طراحی الگوریتم‌ها – مفاهیم پایه ۲

Introduction to Algorithm

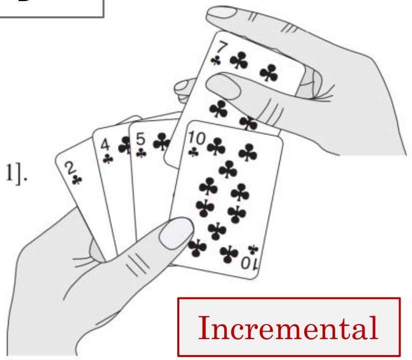
مهدی جوانمردی

۱۴۰۱

مرور مباحث قبل

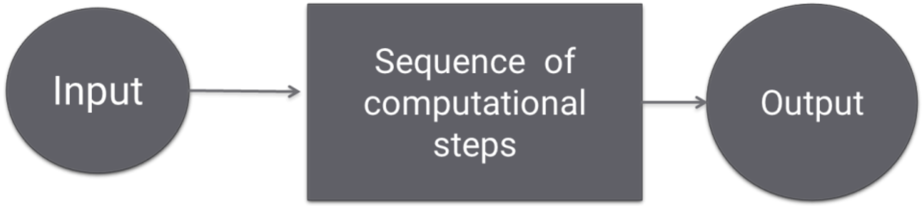
مرتب‌سازی درجی

```
INSERTION-SORT(A)
for j = 2 to A.length
    key = A[j]
    // Insert A[j] into the sorted sequence A[1..j-1].
    i = j - 1
    while i > 0 and A[i] > key
        A[i + 1] = A[i]
        i = i - 1
    A[i + 1] = key
```



Incremental

الگوریتم



اثبات درستی الگوریتم‌های تکرار

Loop Invariant

```
while (condition) {
I"""{
    invariant I
}"""
    code
}
```

- Initialization
- Maintenance
- ☆Termination

اهمیت طراحی اگوریتم

- کامپیوترها سریع هستند ولی نه بینهایت!
- حافظه ارزان است ولی رایگان نیست!

منابع محدود و مشخص

مرتبه زمانی

INSERTION-SORT(A)	cost	times
for $j = 2$ to $A.length$	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1..j-1]$.	0	$n-1$
$i = j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

- Best case
- Worst case
- Average

مرتب‌سازی

Ex. Input: sequence 31, 41, 59, 26, 41, 58

Output: sequence 26, 31, 41, 41, 58, 59

تحلیل زمانی مرتب‌سازی درجی

INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i = j - 1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)\,.$$

مثال مرتب‌سازی

Ex. Input: sequence 31, 41, 59, 26, 41, 58

Output: sequence 26, 31, 41, 41, 58, 59

تحلیل بهترین حالت

Best case	The array is already sorted

1	2	3	4	5	6
1	2	3	4	5	6

INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
$\quad key = A[j]$	c_2	$n-1$
$\quad$ // Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0	$n-1$
$\quad i = j-1$	c_4	$n-1$
$\quad$ **while** $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$\qquad A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$\qquad i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$\quad A[i+1] = key$	c_8	$n-1$

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\
&= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)\ .
\end{aligned}
$$

تحلیل بدترین حالت

Worst case	The array is in reverse sorted order

1	2	3	4	5	6
6	5	4	3	2	1

INSERTION-SORT(*A*)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i = j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$$

and

$$\sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) \\ &\quad + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\ &= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right) n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right) n \\ &\quad - (c_2 + c_4 + c_5 + c_8)\,. \end{aligned}$$

بدترین زمان اجرا و زمان اجرای متوسط

- **Worst-case running time:**
  - the longest running time for any input of size n:
    - Upper bound on the running time for any input
    - For some algorithms, the worst case occurs fairly often
    - The "average case" is often roughly as bad as the worst case
- **Average-case or expected running time:**
  - Technique of probabilistic analysis
  - Assume that all inputs of a given size are equally likely
  - Difficult to analyze

☆ متوسط زمان اجرای مرتب سازی درجی؟

تکنیک‌های طراحی الگوریتم و الگوریتم‌های تقسیم و حل

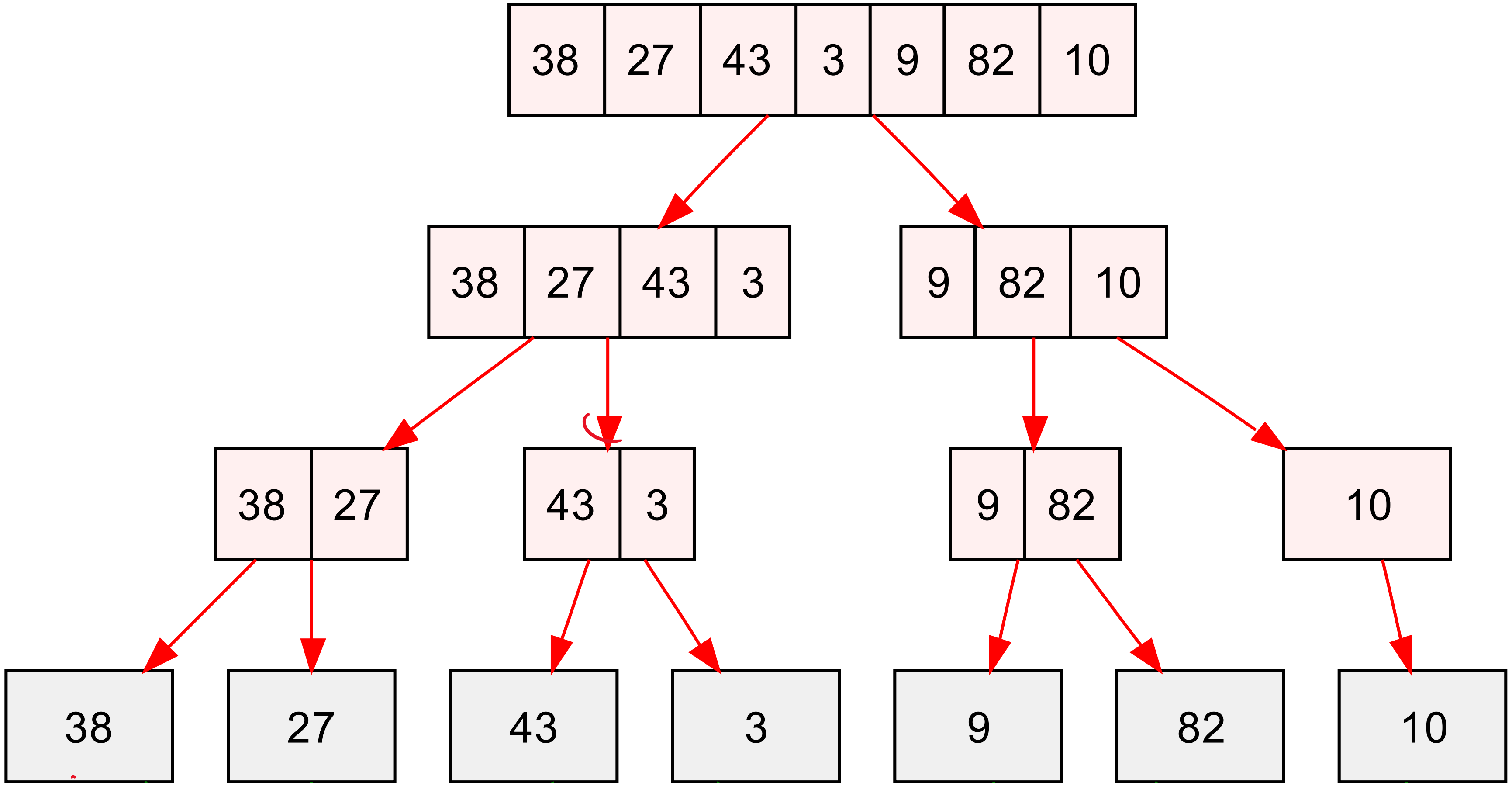
- Insertion sort: **Incremental approach** A[1 .. j-1] A[j] → A[1 .. j]
- **The divide-and-conquer approach**:
  - **Divide** the problem into a number of subproblems (similar to original problem).
  - **Conquer** the subproblems by solving them recursively. If the subproblem sizes are small enough, however, just solve the subproblems in a straightforward manner.
  - **Combine** the solutions to the subproblems into the solution for the original problem.
- **Recursive structure:** to solve a given problem, they call themselves recursively one or more times to deal with closely related subproblems.

مثال تقسیم و حل: Merge Sort

Divide

Conquer

Combine

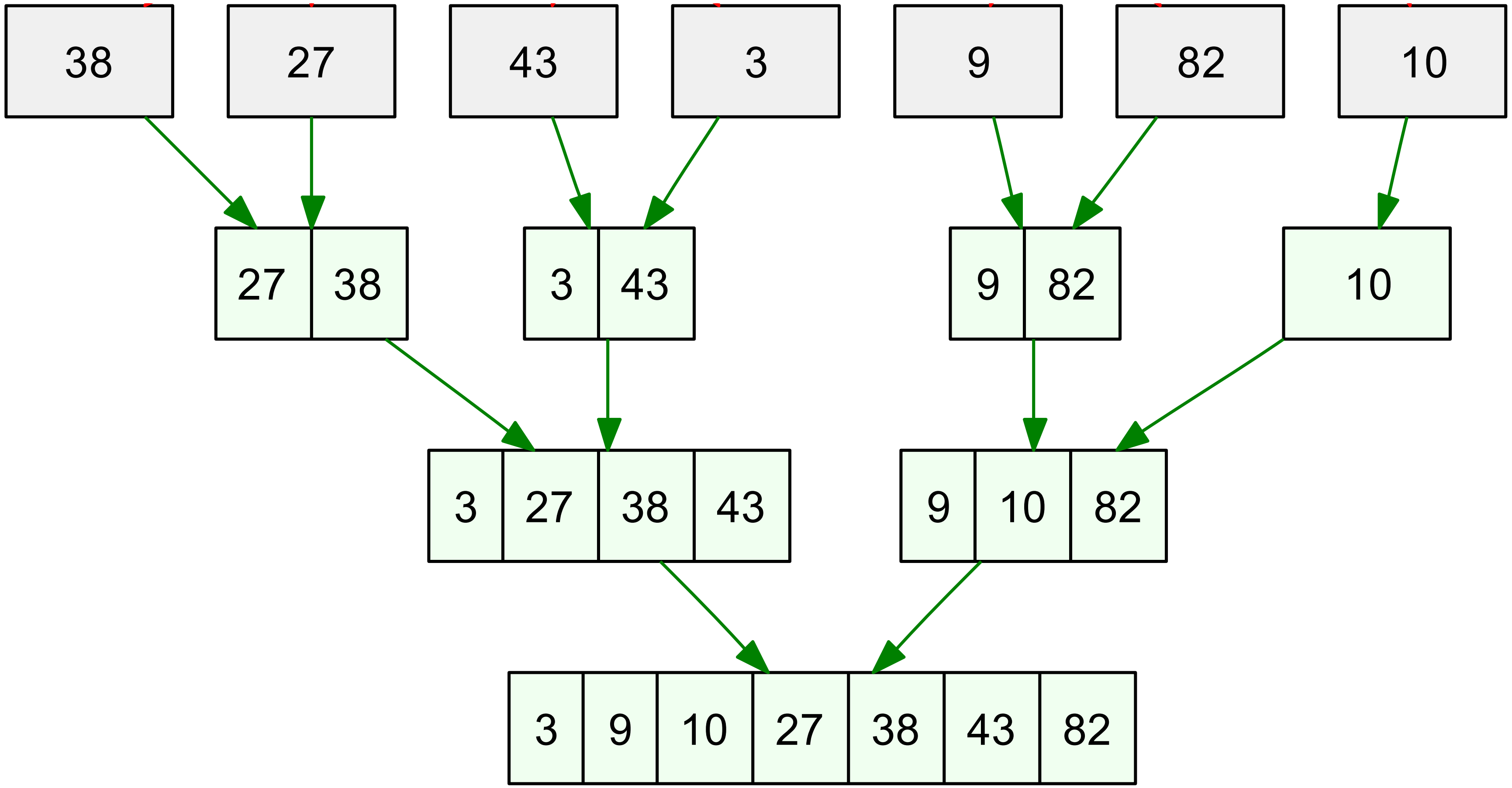


مثال تقسیم و حل: Merge Sort

Divide

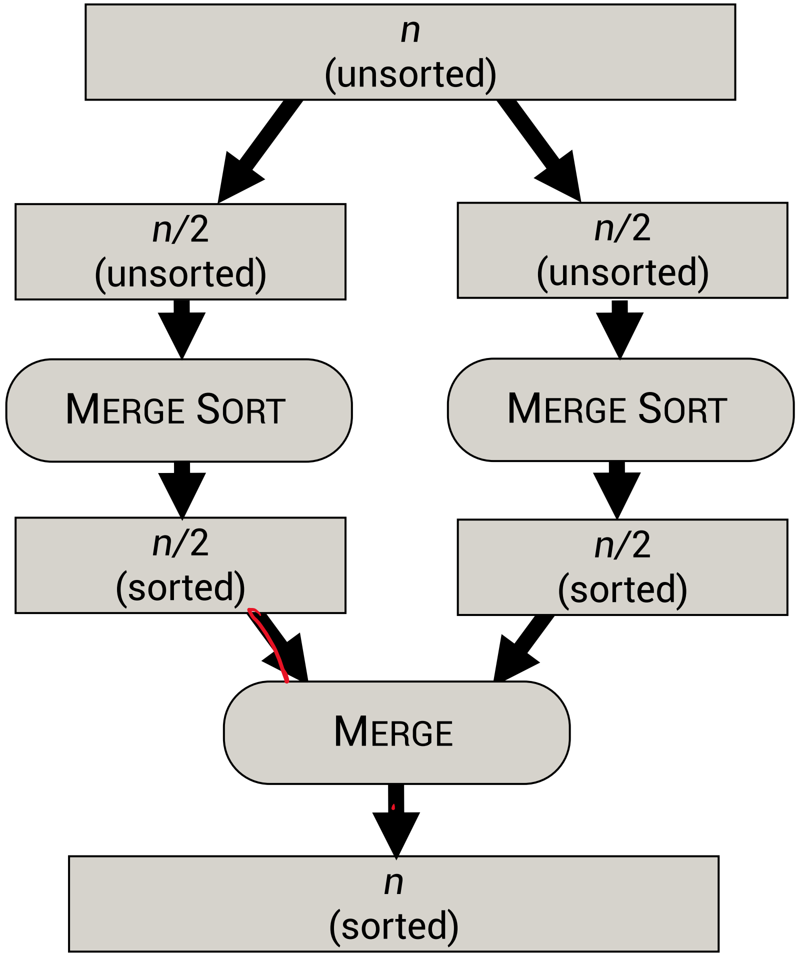
Conquer

Combine



Merge Sort | الگوریتم مرتب‌سازی ادغامی

- **Divide:** Divide the n-elements sequence to be sorted into two subsequences of n/2 elements each
- **Conquer:** Sort the two subsequences recursi -vely using merge sort
- **Combine:** Merge the two sorted subsequences to produce the sorted answer.

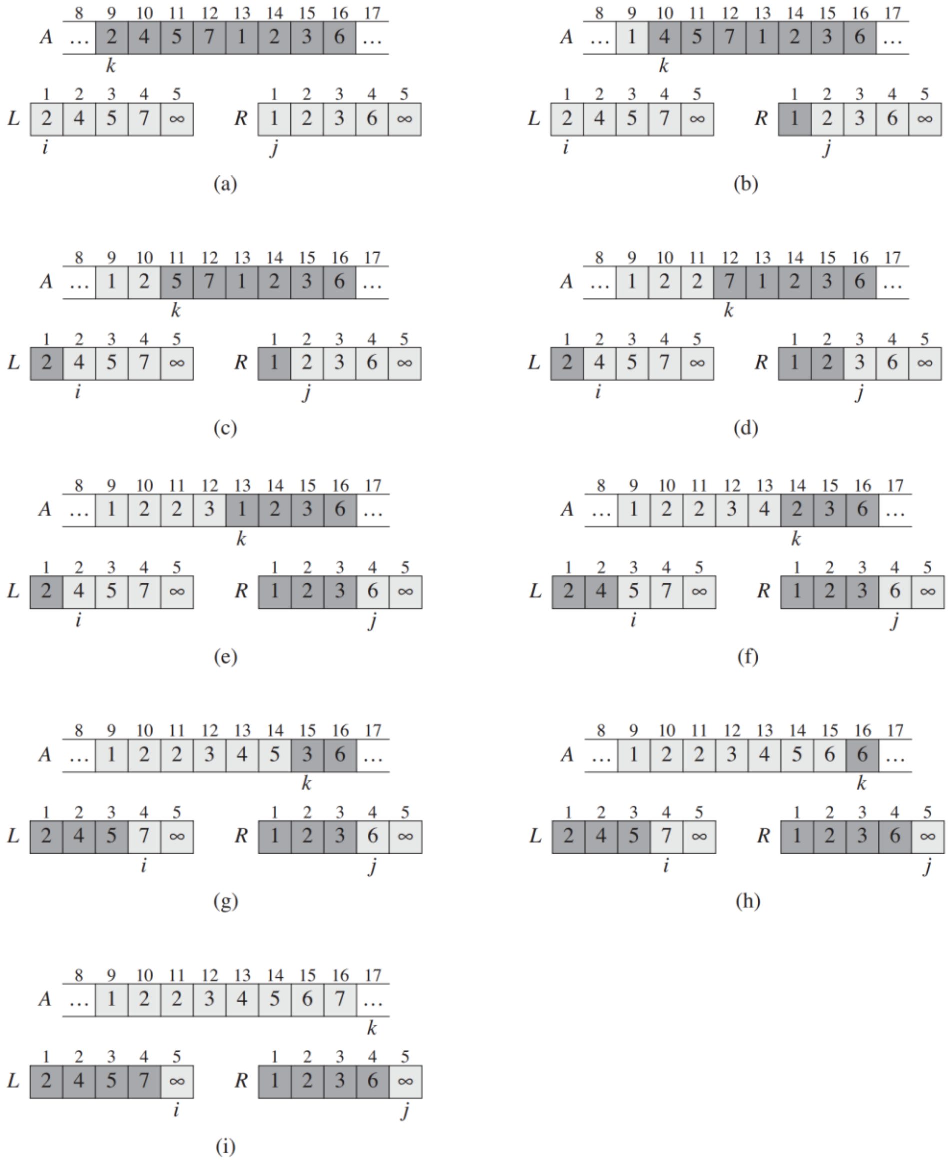



```
MERGE-SORT(A, p, r)
if p < r
    q = ⌊(p + r)/2⌋
    MERGE-SORT(A, p, q)
    MERGE-SORT(A, q + 1, r)
    MERGE(A, p, q, r)
```

Divide
Conquer
Conquer
Combine

Merge Sort | الگوریتم مرتب‌سازی ادغامی

```
MERGE(A, p, q, r)
n1 = q − p + 1
n2 = r − q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i − 1]
for j = 1 to n2
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```



Merge Sort | الگوریتم مرتب‌سازی ادغامی

MERGE(A, p, q, r)

```
n₁ = q − p + 1
n₂ = r − q
let L[1 .. n₁ + 1] and R[1 .. n₂ + 1] be new arrays
for i = 1 to n₁
        L[i] = A[p + i − 1]
for j = 1 to n₂
        R[j] = A[q + j]
L[n₁ + 1] = ∞
R[n₂ + 1] = ∞
i = 1
j = 1
for k = p to r
        if L[i] ≤ R[j]
                A[k] = L[i]
                i = i + 1
        else A[k] = R[j]
                j = j + 1
```

Loop Invariant

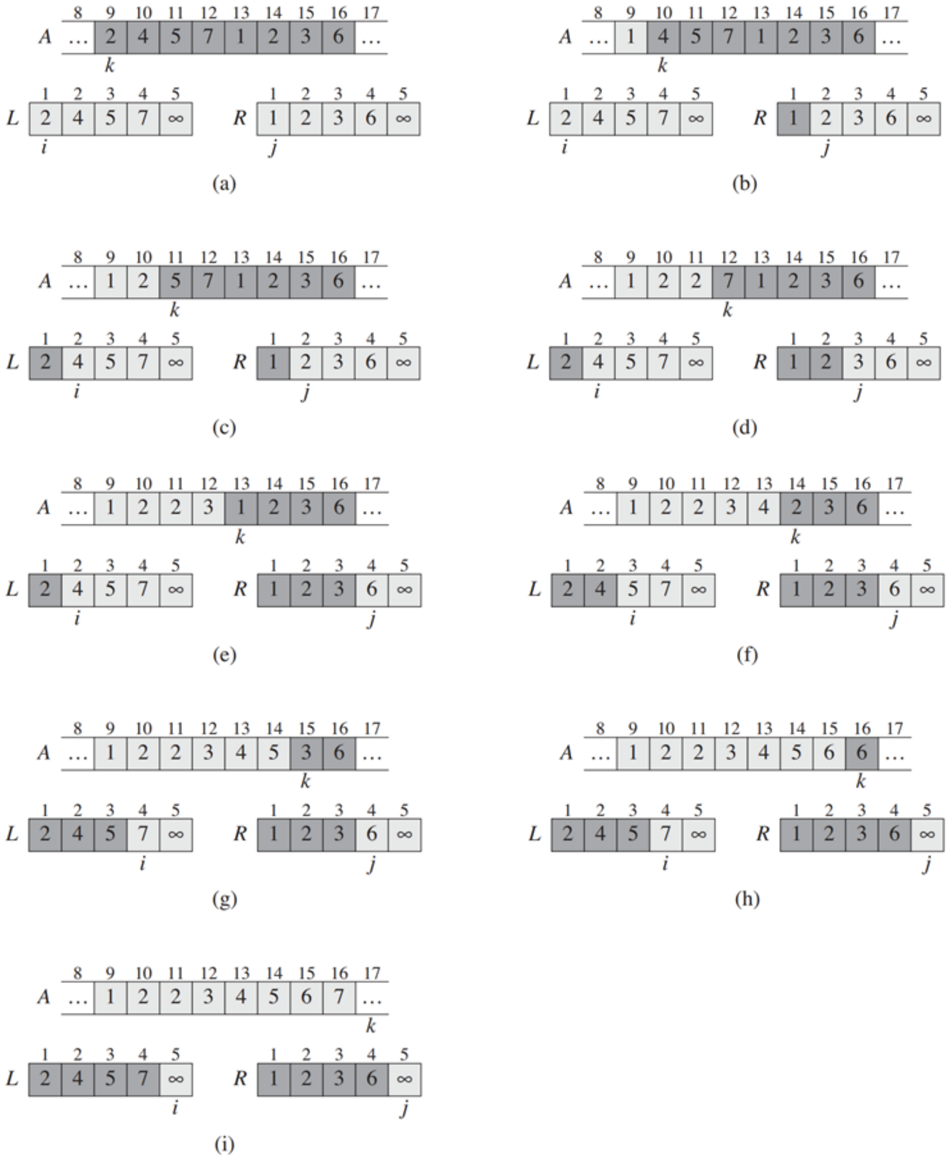
$A[p .. k-1]$

$L[i]$

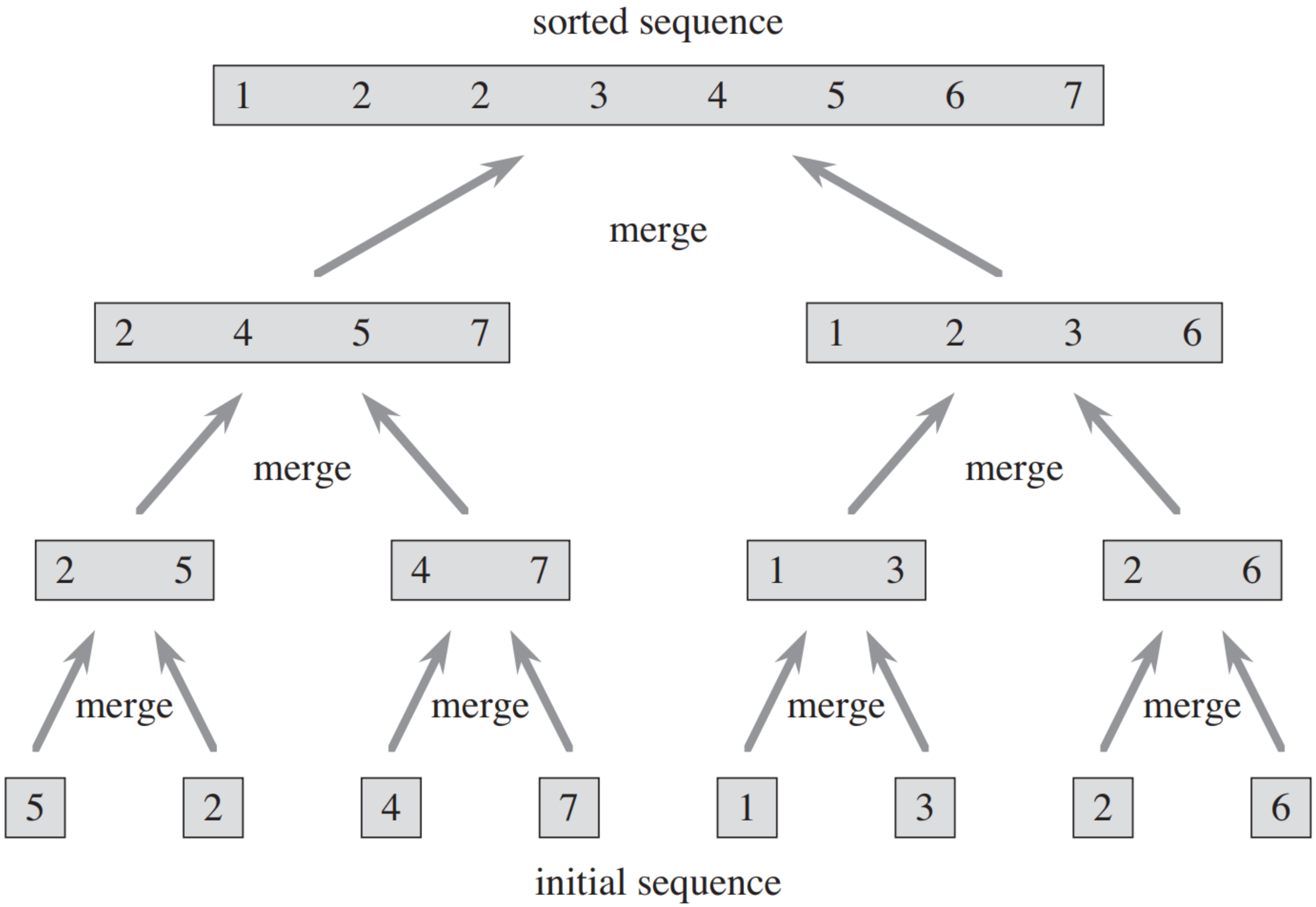
$R[j]$

$L[1 .. n_1+1]$

$R[1 .. n_2+1]$



مثال مرتب‌سازی ادغامی



تحلیل زمانی الگوریتم‌های تقسیم و حل

- **Divide:** D(n) = Θ(1).
- **Conquer:** solve two subproblems, each of size n/2, which contributes 2T (n/2) to the running time.
- **Combine:** the MERGE procedure on an n-element subarray takes time Θ(n), so C(n) = Θ(n).

$$T(n) = \begin{cases} \theta(1) & \text{if } n = 1 \\ 2\,T(n/2) + \theta(n) & \text{if } n > 1 \end{cases}$$

Merge(A,p,q,r)

Θ(1)
```
n1 = q − p + 1
n2 = r − q
```
Θ(n)
```
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i − 1]
for j = 1 to n2
    R[j] = A[q + j]
```
Θ(1)
```
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
```
Θ(n)
```
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

MERGE-SORT(A,p,r)

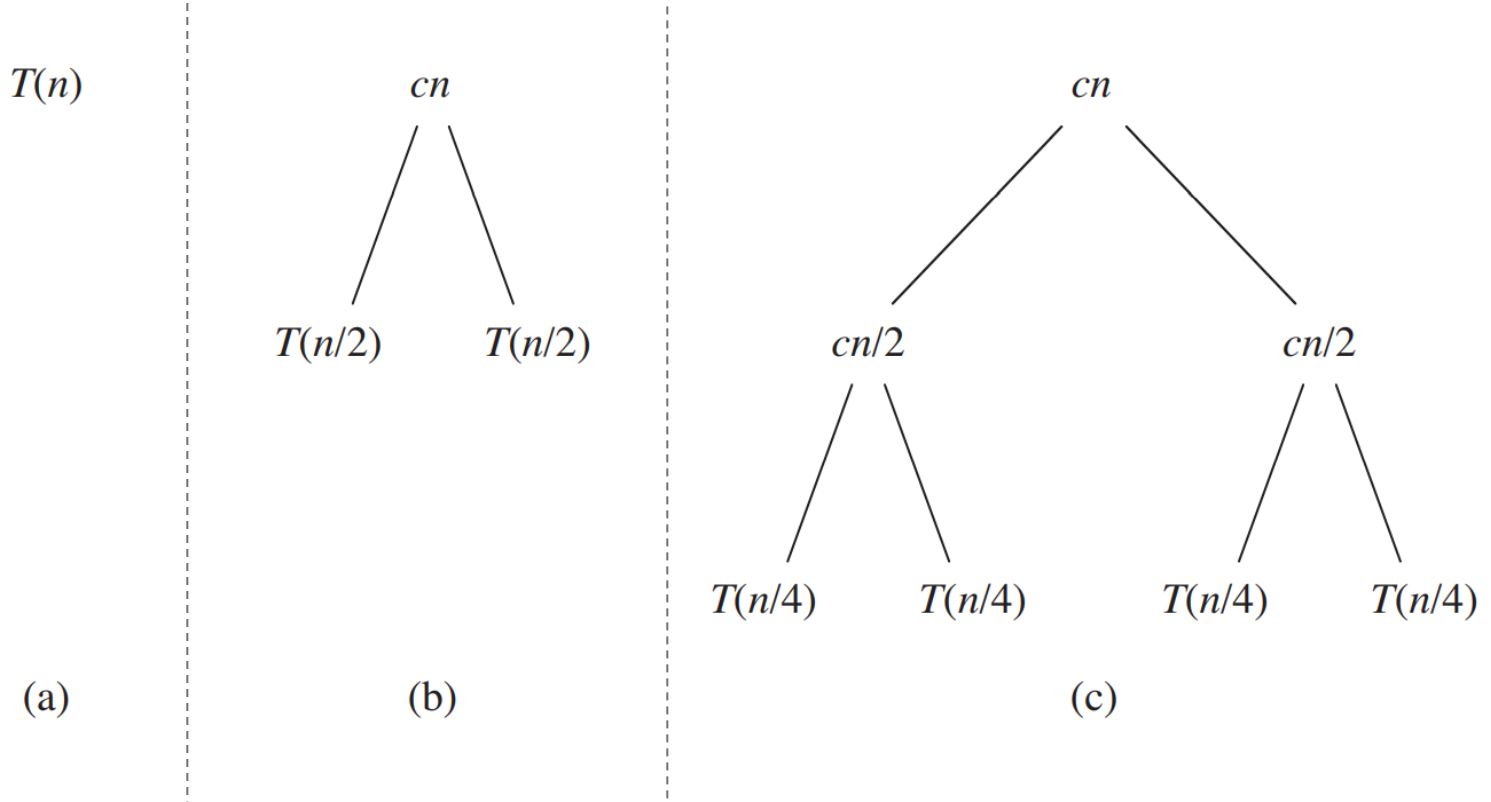
```
if p < r
    q = ⌊(p + r)/2⌋              Θ(1)
    MERGE-SORT(A, p, q)          T(n/2)
    MERGE-SORT(A, q + 1, r)      T(n/2)
    MERGE(A, p, q, r)            Θ(n)
```

تحلیل زمانی با درخت بازگشتی

$$T(n) = \begin{cases} \theta(1) & \text{if } n = 1 \\ 2\,T(n/2) + \theta(n) & \text{if } n > 1 \end{cases}$$

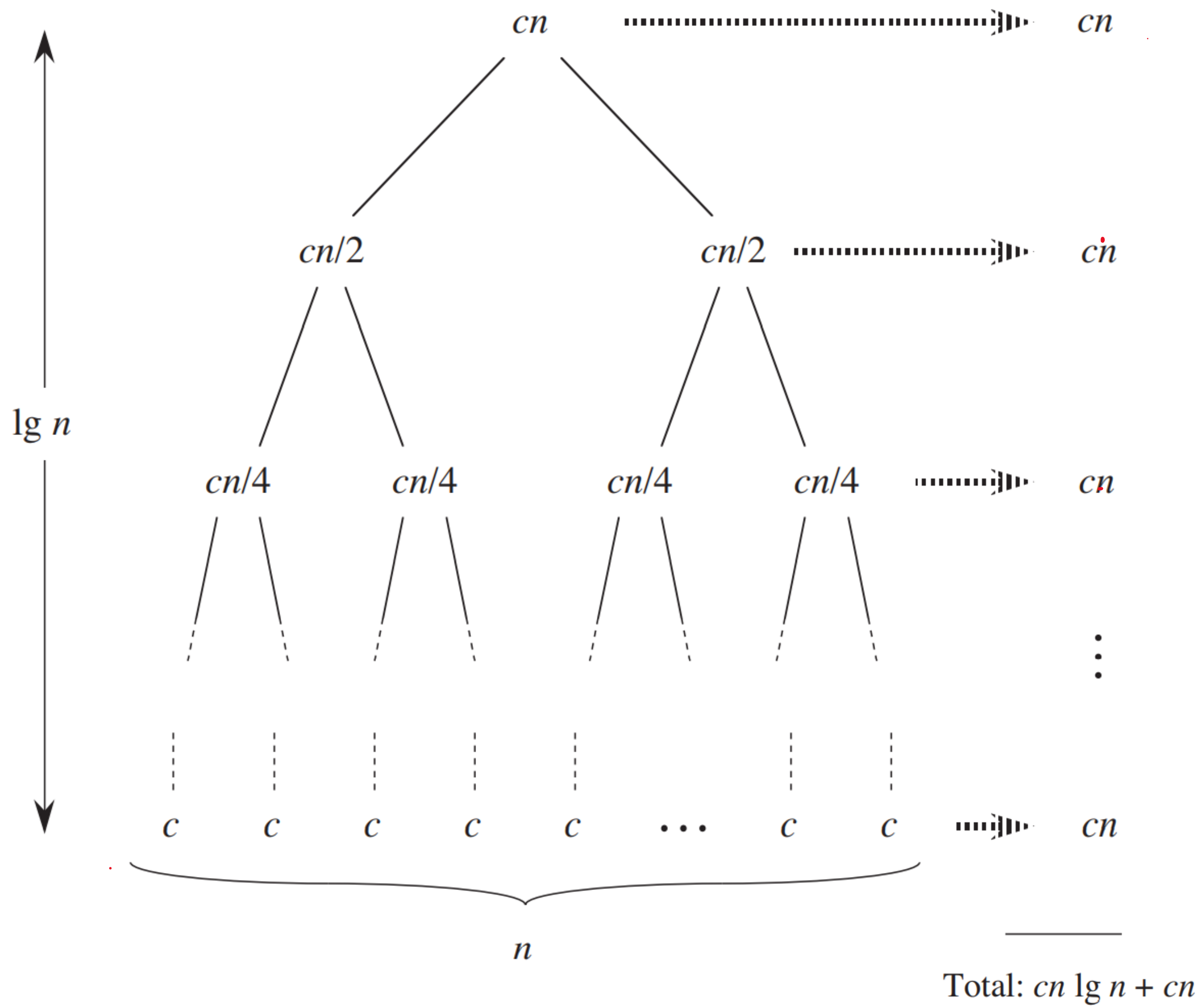
$\Rightarrow$

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2\,T(n/2) + cn & \text{if } n > 1 \end{cases}$$



تحلیل زمانی با درخت بازگشتی

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2\,T(n/2) + cn & \text{if } n > 1 \end{cases}$$



تمرین: تحلیل زمانی روش تقسیم و حل

Use mathematical induction to show that when n is an exact power of 2, the solution of the recurrence

$$T(n) = \begin{cases} 2 & \text{if } n = 2\,, \\ 2T(n/2) + n & \text{if } n = 2^k, \text{ for } k > 1 \end{cases}$$

is $T(n) = n \lg n$.

راه حل تمرین

$$T(n) = \begin{cases} 2 & \text{if } n = 2\,, \\ 2T(n/2) + n & \text{if } n = 2^k, \text{ for } k > 1 \end{cases} \longrightarrow T(n) = n \lg n$$

The base case is when $n = 2$, and we have $n \lg n = 2 \lg 2 = 2 \cdot 1 = 2$.

For the inductive step, our inductive hypothesis is that $T(n/2) = (n/2)\lg(n/2)$. Then

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2(n/2)\lg(n/2) + n \\ &= n(\lg n - 1) + n \\ &= n \lg n - n + n \\ &= n \lg n\,, \end{aligned}$$

which completes the inductive proof for exact powers of 2.