Algebraic Topology 1 Knots & Braids Paper

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1 Introduction

Knot theory had its origins in physics, with Lord Kelvin's (incorrect) model of the different types of atoms as emerging from the different "stable knots" in a "luminiferous aether", with the unknot corresponding to hydrogen, and the trefoil corresponding to the trefoil. This led to an effort to understand, classify, and tabulate knots, until Lord Kelvin's theory was disproved by the discovery of the electron. However, at this point the study of knots was of enough mathematical interest that the mathematical community continued to develop the field. In 1923, Alexander discovered the Alexander polynomial, a useful algebraic tool for telling knots apart, which kick-started the search for more powerful knot invariants. Now, knot theory is consider it's own field of study, and has been connected to numerous other fields of math and physics, ranging from characterizing manifold decompositions to quantum field theory.

Knots are defined as topological objects, twisted loops in space that cannot be pulled through themselves (as everything would then be untied into a circle). However, knots can be arbitrarily complex, so it can be difficult to quickly determine if two knots can be re-arranged into each other, and so are actually the same knot. This leads to the development of *knot invariants*, mathematical objects associated to knots which remain unchanged when knots are re-arranged. If two knots disagree on an invariant, they are not the same knot.

These invariants are powerful, but may still be difficult to compute if they rely on purely topological properties. This is solved by drawing a connection to *braids*, a formalization of multiple strands intertwined around each other, fixed at their ends. Braids naturally form a group structure and so can be quickly written down or entered into a computer. It is proven that every knot can be described using a corresponding braid, which leads to a natural algebraic way to compute with knots.

The final piece of our puzzle is representation theory. The Burau representation was developed as a way to map braids to matrices, in such a way that we can extract the Alexander polynomial of the corresponding knot from the determinant of this matrix. This finally gives us an entirely algorithmic process to compute the Alexander polynomial of a knot, making it much easier to use this invariant.

2 Knots

Definition 2.1. A *knot* K is a smooth embedding of S^1 into $\mathbb{R}^3$. That is, K is a smooth injection $S^1 \to \mathbb{R}^3$.

This definition makes sense with other choices of codomain, $\mathbb{R}^3$ just gives the most natural geometric intuition. We will often consider knots into S^3 instead, however these two codomains are interchangeable. Think of S^3 as the one-point compactification of $\mathbb{R}^3$. Then we may take any knot in S^3 and remove a point somewhere far from the knot to get an equivalent knot in $\mathbb{R}^3$, or similarly take a knot in $\mathbb{R}^3$ and add a point at infinity to get an equivalent knot in S^3.

We can generalize the notion of knot to a *link*

Definition 2.2. A *link* with n components is a smooth embedding of the disjoint union $\bigsqcup S^1$ into $\mathbb{R}^3$.

A knot can be considered a link with one component. Much of the following discussion of equivalent, complements, and invariants generalizes to links, but we will omit this for brevity. We introduce links because they are the natural environment for our later material on braids.

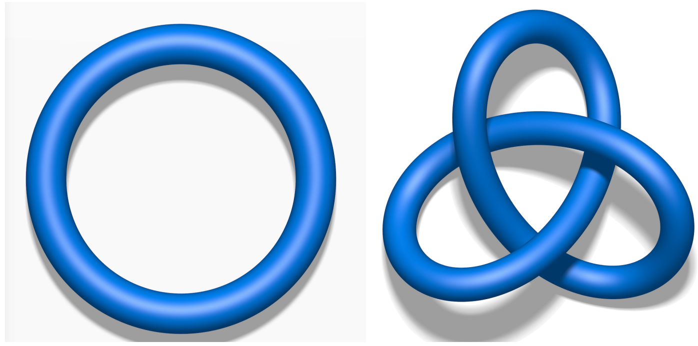

Figure 1: The unknot (left), and trefoil knot (right).

Definition 2.3. A knot is called *tame* if it is piecewise-linear. A knot that is not tame is called *wild*.

Wild knots can exhibit pathological behavior, so in this paper we will only consider tame knots, and will omit the adjective as it is assumed.

The two simplest knots are the *unknot*, given by the function $K(t) = (\sin(2\pi t), \cos(2\pi t), 0)$, and the *trefoil knot*, given by the function $K(t) = (\sin(2\pi t) + 2\sin(4\pi t), \cos(2\pi t) - 2\cos(4\pi t), -\sin(6\pi t))$.

We consider two knots to be equivalent if we can deform one into another. Precisely,

Definition 2.4. Given two knots $K_1, K_2 : S^1 \hookrightarrow W^3$, we say K_1 and K_2 are *isotopic* if there is a smooth $\Phi : S^1 \times [0,1] \to \mathbb{R}^3$ such that $\Phi_t := \Phi|_{S^1 \times t}$ is a knot for each t, and $\Phi_0 = K_0$ and $\Phi_1 = K_1$.

We can visually see that the unknot and trefoil knot are not isotopic, but it is in general difficult to prove two knots are not isotopic. To show that an isotopy does not exist we would have to check all infinitely many possible functions, so this calls fore more powerful tools.

3 Knot Invariants

Any property of a knot which is preserved under isotopy is called a *knot invariant*. If two knots disagree on one of these invariants, then those knots must be non-isotopic. This gives us a tool to prove that *some* knots are not isotopic. However, an invariant may be imperfect, meaning two non-isotopic knots may agree on the invariant, just isotopic knots cannot disagree.

A first instinct is to consider the fundamental group π_1 or the homology groups H_n on the image of the knot, this is always homeomorphic to S^1, so no topological property of the images of knots is an invariant. However, we can consider the space *around* the knot.

Definition 3.1. Give a knot K, define the *knot complement* $M_k = S^3 \setminus \text{im}\,(K)$.

For the knot complement to be an invariant, it must be preserved under isotopy.

Proposition 3.2. *Given isotopic knots K_1 and K_2, M_{K_1} and M_{K_2} are homeomorphic.*

The proof of this fact requires some differential topology we have not built up in this course, see page 13 of Rasmussen [3].

In fact, the opposite is also true. If K_1 and K_2 have homeomorphic knot complement[1], then K_1 and K_2 are isotopic. This fact is called the Gordon–Luecke theorem and is difficult to prove, but we do not need it for our result. This is also not true for links, J. H. C. Whitehead proved that there are infinitely many different links with the same complement.

From this, we see that any topological invariant of the knot complement is a knot invariant. Again our instinct is to try the fundamental group. However, knots may be very complex so the fundamental group

[1]Technically, this would not distinguish between knots which are mirrors, so we need an *orientation-preserving* homeomorphism.

of the knot complement may be hard to calculate. In addition, most methods for finding the fundamental group of a space give a result as a group presentation, and there is no algorithm to determine if two group presentations are the same.

It is easier to determine the equivalence of abelian groups so we might look at the abelianization of the fundamental group. Given any knot K, We know $\pi_1(M_K)^{\mathrm{ab}} \cong H_1(M_K) \cong \mathbb{Z}$, so the abelianization also appears to be a dead end.

Proposition 3.3. *Given a knot K, $H_n(M_K) \cong \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n \geq 2 \end{cases}$*

Proof. Because K is tame, we may take a tubular neighborhood around the image, $N(K)$. See that K is homotopic to $N(K)$ so $S^3 \setminus K$ is homotopic to $S^3 \setminus N(K)$. We can see that $S^3 \setminus N(K)$ is homotopic to the solid 2-torus T^2. Recall $T^2 = S^1 \times D^2$ but D^2 is contractible, so $H_n(M_K) \cong H_n(S^1)$. $\square$

However, we still have a path forward by creating a covering space where both π_1 and H_1 are nontrivial. For a knot K, denote the abelianization map $|\cdot| : \pi_1(M_K) \to H_1(M_K) \cong \mathbb{Z}$. By our Galois-style correspondence, any subgroup of $\pi_1(M_k)$ corresponds to a covering space of M_k. We consider the subgroup $\ker|\cdot| \leq \pi_1(M_k)$.

Definition 3.4. The covering space $p : \tilde{M_K} \to M_K$ where $\pi_1(\tilde{M_K})$ corresponds to the subgroup $\ker|\cdot| \leq \pi_1(M_K)$ is the *infinite cyclic cover* of M_k.

Notice that the set of deck transformations $G(\tilde{M_K}) \cong \pi_1(M_K)/\ker|\cdot| \cong \mathbb{Z}$. This is why we call $\tilde{M_K}$ the infinite cyclic cover. We hope that $H_1(\tilde{M_K})$ is nontrivial, so that it makes a useful knot invariant.

We will work with the *rational homology*, denoted $H_n(\cdot; \mathbb{Q})$, which is constructed similarly to the familiar *integer homology.* When constructing the integer homology of a space X, we define the group of p-chains $C_p(X)$ as the group of formal sums $\sum_{i=1}^{k} x_i \cdot \sigma_i$ where σ_i is a p-simplex and $x_i \in \mathbb{Z}$. We define the group of p-chains over the rationals $C_p(X; \mathbb{Q})$ the exact same, with the only difference being each $x_i \in \mathbb{Q}$. From here we define the boundary maps ∂_p the same, and then $H_p(X; \mathbb{Q}) := \ker \partial_p / \operatorname{im} \partial_{p+1}$ like usual.

Definition 3.5. Let K be a knot, M_K be its knot complement, and $\tilde{M_K}$ be the infinite cyclic cover. Let t be a generator of $G(\tilde{M_K}) \cong \mathbb{Z}$. Denote the ring of Laurent polynomials $\Lambda = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t^{\pm 1}]$, and so we have an action of Λ on the rational homology group $H_1(\tilde{M_K}; \mathbb{Q})$. Thus, we can define the *Alexander module* of K to be $H_1(\tilde{M_K}; \mathbb{Q})$ seen as an Λ-module.

Using the rational homology instead of standard homology means we now have a principal ideal domain, so we may use the Structure Theorem for finitely generated modules over a PID, so we know $H_1(\tilde{M_K}; \mathbb{Q}) \cong \Lambda^k \oplus \Lambda/p_1 \oplus \cdots \oplus \Lambda/p_n$ for some polynomials $p_1, \ldots, p_n$ in Λ.

By breaking down the algebraic structure of the chain complexes of $\tilde{M_K}$ (see [3] p.16) we can see that $H_1(\tilde{M_K}; \mathbb{Q})$ is a torsion module over Λ, aka $k = 0$.

This allows us to define an invariant polynomial, by taking the product of the polynomials in our decomposition.

Definition 3.6. Given a decomposition $H_1(\tilde{M_K}; \mathbb{Q}) = \Lambda/p_1 \oplus \cdots \oplus \Lambda/p_n$, we define the *Alexander polynomial* of K to be the Laurent polynomial $\prod_{i=1}^{n} p_i$.

Note that the Structure Theorem decomposition is unique, up to multiplication by a unit. A multiplicative unit in Λ is just ct^k where $c \in \mathbb{Q}$. While it is in general impossible to tell whether two group presentations are the same, it is easy to tell whether two polynomials differ by multiplication by a unit. This makes the Alexander polynomial a much more useful knot invariant.

However, is still quite difficult to compute. Computing the Alexander polynomial relies on knowing the topological structure of the knot complement, which could be very complex with a complicated knot, and does not lend itself to an easy algorithm. This leads us to develop braids and the Burau representation as a way to give knots an algebraic structure for easier computations.

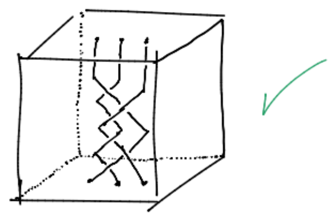

Figure 2: An example of a braid, drawn as piecewise linear.

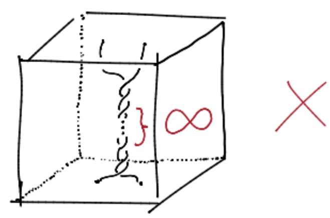

Figure 3: A "wild" braid, unrepresentable as piecewise linear.

4 Braids

Definition 4.1. Let $\mathbb{D} = [0,1]^3$ be the unit cube in $\mathbb{R}^3$. Fix some positive integer n, and let

$$A_i = \left(\frac{1}{2}, \frac{i}{n+1}, 1\right),$$
$$B_i = \left(\frac{1}{2}, \frac{i}{n+1}, 0\right),$$

for each $i = 1, \ldots, n$. Let $d_1, \ldots, d_n$ be a sequence of n piecewise linear curves such that

(i) The images of the d_i are all disjoint,

(ii) For each i, $d_i(0) = B_i$ and $d_i(1) = A_j$ for some $j = 1, \ldots, n$.

(iii) Each of the level planes $E_t := [0,1] \times [0,1] \times \{t\}$ of $\mathbb{D}$ for $t \in [0,1]$ intersects with each of the d_i at exactly one point.

We call the d_i "braid strings," and call their union d a braid.

Example 4.2. Consider the following:

(a) Figure 2 is an example of a braid. When depicting a braid, we often smooth out the polygonal edges of the strings. However, one must always remember that the strings are truly piecewise linear. One encounters trouble when working with smooth braids.

(b) The "wild" braid in figure 3 is *not* an example of a braid, as there is no way to represent infinitely many crossings as a piecewise linear curve in a bounded space. Additionally, we will soon define the braid group $\mathbf{B}_n$, and allowing for a "wild" 2-braid with infinitely many crossings as with figure 3 below would lose our wonderfully simple presentation of $\mathbf{B}_n$.

We define braid equivalence just as with knots, in terms of isotopy. Let $[\beta]$ denote the equivalence class of β, and $\sim$ denote braid equivalence.

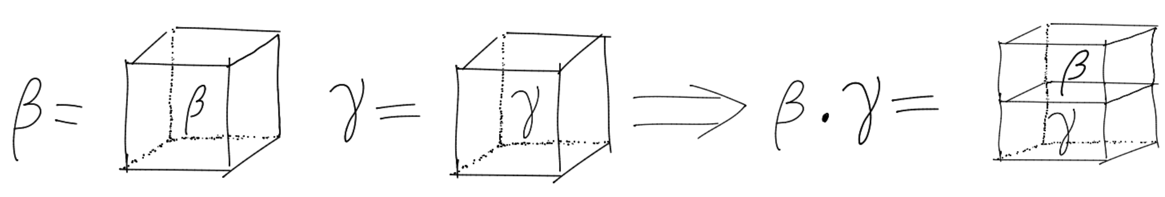

Figure 4: A diagram exhibiting how we form the product.

Definition 4.3. For two n-braids β, γ, we define the product $\beta \cdot \gamma$ by gluing the bottom of a $\mathbb{D}$ containing β to the top of a $\mathbb{D}$ containing γ, and then crushing the resulting cube to unit size. The process is shown diagrammatically in figure 4.

It turns out that $[\beta \cdot \gamma] \sim [\beta] \cdot [\gamma]$. In fact, we can define a group on the set of equivalence classes of n-braids. We call this group the braid group on n strings, and denote it by $\mathbf{B}_n$. We denote the generators of $\mathbf{B}_n$ as σ_i, where each σ_i is a braid where the i'th strand and the $i+1$'th strand swap, with the left strand crossing over the right. Fortunately for us, $\mathbf{B}_n$ has the convenient presentation

$$\begin{aligned} \mathbf{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \, | \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}, \; j - i > 2, \\ \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}, \; 1 \leqslant i \leqslant n-2 \rangle. \end{aligned}$$

We refer the reader to chapter 2 of Murasugi [2] for a full exposition on the braid group, and a proof of this presentation.

5 Braid Closure

One can easily see that, by identifying A_i with B_i for each i, we can derive a link from a given braid β. That any link can be derived in this manner is not nearly as obvious, and takes some effort to prove. However, it is possible, and in fact an algorithm exists to find a such a braid given a link.

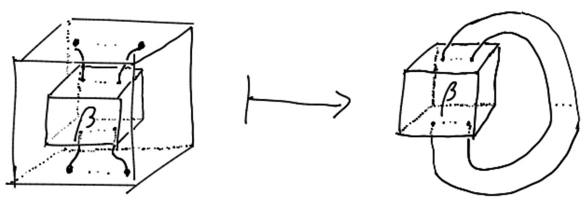

Figure 5: A diagram exhibiting how we form the braid closure.

Theorem 5.1 (Alexander)**.** *Any oriented link can be represented as the closure of a braid.*

Essentially, this says that the obvious map from braids to links is surjective. For a proof of this theorem, we refer to [2], Chapter 8, Section 4.

Let $\beta \in \mathbf{B}_n$. Since we can associate a braid to any given knot, we wish to know when two braids have isotopic closure as oriented links. To start, we can see what operations preserve braid closure up to isotopy.

First, the closure of β must be isotopic to the closure of $\gamma\beta\gamma^{-1}$ as an oriented link. This is because the closure of $\beta\beta'$ is isotopic to the closure of $\beta'\beta$, so in particular it holds for $(\gamma\beta)\gamma^{-1}$. Additionally, we can consider multiplication by $\sigma_n \in \mathbf{B}_{n+1}$, viewing $\mathbf{B}_n$ as a subset of $\mathbf{B}_{n+1}$. Figure 6 illustrates how this leaves the closure of the braid unchanged. Thus, the closure of $\beta \in \mathbf{B}_n$ is isotopic to the closure of $\beta\sigma_n^{\varepsilon}$ for $\varepsilon = \pm 1$

We call these operations *Markov Moves*, and denote them M_1 and M_2 respectively. Surprisingly, Markov moves are really all we can do!

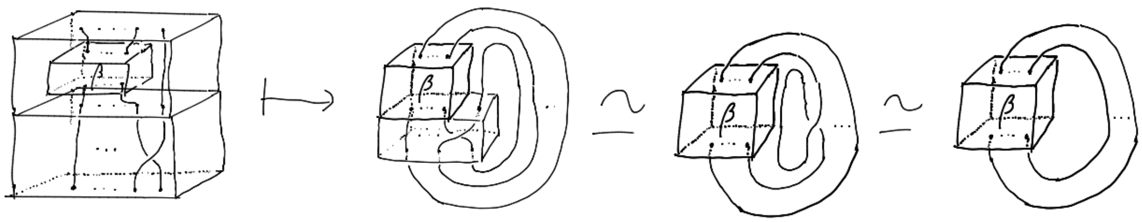

Figure 6: An illustration of the effect of multiplying by σ_n.

Theorem 5.2 (Markov). *Two braids $\beta \in \boldsymbol{B}_n$, $\beta' \in \boldsymbol{B}_m$ have isotopic closures as oriented links if and only if there exists some sequence of braids*

$$\beta = \beta_0 \to \beta_1 \to \cdots \to \beta_k = \beta'.$$

such that β_{i+1} is obtained from β_i by way of a Markov move M_1 or M_2 for each $i = 0, 1, \ldots, n-1$. Whenever such a sequence exists, we say that β and β' are Markov equivalent, and denote it $\beta \sim_M \beta'$.

Once again, we refer to [2] for the proof. We leave as an exercise for the reader the proof that $\sim_M$ is in fact an equivalence relation. Now, alongside Alexander's theorem, this gives us a useful characterization for the isotopy classes of links in $\mathbb{R}^3$ in terms of braids.

Corollary 5.3. *Let $\mathcal{L}$ denote the set of all isotopy classes of nonempty oriented links in $\mathbb{R}^3$. The mapping $\amalg_{n\geqslant 1}\boldsymbol{B}_n \to \mathcal{L}$ assigning a braid to its closure induces a bijection from the quotient $(\amalg_{n\geqslant 1}\boldsymbol{B}_n)/\sim_M$ onto $\mathcal{L}$.*

With this corollary from [1], we can identify isotopy invariants of oriented links with functions on $\amalg_{n\geqslant 1}\mathbf{B}_n$ which are constant on $\sim_M$ equivalence classes. This allows for a slew of new knot and link invariants! Noting that a function f on $\amalg_{n\geqslant 1}\mathbf{B}_n$ can be identified with a sequence of functions f_n on $\mathbf{B}_n$, we motivate the following definition.

Definition 5.4. A Markov function with values in a set E is a sequence of set maps $\{f_n : \mathbf{B}_n \to E\}_{n\geqslant 1}$ satisfying the following conditions:

(i) for all $n \geqslant 1$ and all $\alpha, \beta \in \mathbf{B}_n$,

$$f_n(\alpha\beta) = f_n(\beta\alpha);$$

(ii) for all $n \geqslant 1$ and all $\beta \in \mathbf{B}_n$,

$$f_n(\beta) = f_{n+1}(\sigma_n\beta) = f_{n+1}(\sigma_n^{-1}\beta).$$

It follows easily from our definition of Markov equivalence that each f_n is constant on $\sim_M$ classes. Thus, to each Markov function $\{f_n : \mathbf{B}_n \to E\}_{n\geqslant 1}$ we can assign a link invariant $\widehat{f} : \mathcal{L} \to E$ by $\widehat{f}(L) = f_n(\beta)$ for $\beta \in \mathbf{B}_n$ a knot with closure isotopic to L. This function is well defined, and thus an invariant on $\mathcal{L}$. We now embark on a journey to find an explicit example of a Markov function, which just so happens to take us back to a quite familiar link invariant.

6 Burau Representation

By slight abuse of notation, we now denote by Λ the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$,[2] and let

$$U = \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}.$$

Additionally, we take $I_n \in \mathrm{GL}_n(\mathbb{Z}) \subset M_n(\Lambda)$ to be the identity matrix for $n \geqslant 1$, and I_0 to be a notational convenience to denote an empty spot in a block matrix.

[2] Earlier we defined Λ as a $\mathbb{Q}$ vector space for the sake of rational homology. Here we need only consider elements with integer coefficients, and thus restrict our definition as so.

Definition 6.1. Let $\mathbf{B}_n$ be the standard braid group on n strands. We define the Burau representation

$$\psi_n : \mathbf{B}_n \to M_n(\Lambda)$$

by

$$\psi_n(\sigma_i) = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

Proposition 6.2. *The map ψ_n defined above is, in fact, a group representation.*

Proof. Note that, though we have defined ψ_n on the generators of $\mathbf{B}_n$, we have yet to show that it is a homomorphism. To do so, it suffices to show that $\psi_n(r) = I_n$ for each relation r of $\mathbf{B}_n$.

It's easy to see that $\psi_n(\sigma_i\sigma_j) = \psi_n(\sigma_j\sigma_i)$ for $j - i \geqslant 2$. Then

$$\begin{aligned} \psi_n(\sigma_i\sigma_j) &= \psi_n(\sigma_i)\psi_n(\sigma_j) \\ &= \begin{pmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & U & 0 & 0 & 0 \\ 0 & 0 & I_{j-i-2} & 0 & 0 \\ 0 & 0 & 0 & U & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-1} \end{pmatrix} \\ &= \psi_n(\sigma_j)\psi_n(\sigma_i) \\ &= \psi_n(\sigma_j\sigma_i). \end{aligned}$$

Now, we wish to show that

$$\psi_n(\sigma_i\sigma_{i+1}\sigma_i) = \psi_n(\sigma_{i+1}\sigma_i\sigma_{i+1})$$

it suffices by a similar argument to show that

$$\psi_3(\sigma_1\sigma_2\sigma_1) = \psi_3(\sigma_2\sigma_1\sigma_2).$$

This is a simple matrix computation, and is thus left to the reader.

Finally, we must show that each $\psi_n(\sigma_i) = U_i$ is invertible. It suffices to show that U is invertible. Cayley's theorem tells us that U must satisfy its characteristic polynomial $P_U(\lambda) = \lambda^2 - \operatorname{tr}(U)\lambda + \det(U)$. That is,

$$U^2 - (1-t)U - tI_n = 0.$$

Some simple algebra gives us that

$$U(t^{-1}(U - (1-t)I_n)) = I_n,$$

so

$$\begin{aligned} U^{-1} &= t^{-1}(U - (1-t)I_n) \\ &= \begin{pmatrix} 0 & 1 \\ t^{-1} & 1 - t^{-1} \end{pmatrix} \in M_2(\Lambda). \end{aligned}$$

Since U^{-1} exists, we have

$$U_i^{-1} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & U^{-1} & 0 \\ 0 & 0 & I_{n-j-1} \end{pmatrix} \in M_n(\Lambda).$$

Since each U_i is invertible, $\operatorname{im}\psi_n \subset \mathrm{GL}_n(\Lambda)$. We conclude that ψ_n is a homomorphism from $\mathbf{B}_n$ into $\mathrm{GL}_n(\Lambda)$, i.e. a representation of $\mathbf{B}_n$. $\square$

7 Reduced Burau Representation

Although we have found a representation of $\mathbf{B}_n$, it turns out it is reducible. We will now show that fact.[3]

Theorem 7.1. *Let $n \geqslant 3$ and $V_1, \ldots, V_{n-1}$ be the $(n-1)\times(n-1)$ matrices over Λ given by*

$$V_1 = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, \; V_2 = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix},$$

and, for $1 < i < n-1$,

$$V_i = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}.$$

Then, for all $i = 1, \ldots, n-1$,

$$C^{-1}U_iC = \begin{pmatrix} V_i & 0 \\ *_i & 1 \end{pmatrix},$$

where C is the upper triangular $n \times n$ matrix (c_{ij}), where $c_{ij} = 1$ if $i \leqslant j$ and 0 otherwise, and $*_i$ is the row of length $n-1$ equal to the $\vec{0}$ if $i < n-1$ and $(0, \ldots, 0, 1)$ if $i = n-1$.

Proof. For $i = 1, \ldots, n-1$, set

$$V_i' := \begin{pmatrix} V_i & 0 \\ *_i & 1 \end{pmatrix}.$$

We need only prove that $U_iC = CV_i'$ for all i. Fixing i, the kth column of U_iC is the sum of the first k columns of U_i. Recalling our definition for U_i, we see that U_iC is C, but with $(U_iC)_{ii} = 1 - t$ and $(U_iC)_{(i+1)t} = 1$. Similarly, we see that the kth row of CV_i' is the sum of the last k columns, which happens to give us the same matrix. The reader is encouraged to check this by hand for $n = 3$. ☐

Since the U_i's satisfy the braid relations, conjugates thereof such as the V_i' satisfy the braid relations. Because of our block matrix definition of V_i' in terms of V_i, the V_i's satisfy the braid relation as well. A brief computation shows that the V_i all lie in $\mathrm{GL}_{n-1}(\Lambda)$. Thus, the map $\psi_n^r(\sigma_i) := V_i$ defines a representation $\psi_n^r : B_n \to \mathrm{GL}_{n-1}(\Lambda)$ for all $n \geqslant 3$. We call it the *reduced Burau representation.* For $n = 2$, we set $\sigma_1 \mapsto (-t) \in M_{1\times 1}(\Lambda)$, so that our conjugacy relationship is still satisfied. That is,

$$C^{-1}\psi_n(\beta)C = \begin{pmatrix} \psi_n^r(\beta) & 0 \\ *_\beta & 1 \end{pmatrix}$$

for all $n \geqslant 2$, with $*_\beta$ a function of β. It turns out that no extra information is lost, as lemma 3.10 of [1] shows that $\ker \psi_n = \ker \psi_n^r$ for each $n \geqslant 2$.

8 A New(?) Knot Invariant!

Now that we've defined the reduced Burau representation ψ_n^r, we can finally give an explicit example of a Markov function. We'll define our functions with image in $\Lambda' := \mathbb{Z}[s, s^{-1}]$. We change variables to help differentiate between elements of Λ arising from ψ_n^r and elements of the image. Let

$$g : \Lambda \to \Lambda'$$

be the ring homomorphism sending t to s^2. For a braid β on $n \geqslant 2$ strings, we define the following rational function in s with integral coefficients:

$$f_n(\beta) = (-1)^{n+1}\frac{s^{-\langle\beta\rangle}(s - s^{-1})}{s^n - s^{-n}} g(\det(\psi_n^r(\beta) - I_{n-1}),$$

[3]The following two sections are largely adapted from [1].

where $\langle \beta \rangle \in \mathbb{Z}$ is the image of β under the homomorphism $B_n \to \mathbb{Z}$ sending the generators $\sigma_1, \ldots, \sigma_{n-1}$ to 1. For example, for $n = 2$ and $k \in \mathbb{Z}$,

$$f_2(\sigma_1^k) = -s^{-k}(s + s^{-1})^{-1}((-s^2)^k - 1).$$

In particular, $f_2(\sigma_1) = f_2(\sigma_1^{-1}) = 1$. The closure of $\sigma_1 \in \mathbf{B}_2$ is the unknot, so we find that our invariant coincides with the Alexander polynomial for the unknot. I wonder if that's a coincidence... Regardless, we now must show that it actually defines an invariant.

Lemma. *The mappings* $\{f_n : \boldsymbol{B}_n \to \Lambda'\}_{n \geqslant 1}$ *form a Markov function.*

Proof. Fix $\beta \in B_n$. For any $\gamma \in B_n$,

$$\langle \gamma\beta \rangle = \langle \gamma \rangle + \langle \beta \rangle = \langle \beta \rangle + \langle \gamma \rangle = \langle \beta\gamma \rangle$$

To show that $f_n(\gamma\beta) = f_n(\beta\gamma)$, it suffices to note that

$$\begin{aligned}
\det(\psi_n^r(\gamma\beta) - I_{n-1}) &= \det(\psi_n^r(\gamma)\psi_n^r(\beta\gamma)\psi_n^r(\gamma)^{-1} - I_{n-1}) \\
&= \det(\psi_n^r(\gamma)(\psi_n^r(\beta\gamma) - I_{n-1})\psi_n^r(\gamma)^{-1}) \\
&= \det(\psi_n^r(\beta\gamma) - I_{n-1}).
\end{aligned}$$

Then f_n satisfies the first condition of being a Markov function.

Unfortunately, the proof that $f_n(\beta) = f_{n+1}(\beta\sigma_n) = f_{n+1}(\sigma_n\beta)$ is as unenlightening as it is laborious, mostly consisting of long block matrix calculations. We refer the curious reader to lemma 3.12 of [1] for the rest of the proof. ☐

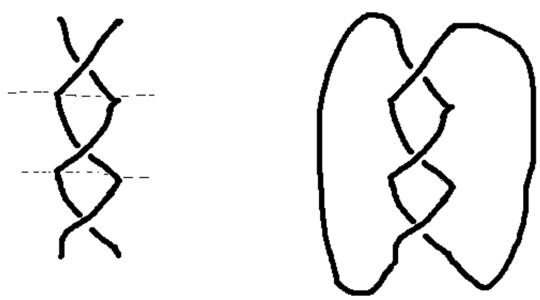

Figure 7: The 2-strand braid σ_1^3 (left) and it's closure (right).

9 Computing the Alexander Polynomial

It turns out that our Markov function from above always coincides with the earlier topological definition of the Alexander polynomial (up to a unit), and some texts even define the Alexander polynomial using this representation, rather than topologically. For a discussion on this connection, see chapter 3 of [1]. Now, we will compute the Alexander polynomial of the trefoil knot.

As we can see from Figure 7, the trefoil knot is the closure of the two-stranded braid σ_1^3. Tugging the bottom left corner of the closure up to the top right corner makes the equivalence clearer.

We already know the formula for f_2 from earlier, so we can calculate

$$\begin{aligned}
f_2(\sigma_1^3) &= -s^{-3}(s + s^{-1})^{-1}((-s^2)^3 - 1) \\
&= \frac{(s^{-2} - s^{-4})(s^6 + 1)}{s^2 - s^{-2}} = \frac{s^4 - s^2 + 1}{s^2} = s^2 - 1 + s^{-2}
\end{aligned}$$

Undoing our ring homomorphism g[4], we get the Alexander polynomial of the trefoil knot as $t - 1 + t^{-1}$. We saw earlier that the Alexander polynomial of the unknot is 1, which is not a unit multiple of $t - 1 + t^{-1}$, so we can now say definitively that the trefoil knot is not the unknot, and have demonstrated the power of the Burau representation.

References

[1] Vladimir Turaev Christian Kassel. *Braid Groups.* Graduate Texts in Mathematics. Springer, 2008. ISBN: 978-0-387-33841-5.

[2] Bohdan I. Kurpita Kunio Murasugi. *A Study of Braids.* Mathematics and Its Applications. Kluwar Academic Publishers, 1999. ISBN: 0-7923-5767-1.

[3] Jacob Rasmussen. "Knots, polynomials, and categorification". In: *Quantum Field Theory and Manifold Invariants.* IAS/Park City Mathematics Series. American Mathematical Society, Dec. 2021, pp. 77–170. ISBN: 9781470461232.

[4] The map $s^2 \mapsto t$ is not a ring homomorphism $\Lambda' \to \Lambda$ as we defined them in this section, but recall that in section 3 we defined the Alexander polynomial with rational powers, so $t^{\frac{1}{2}}$ is okay.