Combinatorial Species and their Generating Series

Noah Parker

April 2025

1 Introduction

The theory of generating functions is both useful and beautiful, but their framing often feels haphazard. We remedy this by giving a foundation for the systematic study of combinatorial objects, their generating functions, and the functional equations arising from certain bijections through the use combinatorial species. Though the study of combinatorial species finds its origins in category theory, we primarily follow [2] in providing an elementary approach.

2 Combinatorial Species

Definition 2.1. A *combinatorial species* is a mapping F which assigns

1. to each finite set U, a finite set $F[U]$ of *F-structures on U*,
2. to each bijection $\sigma : U \to V$ between finite sets U and V, a function $F[\sigma] : F[U] \to F[V]$.

We call $F[\sigma]$ the *F-transport along σ*. Such functions also satisfy

$$F[\sigma \circ \tau] = F[\sigma] \circ F[\tau],$$
$$F[\mathrm{id}_U] = \mathrm{id}_{F[U]},$$

for all finite sets U, V, W, and bijections $\sigma : U \to V$, $\tau : V \to W$. We say a mapping which respects composition and identities like the above is *functorial*.

Example 2.2. We give a list of combinatorial species.

- The species E of sets, defined by $E[U] = \{U\}$, and $E[\sigma](U) = V$. Similarly, we can define $1 = E_0$, $X = E_1$, and in general E_k by
$$E_k[U] = \begin{cases} \{U\} & |U| = k, \\ \emptyset & \text{otherwise.} \end{cases}$$
- The species $\wp$ of subsets, defined by
$$\wp[U] = \{u \subset U\}$$
$$\wp[U](u) = \{\sigma(x) \in V \ : \ x \in u\}$$
for all bijections $\sigma : U \to V$.
Similarly, we can define a species $\wp^{[k]}$ of k-element subsets by restricting the image.
- The species S of permutations, defined by
$$S[U] = \{\sigma : U \xrightarrow{\sim} U\},$$
$$S[\sigma](\tau) = \sigma \circ \tau \circ \sigma^{-1},$$
where "$\xrightarrow{\sim}$" denotes bijection. We often denote $S[n] := S[[n]]$.
We can define another species C of cyclic permutations by restricting the image to cyclic permutations.

- The species L of linear orders, defined by

$$L[U] = \{(u_1, u_2, \dots, u_n) \in U^n \ : \ u_i \neq u_j \ \forall i \neq j\},$$
$$F[\sigma](u_1, \dots, u_n) = (\sigma(u_1), \dots, \sigma(u_n)),$$

  where $n := |U|$. Technically a linear order is a transitive, reflexive, anti-symmetric, total relation $\leqslant$, But we use this much more manageable definition given above.

Species can also be defined by algorithms, functional equations, and more. Examples of these can be found later in this paper, as well as section 1.1 of [2].

3 Associated Series

Let F be a combinatorial species. Functoriality of F gives us that, for any bijection $\sigma : U \to V$,

$$F[\sigma] \circ F[\sigma^{-1}] = F[\sigma \circ \sigma^{-1}] = F[\mathrm{id}_U] = \mathrm{id}_{F[U]}.$$

Hence, $F[\sigma] : F[U] \to F[V]$ is a bijection, so that $|F[U]|$ depends only on $|U|$. Thus, when enumerating F-structures, we will work primarily with $U = [n] := \{1, \dots, n\}$, and use the shorthand $F[[n]] = F[n]$. We then want a nice way to work with the sequence $(f_n)_{n \geqslant 0}$.

Definition 3.1. Let F be a species, and denote $f_n := |F[n]|$.
We define a formal power series

$$F(x) = \sum_{n \geqslant 0} \frac{f_n x^n}{n!},$$

which we call the *generating series* associated with F.

Example 3.2. Coming back to our species defined in example 2.2,

$$E(x) = e^x$$
$$E_k(x) = \frac{x^k}{k!}$$
$$\wp(x) = e^{2x}$$
$$S(x) = L(x) = \frac{1}{1-x}.$$

It's already clear by $S(x) = L(x)$ that this is a quite lossy conversion. We often want to consider F-structures modulo relabeling, for which we need a notion of isomorphism. We already have a notion of the map of structures induced by a bijection, so we need not search for long.

Definition 3.3. Let $s, s' \in F[U]$ be F-structures. We say that $s \cong s'$ if there exists a permutation $\sigma : U \to U$ such that

$$s' = F[\sigma](s).$$

The reader might expect that this induces an equivalence relation. The reader would be correct.

Definition 3.4. We call a member of the quotient set $T(F_n) := F[n]/\cong$ an *isomorphism type of order n*, or an *unlabeled F-structure of order n*.

Often in enumeration problems, we only care to count the isomorphism types of a structure. Thus, it is quite natural to define a generating series for unlabeled F-structures.

Definition 3.5. Let F be a species, and denote $\widetilde{f}_n := |T(F_n)|$.
We define a formal power series

$$\widetilde{F}(x) = \sum_{n \geqslant 0} \widetilde{f}_n x^n,$$

which we call the *type generating series* associated with F.

Example 3.6. Recall our examples once more.

$$\begin{aligned}
\widetilde{E}(x) = \widetilde{L}(x) &= \frac{1}{1-x} \\
\widetilde{S}(x) &= \prod_{j\geqslant 1} \frac{1}{1-x^j} \\
\widetilde{E}_k(x) &= x^k \\
\widetilde{\wp}(x) &= \frac{1}{(1-x)^2}
\end{aligned}$$

We will elaborate on the derivation of $\widetilde{S}(x)$ in due time.

For simplicity, we will use the following notation for coefficients of power series.

Definition 3.7. If

$$G(x) = \sum_{n_i \geqslant 0 \ \forall i \geqslant 1} g_{n_1,n_2,\dots} x_1^{n_1} x_2^{n_2} \cdots$$

is a generic power series in arbitrarily many variables, then we denote

$$[x_1^{n_1} x_2^{n_2} \cdots] G(x) := g_{n_1,n_2,\dots}.$$

With the above notation,

$$\begin{aligned}
f_n &= n!\,[x^n]F(x), \\
\widetilde{f}_n &= [x^n]\widetilde{F}(x).
\end{aligned}$$

Experience shows that, normally, we want to find the type generating series of a given species. Enumerating isomorphism classes is usually harder, but more useful than simply enumerating labeled objects. However, this difficulty carries over to computating the type generating functions.

We'll introduce a number of operation on species that work quite nice with the generating series of a species, while having a much more complicated effect on the type generating series at all. To compute how these operations effect the type generating function, we will have to work with a far more general series, from which regular and type generating series fall out.

To do so, we will have to introuce a little more notation.

Definition 3.8. Let U be a finite set of order n.
We define two set functions $\mathrm{Fix} : [U] \to U$ and $\mathrm{fix} : S[U] \to [n]$ by

$$\begin{aligned}
\mathrm{Fix}\,\sigma &= \{x \in U \ : \ \sigma(x) = x\}, \\
\mathrm{fix}\,\sigma &= |\,\mathrm{Fix}\,\sigma|.
\end{aligned}$$

We often use multiplication notation as shorthand for permutation composition, and define $\mathrm{ord}\,\sigma$ as the smallest integer such that $\sigma^k = \mathrm{id}_U$, and $i < k$ implies $\sigma^i \neq \mathrm{id}_U$. This allows us to express the set of cycles as

$$C[U] = \{\sigma \in S[U] \ : \ \mathrm{ord}\,\sigma = n - \mathrm{fix}\,\sigma\}.$$

Note that any permutation can be decomposed into a set of disjoint cycles. There is a greedy algorithm to do so, which the reader is encouraged to determine themselves if they are unfamiliar with it.

Definition 3.9. Let $\sigma \in S_U$ be a permuation of a finite set U of size n. We associate a vector $(\sigma_1, \dots, \sigma_n)$ of natural numbers with σ, where σ_k is the number of k-length cycles in the cycle decomposition of σ.

Note that, since the cycles are disjoint, $\sum_{k=1}^n k\sigma_k = n$. This allows us to define one of our key objects of study.

Definition 3.10. Let F be a species. We define a formal power series

$$Z_F(x_1, x_2, \ldots) := \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\sigma \in S[n]} (\operatorname{fix} F[\sigma]) p_\sigma,$$

which we call the *cycle index series of* F, where

$$p_\sigma := x_1^{\sigma_1} x_2^{\sigma_2} \cdots$$

with σ_i from the cycle decomposition of σ.

This definition may seem rather obtuse, but we can already see how our simpler series can be derived.

Proposition 3.11. *Let F be a combinatorial species. Then*

$$\begin{aligned} F(x) &= Z_F(x, 0, 0, \ldots), \\ \widetilde{F}(x) &= Z_F(x, x^2, x^3, \ldots). \end{aligned}$$

Proof. We begin by proving the former equality. For a term of the form $\operatorname{fix} F[\sigma] x^{\sigma_1} 0^{\sigma_2} 0^{\sigma_3} \cdots$ to be nonzero, we must have that $\sigma_i = 0$ for all $i > 1$. This forces $1 \cdot \sigma_1 = n$, so that σ fixes $[n]$, i.e. σ is the identity. In which case, $\operatorname{fix} F[\sigma] = |F[n]| = f_n$. Our series simplifies to

$$Z_F(x, 0, 0, \ldots) = \sum_{n \geqslant 0} f_n x^n = F(x).$$

We move on to the latter equality. Once again, we use $\sum_{k=1}^{n} k\sigma_k = n$ to see that each term in the inner sum is of the form

$$\operatorname{fix} F[\sigma] x^{\sigma_1 + 2\sigma_2 + \cdots + n\sigma_n} = \operatorname{fix} F[\sigma] x^n.$$

We then find that

$$Z_F(x, x^2, x^3, \ldots) = \sum_{n \geqslant 0} \left(\frac{1}{n!} \sum_{\sigma \in S[n]} \operatorname{fix} F[\sigma] \right) x^n.$$

This may seem an intractible form, but the equality

$$|F[n]/\cong| = \frac{1}{n!} \sum_{n \in S[n]} \operatorname{fix} F[\sigma]$$

follows by a lemma of Burnside regarding enumerating objects modulo symmetries.[1] We conclude that

$$Z_F(x, x^2, x^3, \ldots) = \sum_{n \geqslant 0} \widetilde{f}_n x^n = \widetilde{F}(x).$$

□

Because of how much infofrmation they carry, cycle index series computations are, in general, quite involved. Thus, we will restrict ourselves to a single, quite simple example for now.

Example 3.12. Fortunately, there are some manageable computations. Consider

$$Z_{E_k}(x_1, x_2, \ldots) = \frac{1}{k!} \sum_{\sigma \in S[k]} x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_k^{\sigma_k}.$$

The summand depends only on the cycle type of σ, so enumerating the number of elements of a cycle type allows us to sum over cycle types $[\sigma]$. It is an elementary (but not trivial) combinatorial computation to see that

$$|[\sigma]| = \frac{k!}{1^{\sigma_1} 2^{\sigma_2} \cdots k^{\sigma_k} \sigma_1! \sigma_2! \cdots \sigma_k!}$$

[1] A description of the lemma in full generality, as well as a proof, is present in section 4.3 of [4]

permutations possess the cycle type $[\sigma]$. This gives us

$$\begin{aligned} Z_{E_k}(x_1,x_2,\ldots) &= \frac{1}{k!}\sum_{[\sigma]\in T(S_k)} \frac{k!x_1^{\sigma_1}x_2^{\sigma_2}\cdots x_k^{\sigma_k}}{1^{\sigma_1}2^{\sigma_2}\cdots k^{\sigma_k}\sigma_1!\sigma_2!\cdots\sigma_k!} \\ &= \frac{1}{k!}\sum_{[\sigma]\in T(S_k)} \binom{k}{\sigma_1,\sigma_2,\ldots,\sigma_k}\prod_{i=1}^{k}\left(\frac{x_i}{i}\right)^{\sigma_i} \end{aligned}$$

On it's own, this is quite unwieldy and unenlightening for larger k. However, if we recognize it as just the k-th term in Z_E, then we get

$$\begin{aligned} Z_E(x_1,x_2,\ldots) &= \sum_{n\geqslant 0}\frac{1}{n!}\sum_{[\sigma]\in T(S_n)}\binom{n}{\sigma_1,\sigma_2,\ldots,\sigma_n}\prod_{i=1}^{n}\left(\frac{x_i}{i}\right)^{\sigma_i} \\ &= \prod_{n\geqslant 1}\exp\left(\frac{x_n}{n}\right) \\ &= \exp\left(\sum_{n\geqslant 1}\frac{x_n}{n}\right). \end{aligned}$$

As an example of our general calculation for the type generating series in terms of the cycle index series, we note that

$$\begin{aligned} \widetilde{E}(x) &= Z_E(x,x^2,\cdots) \\ &= \exp\left(\sum_{n\geqslant 1}\frac{x^n}{n}\right) \\ &= \exp\left(-\log(1-x)\right) \\ &= \frac{1}{1-x}, \end{aligned}$$

as expected.

Finally, we compute Z_S. We consider the n-th term

$$\frac{1}{n!}\sum_{\sigma\in S[n]}\operatorname{fix} S[\sigma]x_1^{\sigma_1}x_2^{\sigma_2}\cdots x_n^{\sigma_n}.$$

It follows from a simple theorem of abstract algebra[2] that

$$\operatorname{fix} S[\sigma] = \frac{|S[n]|}{|[\sigma]|}.$$

However, this formula clearly depends only on $[\sigma]$, so the n-th term of Z_s is

$$\frac{1}{n!}\sum_{[\sigma]\in T(S_n)}|[\sigma]|\frac{|S[n]|}{|[\sigma]|}x_1^{\sigma_1}\cdots x_n^{\sigma_n} = \sum_{[\sigma]\in T(S_n)} x_1^{\sigma_1}\cdots x_n^{\sigma_n},$$

i.e. the coefficients in

$$\prod_{i\geqslant 1}\frac{1}{1-x_i}$$

whose weighted coefficients add up to n. Summing over all n, then, we find that

$$Z_S = \prod_{i\geqslant 1}\frac{1}{1-x_i}.$$

[2]See Orbit-Stabilizer theorem, and discussion of conjugation action on the symmetric group in [4].

4 Isomorphism of Species

Now that we have a quite strong invariant of species, a good next step is to figure out some sort of "isomorphism of species," denote it $\cong$, where $F \cong G$ implies $Z_F = Z_G$. One might think to consider F and G equivalent if $|F[U]| = |G[U]|$ for each U

Definition 4.1. Species F and G are said to be *equipotent* if, there exists a family $\{\phi_U : F[U] \xrightarrow{\sim} G[U]\}$ of bijections from $F[U] \to G[U]$ for each U.

This definition plays well with the generating series $F(x)$ and $G(x)$. In fact, F and G are equipotent exactly when $F(x) = G(x)$, essentially by definition. However, it fails to preserve cycle index functions, or even type generating functions!

Example 4.2. We consider the species L of linear orders and S of permutations. We see that $|L[n]| = |S[n]| = n!$ for each n, so F and G are equipotent. However, we already see that their type generating functions are not equal.

An isomorphism type of linear orders is just a set of size n, of which only one is contained in $[n]$. Thus,

$$\widetilde{L}(x) = \frac{1}{1-x}.$$

An isomorphism class of permutations is just a conjugacy class of $S[n]$, which we know to be in correspondence with cycle types. Cycle types are in correspondence with integer partitions, so we have that

$$\widetilde{S}(x) = \prod_{i \geqslant 1} \frac{1}{1-x^i}.$$

From this, we conclude further that $Z_L \neq Z_S$, as they agree on input $(x, x^2, x^3, \ldots)$. Hence, equipotent species need not have equal type or cycle index series.

It turns out that equipotence fails because it allows bijections to depend too much on arbitrary choices. In the bijections between linear orders and permutations, we often rely on choice of linear order to define our bijections, like the regular $\leqslant$ when showing $|L[n]| = |S[n]|$. We can formalize the requirement as follows.

Definition 4.3. Let F and G be combinatorial species. We say F and G are *isomorphic*, denoted $F \cong G$, if there exists a family of bijections $\phi_U : F[U] \xrightarrow{\sim} G[U]$, with each ϕ_U natural. That is, for any bijection $\sigma : U \xrightarrow{\sim} V$, the following diagram commutes.

$$\begin{tikzcd}
F[U] \arrow[r, "\phi_U"] \arrow[d, "F[\sigma]"'] & G[U] \arrow[d, "G[\sigma]"] \\
F[V] \arrow[r, "\phi_V"'] & G[V]
\end{tikzcd}$$

In set theoretic terms, $G[\sigma] \circ \phi_U = \phi_V \circ F[\sigma]$.

Due to the strength of this equivalence, we will often treat isomorphic species as indistinguishable, or write $F = G$ to mean $F \cong G$. This strengthened equipotency happens to capture the notion of natural, or non-arbitrary isomorphisms. More importantly, cycle index functions are preserved under isomorphism.

Proposition 4.4. *Let F and G be combinatorial species. Then*

$$F \cong G \implies Z_F = Z_G.$$

Proof. Fix $n \geqslant 0$. It suffices to show that $\operatorname{fix} F[\sigma] = \operatorname{fix} G[n]$ for arbitrary $\sigma \in S[n]$. Let

$$\phi_n := \phi_{[n]} : F[n] \xrightarrow{\sim} G[n]$$

be the bijection supplied by $F \cong G$, and take $\sigma \in S[n]$. Then naturality gives us $G[\sigma] = \phi_n \circ F[\sigma] \circ \phi_n^{-1}$ is a conjugate of $F[\sigma]$, so the cycle types of $F[\sigma]$ and $G[\sigma]$ are the same. In particular, they have the same number of 1-cycles, so

$$\operatorname{fix} F[\sigma] = F[\sigma]_1 = G[\sigma]_1 = \operatorname{fix} G[\sigma]$$

with notation as in our first description of cycle decomposition. Then

$$\begin{aligned} Z_F(x_1, x_2, \ldots) &= \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\sigma \in S[n]} (\operatorname{fix} F[\sigma]) p_\sigma \\ &= \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\sigma \in S[n]} (\operatorname{fix} G[\sigma]) p_\sigma \\ &= Z_G(x_1, x_2, \ldots). \end{aligned}$$

□

Now that we have a good notion of equality, it's finally time to start building combinatorial species from other combinatorial species. To do so, we'll define a number of operations on combinatorial species based on how they affect the respective cycle index series and generating series. Many interesting applications arise from this line of inquiry.

5 Operations on Species

The simplest operations we'll define are the analogues to addition and multiplication.

Definition 5.1. Let F and G be combinatorial species.

1. We define the *sum of F and G*, denoted $F + G$, by

$$(F+G)[U] = F[U] \amalg G[U]$$
$$(F+G)[\sigma](s) = \begin{cases} F[\sigma](s) & s \in F[U] \\ G[\sigma](s) & s \in G[U], \end{cases}$$

where $\amalg$ denotes the disjoint union.

2. We define the *Cauchy Product of F and G*, denoted $F \cdot G$, by

$$(F \cdot G)[U] = \coprod_{\{U_1, U_2\} \vDash U} F[U_1] \times G[U_2]$$
$$(F \cdot G)[\sigma]\big|_{F[U_1] \times G[U_2]}(s_1, s_2) = (F[\sigma|_{U_1}](s_1), G[\sigma|_{U_2}](s_2)),$$

where $\{U_1, U_2\} \vDash U$ means the U_i partition U as a set.

3. We define the *substition of F and G*, denoted $F \circ G$ or $F(G)$, by

$$(F \circ G)[U] = \coprod_{\pi \vDash U} F[\pi] \times \coprod_{p \in \pi} G[p]$$

That is, each $F \circ G$ structure on a set U is a triple (π, φ, γ), for $\pi \vDash U$, $\varphi \in F[\pi]$, and $\gamma = (\gamma_p)_{p \in \pi}$ a vector of G structures on the various blocks of π.

We can define $F \circ G$ transports in a similarly explicit manner to $F + G$ and $F \cdot G$. However, it's quite wordy, and will thus be omitted. The reader is encouraged to read section 1.4 of [2] for the details.

Theorem 5.2. *Let F and G be two species. Then*

$$\begin{aligned} Z_{F+G} &= Z_G + Z_G, \\ Z_{F \cdot G} &= Z_F \cdot Z_G, \\ Z_{F \circ G} &= Z_F \circ Z_G. \end{aligned}$$

However, this statement does not really make sense as it stands. How do you compose two formal power series of infinitely many variables? There are obvious notions of addition and multiplication for multivariate real valued power series that we can borrow, but no such notion exists for composition. To make sense of this result, then, we must define a notion of composition on formal power series.

Definition 5.3. Let $f = f(x_1, x_2, x_3, \ldots)$ and $g = g(x_1, x_2, x_3, \ldots)$ be formal power series. We define the *plethystic substitution* $f \circ g$ of f and g by

$$(f \circ g)(x_1, x_2, x_3, \ldots) = f(g_1, g_2, g_3, \ldots)$$

where $g_k = g(x_k, x_{2k}, x_{3k}, \ldots)$.

It turns out that *this* is the notion in which $Z_{F\circ G} = Z_F \circ Z_G$, as well as the origin of the term "substitution" of species, as opposed to "composition." Fortunately, $g_1 = g$, so this notion of substitution coincides with regular power series composition whenever f and g are only functions of x_1. This leads to the following convenient corollary.

Corollary 5.4. *Let F and G be two species. Then*

$$\begin{aligned}(F+G)(x) &= F(x) + G(x),\\ (F \cdot G)(x) &= F(x) \cdot G(x),\\ (F \circ G)(x) &= F(x) \circ G(x).\end{aligned}$$

and

$$\begin{aligned}\widetilde{(F+G)}(x) &= \widetilde{F}(x) + \widetilde{G}(x),\\ \widetilde{(F \cdot G)}(x) &= \widetilde{F}(x) \cdot \widetilde{G}(x),\\ \widetilde{(F \cdot G)}(x) &= Z_F(\widetilde{G}(x), \widetilde{G}(x^2), \ldots).\end{aligned}$$

The first part of this corollary is quite easy to prove, it turns out. Similarly, the full result is quite easy to prove for sums and products. However, the proof that $Z_{F\circ G} = Z_F \circ Z_G$ is quite in depth, requiring notions of analytic functors and symmetric functions. Thus, we refer the reader to [7], an expository paper detailing the process. Fortunately, our operations also admit quite simple identities.

Proposition 5.5. *Let F be a combinatorial species, and define the trivial species 0 by $0[U] = 0[\sigma] = \emptyset$, as well as $1 = E_0$, $X = E_1$ as in our first example. Then*[3]

$$\begin{aligned}0 + F = {} & F + 0 = F\\ 1 \cdot F = {} & F \cdot 1 = F\\ X \circ F = {} & F \circ X = F.\end{aligned}$$

Example 5.6. Now that we have our operations, it's time to do some computations using them.[4]

(a) First, we consider the species Der of derangements. Each permutation in $S[n]$ consists of $k \leqslant n$ fixed points, and a derangement on the remaining $n - k$ points. Thus,

$$S = E \cdot \mathrm{Der}.$$

We've already fully characterize S and E combinatorially, so we can do the same for Der. That is,

$$\begin{aligned}\mathrm{Der}(x) &= \frac{e^{-x}}{1-x},\\ \widetilde{\mathrm{Der}}(x) &= \prod_{j\geqslant 2} \frac{1}{1-x^j}.\end{aligned}$$

$$Z_{\mathrm{Der}}(x_1, x_2, \ldots) = \exp\left(\sum_{i\geqslant 1} \frac{x_i}{i}\right) \prod_{j\geqslant 1} \frac{1}{1-x_j}.$$

By expanding $\mathrm{Der}(x)$ as a product of power series, we can derive the classic formula for $|\mathrm{Der}[n]|$.

[3] We use equality as shorthand for isomorphism here, as discussed. When working with operations, we will usually do so.
[4] These examples can mostly be found in chapter 2 of [3], though he includes no proofs.

(b) Recall that we listed $\widetilde{\wp}(x) = \frac{1}{(1-x)^2}$ as an example earlier without justification. This is because we can derive this result immediately from a functional equation definition of $\wp$. A subset $u \subset U$ is really an ordered partition of U into u and u^c. Conversely, a two element ordered partition of U can be naturally identified with its first block. The latter species can be naturally identified with $E \cdot E$, so $\wp = E \cdot E$ gives us that

$$\widetilde{\wp}(x) = \widetilde{E}(x)^2 = \frac{1}{(1-x)^2},$$

as desired.

(c) It is clear that a permutation can be naturally identified with its set of cycles. This gives us that $S = E \circ C$. From this, we receive the quite strange identity

$$\prod_{j \geqslant 1} \frac{1}{1-x_j} = \exp\left(\sum_{k \geqslant 1} \frac{1}{k} Z_C(x_k, x_{2k}, \ldots)\right),$$

$$\sum_{j \geqslant 1} \log\left(\frac{1}{1-x_j}\right) = \sum_{k \geqslant 1} \frac{1}{k} Z_C(x_k, x_{2k}, \cdots).$$

Labeling our series as $b = \sum_{j \geqslant 1} -\log(1-x_j)$ and $a = Z_C(x_1, x_2, \ldots)$, with b_k and a_k as in the definition of plethysm, our identity becomes

$$b = \sum_{k \geqslant 1} \frac{1}{k} a_k.$$

Now, it turns out we have some analogue of mobius inversion in this setting. To show that, we first note that

$$b_j = \sum_{k \geqslant 1} \frac{1}{k} a_{jk}$$

for each $j \geqslant 1$. With a stroke of good luck, we decide to compute

$$\begin{aligned}
\sum_{k \geqslant 1} \frac{\mu(k)}{k} b_k &= \sum_{k \geqslant 1} \frac{\mu(k)}{k} \sum_{j \geqslant 1} \frac{1}{j} a_{kj} \\
&= \sum_{k \geqslant 1} \sum_{j \geqslant 1} \frac{\mu(k)}{kj} a_{kj} \\
&= \sum_{m \geqslant 1} \sum_{d \mid m} \frac{\mu(d)}{m} a_m \\
&= \sum_{m \geqslant 1} \frac{a_m}{m} \left(\sum_{d \mid m} \mu(d)\right) \\
&= a,
\end{aligned}$$

since $\sum_{d \mid n} \mu(d) = 1$ if $n = 1$ and 0 otherwise. In our specific case, then, we have that

$$\begin{aligned}
Z_c(x_1, x_2, \ldots) &= \sum_{k \geqslant 1} \frac{\mu(k)}{k} \sum_{j \geqslant 1} \log\left(\frac{1}{1-x_{kj}}\right) \\
&= \sum_{m \geqslant 1} \left(\sum_{d \mid m} \frac{\mu(d)}{d}\right) \log\left(\frac{1}{1-x_m}\right) \\
&= \sum_{m \geqslant 1} \frac{\phi(m)}{m} \log\left(\frac{1}{1-x_m}\right),
\end{aligned}$$

where ϕ is the euler ϕ function, and the last equality follows from

$$\phi(m) = \sum_{d|m} \mu(d) \frac{m}{d}.$$

By computing $\widetilde{C}(x) = \frac{x}{1-x}$ through more direct means, we receive the quite strange identity of

$$\frac{x}{1-x} = \sum_{k \geqslant 1} \frac{\phi(k)}{k} \log\left(\frac{1}{1-x^k}\right).$$

(d) The prior example has generalizes with little effort to showing that, for any species F and F^c with $F = E \circ F^c$,

$$Z_{F^c}(x_1, x_2, \ldots) = \sum_{k \geqslant 1} \frac{\mu(k)}{k} \log Z_F(x_k, x_{2k}, \ldots).$$

Fortunately, these examples arise often. For example, partitions are just sets of nonempty sets, and graphs are just sets of connected graphs. In fact, this notion of a disconnected object being identified with its set of connected components is the source of the notation F^c for the latter species.

Then this result, along with the obvious expression of Z_F in terms of Z_{F^c}, tell us that the problem of counting potentially disconnected objects reduces down to the problem of counting connected components, and vice versa.

6 Conclusion

There are a number of more advanced topics one could cover in the topic of combinatorial species. The most obvious next step is to continue studying the content of [2], which requires similarly little algebra. However, with some knowledge of group theory, the reader is encouraged to study Polya theory.

One can use Polya theory to justify the claim that cycle index series contain essentially all the combinatorial information we might wish to study, which we left unsubstantiated. A species theoretic introduction to Polya theory can be found in [5].

The most unfortunate exclusion from this paper is the category theoretic approach to combinatorial species. Modern study of combinatorial species almost exclusively proceeds from a category theoretic point of view. We hope that this paper will give the reader some intution for the reasoning behind less intuitive category theoretical notions like natural transformations, and allow for further study in modern works like [1], perhaps with some background reading in [6].

References

[1] Marcelo Aguiar and Swapneel Mahajan. "Monoidal Functors, Species, and Hopf Algebras". In: 2010. URL: `https://pi.math.cornell.edu/~maguiar/a.pdf`.

[2] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial Species and Tree-like Structures*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1998. ISBN: 9780521573238.

[3] F. Bergeron, G. Labelle, and P. Leroux. *Introduction to the Theory of Species of Structures*. [Online; accessed 27-April-2025]. 2013. URL: `https://bergeron.math.uqam.ca/wp-content/uploads/2013/11/book.pdf`.

[4] D.S. Dummit and R.M. Foote. *Abstract Algebra 2nd Ed.* Wiley India Pvt. Limited, 2008. ISBN: 978-8-126-51776-3.

[5] Andrew Gainer-Dewar. *Pólya theory for species with an equivariant group action*. 2015. arXiv: `1401.6202 [math.CO]`. URL: `https://arxiv.org/abs/1401.6202`.

[6] E. Riehl. *Category Thoery in Context*. Modern Math Originals. Dover, 2016. ISBN: 978-0-486-80903-8.

[7] F. Zhou. *The Composition Law for Cycle Index Series Via Analytic Species*. [Online; accessed 27-April-2025]. URL: `https://www.math.columbia.edu/~fanzhou/files/notes_analytic_species.pdf`.