1981) and Houzel (1990, 1998), for instance. These narratives document the first stages of sheaf cohomology and usually start with Leray's introduction of both the word "*faisceau*" and a first form of sheaf cohomology, in his 1946 lessons on algebraic topology (Leray 1946). They document the refinement of cohomology techniques—as with Weil's introduction of what would soon become the spectral sequence (Weil 1947)—and the central part played by Henri Cartan's algebraic topology seminar, from 1948 onwards. When moving on into the 1950–1957 period, they have investigated several topics: the first significant achievements in algebraic topology and in the theory of functions of several complex variables, the dissemination of sheaf cohomology techniques beyond the Parisian circles where they originated, the introduction of sheaves and cohomology in algebraic geometry, or the interaction with emerging category theory.

Although the long-term roots of sheaf cohomology do not go unacknowledged, we would like to show that their study can provide new insight into more general questions about the historical development of contemporary mathematics.

We will thus change focus, time-scale and theoretical context. We will focus on the emergence of the sheaf *concept* or *structure* (rather than sheaf *cohomology*) in Henri Cartan's work on the theory of functions of several complex variables (and not in topology). More specifically, we will focus on Cartan's 1940–1944 papers in which he defined a new abstract structure and spelled out a new research programme. So as to better characterise this pivotal moment, we shall locate it at the intersection of two time-lines. The first time-line covers the period from 1883 to 1940 and is devoted to the history of what Cartan called the "Cousin problems", from Poincaré's (1883) paper on the representation of meromorphic functions of two complex variables to Cartan's 1940–1944 structural articulation of a research programme for the handling of all Cousin-like problems. Roughly put, the two main problems (Cousin I and Cousin II) are to find a meromorphic function with given poles and principal parts (Cousin I problem), which is the analogue for several complex variables of what Mittag-Leffler's classic theorem states for the one complex variable case; to find a holomorphic function with given zeroes or zero-locus (Cousin II problem), which is the analogue for several complex variables of a classical Weierstrass theorem in one complex variable. Cartan christened them "Cousin problems" because they were both tackled in a uniform way in Pierre Cousin's 1895 dissertation (Cousin 1895).

Our second time-line will cover the period from 1945 to 1953. The background of Cartan's (1940) paper leads to a historical narrative which differs significantly from those that centre on the topological background and cohomological techniques. Among other things, interpreting the 1945–1952 definitions of the sheaf concept as a *second* generation of definitions will help us highlight the fact that, in addition to great technical efficiency, sheaf cohomology also brought conceptual clarification. We will also stress the fact that, in spite of Cartan's involvement, the sheaf cohomology theory of 1945–1950 was developed independently of Cartan's 1940–1944 structural approach to global problems in several complex variables; it can be argued that the unexpected merger of the two research lines, in the work of Cartan and Serre in the early 1950s, would be the real take-off point for sheaf theory as we know it.

This historical narrative should enable us to touch on two more general issues in the history of mathematics: the various uses of problems in the contemporary period, on the one hand; the nature and time-sequence of the structuralist movement, on the other hand.

The Cousin problems are a rather unusual instance since, to a large extent, their formulation and even the technical tools used to tackle them were reasonably stable over a rather long time span (1883–1940). The stability of core elements allows for long-term comparisons and a close chartering of the more variable elements, such as, for instance, their insertion in various research programmes. Beyond the issue of research programmes, we will endeavour to characterise the various uses of problems (Chemla 2009); along the way, we will see the Cousin problems used as mere tools to tackle a more general problem (as in the 1930s); as paradigmatic problems from which to carve out a general *problem form* and a general structure (as in Cartan's 1940 paper); eventually, as an unexpected field of application for new technical tools (in the 1951–1953 period). Among the variable elements, we will also pay close heed to *problem labels*, so as to provide empirical elements for the reflection on general issues such as: how do mathematicians assess and express the similarity or dissimilarity of problems? How and for what purpose do they constitute problem families?

Cartan's (1940) pivotal paper is also an example of structural transition, or transition to structuralism; as such, it can be used to investigate several aspects of structuralism which, we feel, have not received the attention they deserved. There are of course many ways to look at the "structuralist moment", say, from Emmy Noether's *Idealtheorie in Ringbereiche* (Noether 1921) to Bourbaki's 1942 *L'architecture des mathématiques* (Bourbaki 1950) and Eilenberg and MacLane's (1945) *General Theory of Natural Equivalences* (Eilenberg and MacLane 1945).[1] To name a few, one can strive to document the long-term origins of structuralism (Corry 1996); one can study the 1930–1945 phase on the background of the first, turn of the twentieth-century, structuralist phase, with its theories of abstract spaces (Fréchet), its abstract theories of fields (Steinitz) or of topological spaces and Riemann surfaces (Hausdorff, Weyl); the more philosophically oriented will investigate the links (if any) with foundational issues, axiomatic style, and logicist or formalist views of mathematics; the various attempts at formalising the very notion of structure can be analysed.[2] None of these important issues are directly touched upon in this paper. Rather than the conceptual roots of structuralism or the mathematical formalisation of the structure concept, we propose to investigate it as a specific mathematical practice, and focus on two less well-known aspects on which the emergence of the sheaf concept bears directly.

Hermann Weyl's sceptical view of the structural, axiomatic approach in algebra and topology is well-known. In 1931, he concluded his address to the Swiss association *Gymnasium* teachers with a rather grim forecast:

> But I do not want to conceal from you the growing feeling among mathematicians that the fruitfulness of the abstracting method is close to exhaustion. It is a fact that beautiful general concepts do not drop out of the sky. The truth is that, to begin with, there are definite concrete problems, with all their undivided complexity, and these must be conquered by individuals relying on brute force. Only then comes the axiomatizers and conclude that instead of straining to break in the door and bloodying one's hand one should have first constructed a magic key of such and such shape and then the door would have opened quietly, as if by itself. But they can construct the key only because the successful breakthrough enables them to study the lock front and back, from the outside and from the inside. Before we can generalize, formalize and axiomatize there must be mathematical substance. I think that the mathematical substance on which we have practiced formalization in the last few decades is near exhaustion and I predict that the next generation will face a tough time. (Weyl 1995b, pp. 650–651)

The fact that major landmarks of the structural moment, such as van der Waerden's *Moderne Algebra* (van der Waerden 1930, 1931) or Bourbaki's *Eléments de mathématiques*, are textbooks or encyclopaedias, written in an occasionally dogmatic style, comes as a support to Weyl's unforgiving appraisal: the structural approach is merely a textbook style, a way of tidying-up and streamlining a wealth of mathematical knowledge previously gathered by completely different means; clean but not fruitful, and, to some extent, deluded as to its own epistemic power. However, Cartan's 1940/44

[1] The choice of these landmarks as significant (both for their contents and as dates) for the delineation of something I call the "structuralist moment" calls for justification—this is an understatement. However, the content of this paper does not depend on general assumptions on this structuralist moment, and this introduction is no place to go into that.

[2] As in Corry (1996) and Krömer (2007).

papers give an example of the exact opposite: he introduced a structural approach to find the proper tools to deal with a family of problems that had not been solved by other means. It is indeed true that in the 1952–1953 talks by Cartan and Serre on sheaf cohomology and global problems in several complex variables, the "door" seems to open "quietly, as if all by itself"; but the history of sheaf theory on the 1940–1953 period sheds light on the making of this "magic key".

Structuralism as an *ars inveniendi* and not as a mere streamlining practice; Bourbaki as group of researchers—Ehresmann, Weil, Dieudonné and Schwartz will play a part in our narrative—and not only as a group of self-proclaimed legislators in mathematics. We will eventually shift emphasis as to the part of mathematics concerned, turning away from algebraic structures and their importation into combinatorial topology (resulting in algebraic topology). We will study the emergence of the sheaf concept in a context of the *design of structures in geometry* and, in particular, contrast it with the emergence of the fibre-bundle structure. Both the sheaf and fibre-bundle structures are examples of *mixed-structures*, the design of which require the skilful weaving together of several threads: vector spaces and groups on manifolds for fibre-bundles, rings and modules on topological spaces for sheaves. It takes some time to perfect these delicate structures, and know-how in structure design is accumulated in the process. We shall see that the progressive design of these two highly sophisticated structures entailed trading of concepts (e.g. "sections"), models of formulation (e.g. gluing together), general ideas (e.g. going global) and technical tools (e.g. partitions of unity, cohomology with local coefficients). In this final part of our narrative, technical streamlining will go hand in hand with conceptual *bricolage*.

2 Poincaré's generalisation of a Weierstrass theorem

This first part on Poincaré could have been very short. Indeed, we could have just mentioned the theorem that would later be referred to as the starting point of a specific research line, thus ignoring nineteenth century issues to a large extent. We felt, however, that presenting subsequent developments on the background of a rather detailed description of Poincaré's (1883) theorem was necessary to dispel the illusion that "it was there all along". To reach this goal, we distinguished among various elements—the laying-out of the problem, proof techniques, research programmes (if any) etc.—in order to show that some played no part in later developments, while others played significant but evolving parts. We shall also place emphasis on two elements which would later become parts of general mathematical knowledge, but were not in 1883: first, the use of open coverings and patching-up *site techniques* (a notion we shall define); second, the "passing from local to global" problem label.

2.1 "Same" problem, new *ekthesis*

The starting point of Poincaré's (1883) paper *Sur les fonctions de deux variables* (Poincaré 1883b) lies in a theorem of Weierstrass that can be found, for instance, in the 1876 paper *Zur Theorie der eindeutigen analytischen Functionen* (Weierstrass 1876). In his paper, Weierstrass aimed at classifying single-valued functions of one complex variable according to the localisation and type (polar/essential) of their singular points. One of the major building blocks for this classification is the following:

> Let $f(x)$ be a one-valued function of x with ∞ as its only essential singular point; in that case, it is possible to form a function $G_2(x)$ whose sequence of zeroes is identical to that of function $1/f(x)$ (it may have an arbitrary, even an infinite, number of inessential singular points). Then $G_2(x).f(x)$ is also an entire function of x, and, denoting it by $G_1(x)$, we get $f(x) = G_1(x)/G_2(x)$.[3]

If we focus on the case of an infinite family of poles, it is stated explicitly in other parts of the paper that the "sequence of zeroes" (*Reihe der Nullstellen*) is made up of a pair of sequences, one of complex numbers $(a_1, a_2, \ldots.)$ and one of non-null integers $(m_1, m_2, \ldots)$ indicating the multiplicity of the poles. It is, of course, standard knowledge at the time that poles are isolated in the domain of meromorphy (which, in this particular case, is assumed to be $\mathbf{C}$), so the (a_i) sequences converges towards the point at infinity ∞ for any such meromorphic function. Weierstrass' theorem proves that this necessary condition is also sufficient; hence, the converse of a simple (topological) fact about the locus of zeroes of a meromorphic function turns out to be a significant theorem on functional existence (of a meromorphic function with given zero-locus) and function representation: any function which is meromorphic over $\mathbf{C}$ can be written as the quotient of two power series with infinite convergence radius.[4]

The goal of Poincaré's paper is quite straightforward: to establish the analogous theorem in two complex variables. However, the very statement of the two-variable problem cannot proceed along the same lines as its one-variable counterpart and, this very change of formulation would, by itself, single out Poincaré's paper as the natural starting point for our inquiry. Indeed, in the one-variable case, to denote a locus which *could* be that of an entire functions, two simple sequences would do (one of complex numbers, one of integers), with a simple topological *proviso*: the (a_i) sequence has to go to infinity. The locus of zeroes of a functions in two complex variables, though, is a two-dimension spread in four-dimension space (in real terms): not only does this fact make the problem significantly more difficult to solve, but the very *ekthesis*—the laying out of the problem—is, in itself, thorny: how to refer to a locus that *could* be the zero locus of an entire function (without, of course, assuming it *is* the zero-locus for one such function, which would kill the problem before it is hatched[5]). Poincaré resorts to an *ekthesis* which exhibits a non-standard *problem form*:

[3] "Ist $f(x)$ eine eindeutige Function von x mit der einen wesentlichen singularitäten Stelle ∞, so lässt sich in dem Falle, wo sie ausserdem beliebig viele (auch unendlich viele) ausserwesentliche singuläre Stellen hat, eine Function $G_2(x)$ herstellen, für welche die Reihe der Null-Stellen identisch ist mit der Reihe der Null-Stellen der Function $1/f(x)$. Dann ist $G_2(x).f(x)$ ebenfalls eine ganze Function von x, und man hat, wenn diese mit $G_1(x)$ bezeichnet wird, $f(x) = G_1(x)/G_2(x)$" (Weierstrass 1876, p. 102). Free translation.

[4] Functions of this type (in one or more complex variables), Weierstrass and Poincaré call "entire". We will stick to this terminology.

[5] The twenty-first century mathematical reader may find this question to be rather rhetorical, since she can think of standard "obvious" and "elementary" ways to do that. We claim Poincaré's work played a significant part in the emergence of this "basic know-how", a point which can only be partly substantiated in this paper (for more, see (Chorlay 2007)). For now, it is sufficient to acknowledge that the passage from one to two variables significantly affects the laying out of the problem and not only the tackling of it.

Here is the problem:

> I consider a function of two variables, $F(X, Y)$, and assume that, in the neighbourhood of any point X_0, Y_0 it can be written in the form $\frac{N}{D}$, where N and D are two series (ordered according to the powers of $X - X_0$ and $Y - Y_0$) which converge for sufficiently small values of the modulus of these quantities. Furthermore, I assume that, when the moduli of $X - X_0$ and $Y - Y_0$ remain small enough, the series N and D can vanish simultaneously but at isolated points. I claim this function can be written in the form $\frac{G(X,Y)}{G_1(X,Y)}$, where G and G_1 are everywhere converging series ordered according to the powers of X and Y.[6]

So, instead of a pre-functional pair of sequences which served the purpose of setting out the problem exactly as in the one-variable case, Poincaré's setting is functional from the outset. The laying out exhibits the general problem form "passing from local data to global existence", though Poincaré neither uses any such *meta*-level description, nor points to the generality of this type of problem.

At first sight, this *ekthesis* may seem to be void though: since the balls ("*hypersphères*") over which the local representatives are given may overlap, it seems all the N functions are mere analytic continuations one from the other, and form just one big G function (as D functions do for G_1). Poincaré explains that it is not so: in the overlap of two neighbouring balls $N/D = F = N'/D'$, which does not imply $N = N'$ and $D = D'$; rather, it implies that $N = (D/D')N'$, i.e. $N = \varphi N'$ where φ is a regular function defined in the overlap and which vanishes nowhere in it (since, by hypothesis, D and D' have the exact same zero-locus). Thus, the various Ns (and Ds) that are to be patched up into one G (and one G_1) show some kind of global coherence indeed, but not a straightforward one: in the overlaps, they only coincide up to multiplication by non-vanishing regular functions (Poincaré 1883b, p. 147).

2.2 Same problem, different tools

The problem is to find an entire function K such that KF is also an entire function. Poincaré thoughtfully summed up his proof tactics for us (or rather, for the Paris Academy of Science). He first switched from a complex to a real setting by writing $X = x + \mathrm{i}y$, $Y = z + \mathrm{i}t$ (where x, y, z, and t are real numbers) and calls a real function of four variables (x, y, z, t) a potential ("*fonction potentielle*") if its Laplacian Δ vanishes identically. A non-singular one-valued potential function, he calls holomorphic. The first two (out of four) steps of the proof are those of primary interest to us:

[6] "Voici quel est le problème:
Je considère une fonction de deux variables F(X,Y) et je suppose que dans le voisinage d'un point quelconque X_0, Y_0 on puisse la mettre sous la forme $\frac{N}{D}$, N et D étant deux séries ordonnées suivant les puissances de $X - X_0$ et $Y - Y_0$ et convergeant lorsque les modules de ces quantités sont suffisamment petits. Je suppose de plus que, lorsque les modules de $X - X_0$ et $Y - Y_0$ restent assez petits, les deux séries N et D ne peuvent s'annuler à la fois que pour des points isolés. Je dis que cette fonction peut se mettre sous la forme $\frac{G(X,Y)}{G_1(X,Y)}$, G et G_1 étant des séries ordonnées suivant les puissances de X et Y et toujours convergentes." (Poincaré 1883b, p. 147) The page number refers to the *Œuvres*.

1° This being established, I construct an infinity of hyperspherical regions R_1^0, R_2^0, ... I assume that any point (x, y, z, t) belongs to at least one, and at most five of these regions. I assume these regions to be so chosen that inside R_i^0, for instance, function F can be written in the form $\frac{N_i}{D_i}$.
I also consider those regions R_i^1 consisting in the part that is common to two of the R_i^0 regions, and regions R_i^2, R_i^3, R_i^4 consisting in the part that is common to three, four or five of these.
2° I shall construct a potential function J_i^p with the following properties: it is holomorphic outside R_i^p and tends to 0 as $x^2 + y^2 + z^2 + t^2$ increases indefinitely. The difference J_i^p − log mod D is holomorphic inside J_i^p; eventually, on the border of region R_i^p, J_i^p is holomorphic where D does not vanish.[7]

In order to tackle the problem of building up a potential function whose zero-locus is exactly the polar-locus of F, Poincaré first split it up in problems over a denumerable set of balls in four-dimensional space. In these balls, step 2 is tractable with tools from potential theory: Poincaré relied on explicit integral formulae (though without explicitly referring to single layer potentials ("*potentiels simple couche*")) to solve "*le problème de Dirichlet*" for the unit hypersphere (Poincaré 1883b, p. 150); then he proved a uniqueness theorem (for bounded, nearly singularity-free, potential functions in $\mathbf{R}^4$) (Poincaré 1883b, pp. 154–157). He also replaced the multiplicative problem: find a potential function that has the right zero-locus inside the ball and vanishes nowhere outside the ball; with the additive problem: to find a potential function which is regular outside the ball and has the same singularities as log $|D|$ inside the ball. This makes the problem tractable, since the potential-theory methods give him control over singularities, but not over zeroes.

Now, the $\exp(J_i^0)$ are singularity-free potential functions, and each of them has the right zeroes inside the R_i^0 ball and none outside it. It would seem natural to consider $\prod \exp(J_i^0) = \exp\left(\sum J_i^0\right)$, deal with the thorny convergence problems, then recover a complex analytic function from its real part. Both of these play a part in the proof indeed, as steps 3 and 4, and rely on techniques that we need not go into. But before proceeding to steps 3 and 4, Poincaré had to deal with a multiplicity problem stemming from the fact that the R_i^0 balls overlap: in the overlap of two such balls, $\prod \exp(J_i^0)$ has the right zero-locus, but its multiplicity is not right. To tackle this problem, Poincaré used a techniques of checks and balances: in $R_{i,j}^1 = R_i^0 \cap R_j^0$, the potential theory techniques enabled him to build up a function $J_{i,j}^1$, which is

7 "1° Cela posé, je construis une infinité de régions hypersphériques R_1^0, R_2^0, ... Je suppose qu'un point quelconque (x, y, z, t) appartient au moins à une et au plus à cinq de ces régions. Je suppose que ces régions sont choisies de telles sortes qu'à l'intérieur de R_i^0, par exemple, la fonction F peut se mettre sous la forme $\frac{N_i}{D_i}$.
J'envisage également les régions R_i^1, formées de la partie commune à deux des régions R_i^0 et les régions R_i^2, R_i^3, R_i^4 formées de la partie commune à trois, à quatre ou à cinq de ces régions.
2° Je construirai une fonction potentielle J_i^p jouissant des propriétés suivantes: elle est holomorphe à l'extérieur de R_i^p et tend vers 0 quand $x^2 + y^2 + z^2 + t^2$ croît indéfiniment. La différence J_i^p − log mod D est holomorphe à l'intérieur de J_i^p; enfin, sur la limite de la région R_i^p, J_i^p est holomorphe quand D ne s'annule pas." (Poincaré 1883c, p. 145)

singularity-free outside $R^1_{i,j}$ and such that $J^1_{i,j} - \log|D|$ is singularity-free inside $R^1_{i,j}$. To get the right multiplicity in $R^0_i \cup R^0_j$, the sum $J^0_i + J^0_j$ should be replaced by $J^0_i + J^0_j - J^1_{i,j}$; but this, in turn, might fail to show the right multiplicity when three balls overlap .... The checks and balances process needs not be pursued to infinity though, since Poincaré stated (without proof) that he could start from an open covering in which no six balls overlap. Steps 1 and 2 thus lead to the (partial) conclusion:

> (...) if at whatever point inside one of the $n\,R^0_i$ regions, function F can be written in the form $\frac{N}{D}$, then the function
>
> $$\sum J^0_i - \sum J^1_i + \sum J^2_i - \sum J^3_i + \sum J^4_i - \log \operatorname{mod} D$$
>
> is holomorphic.[8]

This formula might look ambiguous, for lack of multiple indices, but the idea is quite clear.

2.3 Same problem, different research programmes

In spite of the fact that the theorem proved in this paper comes directly from Weierstrass' work on one-valued analytic functions in one complex variable (Weierstrass 1876), Poincaré did not take on Weierstrass' research programme. The goal which Weierstrass set for his investigations on general (as opposed to special, e.g. elliptic, functions) one-valued functions in one complex variable is to assess the extent to which the "arithmetic nature" of the dependence between the function (seen as a dependent variable) and the free variable is determined by a few relevant pieces of information, the most important of which being the locus and nature of singular points. The paradigm is that of rational function: they, as a class, are characterised by the fact that they are regular for all values of the variable,[9] except for a finite number of poles; the explicit formula (which exhibits the "arithmetic nature" of the dependence (Weierstrass 1876, p. 83)) for each function of this class is determined (up to a non-zero multiplicative constant) by the locus and orders of zeroes and poles. In order to deal with more general cases, Weierstrass relied on some topological notions which he defined in his lectures on the foundation of function theory (for instance (Weierstrass 1988, p. 83 and fol.)): neighbourhood of a point, domain (i.e. open, strip-connected subsets of numerical spaces), boundary point and boundary of a domain. For an analytic function, he first defined the domain of continuity A ("*Stetigkeitsbereich*", our domain of holomorphy), whose boundary points he defined to be the singular points. He then distinguished between essential and inessential (polar) singular points and labelled A′ the domain made up of A and the poles (our domain of meromorphy). The

[8] "(...) si en un point quelconque intérieur à l'une des n régions R^0_i, la fonction F se met sous la forme $\frac{N}{D}$, la fonction $\sum J^0_i - \sum J^1_i + \sum J^2_i - \sum J^3_i + \sum J^4_i - \log \operatorname{mod} D$ est holomorphe." (Poincaré 1883b, p. 158)

[9] "All" meaning all "finite" values of variable z, and the infinite value $z = \infty$.

general goal is to classify analytic one-valued functions according to the properties of domain A′ and nature of singular points; the 1876 paper deals with the cases in which A′ is the domain of the independent variable (point at infinity included) save for at most a finite number of essential singular points, and promotes the study of essential singular points with the theorem on the density of the image of any of their neighbourhoods. The theorem which is Poincaré's starting point appears half way through Weierstrass' paper, when functions with at most one essential singular points are studied.[10]

This carefully laid out list of questions is not the only aspect that is not reflected in Poincaré's paper; Weierstrass' careful selections of tools is not either. Weierstrass avoided all "transcendental" methods, and admitted but a small range of expressions: power series as starting points; rational functions and the exponential function (one essential singular point, no zeroes) in the final expressions. Needless to say integral formulae (such as Poincaré's single layer potentials) and partial differential operators (such as the Laplacian, and other relevant operators with no counterparts in the one-variable case) are nowhere to be found in Weierstrass' paper.

The fact that Poincaré changed settings altogether by working three of the four steps of his proof in the setting of real functions with identically vanishing Laplacian (i.e. potential functions), is not ascribable to a mere matter of taste. As he pointed out in the paper (Poincaré 1883b, p. 152), and again in 1901 in his general analysis of his mathematical work (Poincaré 1901, pp. 70–73), the chunking of the pole-locus into a denumerable infinity of loci contained in balls could not possibly have been achieved at the complex analytic level: in order for real function u of four real variables be the real part of a (possibly multi-valued) analytic function in two complex variables, u has to satisfy five independent partial differential equations, only one of which is the vanishing of the Laplacian; this is a striking difference with the one-variable case, in which the complete overlap of the two classes—potential functions in two real variables, real parts of (possibly multi-valued) analytic functions in one complex variable—was the starting point of Riemann's theory of functions of one complex variable. In the two-variable case, the class of potential functions is much larger than that of real parts of analytic functions, hence much more tractable since it satisfies the Dirichlet principle (at least for balls). These tactics, of course, call for additional work later on, in order to recover analytic functions from potential functions (step 4 of Poincaré's proof). The way this change of setting allowed Poincaré to overcome the specific difficulty of the two variable case is what he emphasised as the great idea and great success of this 1883 paper, when reflecting on it in his 1901 overview.

2.4 What context for Poincaré's proof? "Site technique" points to potential theory

Interestingly, though, what Poincaré did not emphasise is the topological part on which we concentrated: the link between spaces of four dimensions and coverings with at most five intersecting balls plays a fundamental role in the proof, yet it is used without

[10] This only essential singular point can be sent at infinity, hence the focus on entire functions.

proof or commentary. This specific engagement with topology and lack of comment on it call for a detailed analysis.

Our main claim is that the technological context of Poincaré's (1883) proof is primarily that of German potential theory. There are several reasons why we wrote "technological" context and not "theoretical" context. As to the theoretical context, we cannot come up with something much more specific than "theory of functions". This is the section in which Poincaré placed this paper in his 1901 overview; to be more specific, he placed it in the paragraph on "*Théorie générale des fonctions de deux variables*" (Poincaré 1901, p. 70), as opposed to general results on functions of one variable, and to results on abelian functions. If we stick to the 1883 paper, an overview of the names of mathematicians Poincaré mentioned points to a twofold context:[11] first, a "*théorie générale des fonctions*" context, with Weierstrass (for the origin of the question tackled and for techniques which help establish the convergence of some power-series in several variables), Mittag-Leffler (for convergence techniques), and Sonya Kowalevksy (for a theorem on analytic continuation in several variables (Poincaré 1883b, p. 154)); second, a potential theory context, with Klein, Kronecker (Poincaré 1883b, p. 148) and Schwarz (Poincaré 1883b, p. 152).

A more specific lead is given by the technical way in which the problem is laid out and tackled. We shall use the term *site techniques*[12] to denote the various ways in which functions are studied in terms of their interaction with a site or domain;[13] this notion does not call for a more technical definition, but we claim it is useful to study the growing awareness in nineteenth century mathematics of the importance of "where functions live", and to document the slow accumulation of techniques and vocabulary used by mathematicians to explore the interaction between functional objects and domains/sites. One may feel that there is no need to coin such an awkward term for what can be (and has been) studied as the rise of topology in late nineteenth century and early 20th century mathematics. This "rise of topology" description seems to be less fit for our purpose, however, for several reasons: one general methodological reason, is that "rise of topology" points to a specific body of knowledge that would emerge later, thus leaving us with the somewhat unattractive task of exploring "vague" or "naive" topological reasoning; this "topology before topology" description rests on the unilluminating back projection of topology as a *theory* (a set of concepts, definitions and theorems), on specific forms of *know-how* (in problem solving) and *awareness* (as to the role of particular aspects in specific problems). Another important reason is that what we describe as late 19th century site techniques would not lead to topology in the strict sense, even if "topology" is thought of as encompassing both point-set and algebraic topology (which is already quite a stretch). Evolutions in site techniques would lead equally well to our very elementary notions of subset intersection and union;[14] to set products also. When studied in context, these techniques cannot be thoroughly separated from the non-topological context: it is common historical knowledge

[11] It is worth mentioning that no French mathematicians are mentioned.

[12] We originally used "*techniques du lieu*" in French.

[13] We shall use "site" rather than "domain", since the latter now has a standard, elementary yet technical meaning in set theory. "Site" is also used in topos theory of course, but confusion seems less likely.

[14] See Chap. 5 in Chorlay (2007).

that the study of real or complex functions led 19th century mathematicians to very different site techniques, the real case roughly leading to point-set topology,[15] the other one to combinatorial then algebraic topology; with its emphasis on know-how and awareness, the "site techniques" approach allows for a more fine-grained tracking of epistemic cultures and research fields (Goldstein and Schappacher 2007, p. 52 and fol.).

We will use this approach in the case of Poincaré's (1883) paper. It exemplifies a specific technique which can (loosely) be described this way: starting from "data" given in an open covering, try to "glue" them together by dealing with whatever may come up in the overlaps; or, in modern notation and when only two open subsets U and V are considered, study $U \cap V$ in order to reach a conclusion about $U \cup V$. For the 21st century reader, this may not sound very specific: the approach is standard in the definition of major classes of objects (manifolds of various types, sheaves etc.) and in the tackling of a great wealth of problems. But this technique was by no means "standard" in 1883; we can give here but general elements to back this claim.[16] Another site technique was becoming standard in the theory of complex functions, which consisted in a coherent family of rationally linked site techniques. After Cauchy and Riemann, cutting techniques had become widely available for the study of multi-valued complex analytic functions. When the multi-valuedness is at most denumerable, the function can be used to "unfold" the cut domain into a new domain; on this unfolded domain, the reciprocal of the former function shows some kind of periodicity, or, more precisely, is left invariant under the action of a group. In this case, the unfolded domain is endowed with a tiling made up of elementary domains for the discrete group action. Considering the analytic continuation along a curve, cutting a domain, crossing a cut, unfolding a domain, tiling a domain ... those are techniques that were available to Poincaré in complex function theory and on which he relied on other occasions, in particular in his work on fuchsian functions (Scholz 1980; Gray 1986). Within this set of techniques, the intersections of two adjacent elementary domains for the group action are of non-null codimension, which entails a completely different approach to that of open overlaps.

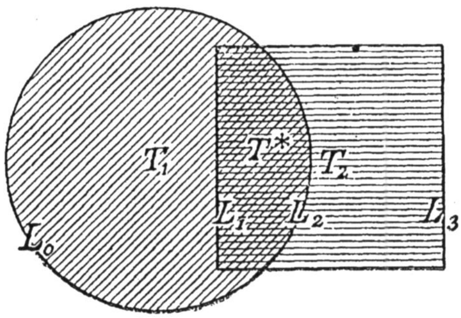
"Adjacent but not overlapping" is also the case in a different, less common site technique, namely that of cellular decomposition. It can be found, in particular, in Poincaré's qualitative work on curves defined by an ordinary (non linear) differential equation (Poincaré 1881, 1885). Other less widespread site techniques can be documented, for instance, in Schwarz's work on conformal equivalence, in which (in modern parlance) an open subset of the complex plane is approximated by a growing sequence of compact unions of finitely many squares (Schwartz 1890, p. 114); a daring variant of this approximation method is used in Poincaré's first proof of his uniformisation theorem for analytic functions (Poincaré 1883a).

To the best of our knowledge, the specific context of the "$U \cap V, U, V, U \cup V$" site technique, is that of potential theory; more precisely: that of the proof methods which H.A. Schwarz and C. Neumann introduced in potential theory around

[15] Though, as we mentioned above, Weierstrass developed notions which would later build up the core of point-set topology in the context of complex analysis.

[16] See Chaps. 1 to 5 in Chorlay (2007).

1870. These methods were developed as alternatives to Dirichlet's and Riemann's minimising proof methods in potential theory, whose weakness had been exposed by Weierstrass (1870). We need not go here into a description of both Neumann's and Schwarz's methods, since the basic principle of what Neumann called his "combinatorial methods" (Neumann 1884, vi) is the same as that of Schwarz's proof by the "alternating process" ("*alternierendes Verfahren*", see, for instance (Schwartz 1890, pp. 133–143)). The problem to be solved is Dirichlet's problem in two real variables: to find a potential function defined inside a plane domain D (assumed to be bounded) whose limit values on the boundary of D coincide with those of a given function. Schwarz endeavoured to show that if the problem can be solved universally (i.e. for arbitrary boundary conditions) on two overlapping domains T_1 and T_2, then it can also be solved for the domain "made up" ("*zusammengesetzt*" (Schwartz 1890, p. 136)) of the two. The *ekthesis* relies on the following figure:



The boundary values are known for L_0 and L_3 and a potential function is to be defined over the domain which Schwarz denotes by $T_1 + T_2 - T^*$.[17] Let k denote the lower bound of the boundary values (k is assumed to be finite). The outline of the "alternating process" is the following: start with potential function u_1, defined in T_1 with boundary conditions given by the original ones on L_0 and k on L_2; consider potential function u_2, defined in T_2 by the original conditions of L_3 and u_1 values on L_1; consider potential function u_3, defined in T_1 by the original conditions of L_0 and u_2 values on T_2 etc. Schwarz establishes that sequence (u_{2n}) tends to a potential function u' in T_1, sequence (u_{2n}) tends to a potential function u'' in T_2, and that $u' = u''$ in T^*. Thus, in spite of the fact that, after any finite number of steps, the potential functions defined over T_1 and T_2 have a non-null difference in T^*, this difference vanishes after an infinite number of steps. Schwarz compares his proof schema to the mechanical principle of a two-cylinder vacuum pump: air is drawn out of a central compartment by the successive actions of two adjacent cylinders.[18]

[17] This testifies to the fact that, for Schwarz, $T_1 + T_2$ would not denote the union but rather a "sum" in which T^* would have multiplicity two.

[18] "In order to prove this proposition, a limiting process can be used which is very analogous to the one with which vacuum is created by a two-cylindre air pump". "*Zum Beweise dieses Satzes kann ein Grenzübergang dienen, welcher mit dem bekannten, zur Herstellung eines luftverdünnten Raumes mittelst einer zweistiefeligen Luftpumpe dienenden Verfahren grosse Analogie hat.*" (Schwartz 1890, p. 136).

Poincaré's finite alternating sum associated with a covering of $\mathbf{R}^4$ by balls is only a variant of this site technique. There is more than a formal analogy: Schwarz's work on potential theory is one of the main body of results on which Poincaré drew in several papers he wrote in 1883 (we mentioned (Poincaré 1883a) above); in the one we are studying, he explicitly mentioned the fact that his proof of Dirichlet's principle for balls in four-dimensional space could be used *à la* Schwarz in order to prove Dirichlet's principle for more general domains (Poincaré 1883a, p. 152). Poincaré would later devise his own general existence theorem for potential functions, with his sweeping-out method (“*méthode de balayage*”), which combines the use of a covering by denumerably many balls (as in his 1883 paper) and a limiting process modelled after Schwarz's and Neumann's alternating processes. The 1883 paper is directly echoed in Poincaré's 1898 paper *Sur les propriétés du potentiel et les fonctions abéliennes* (Poincaré 1898): the aim of the latter paper is to establish the analogous representation results for maximally periodic meromorphic functions in several complex variables (abelian functions). After an *ekthesis* which is exactly the same as that of the 1883 paper (Poincaré 1898, p. 228), and a proof that goes along the very same lines, the resulting theorem contains additional information, because of the periodicity of the meromorphic functions: for a given system of periods, maximally periodic meromorphic functions in several complex variables can be written as quotients of generalised Θ function, i.e. entire functions such that $\Theta(z_i + a_i) = \Theta(z_i)e^U$ (where, (a_i) is a period and U a linear function in (z_i) (Poincaré 1898, p. 241)). As the title of the 1898 paper indicates, this theorem is given as an application of potential theory methods.

2.5 A standard technique when dealing with manifolds?

Our claim about the specific link between this site technique and a coherent family of proof schemes in potential theory may seem questionable to a twenty-first century reader. Is it not, after all, the standard way to define geometric objects of the manifold type? Scholz's work on the emergence of the manifold concept from Riemann to Poincaré shows that dealing with (open) intersections and gluing patches were not standard site techniques in this context, whereas cutting, unfolding, gluing together edges of the fundamental domain of a group action etc. were (Scholz 1980). We claim that, from a historical viewpoint, things went the other way around: our standard site techniques for defining manifolds derive, to a large extent, from site techniques in potential theory, via uniformisation theory. Let us mention but two significant pieces of evidence.

The first is given by Poincaré's 1908 paper *Sur l'uniformisation des fonctions analytiques* (Poincaré 1908). Poincaré's main goal was to give a new and less sketchy proof of his general uniformisation theorem for arbitrary analytic functions in one complex variable; several aspects of the original 1883 proof had met with criticism from Hilbert and W.F. Osgood (Poincaré 1908, §1). The new and improved proof required two significant changes: one in the definition of the auxiliary surface (which would later be called a universal covering of a Riemann-uniformising surface over the plane domain of the multi-valued analytic function), and one in the existence proof of an analytic function mapping this abstract surface onto a plane domain in a one-to-one and unramified manner. The first part of the 1908 paper is dedicated to a general definition of analytic domains "over" the complex plane, generalising Riemann surfaces. To the best of our knowledge, it is the first time that a general theory of Riemann analytic surfaces over the complex plane ("*domaines D*") was expounded in terms of open coverings and gluing rules.[19] On the basis of this definition, Poincaré defines the notion of equivalence of D-domains, the notion of sub-D-domain (Poincaré 1908, §2) and, two paragraphs down, the notion of "*multiple*" of a D-domain (i.e. covering space (Poincaré 1908, §4)). However, this general theory is not what Poincaré is after. In fact, it is completely ancillary to the series of results given in the second part of the text: these existence results for analytic functions are all obtained by the sweeping-out method, and the general theory of D-domains is thoroughly modelled by the requirements of this proof method.

The second piece of evidence comes from Hellmut Kneser's 1926 report on the topology of manifolds (*Die Topologie der Mannigfaltigkeiten* (Kneser 1926)[20]). In this paper, Kneser's goal was to compare the merits of the combinatorial approach and the purely topological ("*rein topologisch*" (Kneser 1926, p. 1)) approach in the definition and study of topological manifolds; also to build bridges between the two. On the purely topological side, he started from Hausdorff's axioms for a general topological space, then added the relevant axioms on the local topological equivalence with an open ball in numerical space (Kneser 1926, pp. 2–3). The main consequence he drew from this definition is a unified way of generating a topological manifold: for an object that satisfies the axioms, a complete description is given by a covering by open subsets U_i (each of which is homeomorphic to a ball V_i), and homeomorphisms between those parts in V_i and V_j which correspond to the open intersection of U_i and U_j; Kneser claims it is clear that this analysis can also lead to synthesis—i.e. that a manifold can be defined from a family of open balls and partial homeomorphisms—, though without using any "gluing" metaphor for this synthesis process (Kneser 1926, p. 4). This description has since become standard since it is the starting point in Veblen and Whitehead's 1932 *Foundations of Differential Geometry* (Veblen and Whitehead 1932). But, in 1926 the description was by no means standard when dealing with the topology of manifolds, as Kneser's comment shows:

> This mode of presentation is none but the "roof-tile covering", which is well-known from function theory. However useful it may be there, it has a drawback with regard to our topological purpose, namely that the mapping between V_{ik} and V_{ki} is, to a large extent, arbitrary. There is no such drawback in another mode of presentation, which can be referred to by the key-word "border pairing".[21]

[19] Of course, one could argue that Poincaré's presentation is a mere elaboration on Weierstrass' definition of analytic functions based on function elements and analytic continuation. This is undoubtedly a core background element, but (1) its relation to the technique we are documenting is only indirect (2) other techniques can be based (and were based) on the idea of analytic continuation.

[20] It evolved from a 1924 talk.

[21] "Diese Darstellungsweise ist nicht anders als die aus der Funktionentheorie bekannte "dachziegelartige Überdeckung". So brauchbar sie dort ist, für topologische Zwecke hat sie den Mangel, dass in sie die Abbildungen zwischen V_{ik} und V_{ki}, also in weitem Masse willkürliche Funktionen eingehen. Frei von diesem Mangel ist eine andere Darstellungsweise, die durch das Schlagwort "Ränderzuordnung" bezeichnet

Open coverings—which entails open intersections—are still referred to the context of uniformisation theory, whereas the more familiar (and, in 1926, efficient) combinatorial viewpoint relies on "cutting" ("*zerlegen*" (Kneser 1926, p. 4)) and identifying edges of generalised polyhedra according to non-"arbitrary" rules (usually derived from a group action).[22]

2.6 Problem form and problem label

To insert Poincaré's (1883) paper on meromorphic functions in two complex variables into our long-term narrative, the questions of the identification of a general problem form, and of the use of a specific problem label remain. No such things are to be found in the 1883 paper itself, and we shall see that this is a general feature of Poincaré's work.

It seems all the more surprising than it is difficult, for the 21st century reader, to study the young Poincaré's work (say, between 1880 and 1885) without using "global" or "from local to global" as core descriptive terms. Along with the paper we focused on, we should mention three other major contributions to global mathematics in this period; we will be brief, since these aspects of Poincaré's work have been studied in detail elsewhere.[23] In 1883, we find Poincaré's paper *Sur un théorème général de la théorie des fonctions* (Poincaré 1883a) which we mentioned before. Consider two complex variables X, Y, linked by an analytic relation which implicitly defines Y as a multi-valued function of X; Poincaré "proved" that there exists an auxiliary complex variable Z such that X and Y are single-valued analytic functions of Z. In a daring generalisation of Riemann's ideas, Poincaré first built up an abstract analytic surface over the X plane, such that (1) Y is a uniform function of the surface generic point, and (2) the surface is simply connected.[24] He then used variants of Schwarz's

footnote 21 continued
werden kann." (Kneser 1926, p. 4) This arbitrariness element, which Kneser deplores, will be the cornerstone of Veblen and Whitehead's idea of manifold structureS (plural): specify a pseudo-group (a notion which they introduced) and you get a specific structure on the manifold you build up (topological, differentiable, analytic, riemannian or pseudo-riemannian, conformal structures etc.).

[22] Our point is not that Kneser's description in terms of open coverings and gluing rules was a major, unprecedented and bewildering breakthrough; his comment helps us point to the fact that this type of description was by no means standard in the 1920s, and that it was still clear at the time that it comes from complex function theory. The fact that this site technique was not standard in topology in the 1920s does not prove, however, that it was not familiar in other theoretical contexts in which "spaces" or "manifolds" play a major part. In the context of differential geometry, it can be shown that this site technique cannot be found in Weyl's or Cartan's papers on differential geometry (cf. in Chorlay 2007, Chap. 11). In the early 1920s, there is one context in which it was standard when defining the differential manifolds, namely that of Birkhoff and Morse's works on dynamical systems and the calculus of variations in the large. This technique can be traced directly to Poincaré, via Hadamard's paper on geodesics on surfaces of negative curvature (cf. Chorlay 2007, Chaps. 8 and 10).

[23] See Gray (1986) and Scholz (1980).

[24] Poincaré gave only a sketch of how this surface is to be defined, but his idea was clearly that of considering classes of paths modulo equivalence relations (one depending on the behaviour of multi-valued function Y, then another one to ensure simple connectedness).

potential-theoretic methods (not the alternating method, however) to prove that the abstract analytic surface is equivalent to an open disc in a complex plane, the generic point of which will be denoted by Z. As we mentioned earlier, several points of this 13-page paper seem questionable; the 1908 paper on the same topic would be 70 pages long, and would rely on different techniques for the construction of the surface and the existence proofs in potential theory; the proof-scheme remains the same, however. A generation later, these uniformisation theorems would be listed among those of a clearly "global" (*im Grossen*) nature.[25]

More precise uniformisation results are reached in the case of algebraic functions and solutions of linear differential equations with rational coefficients, in Poincaré's papers on fuchsian groups (e.g. Poincaré 1884). Not only is the problem of a global nature, but its solution led Poincaré and Klein to a new proof technique that would later become a standard tool when tackling global problems,[26] namely the "*méthode de continuité*" (Klein's "*Continuitätsbeweis*"). To put it briefly:[27] in order to establish that a natural one-to-one map between two (connected) parameter spaces is also onto, Poincaré and Klein attempted to show that its image is both open (using local inversion arguments) and closed (which led Poincaré to "compactify" of the first space, a point which Klein apparently missed).[28]

The third group of works that should be mentioned to establish the fact that Poincaré's early work is epoch-making in the history of global mathematics, is that on the qualitative theory of curves defined by a differential equation.[29] He engaged in 1881 in a series of studies on non-linear ordinary differential equations in two real variables $F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0$ (F a polynomial). To give just one of the first results: starting from his (local) classification of singular points and his definition of their index, Poincaré established a numerical link between the number of singular points of the differential equation and the topological genus the surface defined by equation $F(x, y, z) = 0$ (Poincaré 1885). This work would also later be considered as paradigmatic of what mathematics "in the large" is.[30]

Yet, to the best of our knowledge, Poincaré never pointed to a general common form of these various problems and proof-schemes. Although he often stressed, from

[25] This description of uniformisation theory as tackling a general problem form (namely, to start from uniformisation theorems *im Kleinen* and strive for uniformisation theorems *im Grossen*) can be found in Osgood's Cambridge Colloquim Lectures (Osgood 1899, p. 70) and in Weyl's *Idee der Riemann'schen Fläche* (Weyl 1919, p. 141).

[26] For instance, when Weyl used the same kind of purely topological techniques in 1925, in the context of Lie groups theory, he still referred to the method as "*Kontinuitätsmethode*" (Weyl 1925, p. 629).

[27] See Scholz (1980) and Gray (1986).

[28] This one-sentence summary uses topological vocabulary that was alien to Poincaré and Klein. A less anachronistic description of their proof scheme would be: they use a *reductio ad absurdum*, claiming that if the map was not onto, the boundary of the image subset would be both empty and non-empty.

[29] For a general study on this topic, see Gilain (1991).

[30] For instance, this work is the only example of differential geometry "in the large" that Struik gave in his 1933 MIT talks on the history of differential geometry from the 17th to the 19th century (Struik 1933b, p. 188).

his earliest works, that he would not restrict his study of the behaviour of functions to a neighbourhood of their singular points,[31] he never coined a general term to describe a wide family of problems.

In the period 1901–1912, Poincaré had many occasions to reflect on his scientific career (as in his 1901 overview of his mathematical work), on the nature of mathematics, and on its future problems (as in his 1909 ICM talk on *L'avenir des mathématiques* (Poincaré 1909)). Lines such as:

> My time will not have been wasted (…) if this very fumbling eventually showed me the deep analogy between the problem I just dealt with, and a much more extended class of problems; if they showed me the likenesses and differences; if, in a word, they gave me a glimpse of possibilities for generalisation.[32]

did not refer to global problems. When, in the same 1909 talk, he gave examples of concepts that would clearly play an important role in future mathematics, he first mentioned uniform convergence, then groups and invariants.

As to his views on the nature and role of topology, they would prove stable, from his celebrated 1895 *Analysis situs* paper (Poincaré 1895) to his more epistemological writings on the nature of mathematical knowledge. *Analysis situs* is consistently described as a form of *hypergeometry* (i.e. dealing with spaces and forms of dimension greater than three (Poincaré 1909, p. 181)); and as a *qualitative* geometry, meaning non quantitative (i.e. in which exact measurement is not of the essence) and stable by deformation (hence the idea of a geometry fit for poorly drawn figures) (Poincaré 1901, p. 100).

In 1912, Hadamard would stress the fact that the will to go beyond local properties of functions is the *main* and *central* feature of Poincaré's mathematical work (Hadamard 1912, p. 1922).[33] To Hadamard, this central feature would, in turn, *explain* why *analysis situs* played such an important role in so many parts of Poincaré's work (Hadamard 1912, p. 1864). However obvious all this might have appeared to Hadamard in 1912, it is nowhere to be found in Poincaré's writings.

[31] The non-local nature of the goal is stated in the clearest of ways in the introduction to the series of papers on the qualitative study of curves defined by a differential equation: "*Rechercher quelles sont les propriétés des équations différentielles est donc une question du plus haut intérêt. On a déjà fait un premier pas dans cette voie en étudiant la fonction proposée* dans un voisinage d'un des points du plan. *Il s'agit aujourd'hui d'aller plus loin et d'étudier cette fonction* dans toute l'étendue du plan." (Poincaré 1881, p. 3) (Poincaré's emphasis). Poincaré once explicitly used the term "*problème local*" in his 1907 paper *Les fonctions analytiques de deux variables et la représentation conforme* (Poincaré 1907), but did not go beyond this "local" use.

[32] "Je n'aurai pas perdu mon temps (…) si ces tâtonnements mêmes ont fini par me révéler l'analogie profonde du problème que je viens de traiter avec une classe beaucoup plus étendue de problèmes; s'ils m'en ont montré à la fois les différences et les ressemblances, si en un mot ils m'ont fait entrevoir les possibilités d'une généralisation." (Poincaré 1909, p. 169)

[33] For Hadamard, the page numbers refer to the collected works (Hadamard 1968).

3 The "Cousin problems", from Cousin (1895) to Oka (1938)

3.1 A family of three

In contrast with Poincaré, who singled out and generalised one theorem from Weierstrass' coherent list of results on the theory of single-valued functions defined over the complex plane, Cousin brought together three problems to make one coherent family. The first two—to find a regular function with given zero-locus, and to represent a meromorphic function as a quotient of two regular function—had already been studied together, as it was clear that solving the first was the key to solving the second. Cousin added to the family Mittag-Leffler's famous theorem:

Let there be given:

1° An infinite sequence of distinct values $a_1, a_2, a_3, \ldots$, subject to the condition

$$\lim a_\nu = \infty, \quad \nu = \infty;$$

2° An infinite sequence of entire fractions, rational or transcendental, of variable y, all of which vanish for $y = 0$:

$$\begin{aligned} G_1(y) &= c_1^{(1)} y + c_2^{(1)} y^2 + c_3^{(1)} y^3 \ldots, \\ G_2(y) &= c_1^{(2)} y + c_2^{(2)} y^2 + c_3^{(2)} y^3 \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ G_\nu(y) &= c_1^{(\nu)} y + c_2^{(\nu)} y^2 + c_3^{(\nu)} y^3 \ldots \end{aligned}$$

Then, it is always possible to form an analytic function $F(x)$ with no other singular points than $a_1, a_2, a_3, \ldots$, and such that, for every determinate value of ν, the difference

$$F(x) - G_\nu\left(\frac{1}{x - a_\nu}\right)$$

has, at $x = a_\nu$, a determinate finite value, so that in the neighbourhood of $x = a_\nu$, $F(x)$ can be expressed in the form

$$G_\nu\left(\frac{1}{x - a_\nu}\right) + P(x - a_\nu).$$ [34]

In the introduction to his 1895 thesis *Sur les fonctions de n variables complexes* (Cousin 1895), Pierre Cousin comments:

[34] "Je suppose données:

1° Une suite infinie de valeurs $a_1, a_2, a_3, \ldots$, toutes inégales et assujetties à la condition

$$\lim a_\nu = \infty, \quad \nu = \infty;$$

Two important theorems by Mr. Weierstrass can be regarded as successive consequences of Mr. Mittag-Leffler's theorem; the first one pertains to the existence of an entire function whose zeroes are given in advance; the second one pertains to the expression as a quotient of two entire functions, of a function in one variable whose singular points are but poles. These three theorems make up a set of three propositions which are closely tied together.[35]

There are various ways in which the family of three can be tied together, and Cousin is by no means the first to consider that the first theorem (on singularities) implies the second (on zeroes). For instance, Paul Appell had given in 1883 a generalisation of Mittag-Leffler's theorem to the two complex variables case (Appell 1883): starting with a sequence $f_1(x,y), f_2(x,y), \ldots, f_\nu(x,y)\ldots$ of analytic functions (defined over $\mathbf{C}^2$ but for singularities), whose singular locus goes to infinity,[36] he defined a function f, over $\mathbf{C}^2$, with the same singular locus and such that $f - f_\nu$ is regular where f_ν is singular.[37] Both the *ekthesis*—in terms of singular locus that goes to infinity—and one-page proof—add the f_ν and use Weierstrass' trick to deal with convergence problems—are direct offshoots of Mittag-Leffler's. Appell then formulated the analogous problem: starting from a sequence of entire functions $g_1(x,y), g_2(x,y), \ldots, g_\nu(x,y), \ldots$ whose

footnote 34 continued
2° Une suite infinie de fractions entières, rationnelles ou transcendantes de la variable y, s'annulant toutes pour $y = 0$:

$$\begin{aligned} G_1(y) &= c_1^{(1)}y + c_2^{(1)}y^2 + c_3^{(1)}y^3 \ldots , \\ G_2(y) &= c_1^{(2)}y + c_2^{(2)}y^2 + c_3^{(2)}y^3 \ldots , \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ G_\nu(y) &= c_1^{(\nu)}y + c_2^{(\nu)}y^2 + c_3^{(\nu)}y^3 \ldots \end{aligned}$$

Il est alors toujours possible de former une fonction analytique $F(x)$, n'ayant d'autres points singuliers que $a_1, a_2, a_3, \ldots$, et telle que, pour chaque valeur déterminée de ν, la différence $F(x) - G_\nu\left(\frac{1}{x-a_\nu}\right)$ait, en supposant $x = a_\nu$, une valeur finie et déterminée, de telle sorte que, dans le voisinage de $x = a_\nu$, F(x) puisse s'exprimer sous la forme $G_\nu\left(\frac{1}{x-a_\nu}\right) + P(x - a_\nu)$." (Mittag-Leffler 1882, p. 414). Mittag-Leffler had given a first proof of a slightly less general theorem in 1976. An overview of his work on one-valued analytic functions in one complex variable can be found in *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*, Acta Math. **4** (1884), pp. 1–79.

35 "Deux théorèmes importants de M. Weierstrass peuvent être considérés comme des conséquences successives du théorème de M. Mittag-Leffler; le premier est relatif à l'existence d'une fonction entière admettant pour zéros des points donnés à l'avance; le second est relatif à l'expression sous forme d'un quotient de deux fonctions entières d'une fonction d'une variable n'admettant comme points singuliers que des pôles. Ces trois théorèmes forment un ensemble de trois propositions intimement reliées entre elles." (Cousin 1895, p. 1)

36 Appel expresses this condition in the following way: " (...) for all values of ν greater than a positive integer μ, a positive number a_ν can be assigned so that function $f_\nu(x,y)$ remains holomorphic as long as the moduli of x and y remain less than a_ν; moreover, number a_ν increases indefinitely with ν." " (...) *pour toutes les valeurs de ν supérieures à un entier positif μ,l'on peut assigner un nombre positif a_ν tel que la fonction $f_\nu(x,y)$ reste holomorphe tant que les modules de x et y restent inférieurs à a_ν;de plus ce nombre a_ν augmente indéfiniment avec ν*." (Appell 1883, p. 71)

37 At least if f_ν is the only function in the sequence which is singular at this point (Appell 1883, p. 72).

zero-locus goes to infinity, and a sequence $k_1, k_2, \ldots, k_\nu, \ldots$ of natural numbers, find an entire function G with the same zero-locus and such that $G/f_\nu^{k_\nu}$ is regular and non-null where f_ν vanishes.[38] As to the proof, one line is enough:

To prove these propositions, it is enough to apply the theorem from § 1 to the function

$$\frac{\partial \log G(x, y)}{\partial x}$$

with

$$f_\nu(x, y) = k_\nu \frac{\partial \log g_\nu(x, y)}{\partial x}.^{39}$$

As we saw earlier, the idea of using logarithms to pass from theorems on singular-loci to theorems on zero-loci had been used by Poincaré (in a different technical context, however); it would be used by Cousin, who explicitly referred to both Poincaré and Appell.

What unites the three theorems, in Cousin's thesis, though, is not that the second is a straightforward consequence of the first (on simple domains such as $\mathbf{C}^2$), but that both are consequences of a single fundamental theorem ("*théorème fundamental*").

3.2 A "fundamental theorem" with three corollaries

The proof of this fundamental theorem would make up the first two of the four paragraphs of the thesis. Before we state it, let us make clear what functions and domains Cousin chose to deal with. The domains are what would later be called polycylinders: the $n + 1$ complex variables are independent; each of them varies its own complex plane, in domains bounded by $n + 1$ "*contours fermés, simples ou complexes*" (Cousin 1995, p. 4). As to functions, they do not have to be regular throughout the domain, but they are subject to two constraints: they are "*monotropes*", which means their value at a point is left unchanged after analytic continuation around a loop in the domain;[40] they are "*sans espace lacunaire*":

[38] Again: at least if no other function in the sequence vanishes there.

[39] "Pour démontrer ces propositions, il suffit d'appliquer le théorème du § 1 à la fonction $\frac{\partial \log G(x,y)}{\partial x}$ en prenant $f_\nu(x, y) = k_\nu \frac{\partial \log g_\nu(x,y)}{\partial x}$." (Appell 1883, p. 74)

[40] The term «*monotrope*» comes from the standard French textbooks by Briot and Bouquet. For instance: "Let us assume that point z is bound to stay in a determinate part of the plane; if all paths from initial point z_0 to any point z (fig. 4), located in that part, lead to the same value for the function, we shall say that the function is monotropic in that part of the plane"; "*Supposons que le point z soit astreint à rester dans une partie du plan déterminée; si tous les chemins qui vont du point initial z_0 à un point quelconque z (fig. 4) situé dans cette partie conduisent à la même valeur de la fonction, nous dirons que la fonction est*monotrope*dans cette partie du plan.*" (Briot and Bouquet 1875, p. 10)

> The later condition means precisely that, if M denotes any point in the region at hand, there exists a point which is as close to M as one wills, for which the function is regular.[41]

The fundamental notion is that of *equivalent* functions:

> If two such functions are simultaneously defined inside the same partial area, and if their difference is regular at all points in this area, I shall say that the two functions are *equivalent* in this partial area; if their difference is regular at a point, the two functions are *equivalent* at this point.[42]

Cousins considered functions in $n+1$ variables; the first n variables $x_1, .., x_n$ (collectively denoted by x) have domain Γ, which is the product of n plane domains bordered by closed curves $\gamma_1, \ldots, \gamma_n$. The last variable is denoted by y, its domain is denoted by S and is cut in a finite number of subdomains $R_1, \ldots, R_p$, each of which being bordered by a simple (i.e. non self-intersecting) closed curve.[43]

> To each region R_p I associate a function $f_p(x, y)$ of the $(n+1)$ complex variables $x_1, \ldots, x_n, y$, defined for x interior to Γ and for y interior to a boundary line $\mathfrak{R}_p$ enclosing $R_p(\ldots)$; one also assumes the following: if R_p and R_n are two contiguous regions, $\mathfrak{R}_p$ and $\mathfrak{R}_n$ have in common a partial area where both functions $f_p(x, y)$ and $f_n(x, y)$ are defined; one assumes that these two functions are *equivalent* in the partial area where both are defined;[44]

We are now ready to state the

> Fundamental theorem. There exists a function $F(x, y)$, monotropic and without lacunary space, defined for x interior to Γ and y interior to S, and which, at every point interior to S, is equivalent to the function associated with that point.[45]

Although the theorems to be proved concern functions in several variables and open coverings, Cousin relies solely on single-variable techniques (which is possible

[41] "Cette dernière condition signifie d'une façon précise que si M désigne un point quelconque de la région considérée, il existe un point aussi voisin que l'on veut de M pour lequel la fonction est régulière." (Cousin 1895, p. 10)

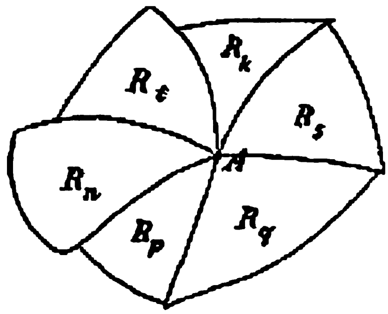
[42] "Si deux fonctions de cette nature se trouvent définies simultanément à l'intérieur d'une même portion d'aire et si leur différence est régulière en tout point de cette aire, je dirai que les deux fonctions sont *équivalentes* dans la portion d'aire considérée; si leur différence est régulière en un point, les deux fonctions seront *équivalentes* en ce point." (Cousin 1895, p. 10)

[43] The R_p domains are explicitly assumed to be simply connected (though Cousin never uses this term).

[44] "A chaque région R_p je fais correspondre une fonction des $(n+1)$ variables complexes $x_1, \ldots, x_n, y$, $f_p(x, y)$ définie pour x intérieur à Γ et pour y intérieur à un contour fermé $\mathfrak{R}_p$ enveloppant R_p (...); on fait de plus l'hypothèse suivante: si R_p et R_n sont deux régions contiguës, $\mathfrak{R}_p$ et $\mathfrak{R}_n$ ont une portion d'aire commune où les deux fonctions $f_p(x, y)$ et $f_n(x, y)$ sont simultanément définies; on suppose que ces deux fonctions sont *équivalentes* dans la portion d'aire où elles sont simultanément définies;" (Cousin 1895, p. 11)

[45] "Théorème fondamental. Il existe une fonction $F(x, y)$ monotrope sans espace lacunaire définie pour x intérieur à Γ et y intérieur à S et qui, en chaque point intérieur à S, est équivalente à la fonction correspondant à ce point." (Cousin 1895, p. 11)

in polycylinders), and on somewhat standard site techniques such as cuttings.[46] The relative positions of the R_p subdomains in the y-plane is described in terms of common boundary curves, l_{np} denoting the curve (or curve components) bordering R_n and R_p; the standard situation is given by the following figure (Cousin 1895, p. 13):



In a neighbourhood of l_{np}, both f_n and f_p are defined and their difference $f_n - f_p$ is regular. Cousin's tactics was to define, on what we would denote $\Gamma \times S$, a multi-valued function Φ that is single-valued[47] in each $\Gamma \times R_p$ and such that its analytic continuation when cut l_{np} is crossed undergoes a jump of $f_p - f_n$. Adding this difference-balancing function Φ to the f_p gives the required single-valued function F. Now, paragraph one is devoted to expounding the techniques for constructing difference-balancing functions of this type. Cousin used standard single-variable techniques by combining jump-functions defined by a parameterised line integral: first, the classic

$$\varphi(x, y) = \frac{1}{2i\pi} \int_{l_{np}} \frac{dz}{z - y},$$

which undergoes a unit jump when the l_{np} "cut" is "crossed", then

$$I_{np}(x, y) = \frac{1}{2i\pi} \int_{l_{np}} \frac{f_p(x, z) - f_n(x, z)}{z - y} dz.$$

Special attention is paid to the behaviour of such functions at vertices (such as point A in the above figure).

After completing the proof of the fundamental theorem, Cousin showed how a slight change in the proof can lead to the solution of a different problem:

[46] In his 1914 report on the theory of functions of several complex variables, Osgood wrote that Cousin's proof was « more elementary » than Poincaré's (Osgood 1914, p. 44).

[47] To be more precise: multi-valued but with separate branches.

> The theorem we have just proved pertains to monotropic functions with no lacunary space; it can be extended to non-monotropic functions of a specific nature, namely these functions which are logarithms of regular functions.[48]

If to each R_p subregion one ascribes a function $u_\mathrm{p}(x, y)$, regular in $\Gamma \times R_\mathrm{p}$, such that, in the overlaps, quotient $u_\mathrm{p}/u_\mathrm{n}$ is regular and non-vanishing, the former proof applies to $f_\mathrm{p} = \log u_\mathrm{p}$. Every step of the way (even in the definition of f_p), one works up to additive constants of type $2ik\pi$ (where k is an integer). It does not matter since, in the last step, only the exponentials appear. Thus, it is established that:

> There exists a regular $U(x, y)$ which is regular at every point (x, y) interior to (Γ, S), and such that at any point interior to S its quotient by the function $u_\mathrm{p}(x, y)$ associated with this point is regular and different from 0 at that point.[49]

The correctness of this conclusion would be questioned by Gronwall, as we shall see below.

Once the fundamental theorem and its logarithmic variant have been proved, Cousin moves on to establish the n-variable, polycylindral analogues of Mittag-Leffler's theorem (on singularities), Weierstrass' theorem (on zeroes) and Poincaré's theorems (on the representation of meromorphic functions). Since—once the logarithmic variant of the fundamental theorem is granted—the proofs of theorems 2 and 3 are just variants of the first one,[50] we need only to present the laying-out and the proving of the analogues of Mittag-Leffler's theorem.

Just like Poincaré's, Cousin's *ekthesis* exhibits what would later be called the "from local to global" problem form; S denotes an open polycylinder, $\Gamma_{a,b,\ldots,c,d,e}$ denotes a product of disks centered on point $(a,b,\ldots,c,d,e)$ and interior to S:

> I intend to show that if [...], at every point $(a, b,\ldots,c, d, e)$ interior to S, a function $f_{a,b,\ldots,c,d,e}(x, y,\ldots,z,t, u)$ is given, which is monotropic and without lacunary space inside $\Gamma_{a,b,\ldots,c,d,e}$; and if the given functions satisfy the above-stated condition, then there exists a function $F(x, y, \ldots, z, t, u)$, monotropic and without lacunary space inside S, which, at every point interior to S, is equivalent to the function given at this point.[51]

The condition being

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[48] "Le théorème qui vient d'être démontré relativement à des fonctions *monotropes et sans espaces lacunaires*, peut être étendu à des fonctions non monotropes d'une nature particulière: je veux parler des fonctions qui sont les logarithmes de fonctions régulières." (Cousin 1895, p. 16)

[49] "Il existe une fonction $U(x, y)$ régulière en tout point (x, y) intérieur à (Γ, S) et telle que en un point quelconque intérieur à S son quotient par la fonction $u_\mathrm{p}(x, y)$ qui correspond à ce point est régulier et différent de 0 au point considéré." (Cousin 1895, p. 20).

[50] This does not mean they show no real extra content, quite the contrary. For instance, when dealing with the representation of meromorphic functions as quotient of regular functions, Cousin uses Weierstrass' *Vorbereitungssatz* and proves the existence of an irreducible representation (Cousin 1895, p. 32).

[51] "Je me propose de montrer que si (...) on se donne pour chaque point $(a, b,\ldots,c, d, e)$ intérieur à S, une fonction $f_{a,b,\ldots,c,d,e}(x, y,\ldots,z, t, u)$ monotrope et sans espace lacunaire à l'intérieur de $\Gamma_{a,b,\ldots,c,d,e}$, et si les fonctions données satisfont à la condition qui vient d'être explicitée, il existe une fonction $F(x, y, \ldots, z, t, u)$ monotrope et sans espace lacunaire, définie à l'intérieur de S, et qui, en chaque point intérieur à S est équivalente à la fonction donnée en ce point." (Cousin 1895, p. 21)

If the point $(a',b',\ldots,c',d',e')$ is close enough to $(a,b,\ldots,c,d,e)$ so as to be interior to circle $\Gamma_{a,b,\ldots,c,d,e}$, the two functions

$$f_{a,b,\ldots,c,d,e}(x,y,\ldots,z,t,u) \quad \text{and} \quad f_{a',b',\ldots,c',d',e'}(x,y,\ldots,z,t,u)$$

must be equivalent at the point $(a',b',\ldots,c',d',e')$.[52]

To prove this result and its variants, Cousin had to overcome the following difficulty: the fundamental theorem deals with domains whose components are compact,[53] but the results he aimed at should hold for domains whose components need be neither closed nor bounded. He proceeded in three steps: first, in two variables, he proved his existence theorems over bounded polycylinders interior to S; step two is a straightforward generalisation to any number of variables; step three deals with the non-compact case and mainly relies on uniform approximation over compact subsets by rational functions (Cousin 1895, p. 38 and fol.).

3.3 Gronwall finds fault with Cousin's reasoning

In 1913, T.H. Gronwall presented to the American Mathematical Society a talk[54] on which he would base a later paper *On the Expressibility of a Uniform Function of Several Complex Variables as the Quotient of Two Functions of Entire Character* (Gronwall 1917). In this paper, he gave a critical presentation of Cousin's work (§1 and 2) and presented some personal results (§3). For the first part, Gronwall had worked with W.F. Osgood—who was working on his general survey on the theory of functions of several complex variables (Osgood 1914)—to locate the flaw in Cousin's reasoning: "The author wishes to acknowledge his indebtedness to Professor Osgood, to whom he communicated the example of §3 in June, 1913, for material assistance in locating the gap in Cousin's proof". (Gronwall 1917, p. 53).[55] So it seems that Gronwall's work on functions of several complex variables first gave him an example that contradicted Cousin's theorem on the representation of meromorphic functions, then worked with Osgood to locate the flaw and devise a proof for a restricted theorem.

Gronwall first summarised Cousin's results, labelling his theorems A, B, and C: A for the existence of an analytic function with given singularities, B for the existence of a holomorphic function with given zeroes, C for the representation of a meromorphic function as a quotient of relatively prime holomorphic functions.[56] He acknowledged

52 "Si $(a',b',\ldots,c',d',e')$ est un point assez voisin de $(a,b,\ldots,c,d,e)$ pour être intérieur au cercle $\Gamma_{a,b,\ldots,c,d,e}$, les deux fonctions $f_{a,b,\ldots,c,d,e}(x,y,\ldots,z,t,u)$ et $f_{a',b',\ldots,c',d',e'}(x,y,\ldots,z,t,u)$ doivent être équivalentes au point $(a',b',\ldots,c',d',e')$." (Cousin 1895, p. 21)

53 "Compact" is of course not part of Cousin's terminology.

54 Cf. (Cole 1914, pp. 173–174).

55 The corrected version of Cousin's proof which Gronwall gave was devised in collaboration with Osgood (Gronwall 1917, p. 54).

56 A conclusion which Gronwall formulated this way: "(…) this function may be expressed as a quotient of two uniform functions holomorphic in $S_1,\ldots,S_n$ and without common zero manifolds of higher dimension than $2n-2$." (Cole 1914, p. 73). The presentation is more algebraic in Gronwall (1917).

the fact that Cousin's proof of theorem A is perfectly rigorous. Since he had a counter-example to theorem C, and C is an easy consequence of B, there has to be a flaw in theorem B. He and Osgood eventually located it in the logarithmic variant of Cousin's "fundamental theorem". The proof they gave is *exactly* Cousin's proof. They simply pointed to the fact that, when starting with $f_{\mathrm{p}} = \log u_{\mathrm{p}}$, the jump-balancing function Φ need not be uniform (i.e. single-valued) in every $\Gamma \times R_{\mathrm{n}}$ if Γ is not assumed to be simply connected; they actually come up with an example (in the product of two ring-shaped domains $1/2 < |z| < 2$) where $G = \mathrm{e}^{\Phi}$ is multi-valued. Hence what Cousin's proof proves, according to Gronwall, is

> Let a function $u_{\mathrm{p}}(x, y)$ be given for every region R_{p}, uniform and holomorphic in (S, R_{p}), boundaries included, and such that for any two adjacent regions R_{n} and R_{p}, the quotient
>
> $$\frac{u_p(x, y)}{u_n(x, y)} = g_{\mathrm{np}}(x, y)$$
>
> is holomorphic and different from zero in (S, T_{np}).[57] Then there exists a function $G(x, y)$ holomorphic in (S, S'), uniform in (Σ, S'), where Σ is any simply connected part of S, and such that in (S, R_{p}) (boundaries included except those y which are end points of an l_{np} and lie on the boundary of S') the quotient
>
> $$\frac{G(x, y)}{u_p(x, y)}$$
>
> is holomorphic and different from zero. (Gronwall 1917, p. 53)

Cousin thought he had established this fact for any polycylinders but he had proved it only for those polycylinders in which at most one of the components is not necessarily simply connected.[58] Of course, this restriction as to what Cousin's proof actually proves does not imply that what he thought he had proved is not the case. This is where Gronwall's counter-example comes into play: in the product of two ring-shaped domains, he exhibited a meromorphic function that cannot be the quotient of two relatively prime (single-valued) holomorphic functions;[59] if theorem C fails to hold for such domains, so does theorem B. The final conclusion is

[57] T_{np} denotes an open neighbourhood of curve l_{np}.

[58] Cousin's requirement that the functions be "*sans espace lacunaire*" does not mean he considered only simply connected components. He explicitly refers to "*contours (...) à connexion multiple*" (Cousin 1895, p. 38). He gets more precise a few pages down the paper, in the two-variable case: "*S désigne une aire connexe prise sur le plan de la variable x et limitée par n + 1 contours fermés simples $C_0, C_1, C_2, \ldots, C_n$, dont le premier C_0 enveloppe S et chacun des n autres est enveloppé par S (...). Les notations S', $C'_0, C'_1, \ldots, C'_m$ (...) ont une signification analogue relativement au plan de la variable y.*" (Cousin 1895, p. 50). See also (Cousin 1895, pp. 55–56).

[59] It can be written as the quotient of two, not relatively prime, one-valued holomorphic function, however.

Theorems B and C are valid when, and only when, $n-1$ of the n regions $S_1, S_2, \ldots, S_n$ are simply connected; the remaining region may be simply or multiply connected. (Gronwall 1895, p. 53)

Cousin's proof relied on a patching argument that paid attention to only one of the components of the polycylinder, and it seems his "up to a harmless integer multiple of $2i\pi$" argument was flawed; however, Gronwall chose not to pinpoint *exactly* where the hitch lay in Cousin's proof, and formulated restrictive domain hypotheses which enabled him to avoid any sort of multi-valuedness at any step of the proof. This paper also clearly shows that topological properties play a part in theorems B and C which they do not play for theorem A (valid in any polycylinder): what part *exactly*? Is the family-likeness of theorems A and B altogether misleading? That would be investigated later, as we shall see.

3.4 Enduring problems, indirect use

As to Cousin's family of three problems, there is a striking contrast between the difficulty of the problems and the ease of access to the questions. These questions are fairly straightforward (at least when put informally, the precise *ekthesis* takes a little more work, but not that much), clearly important, and have well-known one-dimensional analogues. They are part of the standard list of topics that the theory of functions of several complex variables addresses, as is shown by their integration in the classical monographs on the field: in Osgood's 1901 introductory overview on complex variables in Klein's *Encyclopädie* (Osgood 1901, p. 112), in Osgood's 1914 AMS *Colloquim Lecture* (Osgood 1914, p. 44 and fol.), in Behnke and Thullen's 1934 *Theorie der Funktionen mehrerer komplexer Veränderlichen* (Behnke and Thullen 1934, Chap. 5). Yet, no significant breakthrough took place in the period from 1895 to 1934; the only event is of a restrictive nature, when Gronwall unexpectedly hit upon a counter-example to problem C and, for theorems B and C, made it clear what topological hypotheses should be imposed on the polycylinders for Cousin's conclusions to hold.

Before expounding what would later be seen as Oka's breakthrough, let us mention two early 1930s papers that dealt with the topic.

The first paper is Cartan's 1934 note to the French *Académie des Sciences* on *Les problèmes de Poincaré et de Cousin pour les fonctions de plusieurs variables complexes* (Cartan 1934). This three-page note does not include proofs, but it shows that the three problems can be used as *tools* to tackle another family of questions. Cartan first stated the problems in a way that has not significantly changed since Cousin,[60] and used the following labels: what Gronwall called "theorem A", Cartan

[60] For instance: "First Cousin problem.- Let us assume that the domain under study, D, is covered by an infinity of partial domains D_i, interior to D, and that, in every D_i, a meromorphic function f_i is defined; it is also assumed that, whenever two domains D_i and D_j have a common part D_{ij}, the difference $f_i - f_j$ is holomorphic in D_{ij}. We intend to find a function F, meromorphic in D, and such that in every D_i the difference $F - f_i$ is holomorphic"; "*Premier problème de Cousin.- On suppose que le domaine considére D est recouvert à l'aide d'une infinité dénombrable de domaines partiels D_i intérieurs à D, et que, dans chaque D_i, on a défini une fonction méromorphe f_i; on suppose en outre que, chaque fois que deux*

calls "Cousin's first problem" (the additive problem, for meromorphic functions, later denoted as "Cousin I"); Gronwall's "theorem B" is "Cousin's second problem" (the multiplicative problem, "Cousin II"); Gronwall's "theorem C" is called "Poincaré's problem" (quotient representation of meromorphic functions). If, for a given domain D, Cousin's first problem has a solution *whatever* the data are, one says that Cousin's first theorem holds for D. In the list of results stated in this note, only one is a *direct* extension of Gronwall's results; by *direct* extension we mean the following: either a proof that Cousin I holds for a class of domains more general than that of polycylinders; or a proof that Cousin II and Poincaré's theorem hold for a class of domains more general than that of polycylinder with at most one not simply-connected component. All but one of Cartan's 1934 are *indirect* results, which shows a shift of emphasis: the Cousin and Poincaré problems are used as *tools* in order to help classify the various domains that the theory of functions of several complex variable has come to deal with since Hartogs' work. This indirect use is clear in the following two examples, in two complex variables:

> Theorem 1. If the first Cousin theorem holds for D, D is a domain of holomorphy (that is, the total domain of existence for some holomorphic function).[61]
> Theorem 3. Let D be a domain in which the first Cousin theorem holds (...). If, moreover, D is star, or if D is a Hartogs domain, then the second Cousin theorem holds for D.[62]

These theorems show no direct breakthrough, in the sense defined above, but show that the Cousin and Poincaré *properties* have tractable relations with those of important classes of domains: domains of holomorphy, domains of meromorphy, Hartogs domains, domains with convexity properties (with respect to polynomial functions, rational functions, holomorphic functions etc.), star domains, circle domains etc. This issue of domain classification is what is at the heart of Cartan's research programme in the 1930s, and it is not, as we shall see, a personal quirk. The background for the 1934 note to the *Académie des Sciences* is Cartan and Thullen's 1932 paper *Zur Theorie des Singularitäten der Funktionen mehrerer komplexen Veränderlichen—Regularitäts- und Konvergenzbereiche* (Cartan and Thullen 1932). It is devoted to the definition and study of the holomorphic hull of a domain (*Regularitätshulle*) and to its generalisation by the notion of convexity relative to a R-function class (such classes

footnote 60 continued
domaines D_i et D_j ont une partie commune D_{ij}, la différence $f_i - f_j$ est holomorphe dans D_{ij}. On se propose de trouver une fonction F, méromorphe dans D, et telle que dans chaque D_i, la différence $F - f_i$ soit holomorphe." (Cartan 1934, p. 1285).

61 Cartan showed in 1938 that the analogous theorem does not hold in three variables: he exhibited a domain that has the Cousin I property but is not a domain of holomorphy, namely tricylindre $|z_1| < 1, |z_2| < 1, |z_3| < 1$ with point $z_1 = z_2 = z_3$ removed (Cartan 1938).

62 "Théorème 1. Si le premier théorème de Cousin est vrai pour D, D est un domaine d'holomorphie (c'est-à-dire le domaine total d'existence d'une certaine fonction holomophe)." (Cartan 1934, p. 1286)
"Théorème 3. Soit D un domaine pour lequel le premier théorème de Cousin est vrai (...). Si en outre D est étoilé, ou encore si D est un domaine de Hartogs, le deuxième théorème de Cousin est vrai pour D." (Cartan 1934, p. 1286)

being defined by stability under partial derivation and raising to an integer power) (Cartan and Thullen 1932, p. 627 and 629).

A similar but more detailed overview is given in 1937, in Behnke and Stein's paper *Analytischen Funktionen mehrerer Veränderlichen zu vorgegebene Null- und Pollstellenflächen* (Behnke and Stein 1937). Though this report to the Union of German Mathematicians (D.M.-V.) mentions two recent direct breakthroughs with Cousin's problems—one by Cartan and one by Oka—its greatest part is devoted to what we called an indirect use: theorems, examples and counter-examples are given in order to show how "Cousin properties" interact with various domain classifying properties. For instance, Behnke and Thullen give the example of a domain in which Cousin II holds but not Cousin I (Behnke and Stein 1937, p. 186). We saw in Cartan a link between Cousin I and domains of holomorphy; Behnke and Stein comment on a result by Thullen which exhibits a (subtler) link between Cousin II and holomorphy domains:

> (...) those domains in which the second Cousin statement holds, but which are not regularity domains, differ from regularity domains by parts of a lesser dimension.[63]

What sums up Behnke and Stein's review is the following table:

	Cousin 1	Cousin 2	Poincaré	Possible ?
1	+	+	+	+
2	+	+	−	−
3	+	−	+	−
4	+	−	−	+
5	−	+	+	+
6	−	+	−	−
7	−	−	+	+
9	−	−	−	+

(Behnke and Stein 1937, p. 192)

For instance: combination 1 is the case in polycylinders made up of discs; combination 2 is impossible, since Cousin II implies "Poincaré" in any domain; Gronwall's example shows that 4 is the case for some domains.

3.4.1 Cousin I holds for more general domains than polycylinders: the (in)dispensable Weil integral

The only direct result mentioned (but not proved) in Cartan's 1934 short note to the Paris Academy is

> Theorem 2. If some domain D is convex relative to polynomials or rational functions in x, y, the first Cousin theorem holds for D.[64]

63 "(...) die Nicht-regularitätsbereiche, in denen die zweite Aussage von Cousin gilt, nur um niederdimensionale Stücke von Regularitätsbereiche unterscheiden." (Behnke and Stein 1937, p. 187)

64 "Théorème 2.- Si un domaine D est convexe par rapport aux polynômes ou aux fonctions rationnelles en x, y, le premier théorème de Cousin est vrai pour D." (Cartan 1934, p. 1286)

Cartan gives a small hint as to his proof method:

> This result, which goes far beyond Cousin's, is attained by a method analogous to his; but *André Weil's integral* (for functions of several variables) needs be used, whereas Cousin only used the classical Cauchy integral (for functions of one variable).[65]

It was clear that Cousin's proof depended essentially from the fact that his domain was a polycylinder, in which all variables could be studied (somewhat) independently with methods from the theory of functions of a single complex variable; in particular, the proof of the "fundamental theorem" relied on the properties of a function defined by a parameterised line integral along a cut in the plane of one of the variables. As a consequence, this proof-method was ill-suited for non polycylindral domains. However, in 1932, André Weil had devised an integral formula which generalised Cauchy's integral formula for the representation of holomorphic functions. This paper was written while Cartan and Thullen were writing their 1932 paper, and Weil wrote that it was his contribution to the study of R-convex domains in which his friends had engaged. His theorem goes as follows:

> Let $X_1, X_2, \ldots, X_n$ be n polynomials in x, y, and let D be the domain (if it exists) defined by the inequalities
>
> $$|X_i(x, y)| \leq 1 \quad (i = 1, 2, \ldots, n). \tag{1}$$
>
> Let σ_{ij} (if it exists) be the two-dimensional variety, lying on the boundary of D, which is defined by the equalities and inequalities
>
> $$|X_{\mathrm{i}}(x, y)| = |\mathrm{X}_{\mathrm{j}}(x, y)| = 1, \quad |X_{\mathrm{k}}(x, y)| \leq 1 (k \neq i, j).$$
>
> One can then determine polynomials $\Phi_{ij}(\xi, \eta; x, y)$ in ξ, η, x, y, so that, for any $f(x, y)$ holomorphic in closed domain D,
>
> $$\frac{1}{(2i\pi)^2} \sum_{(i,j)} \int\int_{\sigma_{ij}} \frac{\Phi_{ij}(\xi, \eta, x, y)\, f(\xi, \eta) d\xi d\eta}{[X_i(\xi, \eta) - X_i(x, y)]\left[X_j(\xi, \eta) - X_j(x, y)\right]} = f(x, y)$$
>
> if (x, y) is interior to D;
> if (x, y) is exterior to D.[66] $= 0$.

[65] "Ce résultat, qui dépasse de beaucoup celui de Cousin, s'obtient par une méthode analogue à la sienne; mais il faut se servir de l'*intégrale d'André Weil* (pour les fonctions de plusieurs variables) tandis que Cousin utilisait seulement l'intégrale classique de Cauchy (pour les fonctions d'une variable)." (Cartan 1934, p. 1286)

[66] "Soient $X_1, X_2, \ldots, X_n n$ polynômes en x, y, et soit D le domaine (s'il existe) qui est défini par les inégalités (1) $|X_i(x, y)| \leq 1 (i = 1, 2, \ldots, n)$. Soit σ_{ij} (si elle existe) la variété à deux dimensions, située sur la frontière de D, qui est définie par les égalités et inégalités $|X_{\mathrm{i}}(x, y)| = |X_{\mathrm{j}}(x, y)| = 1$, $|X_{\mathrm{k}}(x, y)| \leq 1(k \neq i, j)$. On peut alors déterminer des polynômes $\Phi_{ij}(\xi, \eta; x, y)$ en ξ, η, x, y, de telle sorte

However, Cartan did not publish the proof of his "*théorème 2*" based on Weil's integral, since it occurred to him that Weil's reasoning could not be directly extended to the general case of domains of holomorphy: there was no existence theorem for the generalised Φ functions.[67] Weil would acknowledge this fact in his 1935 paper on *L'intégrale de Cauchy et les fonctions de plusieurs variables* (Weil 1935).[68] We will see that this need for an existence theorem securing the generalised Weil integral formula would play an important part in Cartan's 1940 paper.

Back to Japan after a sabbatical leave which took him to Paris in 1929 to work on the theory of complex function with Gaston Julia,[69] Kyoshi Oka managed to devise a proof of what Cartan had called his "*théorème* 2", first for rationally convex domains, in 1936 (Oka 1936), then for bounded domains of holomorphy in 1937 (Oka 1937). The general result relies on the first: domains of holomorphy are approximated by analytically convex domains which contain it (Oka refers to Cartan and Thullen's 1932 paper (Oka 1984, p. 2)), and analytically convex domains are approximated by rationally-convex domains (Oka 1984, p. 11). In order to tackle the rationally convex case, Oka devised a trick which enabled him to avoid using the yet loosely founded Weil integral, and to resort to standard Cauchy formulae in each component of a polycylinder. The trick goes as follows: replace domain Δ, defined by conditions $x_i \in X_i$ and $R_j(x_1, \ldots, x_n) \in Y_j$ (where the R s denotes ν rational functions in n variables, the X_i and Y_j denote bounded domains), by a rational variety Σ defined

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footnote 66 continued
que l'on ait, pour toute fonction $f(x, y)$ holomorphe dans le domaine fermé D, $\frac{1}{(2i\pi)^2} \sum_{(i,j)} \int\int_{\sigma_{ij}} \frac{\Phi_{ij}(\xi,\eta,x,y) f(\xi,\eta) d\xi d\eta}{[X_i(\xi,\eta)-X_i(x,y)][X_j(\xi,\eta)-X_j(x,y)]} = f(x, y)$ si (x, y) est intérieur à D; $= 0$ si (x, y) est extérieur à D." (Weil 1932, p. 1304)

67 Weil's paper was but a note to the Paris Academy, hence presented the results with a barely sketched proof. In this paper, he stated his theorem in the case of domains defined by a finite number of polynomial inequalities, but hinted at extensions to much more general cases (Weil 1932, p. 1305). In a 1973 overview of his mathematical work, Cartan wrote : "We now know that the additive problem can always be solved for a domain of holomorphy, and, more generally, for a "Stein manifold". This result was first proved by K. Oka. Before Oka, I had seen that the additive problem can be solved by using André Weil's integral, but at that time some techniques were lacking which allow the application of Weil' integral to the general case of domains of holomorphy; I thus forwent publishing my proof."; "*On sait aujourd'hui que le problème additif est toujours résoluble pour un domaine d'holomorphie, et plus généralement pour une "variété de Stein". Ce résultat a été prouvé pour la première fois par K. Oka. Avant Oka, j'avais vu que le problème additif pouvait se résoudre en utilisant l'intégrale d'André Weil, mais comme à cette époque il manquait certaines techniques permettant d'appliquer l'intégrale de Weil au cas général des domaines d'holomorphie, je renonçai à publier ma démonstration.*" (Cartan 1979: XI)

68 "(...) I will make the following assumption, which I could not free myself from: I will assume that to every function $X_i(x, y)$, one can associate two functions $P_i(x, y; x_0, y_0)$, $Q_i(x, y; x_0, y_0)$, holomorphic in x, y, x_0, y_0 when $(x, y) \in$ D, $(x_0, y_0) \in$ D, in such a way that : $X_i(x, y) - X_i(x_0, y_0) = (x - x_0)P_i + (y - y_0)Q_i$ holds identically."; "(...) *Je ferai l'hypothèse suivante, dont je ne suis malheureusement pas arrivé à m'affranchir: Je supposerai qu'à chacune des fonctions $X_i(x, y)$ l'on puisse faire correspondre deux fonctions $P_i(x, y; x_0, y_0)$, $Q_i(x, y; x_0, y_0)$ holomorphes en x, y, x_0, y_0 quand $(x, y) \in$ D, $(x_0, y_0) \in$ D, de façon que l'on ait identiquement: $X_i(x, y) - X_i(x_0, y_0) = (x - x_0)P_i + (y - y_0)Q_i$.*" (Weil 1935, p. 179)

69 As a consequence, many of his early papers would be published in French (or, occasionally, Oka's own brand of French), though in Japanese journals. We will quote from the English translation of his Collected Papers (Oka 1984).

in $(n+\nu)$-dimensional space by ν equations $y_j = R_j(x_1, \ldots, x_n)$; the variety Σ is studied in polycylinder (C) defined by $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_\nu$. In order to use this trick to study the existence problem for meromorphic functions in Δ with given poles, one first has to show that the data can be transferred from the original setting to the new, polycylindrical, one. To warrant this smooth change of setting, Oka first solved (by induction on ν) the following problem: starting from a holomorphic function f in Δ,

> (...) construct a holomorphic function defined on (C′) having the value $f(M)$ for any point M on the portion of Σ lying inside (C′), (C′) being a given domain in the interior of (C). (Oka 1984, p. 2)

The successful change of setting allows for a direct extension of Cousin's method: "But now, the classical methods being applicable almost literally, we shall content ourselves with referring the reader to this". (Oka 1984, p. 6).

3.4.2 Cousin II is purely *topological (for domains of holomorphy)*

After, from Gronwall's paper it had constantly been remarked that, contrary to Cousin I, Cousin II had a topological side to it. Yet, this fact had not been explored in the 1917–1937 period, partly because mathematicians focused on non-topological properties of domains (R-convex domains, star or circle domains etc.). To some extent, it could be argued that neither Gronwall nor his successors had any real tools to tackle the topological side of the question, and that Gronwall's restriction to simply connected polycylinders (save for maybe one component) was the simplest way to set topological questions aside.

After his success with Cousin I, Oka published in 1938 a paper *Sur les fonctions analytiques de plusieurs variables—III Deuxième problème de Cousin* (Oka 1938). This paper plays an important part in our historical narrative, for at least two reasons. First, Oka would establish that for Cousin II (in domains of holomorphy), topology does not play only *some* part, but that the problem is of a purely topological nature. Second, his main proof technique would rely on what we called the "$U \cap V$, U, V, $U \cup V$" site technique, after the 1917–1937 period in which approximation techniques prevailed; this change of focus in terms of site technique is a fundamental background element to Cartan's (1940) structural paper, as we shall see.

To study the role of the topological element in Cousin II, Oka defined its purely topological analogue, which he called the "generalised Problem" ("*problème généralisé*" (Oka 1984, p. 27)): "We shall extend the second Cousin problem to the domain of non-analytic functions" (Oka 1984, p. 25). Consider a domain D in a real space; two continuous functions f_1 and f_2 are said to be equivalent on a subdomain D', if throughout D' we have $f_1 = \lambda f_2$, where λ is continuous and non-vanishing in D'; associate to every point of D an open ball on which a continuous function f is given, so that equivalence holds in the overlaps:

> Under these conditions we ask to find a continuous function F on D equivalent at every point P of the domain to the function attached to P. We shall say that the function F, if it exists, has the given zeros. (Oka 1984, p. 25)

In order for the topological problem to be the analogue of the analytical one, it is also assumed that the zero-locus defined by the zeroes of the f functions is of empty interior: "we abandon, once for all, all cases in which the given zeros fill out a whole portion of the space of the variables" (Oka 1984, p. 26). The three main results of Oka's paper are the following: first, in complex polycylinders, the generalised problem is solvable (for any continuous data) if and only if all components (except at most one) are simply connected. Second, in a domain of holomorphy, if the generalised problem is solvable[70] then so is the analytic problem. Third, an explicit topological condition on the zero-locus is given for the problem to be solvable (Oka 1984, pp. 33–34).

The fact that the topological problem cannot always be solved in a polycylinder with more than one non simply connected component is proved by exhibiting an example, with a complex function defined in the product of two ring-shaped domains. The proof of the topological existence theorem—when at most one component is not simply-connected—depends on a fundamental lemma: let $(x_1, x_2, \ldots, x_n)$ be n complex variables,

> On the plane x_1, we draw a rectangle R which we decompose into equal small rectangles (ω_i) by two systems of straight lines parallel to the sides of R. Let A_1 be a closed domain consisting of one or more of the (ω_i) and their boundaries, and A_2 a closed domain consisting of one of the (ω_i), contiguous to A_1, and its boundary. We shall denote the common part of the boundaries of A_1 and A_2 by L, and the closed domain consisting of A_1 and A_2 by A. Let us take a domain B in the space $(x_2, \ldots, x_n)$. (Oka 1984, p. 27)

With its work in the plane of one of the first variables and its simplicial decomposition of a compact subset, this is reminiscent of Cousin's "fundamental theorem". What follows, however, is a patching-up argument with no analogue in Cousin: given two continuous functions f_1 and f_2, defined over neighbourhoods of (A_1, B) and (A_2, B) (respectively), and globally equivalent in a neighbourhood of L;

> Under these conditions, one can find a continuous function of the n variables x_i, defined in a neighbourhood of A, and equivalent to at least one of the two given functions f_1, f_2 at any point of A, *if L does not coincide with the entire boundary of the rectangle A_2 and if B is simply connected.* (Oka 1984, p. 27)

These topological conditions are necessary and sufficient for a non-vanishing $\lambda = f_1/f_2$, defined over a neighbourhood of L, to be continuously extended over A_2 without vanishing; if $\overline{\lambda}$ denotes the extended function, f_1 (defined over A_1) and $\overline{\lambda} f_2$ (defined over A_2) coincide over a neighbourhood of L, hence a continuous function with the right zero-locus can be defined over $A = A_1 \cup A_2$. Though purely topological in nature, the key step is still expressed by Oka in terms of single-valuedness of $\log\lambda$ in the neighbourhood of L.

As to the analytic part, the existence results derive from Oka's proof of Cousin I for domains of holomorphy; he simply uses the traditional logarithmic transformation to pass from a multiplicative problem to an additive problem. The fact that Cousin II

[70] That is: if for some analytic data, the continuous problem has a continuous solution.

is not solvable for *all* zero-loci in general domains was clear as from Gronwall. Now Oka started investigating the conditions on the zero-locus for Cousin II to be solvable *in this particular case*. He proves that, in domains of holomorphy, the solvability of Cousin II depends only on a topological property of the zero-locus: if D denotes a such domain, Cousin II is solvable for a given zero-locus Σ if and only if this locus can be "swept out" ("*balayable*" (Oka 1984, p. 29)), that is (in modern parlance),[71] if there is a continuous zero-locus in $D \times [0, 1]$ such that $D \times \{0\} = \Sigma$ and $\mathrm{D} \times \{1\} = \varnothing$.

4 Cartan's structural transition

The pivotal point in our narrative is Cartan's pair of papers: *Sur les matrices holomorphes de n variables complexes* in 1940 (Cartan 1940) and *Idéaux de fonctions analytiques de n variables complexes* in 1944 (Cartan 1944). These are actually the two parts of a single work, the delay being ascribable to the prevailing adverse circumstances. The first paper is devoted to the proof of a patching-up theorem that is a direct generalisation of Cousin's "fundamental theorem", which, as we have seen, was also an essential part of Gronwall and Oka's contributions. In 1940, Cartan clearly inserted his generalised theorem in a new research programme,which would be laid out in detail in 1944:

> It seems that our theorem can play an important part in the *global study of ideals of holomorphic functions*. In this regard, it should be noted that the "second Cousin problem" pertains to the global study of ideals which have, in the neighbourhood of every point, a base made up of *only one* function. Apart from this particular case, it seems to me that the global study of ideals has not been tackled. That we will do systematically, in a forthcoming Memoir.[72]

4.1 An "ideal" generalisation of Cousin data and Oka lemma

The key to the generalisation of Cousin II lies in recasting it in terms of ideals:

> (...) *To construct a holomorphic function with given zeroes in a given domain.* It is, of course, necessary to clarify what "given zeroes" means. We shall call "Cousin datum" in a domain D, the following datum: for each point x in D a function f_x is given, which is holomorphic at x; these functions must meet the following requirement: each point a of D has a neighbourhood V where f_a is holomorphic, and at every point of which the quotient f_x/f_a is holomorphic and $\neq 0$. The later condition expresses the fact that, in the ring of holomorphic functions at x, f_{x} and f_{a} generate the same *ideal*. Thus, Cousin's problem is

[71] This is Cartan's rephrasing, cf. (Oka 1984, p. 35)

[72] "Notre théorème semble susceptible de jouer un rôle important dans l'*étude globale des idéaux de fonctions holomorphes*. Remarquons à ce propos que le "deuxième problème de Cousin" se rapporte à l'étude globale des idéaux qui ont, au voisinage de chaque point, une base formée d'*une seule* fonction. En dehors de ce cas particulier, on n'a pas encore abordé, me semble-t-il, l'étude globale des idéaux. C'est ce que nous ferons systématiquement dans un Mémoire ultérieur." (Cartan 1940, p. 2)

> as follows: for any *Cousin datum* in domain D, is there a function f which is *holomorphic in* D, and such that for every point x of D, the quotient f/f_x is holomorphic and $\neq 0$.[73]

At any point x, the formula $f_x = g_x\lambda$, with $\lambda(x) \neq 0$, is now read in terms of ideals: in the ring of analytic regular functions at x (the precise definition would be given in 1944), λ is an invertible element; hence functions f_x and g_x generate the same principal ideal; exhibiting the function f_x gives only an extrinsic representation of an intrinsic objet, which is an ideal. But ideals with more than one generator (at each point) can be considered just as well, Cartan argued. If two sets of p functions at point x are given by $f_1, \ldots, f_p$, and $g_1, \ldots, g_p$, the fact that they generate the same ideal at x can be expressed by the existence of a *invertible* square matrix of holomorphic functions at x expressing the change of basis.[74] In the Cousin II case, which is now seen as the principal ideal case, the fundamental step of Oka's proof was the extension of non-vanishing function λ (defined in the neighbourhood of $L = A_1 \cap A_2$) over A_2, so that the continuous function with the right zero locus could be extended from A_1 to $A_1 \cup A_2$. Cartan first reminded the reader of this "*opération élémentaire*", in a more symmetrical formulation:

> (...) the principle of the proof is the following: to pass from *local data* to a *global existence*, parts are assembled in turn. Each assembling step consists in what we will call an *elementary operation*. For instance, here is the elementary operation that leads to Cousin's theorem:
> Let two compact polycylinders Δ', Δ'' be given, whose components in the planes of the $n-1$ last complex variables are the same, and whose intersection $\Delta' \cap \Delta''$ is *simply connected*; let a function $f(x)$ be given, holomorphic and $\neq 0$ at all points of $\Delta' \cap \Delta''$; the point is to write function f as a quotient f'/f'', where f' is holomorphic and $\neq 0$ at every point of Δ', and f'' is holomorphic and $\neq$ 0 at every point of Δ''.[75]

73 "(...) *Construire une fonction holomorphe ayant des zéros donnés dans un domaine donné*. Il faut, bien entendu, préciser ce qu'on entend par « zéros donnés ». Nous appellerons *donnée de Cousin* dans un domaine D la donnée, en chaque point x de D, d'une fonction f_x holomorphe au point x, ces fonctions satisfaisant à la condition suivante: tout point a de D possède un voisinage V dans lequel f_a est holomorphe et en tout point x duquel le quotient f_x/f_a est holomorphe et $\neq 0$. Cette dernière condition exprime que, dans l'anneau des fonctions holomorphes au point x, f_x et f_a engendrent le même *idéal*. Le problème posé par Cousin est alors: pour toute *donnée de Cousin* dans le domaine D, existe-t-il une fonction f, *holomorphe dans*D, telle que, pour tout point x de D, le quotient f/f_x soit holomorphe et $\neq 0$." (Cartan 1944, p. 149)

74 Cartan uses « base » of an ideal or a module to denote any (finite) set of generators.

75 "(...) le principe de la démonstration est le suivant: pour passer de données locales à une existence globale, on procède par assemblages successifs de morceaux. Chaque stade d'assemblage consiste en ce que nous appellerons une opération élémentaire. Voici par exemple en quoi consiste l'opération élémentaire qui conduit au théorème de Cousin: Etant donnés deux polycylindres compacts Δ' et Δ'' qui ont respectivement mêmes composantes dans les plans des $n-1$ dernières variables complexes, et dont l'intersection $\Delta' \cap \Delta''$ est simplement connexe, étant donnée d'autre part une fonction $f(x)$ holomorphe et $\neq 0$ en tout point de $\Delta' \cap \Delta''$, il s'agit de mettre cette fonction f sous la forme d'un quotient f'/f'', f' étant holomorphe et $\neq$ 0 en tout point de Δ', et f'' étant holomorphe et $\neq 0$ en tout point de Δ''." (Cartan 1944, p. 152)

The main theorem of the 1940 paper gives the analogous result for invertible matrices:

> Theorem I. In the space of n complex variables, let Δ' and Δ'' be two compact polycylinders whose components are the same in the planes of all the variable but one, and whose intersection $\Delta' \cap \Delta'' = \Delta$ is simply connected. Every holomorphic and invertible matrix on Δ can be written, on Δ, as $A = A'^{-1}A''$, where A' is a holomorphic and invertible matrix on Δ', and A'' is a holomorphic and invertible matrix on Δ''.[76]

In 1944, Cartan would refer to theorem I as the "*généralisation du Lemme de Cousin*"; as we saw, this direct generalisation of Oka's proof for Cousin II is reminiscent of the form of Cousin's fundamental theorem, but not of Cousin's proof techniques.

What this theorem warrants is the patching-up of ideals rather than the extension of functions:

> Theorem II. Δ', Δ'' and Δ denoting the same as in theorem I, let us consider, on Δ' and Δ'', respectively, two ideals I' and I'' with finite bases. For I' and I'' to have a common holomorphic basis on union $\Delta' \cup \Delta''$, it is necessary and sufficient that I' and I'' generate the same ideal on the intersection Δ.[77]

It should be noted that even in the case of principal ideals, this type of theorems would not necessarily lead to a solution to the classical Cousin II problem. In the classical problem, a global *function* was aimed at, whereas Cartan's new theorems can, at best, prove the existence of a global function *ideal* (with, hopefully, a finite set of generators). Among other things, the new setting thus affects the meaning of what is to be called a *solution* to Cousin II; the new theory may strive for a solution only in a weak sense, but Cartan's successful generalisation of Cousin and Oka's lemma shows that the *ideal* variant is more tractable. This distinction between a *classical* Cousin II problem and an *ideal* Cousin II problem is not stressed in Cartan's 1940–1944 problems, but it will be later (Cartan 1950b).

4.2 Mixed structure, inner concepts, and problems

Some aspects are only hinted at in 1940 that would be tackled explicitly in 1944; for instance, the following was only a footnote in 1940:

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[76] "Théorème I.- Soient, dans l'espace de n variables complexes, deux polycylindres compacts Δ' et Δ'' qui ont mêmes composantes dans les plans de toutes les variables sauf une, et dont l'intersection $\Delta' \cap \Delta'' = \Delta$ est simplement connexe. Toute matrice holomorphe et inversible sur Δ peut être mise, sur Δ, sous la forme $A = A'^{-1}A''$, A' étant une matrice holomorphe et inversible sur Δ', et A'' une matrice holomorphe et inversible sur Δ''." (Cartan 1940, p. 9)

[77] "Théorème II.- Δ', Δ'' et Δ ayant la même signification qu'au théorème I, considérons sur Δ' et Δ'' respectivement, deux idéaux I' et I'' de bases finies. Pour que I' et I'' admettent une même base holomorphe sur la réunion $\Delta' \cup \Delta''$, il faut et il suffit que I' et I'' engendrent le même idéal sur l'intersection Δ." (Cartan 1940, p. 15)

> The ideal "generated" on set Δ by an ideal I' on Δ' (when $\Delta \subset \Delta'$) is made up of finite linear combinations of functions of I', with coefficients holomorphic on Δ.[78]

Considering that the right objects to patch together are ideals (or, in 1944, modules) is one thing, but in this case, the algebraic vocabulary fails to capture an essential feature. The new structure is not just an algebraic structure (whose elements are functions); the objects are rings, ideals or modules *on* domains or polycylinders. It is the interconnections between ideals and domains that are to be explored; to characterise and investigate this situation, new concepts are to be defined, and a wealth of new terms is to be coined. This laying out of the basic notions for the theory of this new structure makes up the bulk of the 1944 paper. In a very pedagogical style, Cartan first reminds his reader of the (purely algebraic) definition of an ideal in a ring, then remarks:

> But the aforegoing definition remains vague if one fails to specify in what regions the functions are considered. The notion of an ideal will always be relative to a determinate set E in the space of *n* complex dimension. By definition, an *ideal* on E will be an ideal of the ring O_E of *holomorphic* functions on E; I call a holomorphic function on E any function defined and holomorphic in a neighbourhood of E (this neighbourhood is not fixed beforehand, it is relative to the function); two functions are regarded as *identical* if there is a neighbourhood of E on which they coincide.[79]

An ideal is said to be "*ponctuel*"[80] if E is just a point. As we can see, no fancy notion of direct limit was used by Cartan in 1944.

It must be mentioned that Oka defined the same structure in a paper that was published in 1950 but written in 1948, without Oka knowing Cartan's 1944 paper. Oka had read the 1940 paper, however, in which the research programme of "global theory of function ideals" was stated very clearly, and the major notions hinted at. Oka's definition is slightly more precise, and uses what he calls the "arithmetic" notions of congruence: he writes $\varphi \equiv 0 \pmod{f_i}$ to express the fact that φ belongs to the ideal generated by the f_i. The new structure is that of "holomorphic ideal with indeterminate domains" ("*idéal holomorphe de domaines indéterminés*" (Oka 1984, p. 85):

> Consider domains (δ) (connected or not) and holomorphic functions $f(x)$ on (δ); we consider a set (I) of pairs (f, δ). Instead of saying that $(f, \delta) \in (I)$, we

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78 "L'idéal «engendré» sur un ensemble Δ par un idéal I' sur Δ' (lorsque $\Delta \subset \Delta'$) se compose des combinaisons linéaires finies de fonctions de I' à coefficients holomorphes sur Δ." (Cartan 1940, p. 15)

79 "Mais la définition précédente reste vague si l'on ne précise pas dans quelles régions sont envisagées les fonctions. La notion d'idéal sera toujours relative à un ensemble E déterminé de l'espace à *n* dimensions complexes. Un *idéal* sur E sera, par définition, un idéal de l'anneau O_E des fonctions *holomorphes*sur E; j'appelle fonction holomorphe sur E toute fonction définie et holomorphe dans un voisinage de E (ce voisinage n'étant pas fixé à l'avance mais dépendant de la fonction); deux fonctions sont considérées comme *identiques* s'il existe un voisinage de E dans lequel elles coïncident." (Cartan 1944, p. 153)

80 We will translate the French "ponctuel" by "punctual", i.e. attached to a point. It sounds slightly odd in French as well, but the meaning is unambiguous.

> shall sometimes say that $f \in (I)$ on δ. Suppose that this set has the following two properties:
>
> 1° If $(f, \delta) \in (I)$ and $\alpha(x)$ is a holomorphic function of the domain δ' (connected or not), then, we have, $\alpha f \in (I)$ on $\delta \cap \delta'$;
>
> 2° If $(f, \delta) \in (I)$ and $(f', \delta') \in (I)$, then $f + f' \in (I)$ on $\delta \cap \delta'$.
> We shall call (I) a *holomorphic ideal with indeterminate domains*. (Oka 1984, p. 84)

In the 1944 paper, once the basic notion is laid out, Cartan presented the central problem for the theory of this new structure:

> Any holomorphic function on E can be regarded as a holomorphic function on any set E′ included in E. As a consequence, *every ideal on* E *generates an ideal on* E′, when E′ ⊂ E; it is important not to confuse these two ideals: the second one is made up of all finite linear combinations *with coefficients holomorphic on* E′, of functions from the first ideal. Thus, one ideal potentially carries a multitude of ideals, one on every subset of E.[81]

More generally, to any module M defined over domain E, an infinity of punctual modules M_x is associated, one at each point x of E. Now, things can also be done the other way around: given a family of punctual ideals M_x on E, Cartan defined the "associated module" M to be the module of functions defined on E "which belong to M_x at every point x of E" ("*qui appartiennent à* M_x *en tout point x de E*" (Cartan 1944, p. 157)). The main problem is to characterise associated modules, and to some extent, the 1944 formulation of the generalised Cousin problem is as follows: on a compact polycylinder, does any module coincide with its associated module?[82] (Cartan 1944, p. 158). As Cartan pointed out very clearly, the difficulty lies in the fact that the module associated with a subset of the domain is made not only of restrictions of functions, but also of linear combinations of those restrictions with a larger coefficient ring. To show the relevance of this fact, he relied in 1944 on a case he had studied in 1940: consider a family $f_1, \ldots, f_q$ with no common zero in E; at every point x of E, they generate the unit ideal (i.e. the whole ring), since at least one of them does not vanish (hence is invertible at x). However, it is not clear *at all* that the associated ideal over E is that defined by $f_l, \ldots, f_q$; or, to put it in terms of functions, it is not clear at all that there is a family $c_1, \ldots, c_q$ of holomorphic functions on E, such that $\Sigma c_i f_i = 1$ identically on E (Cartan 1944, p. 158).

Tackling the question of associated modules requires that a new concept be introduced. Again, Cartan made its origin clear by his rewording of Cousin II: as we saw earlier, he stressed the fact that the generators f_x of the principal punctual ideals were such that every point a has a neighbourhood V and a generator f_a, defined over V,

81 "Toute fonction holomorphe sur E peut être considérée comme une fonction holomorphe sur n'importe quel ensemble E′ contenu dans E. Il en résulte que tout idéal sur E engendre un idéal sur E′, lorsque E′ ⊂ E; il importe de ne pas confondre ces deux idéaux: le second se compose de toutes les combinaisons linéaires finies, à coefficients holomorphes sur E′, des fonctions du premier idéal. Ainsi, un idéal porte en puissance une foule d'idéaux, un sur chaque sous-ensemble de E." (Cartan 1944, p. 153)

82 Or rather, to be precise: does any module M on coincide with the module associated with the system of punctual modules M_x.

such that f_a also generates the ideal at every point x of V. This concrete property of "Cousin data" is turned into an abstract property which particular instances of the new structure may or may not have:

> Definition. Let E be any set of the space of n complex dimensions, and q an integer $\geq$ 1, given once for all. Let us assume that to every point x of E a q-dimensional module of holomorphic functions (at x) has been associated. We shall say that the punctual modules M_x make up a *coherent system* if every point a of E has a neighbourhood V on which there exists a q-dimensional module which generates the punctual module M_x at every point of the intersection V ∩ E.[83]

Oka defined a similar notion in his 1950 paper, coining the term "local pseudo-basis". Cartan would describe the general role of this concept in 1950 in these terms: "(...) before we pass from local to global, we need to go into local properties in depth, that is, to see how point-wise properties are locally organised".[84]

Both the concepts of a coherent system of punctual modules (there is no "sheaf" word yet!) and that of module over E associated with such a system, are central concepts for the study of the new structure. The second one expresses the globalisation problem and, as such, was clearly there from the beginning: this is what the "global theory of ideals of holomorphic functions" is about. As to the first one, it seems that it took some time to Cartan to grasp its problematic nature fully, maybe due to the fact that it is not problematic in the Cousin case. In 1944, he had to confess that, in order to prove his main theorem (theorem IV, dealing with modules over $\Delta = \Delta' \cap \Delta''$, Δ', Δ'', and $\Delta' \cup \Delta''$), he had to assume something he has not been able to prove:

> (...) problem IV raises a preliminary question: the f_k being holomorphic functions on Δ, let us associate to every point x of Δ the punctual module $M(f_k, x)$ made up of systems of p functions c_k (holomorphic at x) such that $\sum_k c_k f_k = 0$; *do these punctual modules form a coherent system*? Now that is a question which I have not yet managed to settle.[85]

The question became all the more pressing when Cartan showed that regardless of whether this module of relations coincides with its associated modules is essential for the whole theory (Cartan 1944, 1959). This problem of coherence for the sheaf of relations would be solved by Oka in 1950. In the 1944 paper, Cartan also stated the problem of coherence for the ideal of an analytic subvariety (Cartan 1944, p. 187), a problem which he would solve in his 1950 *Idéaux et modules de fonctions analytiques*

83 "Définition.- Soit E un ensemble quelconque de l'espace à n dimensions complexes, et soit q un entier $\geq$ 1 donné une fois pour toute. Supposons qu'à chaque point x de E ait été attaché un module M_x (à q dimensions) de fonctions holomorphes au point x. Nous disons que les modules ponctuels M_x forment un système cohérent, si tout point a de E possède un voisinage V sur lequel existe un module (à q dimensions) qui, en tout point x de l'intersection V ∩ E, engendre le module ponctuel M_x." (Cartan 1944, p. 156)

84 "(...) avant de pouvoir faire le passage du local au global, il faut approfondir les propriétés locales, c'est-à-dire voir comment les propriétés ponctuelles s'organisent localement" (Cartan 1950a, p. 30).

85 "(...) le problème IV soulève une *question préliminaire*: les f_k étant holomorphes sur Δ, associons à chaque point x de Δ le module ponctuel $M(f_k, x)$ formé des systèmes de p fonctions c_k (holomorphes au point x) telles que $\sum_k c_k f_k = 0$; *ces modules ponctuels forment-ils un système cohérent*? Or c'est là une question que je ne suis pas encore parvenu à résoudre." (Cartan 1944, p. 160)

de variables complexes (Cartan 1950a). The very same year, Oka gave the example of a non-coherent "ideal" (Oka 1984, p. 86). In 1950, Cartan admitted that in 1944 he had not noticed that the following fact needed to be proved: if M is a module on E, the system of punctual modules M_x it generates is coherent (Cartan 1950a, p. 30).

4.3 Algebraic transition or structural transition?

At first sight, one could argue that very few algebraic concepts are actually imported into the new framework. There is not much beyond the definitions: ring (Cartan denotes by O_E the ring of holomorphic functions of E), ideal, module; in the case of modules, only submodules of finites powers O_E^q of the ring are concerned. For the module of relations, Cartan coined the term "derived module" ("*module dérivé*" (Cartan 1944, p. 167)) and no notion of kernel appears. In fact, no notion of homomorphism between modules is ever used or even alluded to. Needless to say, no idea of a quotient structure is present in 1944: the fact that Cousin data, being defined "up to something" (additively in Cousin I, multiplicatively in Cousin II) "clearly" indicates that they are quotients of more elementary structures, would not strike Cartan until a few years later, as we shall see. In fact, the ideals and modules that Cartan considers are not *abstract* ideals and modules, in so far as their elements are always complex functions in the classical sense; the elements of a quotient would be of a different nature, a nature that would not be considered seriously until 1952.

In terms of ring and module properties, Cartan did not go beyond finiteness properties (specifically, the noetherian property); no "fancy" notion such as that of a local ring is ever used, despite the fact that this notion had been defined and studied by Krull: his 1938 *Dimensionstheorie in Stellenringe* was already a survey paper (Krull 1938). As far as algebraic theorems are concerned, only two play a part in Cartan's 1944 paper. In a footnote (Cartan 1944, p. 166) Cartan made use of the decomposition of an ideal as intersection of primary ideals, and refers the reader to the second edition of van der Waerden's *Moderne Algebra*. The other theorem plays a more visible part, since it is mentioned in the introduction and a proof is given in an Appendix to the paper: "any punctual ideal has a base" ("*un idéal ponctuel possède toujours une base*" (Cartan 1944, p. 153)), i.e. the ring of convergent power series in several complex variables is noetherian. In this regard, Cartan refers his reader to Rückert's 1933 paper *Zur Eliminationsproblem der Potenzreihenideale* (Rückert 1933).

Not much algebra, maybe, but a recasting of enduring problems in a new and algebraically flavoured setting. But if algebra does not actually play a major part, why bother? It seems that an answer was given earlier: the notion of "ideal" enabled Cartan to come up with a new interpretation of what the second Cousin problem was about, and the class of *ideal* problems proves both larger and more tractable than the classical one. However, this shows *how* the algebraic import proved seminal, but it gives no clue as to why Cartan turned to algebra in the first place.

Part of the answer lies in the type of problems which Cartan set out to tackle, rather than in any specific algebraic result or concept. These problems are exemplified in the introduction of the 1944 paper:

As we mentioned earlier, Cousin's results allow for the *global* study of complex analytic varieties of $n - 1$ dimensions in the space of n dimensions, as well as for the global study of holomorphic functions on these varieties. But it seems no attempt was ever made at a global study of complex analytic varieties in arbitrary dimensions. In this work, we intend to partially fill this gap. Let us analyse the problem more closely: we shall say that, in a domain D, a set E is a complex analytic variety (...) if every point a of D has a neighbourhood in which set E can be defined as the set of zeroes that are common to a finite number of holomorphic functions. Can such a variety be defined *globally* as the set of zeroes that are common to a finite (or even infinite) family of functions which are holomorphic in D? (...) Here is another problem: given an analytic variety E in D, and, at every point x of E, a function f_x which is holomorphic at that point, in such a way that every point a in E has a neighbourhood V such that, for every x on the intersection $V \cap E$, f_x and f_a are equal at every point of E sufficiently near x; is there a function f, *holomorphic in* D, which coincides in the neighbourhood of any point x of E with the function f_x associated with that point?[86]

The first problem is a generalisation of Cousin II for varieties of (complex) codimension different from one. The case in which the function of a finite family $(f_1, \ldots, f_p)$ have no common zeroes is now seen as a particular case of this generalised Cousin II problem! Its solution, namely that (in the right domains) there are functions $c_1, \ldots, c_p$ such that

$$\Sigma c_i f_i = 1 \quad \text{(formula } \alpha)$$

is all the more important in that a slightly modified version provides an existence theorem that warrants the Weil integral formula (Cartan 1944, p. 186). As to the second problem that Cartan mentioned in this passage, it is not a Cousin problem, but we came across it earlier, namely in Oka's proof of Cousin I: to implement his trick (replace a rational domain by a rational subvariety in a polycylinder), Oka had to tackle the question of prolongation to a polycylinder of holomorphic functions that are defined on a subvariety; Cartan wrote:

86 "Les résultats de Cousin, nous l'avons rappelé, permettent l'étude *globale* des variétés analytiques complexes à $n - 1$ dimensions de l'espace à n dimensions, ainsi que l'étude globale des fonctions holomorphes sur ces variétés. Mais rien ne semble avoir été tenté *pour l'étude globale des variétés analytiques complexes à un nombre quelconque de dimensions*. Nous nous proposons, dans ce travail, de combler partiellement cette lacune. Analysons de plus près le problème: nous dirons que, dans un domaine D, une ensemble E est une variété analytique complexe (...) si chaque point a de D possède un voisinage dans lequel l'ensemble E peut être défini comme l'ensemble des zéros communs à un nombre fini de fonctions holomorphes. Une telle variété peut-elle être définie *globalement* comme l'ensemble des zéros communs à un nombre fini, ou même infini, de *fonctions holomorphes* dans D? (...) Voici un autre problème: étant donnée une variété analytique E dans D, et, en chaque point x de E, une fonction f_x holomorphe en ce point, de manière que tout point a de E possède un voisinage V tel que, pour tout point x de l'intersection $V \cap E$, f_x et f_a soient égales en tout point de E suffisamment voisin de x, existe-t-il une fonction f *holomorphe dans* D et qui, au voisinage de tout point x de E, coïncide sur E avec la fonction f_x relative à ce point ?" (Cartan 1944, p. 152)

> Besides, this is the very goal which led me, a few years ago, to undertake in a systematic fashion the *global study of ideals of holomorphic functions*.[87]

Here, the link with algebra is one of analogy: in "modern algebra", the theory of polynomial ideals deals with these very questions, but with polynomials instead of analytic functions. In particular, this theory is the new setting for classical results of elimination theory, such as Hilbert's *Nullstellensatz*; quoting from van der Waerden's *Moderne Algebra*

> If f is a polynomial in $K[x_1, \ldots, x_n]$ which vanishes at all common zeroes of polynomials $f_1, \ldots, f_r$, then the following congruence holds
>
> $$f^\rho \equiv 0\ (f_1, \ldots, f_r)$$
>
> for some natural number ρ (and conversely).[88]

In particular, when the polynomials have no common zeroes, the classical algebraic analogue of formula α goes:

> If the polynomials $f_1, \ldots, f_r$ of $K[x_1, \ldots, x_2]$ have no common zeroes in any algebraic field over K, then the following relation holds in $K[x_1, \ldots, x_n]$
>
> $$1 = A_1 f_1 + \cdots + A_r f_r.$$ [89]

Some more specific questions which Cartan tackled as from 1940 are also direct analogues of standard questions in the theory of polynomial ideals (in its *Eliminationstheorie* part): the actual transformation of a given basis into another one (Cartan 1940, p. 18 and fol.), the representation of subvarieties of codimension q by exactly q equations (Cartan 1944, §8).

There is no textual indication that Cartan also considered that there was some kind of analogy between changes of base rings in his theory (where they result from changes of domains) and changes of base fields in algebra. What is clear, is that Cartan saw that several problems (or tricks, or proof methods, or loosely-founded formulae) in the theory of several complex variables called for a general theory of *analytic* submanifolds, and that a significant change of setting had occurred a few years earlier in the theory of *algebraic* manifolds. Cartan's introduction of the language of ideal theory in complex function theory echoes the opening claim of van der Waerden's 1927 *Zur Nullstellentheorie der Polynomideale*:

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87 "C'est d'ailleurs dans ce but que j'ai été amené, il y a quelques années, à entreprendre systématiquement l'*étude globale des idéaux de fonctions holomorphes* (...)." (Cartan 1944, p. 152)

88 "Ist f ein Polynom in $K[x_1, \ldots, x_n]$, das in allen gemeinsamen Nullstellen der Polynome $f_1, \ldots, f_r$ verschwindet, so gilt eine Kongruenz $f^\rho \equiv 0\ (f_1, \ldots, f_r)$ für eine natürliche Zahl ρ (und umgekehrt)." (van der Waerden 1931, p. 11)

89 "Wenn die Polynome $f_1, \ldots, f_r$ aus $K[x_1, \ldots, x_n]$ in keinem algebraischen Körper über K eine gemeinsame Nullstelle haben, so gilt in $K[x_1, \ldots, x_n]$ eine Relation $1 = A_1 f_1 + \cdots + A_r f_r$." (van der Waerden 1931, p. 10)

> The exact foundation of the theory of algebraic manifold in n-dimensional space can only be reached with the help of the theory of ideals, for even the definition of an algebraic manifold leads directly to a polynomial ideal.[90]

Van der Waerden saw his work as an import, into algebraic geometry, of the abstract methods that Emmy Noether had developed.[91] A similar import and a similar claim is to be found in Rückert's theory of rings of convergent power series (with complex coefficient):

> In the elimination theory for convergent power series of several complex variables, the common zeroes of a system a functions vanishing at point zero is investigated. The main result of the theory is the following theorem of Weierstrass [the preparation theorem] (...). Up to now, all proofs of this fundamental result rely mainly on tools from function theory. (...) In this work, we will show that a proper treatment of the elimination problem (up to formulae (2) and (3)) requires but formal methods, hence no function-theoretic tool. The general theory of ideals and the general theory of fields prove to be such methods.[92]

The analogy of problems does not entail one of methods: Cartan's methods are purely "*funktionentheoretisch*", however, and his change of setting looks nothing like van der Waerden's *idealtheoretisch* foundation of (affine) algebraic geometry, or Rückert's work on local analytic geometry. It can also be even argued that Cartan had always considered possible parallel developments in algebra and in complex analysis, and that he had already imported questions from the theory of polynomials to complex analysis before he was even aware of ideal-theoretic formulations. For instance, in his 1931 paper *Sur les variétés définies par une relation entière* (Cartan 1931), he sketched the following research programme: to study analytic manifolds defined by one regular equation in two variables "*par analogie avec les courbes algébriques*" (Cartan 1931, p. 25). In particular, in the first paragraph of this paper, he relied on the two Cousin theorems (in the product of two planes or two discs) to study whether, given two "entire" functions F, G with no common zeroes, some $UF+VG \equiv 1$ relation

[90] "Die exacte Begründung der Theorie des algebraischen Mannigfaltigkeiten in n-dimensionalen Räumen kann nur mit Hilfsmitteln der Idealtheorie geschehen, weil schon die Definition einer algebraischen Manigfaltigkeit unmittelbar auf Polynomideale führt." (van der Waerden 1927, p. 183)

[91] " The only way to reach the greater ease and (at the same time) greater generality which the theory we are presenting—a theory which was first vividly expounded by E. Noether (Math. Ann. 99) – enjoys when compared to older theories, is to arithmeticise completely all concepts and all operations."; "*Die größere Einfachheit und zugleich größere Allgemeineheit der vorliegenden, stark an E. Noether (Math. Ann. 99, s.o.) anlehnenden Theorie gegenüber den älteren Theorien konnte nur erreicht werden durch äußereste Arithmetisierung aller Begriffe und Operationen.*" (van der Waerden 1927, p. 185).

[92] "In der Eliminationstheorie der konvergenten Potenzreihen mehrerer komplexen Veränderlichen werden die gemeinsamen Nullstellen eines Systems im Nullpunkt verschindender Potenzreihen (...) untersucht. Das Hauptergebnis der Theorie ist das folgende Theorem von Weierstrass (Weierstrass' preparation theorem) (...). Die bisherige Beweise dieses grundlegenden Ergebnisse stützen sich vorwiegend auf Hilfsmittel aus der Funktionentheorie. (...) In dieser Arbeit wird gezeigt, dass eine sachgemässe Behandlung des Eliminationsproblems bis zu der algebraischen Darstellungen (2) und (3) der Gebilde nur formale Methoden, also keine funktionentheoretischen Hilfsmittel benötigt. Als solche Methode erweisen sich die allgemeine Idealtheorie und die allgemeine Körpertheorie." (Rückert 1933, pp. 259–260)

could be proved.[93] The analogy, again, was one of questions rather than methods, and the end of Cartan's introduction stressed this point:

> (...) can a theory of entire curves be fruitful? Will it be possible to approach it in some other way, more algebraic (so to speak) than transcendental (as in this work)? Will this theory, on the contrary, deal with but general points; the latter, however interesting they may be, are more often than not of no help when it comes to solving a specific problem.[94]

The outlook would be brighter in 1940, and the research programme much wider (only plane curves had been considered in 1931). But the new theory would still be based on transcendental methods, and Cartan's "global theory of ideal of holomorphic functions" would be nothing like the purely algebraic Dedekind-Weber theory of algebraic curves.

It may be that Cartan's new theory imported from modern algebra little more than definitions; also, that Cartan had always been aware of strong analogies between standard questions in the theory of polynomials and important questions in complex analysis. It does not mean, however, that nothing significant changed in 1940, or that the model of modern algebra played but a cosmetic part. The 1940–1944 theory is not algebraic (not even "*en quelque sorte algébrique*"), but it displays a *structural practice* for which modern algebra—among others—provided a model; a practice resulting in papers which look nothing like what Cartan had written prior to 1940.

Let us give an overview some of the elements that make up this practice, while paying special heed to the use of problems. First, the use of Cousin problems changes. In the 1930s, we identified two different uses: a direct use, in which Cousin I, Cousin II and "Poincaré" were proved to hold for ever more general domains; an indirect use, in which "Cousin" and "Poincaré" were treated as properties whose interactions with other relevant properties contributed to the charting the fast-growing world of domain-types being investigated in the theory of functions of several complex variables. None of these uses lies at the heart of Cartan's 1940–1944 papers, although direct results may appear among the outcomes. Rather, the Cousin problems have their specific content hollowed out so that a general form may appear. Two general forms actually: first, the notion of "Cousin data"—which appeared in Poincaré and Cousin's *ekthesis*—would serve as a model to define the notion of coherent ideal of holomorphic functions, a new and abstract *structure*; second, the Cousin problems are *samples*[95] of the *problem form "passer de données locales à une existence globale"* (Cartan 1944, p. 152). The other problems that Cartan's research programme covers are not "generalised" Cousin

93 In this case, symbol $\equiv$ denotes the fact that the equality holds at every point of the domain, not congruence!

94 "(...) une théorie des courbes entières peut-elle être féconde? Pourra-t-on l'aborder par une autre voie en quelque sorte algébrique, et non transcendante comme dans ce travail? Restera-t-elle au contraire dans des généralités qui, pour intéressantes qu'elles soient, ne sont souvent d'aucun secours lorsqu'il s'agit de résoudre en particulier un problème précis?" (Cartan 1931, p. 25)

95 The word "sample" is here used in the (somewhat) technical meaning it has in Goodman's *Languages of Art*: an object is exhibited as a sample of a property is it both *has* and *denotes* this property (Goodman 1976, p. 53).

problems, at least not direct generalisations; yet, they are problems which share this form.

A second aspect of this structural practice lies in the interlacing of several types of problem; to be precise, what we are discussing here is problem *functions* rather than problem *types*. There are problems which exhibit the general Cousin form, which Cartan describes as those for which he engaged in the study of ideals of holomorphic functions: representation of zero-loci in *any* codimension as loci of zeroes for finite families of globally defined functions; prolongation of holomorphic functions defined over such a locus to globally defined functions (on a polycylinder, for instance). We shall call these problems *target problems*, as opposed to *template problems* such as Cousin II. In the 1944 paper, these target problems are those that are stated in the introduction, and for which some important results are proved at the very end of the paper. A significantly longer list of target problems would be presented in 1950, in Cartan's *Problèmes globaux en théorie des fonctions analytiques de plusieurs variables complexes* (Cartan 1950b). The bulk of the 1944 paper, however, is devoted to the careful stating and interconnecting of a web of *inner problem*, which appear when the new *abstract* structure starts to be investigated as such. The main problems of this type, as we saw earlier, are problems of coherence for various types of modules on the one hand; and, on the other hand, problems of "associated modules": what can you say about a module defined over a *given* domain from the modules it generates over sub-domains, be they points, open subdomains or compact polycylinders. The emergence of these inner problems constitutes an element that is specific to the structural transition: nothing of the kind had been encountered in the long-term history of the Cousin problems. They go hand in hand with the definition of a new and abstract structure; also with the coining of a wealth of new terms. The latter element of practice is very well documented by the 1944 paper, which ends with an index of 26 terms, most of which were coined by Cartan: *donnée de Cousin, idéal de fonctions holomorphes, idéal ponctuel, système cohérent d'idéaux (ou de modules) ponctuels, module associé à un module, module parfait, module pur, module dérivable, module faiblement dérivable* ... (Cartan 1944, p. 197).

5 The Cousin problems in the setting of sheaf theory

The years from 1945 to 1953 can be divided into two clearly distinct periods, as far as the interactions between Cousin-like problems and the development of sheaf theory are concerned. Year 1951 can serve as dividing line, as two important texts indicate: the 1950–51 volume of talks given at the Cartan topology seminar at the *Ecole Normal Supérieure*, in which an initial standard form is given to sheaf theory; then Cartan's 1950 *Idéaux et modules de fonctions analytiques de variables complexes* (Cartan 1950a), which we mentioned earlier. In the first document, many elements that had emerged in the 1945–1951 period are combined into a new major cohomology theory *in topology*; in the second text, several important questions left pending in the 1944 paper are answered. But in 1951, the two contexts were still clearly distinct; in 1951, topological sheaf cohomology played no part in the furthering of the "*théorie* *globale des idéaux de fonctions holomorphes*". The two contexts merged in 1952, and this resulted in a second complete change of setting for the Cousin problems.

In this part we shall change the form of our narrative, for at least three reasons. The first and obvious one is that we temporarily have to do without the guiding thread of the Cousin problems, since we claim the emergence of sheaf cohomology in the 1945–1950 period is independent of them. The second reason is that we feel the standard focus on Leray's work leaves out of the picture a great wealth of elements that played a significant part in the formulation of the 1950 sheaf cohomology theory: it may be that important ideas (and the term "*faisceau*") came from Leray, and that Cartan's rewriting, simplifying and generalising of Leray's concepts provided the core elements of the new theory; but core elements are not always all there is to it, and the 1945–1950 period is one of intense exchanges of ideas, particularly among Bourbaki members. The 1950 sheaf concept emerged from a period of the design of structures in which concepts, techniques, formulations, theorems, tool-boxes and mottoes quickly passed from one field to another. Hence we opted for a non-linear narrative, in which a variety of elements will be commented upon, all of which played more or less direct parts in the emergence of sheaf cohomology; a bundle of ideas, so to speak. The third reason for our change of pace lies in the fact that for many of the points we will be touching upon, historical work is already available, be it on the general history of algebraic topology, on the development of cohomological methods in topology and group theory, on the first notions of category theory, or on the Bourbaki group; as far as Bourbaki is concerned, it should be stressed that the emphasis is usually laid on Bourbaki as collective author of the *Elements de mathématiques* rather than on Bourbaki as a connected group of researchers.

5.1 "Modern methods" in topology

In 1945, Cartan published a paper which marked the beginning of the topological phase of his work. The very title is interesting: *Méthodes modernes en topologie algébrique* (Cartan 1945): modern methods, not recent developments. This short paper presents no new results, but spells out in a clear and pedagogical style some notions and techniques which, for Cartan, show how topology should be done. The notions and techniques would soon be part and parcel of the topology toolbox on the base of which sheaf cohomology would be designed.

First, there are new methods for the definition of homology (or, rather, cohomology) groups.[96] The first part of Cartan's 1945 paper directly echoes Steenrod's 1936 paper *Universal homology groups* (Steenrod 1936), which, in turn, relied on Čech 1933 *Théorie générale de l'homologie dans un espace quelconque* (Čech 1933). Steenrod made it clear that, in order to obtain all possible homological invariants for topological spaces, it was not necessary to consider all possible groups as coefficient groups. If the space can be described as a complex, the additive group of integers is universal,

[96] The clear distinction between homology and cohomology emerged in the 1940s, and Cartan relied on 1930s papers in which everything is called homology. When directly commenting on such a paper, we will stick to the author's vocabulary.

meaning all homological information can be derived from it; Steenrod shows that for locally compact topological spaces, the multiplicative group **T** of complex numbers of modulo 1 is universal. For these general topological spaces, Steenrod used the (co)homology groups introduced by Čech. Starting from Alexandrov's notion of nerve of an open covering as bridge-building device between general topology and combinatorial topology, Čech defined homology and relative homology groups for compact topological spaces by considering cycles that remain stable through chains of projections associated with covering refinements. Steenrod made it slightly more formal by introducing "*mapping systems and homomorphism systems*" (Steenrod 1936, p. 662). In 1945, Cartan expounded Steenrod's 1936 theory for the homology of compact spaces, and used the following trick for locally compact spaces: the homology groups $\Gamma^r(E)$ of space E are defined as relative homology groups, the main space being the Alexandrov compactification of E and the subspace being the point at infinity. What struck Cartan as specially "modern" in these methods is the fact that they deal directly with topological spaces and open coverings, and do not rely on cellular decompositions and combinatorial structures; he stressed this feature on several occasions, for instance: "it should be noted that our proof (...) never relies on any considerations of triangulation or paving".[97]

Another modern feature is that all results derive from the one and only core theorem ("*théorème fondamental*"), which deals with a long exact sequence of groups. Namely, if E is a locally compact space, F a closed subspace, and U = E − F,

> Thus we have a cascade of so-called canonical representations, from each group in the sequence
>
> $$\ldots\Gamma^{r}(F), \Gamma^{r}(E), \Gamma^{r}(U), \Gamma^{r-1}(F), \Gamma^{r-1}(E), \Gamma^{r-1}(U), \ldots$$
>
> to the next; and these representations have the following fundamental property:

Theorem 5.1 (fundamental theorem) *If* $\Gamma_1, \Gamma_2, \Gamma_3$ *denote any three consecutive groups in the latter sequence,* φ *denoting the canonical representation from* Γ_1 *to* Γ_2*, and* ψ *the canonical representation from* Γ_2 *to* Γ_3*, the composite representation* $\psi o \varphi$ *is* null. *Moreover, when the base group g is* **T**, *the subgroup of* Γ_2 *made up of the elements whose image by* ψ *is null is* exactly the image $\varphi(\Gamma_1)$.[98]

It is clear that our use of the term "exact sequence" was anachronistic; Cartan's formulation is exactly the same as that of Hurewicz and Steenrod (1941) or Ehresmann and Feldbau (1941), when dealing with the long exact sequence of homotopy groups for

[97] "on remarquera que notre démonstration (...) ne fait à aucun moment intervenir de considérations de triangulation ou de pavage" (Cartan 1945, p. 13)

[98] "On a ainsi une cascade de représentations, dites canoniques, de chacun des groupes de la suite ... $\Gamma^{r}(F), \Gamma^{r}(E), \Gamma^{r}(U), \Gamma^{r-1}(F), \Gamma^{r-1}(E), \Gamma^{r-1}(U), \ldots$ dans le suivant; et ces représentations jouissent de la propriété fondamentale suivante: *Théorème 5.1* (Théorème fondamental). $\Gamma_1, \Gamma_2, \Gamma_3$ désignant trois groupes consécutifs quelconques de la suite précédente, φ désignant la représentation canonique de Γ_1 dans Γ_2, et ψ la représentation canonique de Γ_2 dans Γ_3, la représentation composée $\psi o \varphi$ est nulle. En outre, lorsque le groupe de base g est **T**, le sous-groupe de Γ_2 formé des éléments dont l'image par ψ est nulle, est précisément l'image $\varphi(\Gamma_1)$." (Cartan 1945, p. 6)

fibre-bundles. The goal of Cartan's paper is to show how a great wealth of topological results can be established in a uniform and straightforward way by applying this one fundamental theorem to different situations.

After a proof of the theorem of invariance of domain, Cartan gives another example which may strike a bell for us. He shows how the orientation problem for connected topological manifolds of dimension n is to be laid out and tackled. As to the laying out

> An n-dimensional open ball B_n is an n-dimensional manifold. By definition, to *orient* B_n is to pick one of the two isomorphisms from the group $\Gamma^n_T(B_n)$ to the base group **T**. An orientation on B_n induces an orientation on every open subset U of B_n which is homeomorphic to B_n (...). By definition, to orient an n-dimensional manifold E is to orient all subsets of E which are homeomorphic to B_n, so that if U and V are to such subsets, with U $\subset$ V, the orientation of U is induced by that of V.[99]

After setting out the concepts of open covering and induced orientation on subdomains, Cartan used the fundamental theorem to study the patching-up of structures along the "$U \cap V$, U, V, $U \cup V$" scheme:

Theorem 8.2 *Let E be a locally compact, union of two open subsets U_1, and U_2, with intersection V. The subgroup $\Gamma^r(E)$ made up of the elements whose traces in $\Gamma^r(U_1)$ and $\Gamma^r(U_2)$ are null is isomorphic to the quotient of $\Gamma^{r+1}(V)$ by the subgroup generated by the traces (in $\Gamma^{r+1}(V)$) of elements of $\Gamma^{r+1}(U_1)$ and $\Gamma^{r+1}(U_2)$.*[100]

The final result is, of course, that Γ^n is isomorphic to **Z** is E is orientable, to **Z/2Z** if it is not. Again, this result is by no means a breakthrough in topology; the purpose was to show that the question of orientation can be laid out in terms of open coverings, and tackled algebraically with a long exact sequence of groups. Although this laying out clearly echoes what Cartan did in the theory of complex functions, no connection would explicitly be made until 1951/52.

The role of Leray is rather well known, and we refer the reader to Houzel (1990) and Houzel (1998) for detailed analyses on the cohomological aspects. We will concentrate on a few elements which would play a direct part in Cartan's 1950 creation of sheaf cohomology, thus paying little attention to Leray's striking results or to the evolution of his ideas in the 1945–1950 period.

To sum things up, Leray developed his notion of "*faisceau*" in order to get a long exact cohomology sequence associated with a continuous map between topological

[99] "Une boule ouverte B_n de dimension n est une variété de dimension n. Par définition, orienter B_n, c'est choisir l'un des deux isomorphismes du groupe $\Gamma^n_T(B_n)$ sur le groupe de base **T**. Une orientation de B_n induit une orientation pour tout sous-ensemble ouvert U de B_n homéomorphe à B_n(...). Par définition, orienter une variété E de dimension n, c'est orienter chacun des sous-ensembles de E homéomorphe à B_n, de manière que si U et V sont deux tels sous-ensembles satisfaisant à U $\subset$ V, l'orientation de U soit induite par celle de V." (Cartan 1945, p. 9)

[100] "*Théorème 8.2.* Soit E localement compact, réunion de deux sous-ensembles ouverts U_1 et U_2, d'intersection V. Le sous-groupe $\Gamma^r(E)$ formé des éléments dont la trace dans $\Gamma^r(U_1)$ et la trace dans $\Gamma^r(U_2)$ sont nulles, est isomorphe au quotient de $\Gamma^{r+1}(V)$ par le sous-groupe engendré par les traces (dans $\Gamma^{r+1}(V)$) des éléments de $\Gamma^{r+1}(U_1)$ et de $\Gamma^{r+1}(U_2)$." (Cartan 1945, p. 10)

spaces $\pi: \mathrm{E} \rightarrow \mathrm{E}'$, at least when the map is a fibration. He drew on two tools developed by Steenrod in the early 1940s, namely the long exact homotopy sequence associated with a fibre bundle (Hurewicz and Steenrod 1941), and the notion of cohomology with local coefficients (Steenrod 1943). The latter had been introduced in Steenrod's 1942 *Topological Methods for the Construction of Tensor Functions* (Steenrod 1942), in which Steenrod defined the general notion of a tensor bundle,[101] then studied the existence problem for continuous sections in terms of characteristic classes (as Stiefel (1935) and Whitney (1935) had done before).[102] The problem is that, to define these characteristic classes associated with a fibre bundle, the standard (co)homology theories would not usually do. The reason is the following: the chains are defined as formal combinations of simplices with coefficients in the homotopy groups of the fibres. On a simplex, the bundle is trivial so a generic fibre can be used, but there is no natural way to identify the fibres throughout the whole base manifold. Thus the coefficients that appear in the cycle sums belong to different groups; these groups are isomorphic, since all fibres are homeomorphic, but not *naturally* isomorphic, so this homology theory does not boil down to the standard one, in which there is only one group of coefficients (Steenrod 1942, p. 122). Fortunately, there is a natural group morphism between the group associated with a simplex and those associated with its edges, which is enough to define a boundary operator between chain groups. One year later, Steenrod would present a more general theory of this type of cohomology theories, in his paper on *Homology with local coefficients* (Steenrod 1943). The starting point is a direct generalisation of the 1942 particular case:

> We shall say that we have a *system of local groups (rings) in the space R* if (1) for each point x, there is given a group (ring) G_x, (2) for each class of paths α_{xy}, there is a given group (ring) isomorphism $G_x \rightarrow G_y$ (denoted by α_{xy}), and (3) the result of the isomorphism α_{xy} followed by β_{xy} is the isomorphism corresponding to the path $\alpha_{xy}\beta_{xy}$. (Steenrod 1943, p. 611)

Leray's 1946 notion of "*faisceau*" relies on Steenrod's ideas from 1942 and 1943, although homotopy classes of paths play no part in his version of homology with local coefficients. Modules (or rings) are attached not only to points, but to any closed subset as well:

> A *sheaf* of modules (or rings) on a topological space will be defined by the following data: 1° to each closed subset F of points of E, a module (or ring) $\mathbf{B}_\mathrm{F}$ is associated, which is null when F is empty; 2° to each couple of closed subsets of points of E, say f and F, such that f $\subset$ F, a homomorphism from $\mathbf{B}_\mathrm{F}$ to $\mathbf{B}_\mathrm{f}$ is associated, which transforms an element b_F of $\mathbf{B}_\mathrm{F}$ in its *intersection* b_F.f with f; if f' $\subset$ f $\subset$ F and if $\mathrm{b}_\mathrm{F} \in \mathbf{B}_\mathrm{F}$, we must have $(\mathrm{b}_\mathrm{F}.\mathrm{f}).\mathrm{f}' = \mathrm{b}_\mathrm{F}.\mathrm{f}'$.[103]

[101] As well as that of subbundle of a vector fibre-bundle, as in the case of unit-sphere bundles on a riemannian manifold.

[102] De Rham had also done it in his dissertation, but in a particular case.

[103] "Un faisceau de modules (ou d'anneaux) sera défini sur un espace topologique E par les données que voici: 1° à chaque ensemble fermé F de points de E est associé un module (ou un anneau) $\mathbf{B}_\mathrm{F}$, qui est nul quand F est vide; 2° à chaque couple d'ensembles fermés, f et F, de points de E, tels que f $\subset$ F, est associé

A "*faisceau*" is called normal if, to put it anachronistically, the module associated with a given closed set is the direct limit of those associated with its closed neighbourhoods. Chains are defined *à la* Steenrod, and a border operator can be derived from the "*intersection*" morphisms. The aim is to define (co)homological invariants for a map π: E $\rightarrow$ E$'$ and not only for spaces E and E$'$. For any integer p, a "*p-ième faisceau d'homologie*" can be defined on E$'$, by associating with every closed subset F$'$ the (standard) p-th homology group of π^{-1}(F$'$) (Leray 1946, p. 1366). The q-th homology of E$'$ groups with coefficients in that p-th *faisceau* make up the (p, q)-th cohomology group of map π. Needless to say this is also the starting point of the theory of spectral sequences.

A slightly altered theory appeared in print in the proceedings of a 1947 conference on algebraic topology (Leray 1949).[104] It includes a new element that would also play a key role for Cartan, namely *fine* resolutions.[105] To define more general sheaf cohomologies than those he associated with continuous maps, Leray starts from the same notion of *faisceau*, usually of complexes of modules. After the definition of the "*support*" S of a complex automorphism, the notion of fine complex is introduced:

> We shall say that **K** is *fine* when, given a finite open covering of X, $\bigcup_\alpha V_\alpha =$ X(...), maps λ_x from **K** to itself can be found, such that
>
> $$\lambda_\alpha(k - k') = \lambda_\alpha k - \lambda_\alpha k'; \quad \sum_\alpha \lambda_\alpha k = k; \quad S(\lambda_\alpha) \subset V_\alpha.$$ [106]

Sheaves of module complexes with a differential (i.e. square-null) operator are called differential modules, and the ring cohomology is defined from the tensor product O with a "*couverture fine*":

> Definition.- Given a locally compact space X, and **B** a differential sheaf on X, we shall call the homology ring of X relative to **B** (denoted by **H** (XOB)), the ring which is defined up to isomorphism by **H** (XOB), where **X** denotes a fine covering of X.[107]

footnote 103 continued
un homomorphisme de $\mathbf{B}_F$ dans $\mathbf{B}_f$, qui transforme un élément b_F de $\mathbf{B}_F$ en son intersection b_F.f par f; si f$'$ $\subset$ f $\subset$ F et si $b_F \in \mathbf{B}_F$, on doit avoir (b_F.f).f$'$ = b_F.f$'$." (Leray 1946, p. 1366)

104 As Leray make it clear in the introduction, the 1949 paper differs significantly from the 1947 talk, due to work with Cartan between 1947 and 1949.

105 The term " resolutions" is anachronistic.

106 "Nous dirons que **K** est fin quand, étant donné un recouvrement fini, ouvert de X, $\bigcup_\alpha V_\alpha =$ X(...) on peut trouver des applications λ_x de **K** en lui-même, telles que $\lambda_\alpha(\mathrm{k} - \mathrm{k}') = \lambda_\alpha \mathrm{k} - \lambda_\alpha \mathrm{k}'$; $\sum_\alpha \lambda_\alpha k =$ k; $S(\lambda_\alpha) \subset V_\alpha$." (Leray 1949, p. 73)

107 "Définition. Etant donné un espace localement compact X et un faisceau différentiel **B** sur X, nous nommerons anneau d'homologie de X relatif à **B** et nous noterons **H** (XOB) l'anneau, défini à une isomorphie près, par **H** (XOB) où **X** désigne une couverture fine de X." (Leray 1949, p. 77)

5.2 The tool and the theme: beyond topology

There is no denying that the new cohomological techniques in topology—Čech cohomology, long exact sequences, local coefficients—play a major part in the making of Cartan's 1950 sheaf concept: they provide the basic tools and the theoretical context. Yet, it can be shown that fields other than topology, and techniques other than cohomology also play a part. Other specific tools and specific ways of laying out and tackling problems circulated in the Bourbaki milieu and across disciplinary boundaries. We should like to briefly present the case of one of these tools—namely partitions of unity—and the way it helped shape the laying out of problems in very different fields, with a particular emphasis on functional analysis.

To the best of our knowledge,[108] the first paper devoted to the introduction of smooth partitions of unity as a new tool in differential geometry is Bochner's 1937 *Remark on a theorem of Green* (Bochner 1937). The goal of this very short paper is not to prove any new results. Rather, Bochner criticises the standard proof of a well-known theorem—a particular case of our Stokes formula (the Green formula)—and presents a more elegant proof based on a new trick. The standard proof, Bochner writes, "requires a cellular subdivision of R into sufficiently small subregions whose boundary is sufficiently smooth, an application of the local theorem to each subregion, and finally a justification of the mutual cancellation of the boundary terms arising from the artificial cellular partition" (Bochner 1937, p. 334). The introduction of an auxiliary family of function (the supports of which are compatible with a well-behaved open covering) helps devise a less awkward proof. Bochner did not coin a name for this auxiliary set of functions, and actually, a similar trick was used by Whitney in his 1936 paper on *Differentiable Manifolds* (Whitney 1936, p. 67).

It seems that Bourbaki members were not aware of Bochner's handy (and smooth) functions; they had their own brand, however. Topological partitions of unity had been introduced in a very different context by Dieudonné, in a 1937 note *Sur les fonctions numériques continues définies dans un produit de deux espaces compacts* (Dieudonné 1937). As in Bochner's case—and to some extent, as in the case of Cartan's paper on new methods in algebraic topology—the goal was to give a better (more straightforward, more elegant) proof of a well-known theorem. When presenting the theory of integration on product spaces, Dieudonné wrote, one usually proves as a key lemma the fact that continuous functions on the product of two compact spaces can be uniformly approximated by product functions (i.e. $f(x)g(y)$). While the usual proof draws on Weierstrass' approximation theorem, Dieudonné's proof relies on a new Lemma:

Lemma. Let E be a compact space, $A_1, A_2, \ldots, A_n$ a finite number of open sets whose union is identical to E. One can find a finite number m of non-negative numerical

[108] This knowledge is partly based on Dieudonné's account (Dieudonné 1981, p. 698).

functions $\lambda_k(x)$, defined and continuous in E, such that $\lambda_k^{-1}(z) > 0$ is contained in one of the A_i sets ($i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$), and that, identically,

$$\sum_{k=1}^{m} \lambda_k(x) = 1.$$ [109]

These functions were used by Weil in his 1940 book *L'intégration dans les groupes topologiques* (Weil 1940, p. 37). They were quickly introduced by other Bourbaki members in several important textbooks, and thus become standard elements in the topology/geometry toolbox. A generalised version, based on Urysohn's lemma in normal topological spaces, is presented in Bourbaki's *Topologie générale* (Bourbaki 1948, p. 66); the term "*partition continue de l'unité*" is used in this publication, but it had already appeared in print in Chevalley's 1946 book on the *Theory of Lie groups* (Chevalley 1946). As is well known, this book is not merely a monograph on Lie groups. Lie groups are only introduced after a very general study of two other structures, namely topological groups and differentiable manifolds, and Chevalley's presentation of differentiable manifolds was to set the model that has most usually been followed since. He used partitions of unity to define the integration of continuous functions on a smooth, compact, oriented manifold (Chevalley 1946, p. 161). Partitions of unity also play an important part in De Rham's *Variétés différentiables* (de Rham 1955), a part on which De Rham[110] laid the emphasis in the introduction to the book (de Rham 1955, v). Apparently, he was then unaware of Bochner's quick and direct construction of smooth partitions of the unity, since he first used Dieudonné's continuous partitions, then smoothed them out by convolution with mollifiers. De Rham had been using partitions of the unity for several years, and Hermann Weyl had probably heard of them from him, in the 1950 Princeton seminar on harmonic integrals that De Rham and Kodaira organised under Weyl and Siegel's supervision. Weyl uses "*Dieudonnésche Faktoren*" in the third edition of his *Idee der Riemann'schen Fläche* (Weyl 1955, p. 65), and this is part of the overall revamping of the topological toolbox: out with cellular decompositions and combinatorial topology, in with open coverings and algebraic topology.

Bourbakian partitions of unity play a role beyond topology and geometry. For instance, they are also core elements in Laurent Schwartz's (1951) book, the *Théorie des distributions*, although no such things were to be found in the first papers on distributions (1946). In 1951, partitions of unity were closely associated with the introduction of a theme which, again, was absent in the early papers, namely the theme of "local and global"; it could be argued that, at least in this case, the tool and the theme are the two sides of the same coin. In order for local/global to play

[109] "Lemme.- Soit E un espace compact, $A_1, A_2, \ldots, A_n$ un nombre fini d'ensembles ouverts dont la réunion est identique à E. On peut trouver un nombre fini m de fonctions numériques non négatives $\lambda_k(x)$ définies et continues dans E, telles que l'ensemble $\lambda_k^{-1}(z) > 0$ soit contenu dans un des ensembles A_i ($i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$) et que l'on ait identiquement $\sum_{k=1}^{m} \lambda_k(x) = 1$." (Dieudonné 1937, p. 593)

[110] Of course, De Rahm was not a member of Bourbaki in the strict sense, but the connections with Bourbaki members are just too numerous to be listed: friendship (and rivalry) with Henri Cartan, exchanges of ideas on analysis on manifolds with Weil (1947), early use of Schwartz's notion of distribution etc.

an structural role in his theory of distribution, Schwartz first defines the support of a distribution, a notion which is not altogether straightforward, since distributions are not functions defined on $\mathbf{R}^n$ domains (in which case the notion of locus of zeroes is straightforward) but functionals on spaces of real-valued functions defined in $\mathbf{R}^n$ domains:

> A distribution T is said to be null in a open set Ω of $\mathbf{R}^n$ if $T(\varphi) = 0$ whenever $\varphi \in D$ has its support in Ω. Two distributions T_1, T_2 are said to be equal in Ω if $T_1 - T_2$ is null in Ω. This definition allows for a *local viewpoint* on distributions, like the one we have on functions or measures; we will be able to write equalities between distributions for an open set Ω of $\mathbf{R}^n$, without prejudging in the least what holds in $\mathbf{R}^n$ as a whole.[111]

Once this definition allows for a "local viewpoint" on distributions, partitions of unity can be used to prove the fundamental patching-up principle (*"principe du "recollement des morceaux""*):

> "Patching-up" principle.- Theorem IV. Let $\{\Omega_i\}$ be a (finite or infinite) family of open sets, whose union in Ω; moreover, let $\{T_i\}$ be a family of distributions depending on the same family I of indices. Distribution T_i is defined on open set Ω_i; it is furthermore assumed that, if Ω_i and Ω_j have non-empty intersection, T_i and T_j coincide on this intersection. Then there exists one and only one distribution, T, defined in Ω, which coincides with T_i in every open set Ω_i.[112]

The local/global theme is a *leitmotiv* throughout the whole book: in the introduction to the third chapter, Schwartz writes "This chapter will be devoted to the study of the convergence of distributions on the one hand; to their local and global structure, on the other hand".[113] It is shown, for instance, that every distribution locally coincides with that associated with a continuous function, and an example shows that it need not be globally so (Schwartz 1951, p. 83). For several theorems, it is clearly stressed that they exhibit the *local to global* form, for instance: "If division by H is locally possible, it is globally possible".[114]

The same link between the tool (partitions of unity) and the theme (local/global conceptual tension) was clear in Chevalley's *Theory of Lie groups*. As Chevalleyclearly

[111] "On dit qu'une distribution T est nulle dans un ensemble ouvert Ω de $\mathbf{R}^n$ si $T(\varphi) = 0$ toutes les fois que $\varphi \in D$ a son support contenu dans Ω. Deux distributions T_1, T_2 sont dites égales dans Ω si $T_1 - T_2$ est nulle dans Ω. Cette définition permet de considérer les distributions, comme les mesures ou les fonctions, d'un point de vue local; on pourra écrire des égalités entre distributions pour un ouvert Ω de $\mathbf{R}^n$, sans préjuger en rien de ce qui se passe dans l'espace $\mathbf{R}^n$ entier." (Schwartz 1951, p. 25)

[112] "Principe du «recollement des morceaux».- Théorème IV. Soit $\{\Omega_i\}$ une famille finie ou infinie d'ouverts, de réunion Ω; soit d'autre part $\{T_i\}$ une famille de distributions dépendant du même ensemble d'indices I. La distribution T_i est définie dans l'ouvert Ω_i; on suppose de plus que, si Ω_i et Ω_j ont une intersection non vide, T_i et T_j coïncident dans cette intersection. Alors il existe une distribution et une seule, T, définie dans Ω, qui coïncide avec T_i dans chaque ouvert Ω_i." (Schwartz 1951, p. 26)

[113] "Ce chapitre va d'une part étudier la convergence des distributions, d'autre part étudier leur structure locale et globale" (Schwartz 1951, p. 65)

[114] "Si la division par H est possible localement, elle est possible globalement" (Schwartz 1951, p.124).

puts in the introduction, his goal is to make up for the outrageous lack of a basic but thorough exposition of the theory of differentiable manifolds and Lie groups from the global viewpoint:

> Expository books on the theory of Lie groups generally confine themselves to the local aspect of the theory. This limitation was probably necessary as long as general topology was not yet sufficiently well elaborated to provide a solid base for a theory in the large. These days have now passed, and we have thought that it would be useful to have a systematic treatment of the theory from the global viewpoint. (Chevalley 1946, vii)

As far as sheaf cohomology is concerned, fine resolutions would lose their central role in the 1950s, with the introduction of more general notions such as those of projective and injective resolutions, in a new, category-theoretic framework. Yet, in the 1945–1950 period, partitions of unity played an important part in the settings which Bourbaki members devised for both classical theories (as in Chevalley's *Theory of Lie Groups*) and brand new ones (as in Schwartz's theory of distributions). They may look like a mere handy trick (and, in the case of sheaf cohomology, a disposable one), but a trick of somewhat *general* and *architectural* value: general in so far as they are not constrained by specific disciplinary boundaries; architectural in so far as they go hand in hand with a conceptual rewriting that centred on the local-global tension. When, in 1946, Leray started to develop his idea of a long cohomology exact sequence associated with a fibre map, "local and global" was not clearly part of the research programme; it would be in 1949.

In the next two paragraphs, we will present Cartan's 1950/51 theory of sheaf cohomology, and stress the role of two neighbouring theories in Cartan's streamlining of Leray's ideas: fibre-bundle theory and group cohomology.

5.3 The fibre-bundle model

The definition of sheaves that Cartan presented in his 1950/51 topology seminar differs significantly from that of system of local coefficients (Steenrod) and from Leray's notion of *faisceaux*. The presentation draws on notions of base space, total space, projection and sections which are clearly modelled after the presentation of fibre-bundles. The total space is first introduced:

Definition Let K be a commutative ring with unit element (...). A sheaf of K-modules on a (regular) topological space **X** is a set F, endowed with a map p (called the "projection") from F to **X**, and with the two following structures:

(1) for each point $x \in \mathbf{X}$, the inverse image $p^{-1}(x) = F_x$ is endowed with a K-module structure;
(2) F is endowed with a topological structural (usually not Hausdorff) satisfying these two conditions: (α) the laws of composition of F (not everywhere defined) defined by the K-module structures are <u>continuous</u>; (β) projection p is a <u>local</u>

homeomorphism (i.e. every element of F has an open neighbourhood which p maps bi-univocally and bi-continuously on an open subset of **X**).[115]

This definition *via* a total space (*espace étalé*) had been devised by Lazard in 1949/50 (Cartan 50/51a: 1). Continuous sections are then defined, $\Gamma(F, X)$ denoting the K-module of sections of sheaf F over open subset X of base space **X**. Since modules are modules of functions, the definition of the restriction morphism $\Gamma(F, Y) \rightarrow \Gamma(F, X)$ (if $X \subset Y$) is straightforward; it is noted that, for any point x of **X**, the F_x module can be identified with the direct limit of the $\Gamma(F, X)$ when X goes through the directed set of open neighbourhoods of x. This, in turn, gives a method of construction of sheaves:

> Reciprocally: let us assume that, to each open set X of a fundamental system of space **X**, a module F_X is attached, and, to each couple (X,Y) of open sets such that $Y \supset X$ (granted F_Y and F_X are defined), a homomorphism f_{XY} from F_Y to F_X is associated in such a way that, if $X \subset Y \subset Z$, homomorphism f_{XZ} is the composite $f_{XY}f_{YZ}$. These data define a sheaf F as follows: to each point $x \in \mathbf{X}$, let us consider the module F_x, inductive limit of the F_X relative to the open sets containing x; on the union F of the F_x a structure of sheaf is defined, first by defining map p which, to an element u of F, associates the point x such that $u \in F_x$, then by taking as a system of open sets V(X,x) of the topology of F the following sets: arbitrarily take an open set $X \subset \mathbf{X}$ such that F_X is defined, an element $v \in F_X$, and the set V(X,x) of images of v in all the F_x modules relative to the points x of X.[116]

Although a general fibre-bundle structure had been introduced by Seifert and Threlfall (Seifert 1933), the terms "base space" and "total space" were introduced by Whitney in his 1935 definition of sphere-spaces (Whitney 1935); Whitney had also laid emphasis on the notion of sections (which he called projections!) and on the problem of global prolongation of local sections. Later presentations of what fibre-bundles are (and are about), such as that of Steenrod (1942) or Ehresmann and Feldbau (1941),

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[115] "Définition: Soit K un anneau commutatif à élément unité (...). Un faisceau de K-modules sur un espace topologique (régulier) **X** est un ensemble F, muni d'une application p (dite «projection») de F sur **X** et des 2 structures suivantes:

(1) pour chaque point $x \in \mathbf{X}$, l'image réciproque $p^{-1}(x) = F_x$ est munie d'une structure de K-module;
(2) F est muni d'une structure topologique (en général non séparée) satisfaisant aux deux conditions: (α) les lois de compositions de F (non partout définies) définies par les structures de K-module des F_x sont continues; (β) la projection p est un homéomorphisme local (i.e. tout élément de F possède un voisinage ouvert que p applique biunivoquement et bicontinûment sur un ouvert de **X**)." (Cartan 1950–51a, 1)

[116] "Réciproquement: supposons que l'on ait attaché, à chaque ouvert X d'un système fondamental de l'espace **X**, un module F_X, et, à chaque couple (X,Y) d'ouverts tels que $Y \supset X$ et que F_Y et F_X soient définis un homomorphisme f_{XY} de F_Y dans F_X, et cela de manière que, si $X \subset Y \subset Z$ l'homomorphisme f_{XZ} soit le composé $f_{XY}f_{YZ}$. Ces données définissent un faisceau F comme suit: pour chaque point $x \in \mathbf{X}$, on considère le module F_x, limite inductive des F_X relatifs aux ouverts X contenant x; sur la réunion F des F_x, on définit une structure de faisceau, d'abord en définissant l'application p qui, à un élément u de F, associe le point x tel que $u \in F_x$, puis en prenant comme système d'ouverts V(X,v) de la topologie de F les ensembles suivants: on prend arbitrairement un ouvert $X \subset \mathbf{X}$ tel que F_X soit défini, un élément $v \in F_X$, et l'ensemble V(X,v) des images de v dans tous les modules F_x relatifs aux points $x \in X$." (Cartan 1950–51a, 3)

also linked those three elements: (1) the definition of an abstract geometric structure involving a space over another space, (2) the fundamental tension between local[117] but not global triviality of the topological structure (product structure), and (3) the idea that the question of section prolongation is the right way to tackle fibre-bundle problems.

Steenrod added to that scheme a very clear analysis of how problems which were classically formulated in terms of magnitudes defined on a given space could be studied in terms of sections of a fibre-bundle. He used the example of tensors on a differentiable manifold: the standard presentation (as in Weyl's, Elie Cartan's and Schouten's works on connections in the early 1920s) was that tensors were magnitudes that could locally be written with explicit (extrinsic) formulae of some sort (in an arbitrary coordinate system); their intrinsic nature was warranted by the transformation laws according to which these formulae transformed under changes of local maps. Steenrod suggested a completely new setting: he first defined the set of point-tensors at a point of the manifold; he then showed that the set of all point tensors at all points of the manifold can be made into a new differentiable manifold, provided the classical transformation laws are read as patching-up rules; he eventually explained that this new manifold is the total space of a fibre-bundle over the first manifold, and that classical tensors are just sections of this fibre-bundle.[118]

The same ideas (and several others) were presented in a more general setting in the 1941–1943 series of note that Bourbaki-member Ehresmann and his student Feldbau published in the *Comptes rendus de l'Académie des Sciences*. Cartan's twofold presentation of the sheaf structure is strictly modelled after Ehresmann's presentation of the fibre-bundle structure: he first *defined* fibre-bundles in terms of projection map, algebraic structure on the fibres $p^{-1}(x)$ (given by the action of a group) and locally trivial structure;[119] the latter then presented a "*méthode de construction d'un espace fibré*" (Ehresmann 1941, p. 762) in which data over an open covering (U_i) of a base space are patched-up according to rules whose general form is expressed in terms of maps from $U_i \cap U_j$ to the group.[120] Ehresmann also introduced several ideas that have no equivalent in Steenrod's presentation. The most fundamental ones are those of principal fibre-bundle and associated fibre-bundles. They allow him to show that classical algebraic objects (group morphism, group representation) have fibre-bundle analogues.

This quick overview of the early years of fibre-bundle theory show that the links between sheaves (1950/51 style) and fibre-bundles are more varied that one might think. It was clear from the outset that sheaf *cohomology* evolved from Steenrod's cohomology with local coefficients and that Leray aimed for a cohomological analogue of the long homotopy sequence associated with a fibre-bundle. But many aspects of the 1950/51 sheaf *concept* were also modelled after the fibre-bundle concept: twofold presentation (axiomatic description / standard mode of construction), construction of a

117 In this context, "local" means "local on the base".

118 Or of sub-fibre-bundles, such as, for instance, the bundle of non-singular quadratic forms when a pseudo-riemannian metrics is to be defined.

119 This local triviality element is of course not retained in the definition of sheaves.

120 The compatibility conditions over $U_i \cap U_j \cap U_k$ are stated (Ehresmann 1941, p. 762).

total space from local data, formulation of a single core problem (section prolongation). This does not mean, of course, that sheaves are instances of fibre-bundles, since the sheaf structure requires no local product structure. The fact that fibre-bundles are special kinds of sheaves is not clearly stated in the 1950/51 seminar, in which only the example of a trivial bundle (i.e. product space) is given in the list of sheaves examples (Cartan 1950–51a, p. 3).

The notion of fibre-bundle also plays a part in the history of Cousin problems in the 1945–1950 period, a part that is independent from the design of sheaf cohomology in topology. In his 1950 talk on *Problèmes globaux dans théorie des fonctions de plusieurs variables complexes*, Cartan presented the outline of his global theory of ideals of holomorpic functions. After expounding the Cousin II problem in this setting, he stressed that the new ideal-theoretic formulation of the problem fails to capture the original problem: it aims at a global *ideal* that generates the punctual principal ideals; to solve the Cousin problem, you need to find a single *function* that generates these ideals. He then gave a new, fibre-bundle theoretic formulation of Cousin II; B denotes a domain of holomorphy:

> A Cousin datum in B defines the following new topological space E: a point of E will be, by definition, a couple (z, f) made up of a point z of B and a generating element f of the principal ideal I_z attached to point z; we shall identify the couples (z, f) and (z, f') if $z = z'$ and if the quotient f/f' (which is holomorphic and $\neq 0$ at point z) *equals one* at z. Let the multiplicative group $\mathbf{C}^*$ of non-null complex numbers operate on E. (...) In the parlance of modern topology, E is a principal fibre bundle with group $\mathbf{C}^*$, of base space B. The hypothesis according to which the ideals I_z form a coherent system expresses the fact that each fibre has a neighbourhood which is isomorphic to the product U $\times$ $\mathbf{C}*$ of an open set U of B by fibre $\mathbf{C}^*$; this enables us to define on E a structure of complex analytic variety.
> (...) One immediately reckons that a solution to the Cousin problem defines an analytic section of that fibre bundle. (...) Thus, for the Cousin problem to have a solution (...) our fibre bundle E has to be trivial;[121]

This directly echoes Steenrod's construction of tensor-bundles, and draws on Ehresmann's characterisation of the triviality of principal bundles by the existence of a global section. It also gives new insight into the topological nature of Cousin II (on domains

[121] "Une donnée de Cousin dans B définit un nouvel espace topologique E que voici: un point de E sera, par définition, un couple (z, f) formé d'un point z de B et d'un élément générateur f de l'idéal principal I_z attaché au point z; on identifiera les couples (z, f) et (z', f') si $z = z'$ et si le quotient f/f' (qui est holomorphe et $\neq 0$ au point z) est égal à un au point z. Faisons opérer, dans cet espace E, le groupe multiplicatif $\mathbf{C}^*$ des nombres complexes $\neq 0$. (...) Dans le langage de la topologie moderne, E est un espace fibré principal, de groupe $\mathbf{C}^*$, ayant B pour base. L'hypothèse selon laquelle les idéaux I_z forment un système cohérent exprime que chaque fibre possède un voisinage isomorphe au produit U $\times$ $\mathbf{C}*$ d'un ensemble ouvert U de B par la fibre $\mathbf{C}^*$; ceci permet de définir, sur E, une structure de variété analytique-complexe.
(...) On voit aussitôt qu'une solution du problème de Cousin définit une section analytique de cet espace fibré.
(...) Ainsi, pour que le problème de Cousin ait une solution (...) notre espace fibré E doit être trivial;" (Cartan 1950b, p. 161)

of holomorphy), a nature that Oka had clearly established. In fact, this reformulation of Cousin II in terms of principle bundle over an analytic manifold comes from André Weil, as Cartan acknowledged in the 1950 talk. For instance, Weil had expounded the algebraic geometry analogue in 1949, in a talk on *Fibre-spaces in Algebraic Geometry* (Weil 1949). In this context, he used the Zariski topology on an abstract algebraic manifold to show that many standard questions in algebraic geometry can be restated as questions about fibre-bundles; the basic example is that of divisor classes and line-bundles. He also pointed to the fact that this new setting should help define new invariants in algebraic geometry, such as, for instance, the algebraic analogues of Stiefel–Whitney–Chern classes.

5.4 Morphisms, exactness: the general framework of axiomatic (co)homology

In Cartan's 1950/51 introductory seminar talk, the definition of a sheaf of **K**-modules on a topological space is followed by the definitions of a sub-sheaf and a quotient sheaf. The notion of sheaf homomorphism is then defined, and it is remarked that such a homomorphism induces a **K**-module homomorphism between modules of sections over an open subset X: "This defined $\Gamma(F, X)$ as a functor of the sheaf F".[122] The definitions of kernel-sheaf and image-sheaf associated with a sheaf homomorphism are then given, which allow for the definition of the sheaf-theoretic notion of an exact sequence, starting from a short exact sequence $0 \to F' \to F \to F'' \to 0$,

> Given such an exact sequence, the sequence of associated homomorphisms
>
> $$0 \to \Gamma(F', X) \to \Gamma(F, X) \to \Gamma(F'', X)$$
>
> is exact (that stems trivially from the definitions). But, in general, homomorphism $\Gamma(F, X) \to \Gamma(F'', X)$ is not onto, in spite of the fact that $F \to F''$ is onto. We will take up that issue again later.[123]

Indeed, the two other talks which Cartan gave in his 1950/51 topology seminar are devoted to the development of sheaf cohomology. Talk 2 is devoted to the introduction of Φ-families of closed, paracompact subsets of the base set X (Cartan 1950–51b, p. 3), and to lemmas about the behaviour of fine sheaves when sections Γ_Φ with support in a Φ-family are studied. These preliminary lemmas are used in the third talk, in which Cartan presented an axiomatic theory of sheaf cohomology. The ring **K** is assumed to be a principal ideal domain:

> Axioms of a cohomological theory: (...) Let us assume that the following are given:

[122] "Ceci définit Γ(F,X) comme foncteur du faisceau F" (Cartan 1950–51a, p. 5)

[123] "Etant donnée une telle suite exacte, la suite des homomorphismes associés $0 \to \Gamma(F', X) \to \Gamma(F,X) \to \Gamma(F'', X)$ est exacte (trivial à partir des définitions). Mais, en général, l'homomorphisme $\Gamma(F,X) \to \Gamma(F'', X)$ n'est pas sur, bien que $F \to F''$ soit sur. On reviendra plus loin sur cette question." (Cartan 1950–51a, p. 5)

I. For every sheaf F of K-modules (on a space **X**), and for every integer q, a K-module $H^q_\Phi(\mathbf{X}, F)$, called the q-th cohomology module of space X, relative to family Φ and coefficient sheaf F (...)
II. For every sheaf homomorphism $F \to F'$, and for every integer q, a homomorphism $H^q_\Phi(\mathbf{X}, F) \to H^q_\Phi(\mathbf{X}, F')$.
III. For every exact sequence of sheaves and homomorphisms $0 \to F' \to F \to F'' \to 0$, and for every integer q, a homomorphism $H^q_\Phi(\mathbf{X}, F'') \to H^{q+1}_\Phi(\mathbf{X}, F')$.

With respect to the above data, the following six axioms are assumed to hold:
(a) $H^q_\Phi(\mathbf{X}, F) = 0$ for $q < 0$, $H^0_\Phi(\mathbf{X}, F) = \Gamma_\Phi(F)$;
(b) If F is fine, $H^q_\Phi(\mathbf{X}, F) = 0$ for any q;
(c) For an exact sequence $0 \to F' \to F \to F'' \to 0$, the sequence

$$\ldots H^q_\Phi(\mathbf{X}, F') \xrightarrow{\beta_q} H^q_\Phi(\mathbf{X}, F) \xrightarrow{\gamma_q} H^q_\Phi(\mathbf{X}, F'') \xrightarrow{\delta_q} H^{q+1}_\Phi(\mathbf{X}, F') \to \ldots$$

(where β_q and γ_q are the homomorphisms defined at II, and δ_q is the homomorphism defined at III) is an exact sequence;[124]

The three other axioms express the functorial nature of H^q_Φ and of the connecting homomorphisms $H^q_\Phi(\mathbf{X},F'') \to H^{q+1}_\Phi(\mathbf{X},F')$. The bulk of the talk is devoted to the proof of the existence of such a sheaf cohomology theory; "uniqueness" is also proved, after the notion of isomorphism of cohomology theories has been defined.

The style of this presentation looks strikingly new, for instance, when compared with Leray's versions of his sheaf cohomology theory. This fact is accounted for in the clearest of ways, however, by the immediate context. The 1950/51 seminar is devoted to the study of cohomology theories, in topology and beyond. The first four talks deal with group cohomology: the first two talks were given by Samuel Eilenberg and the other two by Cartan. This series of talks starts with an axiomatic presentation of group homology and cohomology, which goes along the very same lines as Cartan's

[124] "Axiomes d'une théorie de la cohomologie : (...) On suppose que l'on s'est donné

I. Pour tout faisceau F de K-modules (sur l'espace **X**), et pour tout entier q, un K-module $H^q_\Phi(\mathbf{X}, F)$, appelé le q-ième module de cohomologie de l'espace **X**, relativement à la famille Φ et aux faisceau de coefficients F (...)
II. Pour tout homomorphisme de faisceau $F \to F'$, et pour tout entier q, un homomorphisme $H^q_\Phi(\mathbf{X}, F) \to H^q_\Phi(\mathbf{X}, F')$.
III. Pour toute suite exacte de faisceaux et d'homomorphismes $0 \to F' \to F \to F'' \to 0$, et pour tout entier q, un homomorphisme $H^q_\Phi(\mathbf{X}, F'') \to H^{q+1}_\Phi(\mathbf{X}, F')$.

Relativement aux données précédentes, on suppose vérifiés les 6 axiomes suivants:
(a) $H^q_\Phi(\mathbf{X}, F) = 0$ pour $q < 0$, $H^0_\Phi(\mathbf{X}, F) = \Gamma_\Phi(F)$;
(b) Si F est fin, $H^q_\Phi(\mathbf{X}, F) = 0$ pour tout q;
(c) Pour toute suite exacte $0 \to F' \to F \to F'' \to 0$, la suite

$$\ldots H^q_\Phi(\mathbf{X}, F') \xrightarrow{\beta_q} H^q_\Phi(\mathbf{X}, F) \xrightarrow{\gamma_q} H^q_\Phi(\mathbf{X}, F'') \xrightarrow{\delta_q} H^{q+1}_\Phi(\mathbf{X}, F') \to \ldots$$

(où β_q et γ_q sont les homomorphismes définis en II, et δ_q est l'homomorphisme défini en III) est une suite exacte;" (Cartan 1950–51c, p. 1) presentation of sheaf cohomology: the role that fine sheaves play in sheaf-cohomology, free abelian groups play for group-homology, and injective groups play for group-cohomology (Eilenberg 1950–51a). Several applications are expounded, for instance, the fact that a homology theory can be associated with functor $-\otimes_{\Pi} A$, and a cohomology theory to functor $\mathrm{Hom}_{\Pi}(-, A)$ (Eilenberg 1950–51b). These echoing talks on group- and sheaf-cohomology are but part and parcel of Cartan and Eilenberg's collaborative work on homology theories, which would result in their celebrated *Homological Algebra* (Cartan and Eilenberg 1956). In this book, written between 1950 and 1953,[125] Cartan and Eilenberg generalised this axiomatic approach (though in algebraic contexts only)[126] and introduced the general notion of derived functors for non left-exact or non right-exact additive functors.

The wish to present the cohomology of sheaves in the larger context of cohomology theories thus led to the formulation of a single core problem: to study the non right-exactness of the section functor; this functorial formulation, in turn, called for an emphasis on sheaf morphisms and short exact sequences of sheaf morphisms. It should be stressed that, until this change of setting, morphisms played only a small part in the theories of cohomology with local coefficients. For instance, there are no morphisms between systems of local groups in Stennrod's 1943 paper on this new type of cohomology theory. Even in Leray's 1949 exposition of his theory of cohomology with coefficients in a *faisceau*, morphisms are just mentioned in passing, after the definition of a sub-sheaf and a quotient sheaf (Leray 1949, p. 70). As we saw earlier, Leray is primarily concerned with morphisms between base spaces, not with morphisms between sheaves on the same space; the latter need to be defined, however, to allow for a definition of the homology sheaf of a sheaf of differential complexes.[127]

One should remember that when Cartan introduced his structure of ideals of holomorphic functions, he mentioned ideals and modules (i.e. submodules of the product of a finite number of copies of the structure sheaf), but never introduced homomorphisms. It could be argued that, if only short exact sequences are to be studied, definitions of sub- and quotient objects can prove adequate. But those are nowhere to be found in Cartan's writings on ideals of holomorphic functions; neither in 1940–1944, nor even in the 1950 paper *Idéaux et modules de fonctions analytiques de variables complexes* (Cartan 1950a). As we mentioned earlier, the latter is mainly devoted to proofs of coherence and to a simplification of terminology. The general notions remain the same as in 1944, with some slight changes of formulations, such as, for instance, the introduction of direct limits (Cartan 1950a, p. 31), and a new definition of coherence. Sheaves of relations are not seen as kernels; the fact the "Cousin data" are systems of objects of a given type, which coincide up to objects of another type, is not described in terms of quotient structure.

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125 See Cartan's 1973 overview of his mathematical work in (Cartan 1979, XVII).

126 The topological contexts are, of course, dealt with in Eilenberg and Steenrod (1952).

127 Since the differential map d is a sheaf morphism.

5.5 The return of Cousin

There is a striking contrast between Cartan's 1950 paper on ideals and modules of holomorphic functions and the last talks of the 1951/52 seminar. The first one remains in the 1940/44 setting, whereas the talks given by Cartan and Serre in the spring of 1952 expound the cohomology theory of analytic sheaves on Stein spaces. This material was presented to the general mathematical audience in March 1953 at a *Colloque sur les fonctions de plusieurs variables* (Cartan 1953; Serre 1953), in Brussels. Remmert reported that, upon attending Serre's talk at that conference, one of the German participants commented: "We have bows and arrows, the French have tanks."[128]

After a 1949/50 seminar devoted to the study of fibre-spaces and homotopy theory, and a 1950/51 seminar devoted to cohomology theories (of groups and sheaves), the 1951/52 Cartan seminar was devoted to analytic functions in several variables. The first 17 talks are devoted to a clear and up-to-date presentation of the topics that had been of interest to Cartan for several years: theory of important classes of space (Kähler manifolds, manifolds defined by various convexity properties, Stein spaces),[129] theta functions and abelian functions (talks by Cartan and Cerf), the Weil integral (M. Hervé); a long series of talks is devoted to the study of local questions which had become central elements in Cartan and Oka's general study of ideals of holomorphic functions: rings of convergent power series (F. Bruhat),[130] ideal of an analytic subvariety at one of its points (Bruhat, Cartan, Frenkel), sheaf of relations between holomorphic functions (Frenkel, Malatian).

The last three talks are devoted to the cohomology of coherent sheaves on Stein spaces, with talks by Cartan stating and proving the main theorems and a final talk by Serre giving some important applications. Aside from proofs, the content of these talks is exactly the same as that of the 1953 conference talks in Brussels.

The two main theorems are labelled A and B:

Theorem A *Let X be a Stein manifold, or a compact subset of a Stein manifold which is identical to its envelope. Let F be a coherent analytic sheaf on X. Then, for any point* $x \in X$*, the image, in the* $\mathbf{O}_x$*-module* I_x*, of the module of sections* $H^0(X,I)$*, generates* I_x *for its* $\mathbf{O}_x$*-module structure.*[131]

To prevent any misunderstanding, Cartan comments that this theorem does not say that any local section has a global prolongation; rather, it says that any local section (say, at x) can be written as a linear combination of global sections, with coefficients in $\mathbf{O}_x$. This aspect is typical of Cartan's global theory of ideals of holomorphic functions: a central feature of the 1940 theory was the idea that two types of restrictions/prolongations could be considered, one in terms of functions, and one in terms

128 „Wir haben Pfeil und Bogen, die Franzosen haben Panzer." (Hilton et al. 1991, p. 277)

129 These talks were all given by Cartan.

130 As in several other talks, the theory is expounded for any non-discrete, complete valued field.

131 "Théorème A: Soit X une variété de Stein, ou un compact d'une variété de Stein identique à son enveloppe. Soit F un faisceau analytique cohérent sur X. Alors, pour tout point $x \in X$, l'image, dans le $\mathbf{O}_x$-module I_x, du module des sections $H^0(X, I)$, engendre I_x pour sa structure de module sur $\mathbf{O}_x$." (Cartan 1951–52, p. 7)

of ideals (or modules) generated by functions. The fact that a change of domains (by restriction or extension) entails a change of base rings for the algebraic structures under study also comes from the 1940/44 theory, and no such element played a part in the 1950/51 sheaf cohomology theory. In spite of its cohomological formulation, theorem A is the final result of the "*théorie globale des idéaux de fonctions holomorphes*" research line. Theorem B, however, had no counterpart in the 1940/44 theory, at least for $q \geq 2$:

Theorem B *Let X be a Stein manifold, or a compact subset of a Stein manifold which is identical to its envelope. Let F be a coherent analytic sheaf on X. Then the cohomology modules $H^q(X,F)$ vanish for all integers $q \geq 1$.*[132]

Although the seminar talks on these topics were given by Cartan and Serre in the spring of 1952, the transcripts were written in the autumn, and Serre warns the reader that this text differs significantly from the actual talk (Serre 51/52: 1). Actually, some information can be drawn from a series of letters from Serre to Cartan, which were later published (Serre 1991); these documents help understand how important problems in the theory of functions of several complex variables were reformulated in the setting of sheaves and fibre-bundles. In a letter (dated April 30 1952), Serre gave the cohomological formulation of the conditions for solving the Cousin problems:

> Let X be a complex analytic variety; we will introduce a number of sheaves on X:
> F_a: sheaf of elements of holomorphic functions at all points of X (this sheaf is endowed with the *addition* of hol. f.)
> F_m: multiplicative sheaf of elements of invertible holomorphic functions.
> G_a: same definition as for F_a, with *meromorphic* functions instead of holomorphic functions.
> G_m: same definition as for F_m, with non-null meromorphic functions instead of non-null holomorphic functions.
> (...) One can thus consider the cohomology groups of space X with values in these various sheaves. One first checks the following:
> *For the additive Cousin pb. to be always solvable in X, it is necessary and sufficient that* $H^1(X, F_a) \rightarrow H^1(X, G_a)$ *be one to one and onto; for the multiplicative problem to be, it is necessary and sufficient that* $H^1(X, F_m) \rightarrow H^1(X, G_m)$ *be one to one and onto.*
> In particular, for the additive Cousin pb. to be solvable, *it is sufficient* that $H^1(X, F_a) = 0$; same for the multiplicative one.[133]

[132] "Théorème B. Soit X une variété de Stein, ou un compact d'une variété de Stein identique à son enveloppe. Soit F un faisceau analytique cohérent sur X. Alors les modules de cohomologie $H^q(X, F)$ sont nuls pour tout entier $q \geq 1$." (Cartan 1951–52, p. 7)

[133] "Soit X une variété analytique complexe; on va introduire un certain nombre de faisceaux sur X:
F_a: faisceau des éléments de fonctions holomorphes en tout point de X (ce faisceau muni de l'*addition* des f.hol.)
F_m: faisceau multiplicatif des éléments de fonctions holomorphes inversibles.
G_a: même définition que F_a, le fonctions *méromorphes* remplaçant les fonctions holomorphes.

However, it seems that theorem B was not known to Serre in May 1952. He writes that if X is a domain of holomorphy, Cartan's work on ideals of holomorphic functions show that $H^1(X, F_a) = 0$, but that it is not known whether $H^n(X, F_a) = 0 (n \geq 2)$ (Serre 1991, p. 279). In the next letter, he draws various conclusion from the hypothesis $H^i(X, F_a) = 0 (i > 0)$, remarking that "to impose this condition is not a priori ridiculous, since we know it holds for cubes, and presumably for many other spaces".[134] This case is the starting point of the proof given in October 1952 (Cartan 1951–52, p. 7).[135]

The restating of the Cousin problems in the new theoretical framework relies on the notion of quotient sheaf. In Cartan 1953 conference talk, **M** denotes the sheaf of meromorphic functions:

> It is clear that **O** is a subsheaf of **M**. Let us interpret the quotient sheaf **M/O**; if $m \in \mathbf{M}_x$, the class of m in $\mathbf{M}_x/\mathbf{O}_x$ is called the *principal part* of m. A section of **M/O**is called a *system of principal parts*. Let us consider homomorphism φ: $\Gamma(X, \mathbf{M}) \to \Gamma(X, \mathbf{M}/\mathbf{O})$; to each meromorphic function in X, φ associates a system of principal parts. The classical *additive Cousin problem* (or first Cousin problem) deals with the characterisation, among systems of principal parts in X, of those that come from a meromorphic function on X; that is, to characterise the image of homomorphism φ.[136]

In the 1940/44 theory, neither homomorphisms nor quotients were considered. This new formulation of the problem clearly derives from its being tackled by cohomological means. If X is a Stein space, theorem B warrants that $H^1(X, \mathbf{O}) = 0$, so φ is onto and the additive Cousin problem is always solvable. It should be noted that the surjectivity of map φ captures the fact the *classical* Cousin I problem is solvable (i.e. a

footnote 133 continued
G_m: même définition que F_m, les fonctions méromorphes non nulles remplaçant les fonctions holomorphes non nulles.
(...) On peut donc parler des groupes de *cohomologie* de l'espace X à valeurs dans ces différents faisceaux. On vérifie d'abord ceci:
Pour que le pb. de Cousin additif soit toujours résoluble dans X, *il faut et il suffit que* $H^1(X, F_a) \to H^1(X, G_a)$ *soit biunivoque; pour que le problème multiplicatif le soit, il faut et il suffit que* $H^1(X, F_m) \to H^1(X, G_m)$ *soit biunivoque.*
footnote 133 continued
En particulier, pour que le pb. de Cousin additif soit résoluble, *il suffit* que $H^1(X, F_a) = 0$, de même pour le multiplicatif." (Serre 1991, p. 278)

[134] "imposer cette condition n'est pas ridicule a priori, puisqu'on sait qu'elle est vérifiée par les pavés, et sans doute par bien d'autres espaces" (Serre 1991, p. 279)

[135] It seems the fact that sheaf cohomology turned out to be exactly the right framework to tackle the 1940/44 research programme really came as a surprise to Cartan: see Cartan (1999, p. 787), and Serre's talk at the Cartan centenary celebration (Serre 2004).

[136] "Il est clair que **O** est un sous-faisceau de **M**. Interprétons le faisceau quotient **M/O**; si $m \in \mathbf{M}_x$, la classe de m dans $\mathbf{M}_x/\mathbf{O}_x$ s'appelle la *partie principale* de m. Une section de **M/O** s'appelle un *système de parties principales*. Considérons l'homomorphisme $\varphi : \Gamma(X, \mathbf{M}) \to \Gamma(X, \mathbf{M}/\mathbf{O})$; à chaque fonction méromorphe dans X, φ associe un système de parties principales. Le classique *problème additif de Cousin* (ou premier problème de Cousin) consiste à caractériser, parmi les systèmes de parties principales dans X, ceux qui proviennent d'une fonction méromorphe dans X; autrement dit, à caractériser l'image de l'homomorphisme φ." (Cartan 1953, p. 46)

global meromorphic *function* can be found), and not only the *ideal* Cousin I problem of 1940/44.

As to Cousin II, Serre's 1953 reformulation reads as follows:

> We shall reformulate this problem in the sheaf language: let **G** be the sheaf of germs of meromorphic functions on X (the composition law being *multiplicative*), **F** the subsheaf of **G** made up of the germs of invertible holomorphic functions (that is, non-null in the neighbourhood of the point at hand), let **D** be the quotient sheaf **G/F**; it is clear that **D** is the sheaf of germs of divisors.[137]

F is a multiplicative group and not a coherent **O**-module, so even on a Stein space it is not necessary that $\mathrm{H}^1(\mathrm{X}, \mathbf{F})$ vanish. To get some information, Serre relies on the sheaf theoretic rewording of the classical log transformation:[138] there is a natural sequence of sheaves of groups $0 \to \mathbf{Z} \to \mathbf{O} \to \mathbf{F} \to 0$, where **Z** is the constant sheaf of integers and $\mathbf{O} \to \mathbf{F}$ is the exponential map $f \to e^{2i\pi f}$; the sequence is exact, since every non-vanishing holomorphic function can be locally written as the exponential of a holomorphic function. Combining the two long exact sequence of cohomology, and using the fact that, on a Stein space, $\mathrm{H}^{\mathrm{q}}(\mathrm{X}, \mathbf{O}) = 0(\mathrm{q} \geq 1)$ gives a natural map $\mathrm{H}^0(\mathrm{X}, \mathbf{D}) \to \mathrm{H}^2(\mathrm{X}, \mathbf{Z})$. This map can be interpreted geometrically: a divisor can be seen as a set of subvarieties (of complex codimension 1 in a complex n-dimensional space X), with integer multiplicities; this is an element of $\mathrm{H}_{2\mathrm{n}-2}(\mathrm{X},\mathbf{Z})$, which is then mapped on its dual. Now, $\mathrm{H}^2(\mathrm{X}, \mathbf{Z})$ depends *only* on the topology of X, a fact that captures Oka's proof that the solvability of Cousin II (on a domain of holomorphy) is a purely topological problem.[139] These aspects could not fit within the 1940/44 global theory of ideals of holomorphic functions, since it only considered sheaves of **O**-modules (whereas the interplay between such sheaves and group-sheaves are of the essence here), and the patching-up arguments did not reach beyond the *first* cohomology groups. As Cartan stressed in his talk:

> The cohomological formulation of theorem B, and the idea of studying not only the $q = 1$ case, but any $q > 0$ case, are both J.-P. Serre's.[140]

Needless to say, Serre also mentions Poincaré's problem (though without paying heed to coprimeness): "On a Stein manifold, any meromorphic function is the quotient of two holomorphic functions."[141]

[137] "Nous allons reformuler ce problème dans le langage des faisceaux: soit **G** le faisceau des germes de fonctions méromorphes sur X (la loi de composition étant la *multiplication*), **F** le sous-faisceau de **G** formé des germes de fonctions holomorphes inversibles (c'est-à-dire non nulles au voisinage du point considéré), **D** le faisceau quotient **G/F**; il est clair que **D** est le faisceau des germes de diviseurs." (Serre 1953, p. 60)

[138] This transformation had been used by Poincaré (to transform zeroes into singularities, in a context where there is control over singularities but not over zeroes) and Cousin (to apply an additive result to a multiplicative problem).

[139] Independently from Cartan and Serre, Karl Stein had investigated these topological issues. See, for instance, Stein (1941) and Stein (1951).

[140] "La formulation cohomologique du théorème B, et l'idée d'étudier non seulement le cas q = 1, mais le cas q > 0 quelconque, sont dues à J.-P. Serre." (Cartan 1953, p. 51)

[141] "Toute fonction méromorphe sur une variété de Stein est quotient de deux fonctions holomorphes" (Serre 1953, p. 63).

6 Conclusion

We presented this narrative as a case study in the history of structuralism in non-algebraic contexts, by focusing on the role of problems in the emergence of new structures (such as the sheaf structure) and theories (such as sheaf cohomology). Though the narrative goes into much more detail on the 1940–1953 period, in which structures are in play, we attempted to seize the opportunity to follow the history of the Cousin problems from Poincaré's (1883) paper.

The following of this lead over a rather long period of time is partly justified by the fact that this historical line is systematically referred to in the works we mentioned, usually in the introduction of the papers. However, presenting the history of a very specific family of problems over such a large period of time (1883–1940) raises questions of method: how much context should you take into account? Should one stick to the mathematical description of successive rational moves, such as, for instance, exhibiting a counter-example, generalising to new domains, defining the topological analogue of an analytic problem? We attempted to give our narrative a more historical turn—though without ever aiming at a general history of the theory of functions of several complex variables—by focusing on what can be specific to the medium-term history of a *problem*, or problem family.

As for Poincaré, we stressed the importance of two elements which would prove enduring throughout the history of the Cousin problems: first, an *ekthesis* which relied on open coverings and differences of a given type in the overlaps; second, a *site technique* of the "step by step patching-up" type, which we also referred to as the "$U \cap V$, U, V, $U \cup V$" site technique. However familiar both may have become, we endeavoured to show (within a limited space) that both this type of *ekthesis* and this site technique were not common in the 1880s, and pointed to a quite specific theoretical and technical context—that of Neumann and Schwarz's proof methods in potential theory—which would not be later associated with the Cousin problems. *Ekthesis* and site technique both differ from what is to be found in Weierstrass' original one-variable theorem; only the *ekthesis* is retained by Cousin, who relied on much more standard site techniques when it came to give proofs. We also showed that Poincaré never considered that this specific problem, along with several others that he tackled between 1880 and 1885, belonged to a family that should be labelled "global problems", or "passing from local data to global existence problems". These problem labels emerged a little later; they did not play a significant part in the history of the Cousin problems until Henri Cartan engaged in the "*théorie globale des idéaux de fonctions holomorphes*", in 1940. So many elements (*ekthesis*, site technique, problem label), so many timelines.

The period from 1895 to 1938 provides an interesting case study, since, in spite of the development of the theory of functions of several complex variables, the theoretical framework does not seem to change dramatically: the Cousin problems are clearly identified as central and difficult problems for the theory and they are tackled with the same tools. The fact that no breakthrough occurred does not mean these problems were just referred to in monographs, to be passed on to the next generation of mathematicians: after hitting on a counter-example, Gronwall (and Osgood) gave a restricted version of Cousin's second theorem, just to be on the safe side; Cartan, Behnke and Stein defined Cousin *properties* and investigated their links with a wealth of new, domain-defining properties; we termed this an indirect use of problems. To make sense of the deep dissimilarity between Cousin I and Cousin II, Oka defined and studied a topological analogue of Cousin II, thus showing its purely topological nature (on domains of holomorphy).

We then investigated in greater detail how the Cousin problems got inserted into two structural contexts: that of Cartan's theory of ideals of holomorphic functions (1940–1950); that of sheaf cohomology, as from 1951.

One could argue that the Cousin problems played only a pedagogical role in Cartan's (1940) exposition of his research programme on ideals of holomorphic functions. Along this line, the description would be the following: emulating van der Waerden's reformulation of elimination theory and algebraic geometry in terms of rings, ideals and modules, Cartan engaged in the study of ideals of holomorphic functions; the Cousin problems were well-known examples of the type of problems which this new theory aimed at tackling, thus serving his expository purposes well; however, Cousin II was really too specific to be much more than an emblem for the new theory, since it is a problem in complex codimension one, whereas the new theory aimed at subvarieties in any dimension (including empty subvarities, in the case of functions with no common zeroes). This new theory is "structural" in so far as it imports concepts and theorems from *moderne Algebra.*

We claim that this is not accurate, and that it fails to capture what gives this case study a more general significance. The links between the Cousin problems and this particular instance of structural transition are numerous and fundamental. First, the main *target problems* which motivated Cartan's (1940) research programme had emerged in the context of recent attempts to tackle the Cousin problems: Cartan's (1934) attempt depended on the Weil integral, which, in turn, depended on the holomorphic analogue of the classical polynomial *Nullstellensatz*; Oka's trick (which, among other things, served to bypass the ill-founded Weil integral) required that it be proved that, in a polycylinder, holomorphic functions on a subvariety (of any codimension) are restrictions of holomorphic functions over the whole space. Second, the structure of "ideal of holomorphic functions" could seem to be a somewhat shallow generalisation—of van der Waerden, or of Rückert's algebraic study of the ring of convergent power series—if it were not for its links with the Cousin problems. The definition of this new structure enables Cartan to give an alternative reading of the Cousin problems: these problems can be seen either in terms of functions (in which case a global function is aimed for) or in terms of function ideals (in which case a global ideal is aimed for). The definition of the new structure is an ideal-theoretic rewriting of Cousin II's *ekthesis*, the latter problem serving here as *template problem*; by this change of setting, Cartan did not only generalise the problem to arbitrary codimension, but also changed the meaning of "being a solution to the Cousin problem". Third, Cartan's matrix generalisation of a fundamental lemma from Oka's 1938 proof (a lemma which was already is Cousin's paper) shows that the patching-up of ideals of functions is more tractable than the patching-up of functions. This new "ideal", generalised (in terms of dimensions) *and* weaker (trading ideals for functions) form of Cousin problems was too weak to solve the classical Cousin II problem, but potent enough to solve the target problems. The move is structural in style and practice, since both a general structure and general *problem form* are abstracted from Cousin data and Cousin problems; the general structure is studied as such, which leads to the investigation of *inner problems*, and to the identification of "coherence" as a new and fundamental abstract property. This style is consistent with the model given by abstract algebra; this practice fits the general description given by Bourbaki in their 1942 *L'architecture des mathématiques* (Bourbaki 1950).

The detailed study of the links between problem families and structural *transition* should help flesh out the very general description of the structural approach that can be found, for instance, in Bourbaki. In 1942, they described mathematics as a "a storehouse of abstract forms" (Bourbaki 1950, p. 231); *forms* which are also *tools*, whose abstract (i.e. context-free), *object*-like definition warrant general applicability: "The "structures" are tools for the mathematician. (...) One could say that the axiomatic method is nothing but the "Taylor system" for mathematics" (Bourbaki 1950, p. 227). The arguments as to why it should be done, and what the epistemic gains result (economy of thought, insight into formal analogies, uniform treatment of seemingly different problems etc.) are quite clear; but no clue is given as to *how* it is done. In particular, Bourbaki stress the fact that the list of important structures is not closed, and that *invention* of structures accounts for numerous recent breakthroughs (Bourbaki 1950, p. 230). They also emphasise the fact that the axiomatic method is at its best when it succeeds in showing that some structure plays an important part in a field where, *a priori*, it seems it played none (Bourbaki 1950, p. 228). Yet, as to how these breakthroughs were made, these new structures invented and these unexpected structures identified in more classical settings, Bourbaki give no clue; or rather, they leave it to intuition: "more than ever does intuition dominate in the genesis of discoveries." (Bourbaki 1950, p. 248)!

Though Bourbaki's 1942 assessment of the axiomatic method differs completely from that of Weyl's 1932 talk, *L'architecture des mathématiques* really makes it all look like the "*magic key*" which Weyl derided. The cohomology theory of analytic sheaves on Stein spaces, which Cartan and Serre presented in their 1953 conference talks on global problems in the theory of functions of several complex variables, seems all the more magical for its combination of two features: an extraordinary efficiency, and an extraordinary coherence. The list of elements is impressive: list of target problems, axiomatic definition of the sheaf structure, cohomology groups, exact sequences (short and long, of sheaves and of groups), coherence results, sections of a projection map between a total space and a base space, left exact but not right-exact functors, quotient sheaves, fine sheaves, fine resolutions, Čech cohomology, convexity properties ...; all the elements appear as necessary and interconnected parts of an indivisible whole. The links are so tight that the very idea of a genesis of the theory is puzzling.

The context for the formulation of this theory differs significantly from that of Cartan's (1940) theory of ideals of holomorphic functions; so is the part played by the Cousin problems. The stages in the perfecting of the cohomological tool have been recounted elsewhere, so we strove for a different kind of picture: one of piecemeal tool-making and unexpected efficiency. Between 1944 and 1950, two research lines evolved with no specific intersection: the "global theory of modules of holomorphic functions" on the one hand, for which the main (thorny) coherence problems took some time solving, and an extension of the local part of the theory in purely algebraic settings was contemplated (replacing **C** by complete, sometimes algebraically closed, valued fields); the cohomology of "*faisceaux*", on the other hand, at the intersection of topology and axiomatic cohomology. Though local coefficients and *faisceaux* had been introduced by Steenrod and Leray with specific targets problems in mind, it can be argued that in the hands of Cartan, sheaf cohomology was partly perfected, simplified and streamlined for the sake of perfecting, simplifying and streamlining, in consistence with the epistemic values promoted by Bourbaki; tapping, also, on *know-how* in the *design of structures* accumulated within Bourbaki (Weil, Ehresmann, Dieudonné) and bourbakian circles (Leray, de Rham, Eilenberg). Concepts, axiom schemes for structures, axiom schemes for cohomology theories, tools and mottoes circulated across blurred disciplinary boundaries. We mentioned the well-known parts played by fibre-bundle theory—with its notions of total space, its focus on the question of section prolongation—and general cohomology—with the parallel between sheaf cohomology and group cohomology, the focus on the question of non-exactness of additive functors etc. We also used the example of partitions of unity, to show that a tool and mottoes (such as: pass from local to global) could, temporarily, be seen as two sides of the same coin, and circulate through various theories: partitions of unity and the sheaf-flavoured setting-out of Schwartz's theory of distribution in 1951; a central part played by fine resolutions in the early years of sheaf cohomology. Streamlining indeed, but *bricolage* as well, with both a *lateral circulation of epistemic elements* (tools, mottoes, site techniques, problem laying-out schemes etc.) across disciplinary boundaries, and replacement of *functionally equivalent* elements at various stages of the theory: the definition of sheaf structures as total spaces (*espace étalé*) would later be replaced by a more category-theoretic formulation; fine resolutions would be replaced by other, more general types of resolutions (be they, injective, projective, flabby etc.); the 1950 definition of coherence is not that of 1944 any more etc.[142]

The Cousin problems were not target problems in the 1945–1950 development of sheaf cohomology, and Cartan's 1952 theorem A is merely a rewording in the language of sheaves of the main result of the research line he initiated in 1940. Yet, it turned out, unexpectedly, that the proof methods used in that context could also prove the cohomological triviality of coherent analytic sheaves on Stein manifolds (theorem B). The Cousin problems played a part in the merger of two research lines, as Serre's 1952 letters to Cartan clearly show. This new change of setting affected the Cousin problems in manifold ways. Among other things, they meant a return to the original *functional* Cousin problems, as opposed to Cartan's (1940) *ideal* problems; a return that Weil had anticipated in the context of fibre-bundles. At a deeper level, new structural elements had to be injected into the 10-year-old "modules of holomorphic functions" structure to make it fit into the cohomological machinery: until then, oddly enough, neither homomorphisms nor quotients played any part in that research line. This led to the reformulation of Cousin data as sections of a quotient sheaf. The merger of the research lines eventually led to exchanges of conceptual elements in the other direction just as well. In particular, the fact that a change of domains entails a change of base

142 Our analysis in terms of *epistemic culture* and *elements of practice* clearly parallels that of K. Chemla. See, for instance Chemla (2009).

rings had been a core element of the "modules of holomorphic functions" programme, whereas changes of base rings played no part in Cartan's 1950/51 topological sheaf cohomology. Later in the 1950s, this feature would be fundamental for the rewriting of abstract algebraic geometry in the framework of sheaf cohomology.

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